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On Optimal Structural Remodeling[1]

N. OLHOFF[2] AND J. E. TAYLOR[3]

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Abstract. We present the problem of remodeling a given structure such as to improve structural performance optimally within a specified available resource. The development pertains to all types of problems where the mode of structural response is governed by an extremum principle. A variational formulation is used, and the idea is illustrated for maximum-stiffness remodeling of single-purpose structures.

Key Words. Optimal remodeling, structural optimization, calculus of variations, optimality conditions, structural mechanics.

1. Introduction

The demonstration in Refs. 1–3 that fundamental variational principles of structural analysis provide an excellent basis for the mathematical formulation of various optimal design problems has greatly contributed to the development of the field of optimization; see Ref. 4. The impact ranges from the derivation of specific optimality criteria to the establishment of basic optimality theorems; see, for example, Ref. 5.

Variational principles of structural analysis also constitute the basis for the problem of optimal structural remodeling to be discussed in the current paper. The label *structural remodeling* identifies the sort of design problem where the objective is to predict an appropriate modification to a given initial design. Optimal remodeling refers to the best form of modification within an allotted resource.

This type of problem seems to be somewhat broader in formulation than conventional optimal design problems. For example, the initial design

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[1] The work of the second author was supported in part by the National Science Foundation.

[2] Associate Professor, Department of Solid Mechanics, The Technical University of Denmark, Lyngby, Denmark. Summer 1977 Visiting Research Associate, Department of Applied Mechanics and Engineering Science, University of Michigan, Ann Arbor, Michigan.

[3] Professor, Departments of Aerospace Engineering, and Applied Mechanics and Engineering Science, University of Michigan, Ann Arbor, Michigan.

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may be regarded as a wide concept. It may be an existing structure to be improved by remodeling, but it may as well be the design skeleton exclusively expressing geometric minimum constraints for a constrained optimization problem, or (for example) the design obtained at a certain stage of an incremental or iterative procedure of optimization.

Our development is written in terms of maximum-stiffness (minimum-compliance) remodeling of one-dimensional or two-dimensional elastic continuum structures. However, the results are available by similar means for discrete as well as continuum structural forms and for various design constraints, e.g., Euler stability, plastic collapse, vibration frequency, or stationary creep.

The latter sort of generalization depends only on the availability of an extremum principle cover for the respective mode of structural response. One may expect that additional considerations such as discontinuous design variable (Refs. 6, 7), self weight (Ref. 8), multiple eigenvalues (Ref. 9), or nonconservative loads (Ref. 10) may be accommodated in an optimal remodeling formulation without difficulty.

The paper consists of two parts. In Section 2, considering remodeling through reinforcement only, we derive the necessary conditions governing the optimal distribution of reinforcement that, within a specified amount of available material, minimizes the structural compliance for a specified load.

In Section 3, the optimal remodeling idea is extended to the case where simultaneous removal and addition of material may take place in separate, unspecified subregions of the structure. For this optimal compound remodeling, a total resource (cost) is specified, which is based on different unit cost factors for removal and addition of material; such factors have been used earlier in the context of other structural optimization problems (e.g., see Refs. 11–13).

The necessary conditions derived for both optimal remodeling formulations presented are shown to be also sufficient conditions for global optimal remodel design, if the structural stiffness depends linearly on the design variable.

2. Formulation for Reinforcement Only

We consider linearly elastic continuum systems with the property that the stiffness $S(x)$ varies with some power $n \geq 1$ of the design variable $D(x)$. Structural response for design $D(x)$ is associated through the familiar extremum principle for such systems with the functional:

$$\Pi = \int_R [S(D)\eta - p\mathbf{u}]\, dx. \tag{1}$$

Here, $\eta[\mathbf{u}(x)]$ represents specific energy,[4] and the integral and its argument quantities are to be interpreted in form according to the particular type of one-dimensional, or two-dimensional systems.

For the optimal remodeling problem allowing only reinforcement, the starting design, say $D_0(x)$, is regarded to be inviolable. In other words, the net design

$$D(x) = D_0(x) + D_r(x)$$

must satisfy

$$D(x) \geq D_0(x),$$

whereby the remodeling is constrained to meet

$$D_r(x) \geq 0. \tag{2}$$

Note that, as a result of this constraint, the degree of statical redundancy in the remodeled design $D_0 + D_r$ will not be less than the degree in the initial design D_0. Constraint equation (2) is interpreted using the slack function $\sigma(x)$:

$$D_r(x) - \sigma^2(x) = 0. \tag{3}$$

The optimal remodeling design problem is stated in the following form: Within a specified amount

$$\int_R D_r \, dx = V$$

of material available for remodeling, determine the distribution D_r that minimizes compliance for a specified load. The optimal remodel and associated response are identified with stationarity of the functional

$$F = \Pi - \Lambda\left[\int_R D_r \, dx - V\right] + \int_R \lambda [D_r - \sigma^2] \, dx. \tag{4}$$

Therefore, the optimal solution must satisfy

$$\delta_u F \equiv \delta_u \Pi = 0, \qquad x \in R, \tag{5-1}$$

$$[\partial S(x)/\partial D_r(x)]\eta(x) - \Lambda + \lambda(x) = 0, \qquad x \in R, \tag{5-2}$$

$$\lambda(x)\sigma(x) = 0, \qquad x \in R, \tag{5-3}$$

$$\int_R D_r \, dx - V = 0, \tag{5-4}$$

[4] Specific energy identifies energy per unit volume.

as well as the constraint equation (3). Solutions of the switching equation (5-3) are associated with regions of remodel ($\sigma \neq 0$, $\lambda = 0$) and of unaltered design ($\sigma = 0$, $\lambda \geq 0$). Weierstrass–Erdmann corner conditions provide additional equations for the boundaries of these regions.

It is to be verified in what follows that, for the case $n = 1$, the additional requirement

$$\lambda \geq 0, \tag{6}$$

together with Eqs. (5) and (3) are necessary and sufficient to predict the optimal remodel design. Taking $(\hat{D}, \hat{u})$ to label the remodel design and associated response which satisfy Eqs. (5) and (3), minimum compliance is expressed in terms of total strain energies as

$$U(D, u) - U(\hat{D}, \hat{u}) \geq 0. \tag{7}$$

Comparison designs D are arbitrary within the boundary-value problem (5-1) and constraint (5-4).

To facilitate the proof,[5] Eq. (7) is reduced as follows. By the governing minimum principle, since response $\hat{u}$ is admissible for structure D, we can write

$$\Pi(D, \hat{u}) - \Pi(D, u) \geq 0.$$

To simplify this expression, expand the first term and note that

$$\Pi(D, u) = -U(D, u).$$

Thus, the semi-inequality becomes

$$U(D, \hat{u}) - \int_R p\hat{u}\, dx + U(D, u) \geq 0.$$

Also,

$$\int_R p\hat{u}\, dx = 2U(\hat{D}, \hat{u}),$$

whereby we obtain

$$U(D, u) - U(\hat{D}, \hat{u}) \geq U(\hat{D}, \hat{u}) - U(D, \hat{u}). \tag{8}$$

Thus, Eq. (7) will be met if it can be proved that

$$U(\hat{D}, \hat{u}) - U(D, \hat{u}) \geq 0. \tag{9}$$

Applying Eqs. (3) and (5-3), Eq. (5-2) may be interpreted as[6]

$$\hat{\eta} = \eta(\hat{u}) = \begin{cases} \Lambda, & \text{for } x \in R_{\hat{r}}, \\ \Lambda - \lambda, & \text{for } x \in R_{\hat{0}}. \end{cases} \tag{10}$$

[5] A similar form of sufficiency argument is presented in a different context by Mróz in Ref. 14.
[6] For brevity the S–D proportionality factor is set equal to unity.

Here,

$$R = R_{\hat{r}} \cup R_{\hat{0}},$$

with $R_{\hat{r}}$ and $R_{\hat{0}}$ representing respectively the unions of subregions in which the remodel function $\hat{D}_r \neq 0$ and $\hat{D}_r = 0$ (at intersection points or curves both $\sigma = 0$ and $\lambda = 0$). R_r and R_0 have similar meaning for the comparison remodel design D_r. Taking $R_{0\hat{0}}$, $R_{0\hat{r}}$, $R_{r\hat{0}}$, $R_{r\hat{r}}$ to represent regions of overlap between R_0 and $R_{\hat{0}}$, R_0 and $R_{\hat{r}}$, R_r and $R_{\hat{0}}$, R_r and $R_{\hat{r}}$, respectively, Eq. (9) is written specifically in the form

$$\int_{R_{0\hat{0}}} \hat{D}\hat{\eta}\, dx + \int_{R_{0\hat{r}}} \hat{D}\hat{\eta}\, dx + \int_{R_{r\hat{0}}} \hat{D}\hat{\eta}\, dx + \int_{R_{r\hat{r}}} \hat{D}\hat{\eta}\, dx$$
$$-\left[\int_{R_{0\hat{0}}} D\hat{\eta}\, dx + \int_{R_{0\hat{r}}} D\hat{\eta}\, dx + \int_{R_{r\hat{0}}} D\hat{\eta}\, dx + \int_{R_{r\hat{r}}} D\hat{\eta}\, dx\right] \geq 0, \quad (11)$$

where the relation $S \propto D$ has been used.

By substitution for D and $\hat{D}$ in terms of D_0, $\hat{D}_r$, D_r, and by application of Eq. (10), Eq. (11) is reduced to

$$\int_{(R_{0\hat{r}}+R_{r\hat{r}})} \Lambda\hat{D}_r\, dx - \left[\int_{(R_{r\hat{0}}+R_{r\hat{r}})} \Lambda D_r\, dx - \int_{R_{r\hat{0}}} \lambda D_r\, dx\right] \geq 0.$$

The first two terms are cancelled using constraint Eq. (5-4), whereby

$$\int_{R_{r\hat{0}}} \lambda D_r\, dx \geq 0. \quad (12)$$

Since $D_r \geq 0$ for $x \in R_{r\hat{0}}$, $\lambda \geq 0$ is sufficient to have Eq. (12) satisfied. Since D_r is arbitrary, the condition $\lambda \geq 0$ is necessary for (12) as well. This follows, since it would always be possible to find a remodel D_r that is zero everywhere in $R_{r\hat{0}}$, except where $\lambda < 0$, thereby violating Eq. (12). We note that the (exceptional) possibility $\lambda \equiv 0$ in $R_{r\hat{0}}$ corresponds to the case where the given design D_0 is already optimal in that region. We note as well that, for $n > 1$, the necessity and sufficiency of Eqs. (3), (5), (6) can be developed in similar form based on variation δD_r, to demonstrate local optimality of the remodel function $\hat{D}_r$.

The proof establishes uniqueness of the remodel design, for resource of available material $V \geq 0$. We note that, for some value (say, $V = V^*$), the remodeling may transform the starting structure to a configuration corresponding to unconstrained optimal design of the entire structure. The value V^* depends on initial design $D_0(x)$. For increment

$$\Delta V = V - V^* > 0,$$

the remodeling reflects a uniform stiffening over the structure, i.e., the remodeled structure is similar to the initial form.

We observe that the maximum-stiffness remodel design for statically determinate (zero redundancy) systems is at the same time the *strongest design*, where the strongest structure is defined as the design of given volume of material that will carry maximum load without violating an upper bound on stress. The bound on (octahedral shear) stress is expressed by

$$\eta \leq \bar{\eta},$$

$\bar{\eta}$ specified. This result is verified simply by noting that the shifting of material from within region $R_{\hat{r}}$ necessary to obtain any design $D_r \neq \hat{D}_r$ results in

$$\eta > (\max \eta) = \Lambda.$$

Example 2.1. For a straight bar with axial force distribution $p(x)$ and deflection $u(x)$, we have

$$\eta = \tfrac{1}{2}u'^2,$$

and (5-1) is identified with the equilibrium equation

$$(ED(x)u')' + p(x) = 0.$$

Taking

$$D_0(x) = A[2 + \cos(5\pi x/2L)],$$

see Fig. 1a, and assuming the bar to be subjected to tensile forces

$$P = 2EA$$

at $x = 0$ and $x = L$, the results of remodeling with

$$V_1 = 0.08AL \qquad \text{and} \qquad V_2 = 0.8AL,$$

respectively, are illustrated by the D_r curves in Figs. 1b and 1c. The solid curve in Fig. 2 indicates the specific energy distribution $\frac{1}{2}u'^2$ before remodeling, while the dashed and the dashed–dotted lines illustrate how the specific energy is decreased to a constant level (given by Λ) in remodeled (sub-) regions. The example structure is taken to be statically determinate; therefore, in nonremodeled subregions, the specific energy is unaltered.

3. Compound Remodeling

A problem of somewhat different form arises if the remodeling should provide for simultaneous removal and addition of material. An interpretation is given first for the case where the amounts of added and removed

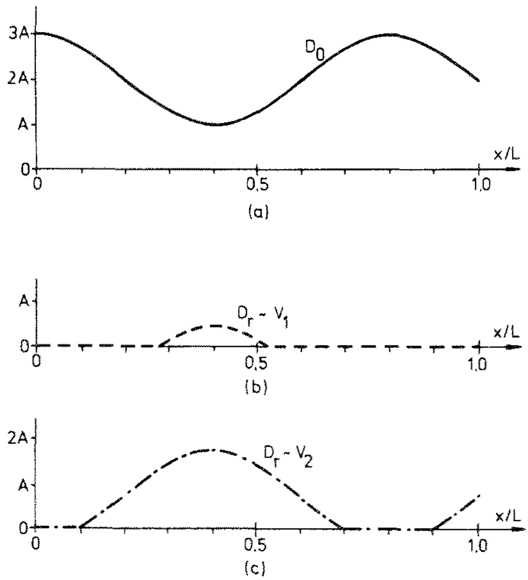

Fig.1. Original design D_0 and optimal reinforcement distributions D_r corresponding to $V_1 = 0.08AL$ and $V_2 = 0.8AL$ in Example 2.1.

material are constrained to be equal. The means to compare this form of modification with the remodeling of Section 2 is established through the introduction of a relative unit cost index; see Refs. 11–13. With the use of this factor, it becomes possible as well to cover the more extensive problem comprised of a sequence of a first modification based on shifting of material as just described (internal remodeling), followed by a stage of stiffening through the addition of virgin material. This combined modification is termed compound remodeling.

Symbols D_{sw} and D_w are designated to represent components of remodeling corresponding respectively to stiffening and weakening of original design D_0. The functions are constrained according to

$$\int_R (D_{sw} - D_w)\, dx = 0, \tag{13}$$

while the net cost of stiffening by this means is expressed as

$$r \int_R D_{sw}\, dx = C_{sw}. \tag{14}$$

The aforementioned relative unit cost index r is introduced here to facilitate subsequent interpretation.

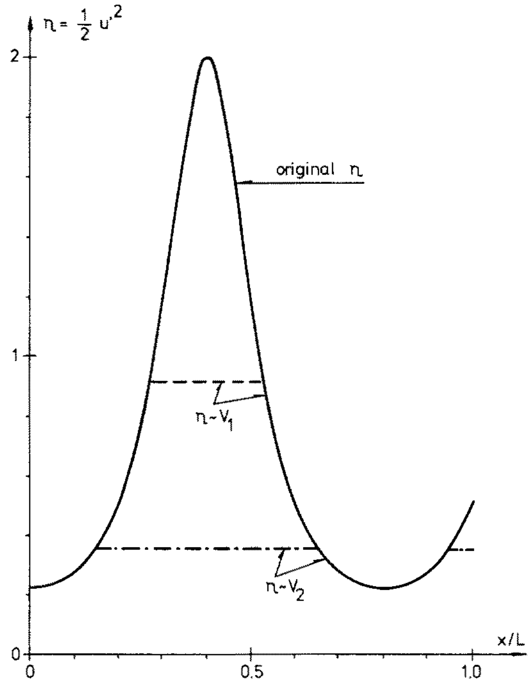

Fig.2. Specific energy functions η for original and remodeled structures in Example 2.1.

Restricting the remodel functions to nonnegative value as before, the optimal internal remodeling is governed by the functional

$$G=\int_R [S(D_0+D_{sw}-D_w)\eta-pu]\,dx-\Gamma\left[r\int_R D_{sw}\,dx-C_{sw}\right]$$

$$-K\left[\int_R (D_{sw}-D_w)\,dx\right]+\int_R [\gamma(D_{sw}-\tau^2)+k(D_w-w^2)]\,dx. \quad (15)$$

Necessary conditions comparable to Eqs. (3), (5) take the form:

$$\delta_u G=0, \quad (16\text{-}1)$$

$$(\partial S/\partial D_{sw})\eta(x)=r\Gamma+K-\gamma(x), \quad (16\text{-}2)$$

$$(\partial S/\partial D_w)\eta(x)=-K-k(x), \quad (16\text{-}3)$$

$$\gamma(x)\tau(x)=0, \quad (16\text{-}4)$$

$$k(x)w(x)=0, \quad (16\text{-}5)$$

$$D_{sw}(x)-\tau^2(x)=0, \quad (16\text{-}6)$$

$$D_w(x)-w^2(x)=0, \quad (16\text{-}7)$$

along with Eqs. (13)–(14). Appropriate solutions of the switching equations (16-4) and (16-5) predict intervals of strengthening, weakening, or no modification; corner conditions apply as mentioned in Section 2. We note also that, if

$$(\partial S/\partial S_{sw}) = -(\partial S/\partial D_w),$$

Eqs. (16-2) and (16-3) lead to

$$k(x)+\gamma(x) = r\Gamma, \tag{17}$$

i.e., the functions k and γ become dependent.

A sufficiency proof for remodeling covered by functional G is available for the case where stiffness is linear in the design function, i.e., $n = 1$. The argument follows the same lines as the proof given in Section 2. To be brief, we simply cite the result:

$$\gamma(x) \geq 0, \qquad k(x) \geq 0, \tag{18}$$

together with the necessary conditions already identified, comprise necessary and sufficient conditions for the optimal internal remodeling.

The constant K in Eq. (16) represents the level of specific energy in subregions where material has been removed in the remodeled structure. In the reinforced subregions, the value of specific energy is given by $r\Gamma + K$. The qualitative behavior of solutions for the subject internal remodeling [within the mild restriction of Eq. (17)] is indicated in Fig. 3; the curves represent initial and remodeled specific strain energies as they would appear for a statically determinate D_0. Separate subregions of weakening and strengthening are labeled R_w and R_s, respectively. If the extent of remodeling is measured in terms of

$$V = \int_R D_{sw}\, dx = C_{sw}/r,$$

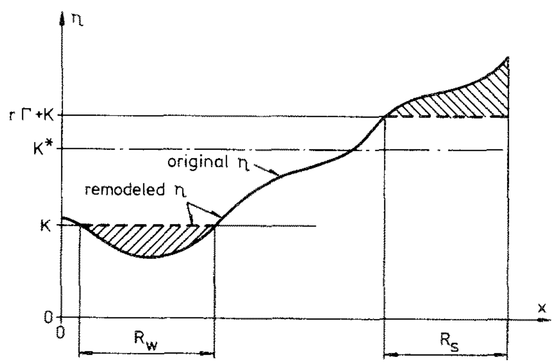

Fig. 3. Specific energy functions η for original and internally remodeled structures.

one might identify a value V^* for internal remodeling corresponding to the V^* defined in Section 2. The value of specific energy corresponding to extent V^* of remodeling is labeled K^* in Fig. 3.

In order to describe compound remodeling, we introduce an expression for the total cost C in the form

$$C_{sv} + C_{sw} = \int_R (D_{sv} + rD_{sw})\, dx = C. \tag{19}$$

Here, C_{sv} and D_{sv} represent respectively the cost and the remodel function for reinforcement only (i.e., virgin material) remodeling. The parameter r measures the relative unit cost of *internal* versus *virgin material* reinforcement. The optimal remodeling will take one form or another, depending on the value of the relative unit cost index r, as follows. For a given initial design D_0, there exists a value, say $\hat{r}$, such that: (i) if $r > \hat{r}$, the optimal remodeling consists of *reinforcement only* modification alone or (ii) if $r < \hat{r}$, the optimal remodeling consists of either *internal* modification alone, or of a sequence comprising *internal* followed by *reinforcement only* remodeling.

The limit value $\hat{r}$ is that value of r which results in equally stiff optimally remodeled structures, for small equal cost increments of remodeling of the two types (i) and (ii). The use of a comparison between equal-cost–equal-merit optimal systems in this context is treated in Ref. 13. We note as well that, for some value of cost, say C^*, the optimal remodeling of D_0 leads to a configuration that corresponds in form to the unconstrained optimal design of the entire structure (for $r < \hat{r}$, C^* is bounded, in contrast to V^* of Section 2). The choice under the option within (ii) depends, for given $C \leq C^*$, on the magnitude of $\hat{r} - r$; for sufficiently large values, the first option results, while $\hat{r} - r$ less than some value leads to the latter form of optimal remodeling.

4. Discussion

This paper demonstrates that structural performance in a deterministic manner can be changed optimally for a specified available resource. The variety of structural types and behavior considered in the references illustrates potential areas for application of the optimal remodeling ideas. In addition to obvious applications for structural elements, the ideas may prove to be useful for large-scale structural systems, where a given set of local improvements may differ greatly from the optimal consumption of a remodel resource.

The particular variational formulation used to express the optimal remodeling problem is an augmented form of the familiar extremum principle from structural analysis. The minimum principle of analysis becomes a max–min problem in the extension for optimal design.

The prediction from remodeling theory for the way that a structure becomes reinforced with ever-increasing resource (quantity of material) is intuitively appealing. We speculate that the analytical support for this result might prove to be useful in the design of solution procedures.

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