Optimal Solution Approximation for Infinite Positive-Definite Quadratic Programming[1]

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Abstract. We consider a general doubly-infinite, positive-definite, quadratic programming problem. We show that the sequence of unique optimal solutions to the natural finite-dimensional subproblems strongly converges to the unique optimal solution. This offers the opportunity to arbitrarily well approximate the infinite-dimensional optimal solution by numerically solving a sufficiently large finite-dimensional version of the problem. We then apply our results to a general time-varying, infinite-horizon, positive-definite, LQ control problem.

Key Words. Time-varying systems, positive-definite costs, infinite-horizon optimization, infinite quadratic programming, solution approximations, LQ control problems.

1. Introduction

Consider a general doubly-infinite quadratic programming problem of the following form:

$$(P) \quad \min \ \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle,$$

$$\text{s.t.} \quad A_{i,i-1} x_{i-1} + A_{ii} x_i = b_i, \quad i = 1, 2, \ldots,$$

$$x_j \in \mathfrak{R}^{n_j}, \quad j = 1, 2, \ldots,$$

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where A_{10} is defined to be 0, Q_j is a symmetric matrix for each j, $b_i \in \Re^{m_i}$ for each i, and $A_{i,i-1}$, A_{ii} are matrices of appropriate sizes for each i. The lower-staircase structure of the constraint system is equivalent to the condition that each constraint contains finitely many variables and each variable appears in at most finitely many constraints. An important special case is the LQ regulator problem (Ref. 1).

As we shall see, we may, without loss of optimality, embed (P) in the infinite-dimensional Hilbert space

$$H = \left\{ (x_j)_{j=1}^\infty : x_j \in \Re^{n_j}, j = 1, 2, \dots ; \sum_{j=1}^\infty \|x_j\|_2^2 < \infty \right\},$$

the real Hilbert sum of the $\Re^{n_j}$. We assume that the linear operator Q, given by

$$Q(x) = (Q_j x_j)_{j=1}^\infty, \qquad x = (x_j)_{j=1}^\infty \in H,$$

is a bounded operator mapping H to itself. Note that Q is a bounded operator if and only if

$$\sup_{1 \le j < \infty} \|Q_j\|_2 < \infty,$$

where $\|\cdot\|_2$ is the matrix norm corresponding to the underlying Euclidean norm, also denoted by $\|\cdot\|_2$. Of course, Q is also self-adjoint since each Q_j is symmetric, $j = 1, 2, \dots$.

Throughout this paper, we will assume that Q is positive definite in the following sense. In conventional operator theory, an operator T is positive definite if $\langle v, T(v) \rangle > 0$, for all v in the underlying Hilbert space K different from zero. Here, Q is positive definite in this sense if and only if Q_j is a positive-definite matrix, for each $j = 1, 2, \dots$. However, we adopt the stronger notion of positive definiteness offered in Refs. 2–3. See also Ref. 4, where the term positive bounded below is used. Thus, we will say that an operator T is positive definite if there exists $\alpha_T > 0$ such that

$$\alpha_T \|v\|^2 \le \langle v, T(v) \rangle, \qquad v \in K.$$

This is a stronger notion, since for example, if K is the space l^2 of square-summable sequences and

$$T(v) = ((1/i)v_i)_{i=1}^\infty, \qquad v \in l^2,$$

then T satisfies the former condition but not the latter. Moreover, it is not difficult to verify that, in general, T is positive definite if and only if there

exists $\alpha > 0$ and a positive-semidefinite operator $R: K \rightarrow K$, i.e.,

$$\langle v, R(v) \rangle \geq 0, \qquad v \in K,$$

such that

$$\langle v, T(v) \rangle = \langle v, R(v) \rangle + \alpha \|v\|^2, \qquad v \in K.$$

In terms of the Q_j, it can be shown that our operator Q is positive definite (in the stronger sense) if and only if each Q_j is positive definite, for $j = 1, 2, \ldots$, and the α_{Q_j} can be chosen so that

$$\inf_{1 \leq j < \infty} \alpha_{Q_j} > 0.$$

We also assume that the feasible region X, where

$$X = \{ x \in H : A_{i,i-1} x_{i-1} + A_{ii} x_i = b_i, i = 1, 2, \ldots \},$$

is nonempty. As we shall see in Section 2, there is no loss of optimality in requiring that $X \subseteq H$, i.e., requiring that

$$\sum_{j=1}^{\infty} \|x_j\|_2^2 < \infty$$

be included among the constraints. Of course, X is closed and affine in H, and for each $x \in H$, we have

$$\langle x, Q(x) \rangle = \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle.$$

For Q positive definite, it is well known that $\langle \cdot, Q(\cdot) \rangle$ defines an inner product $\langle \cdot, \cdot \rangle_Q$ on H, with associated norm $\| \cdot \|_Q$ given by

$$\|x\|_Q^2 = \langle x, Q(x) \rangle, \qquad x \in H.$$

Since

$$\alpha_Q \|x\|^2 \leq \|x\|_Q^2 \leq \|Q\| \, \|x\|^2, \qquad x \in H,$$

it follows that $(H, \langle \cdot, \cdot \rangle_Q)$ is a Hilbert space H_Q, which is the same set as H and which is equivalent to H as a normed space. Thus, (P) may be reformulated as follows:

$$\text{(P)} \quad \min_{x \in X} \|x\|_Q^2.$$

Consequently, an optimal solution to (P) is simply a best approximation in X to the zero element of H, where X is closed, convex, and nonempty. Such is well known to exist and be unique (Ref. 5). We let $x^* = (x_j^*)_{j=1}^{\infty}$ denote the unique solution to problem (P) in H.

Our main objective in this paper is to approximate this optimal solution by the unique optimal solutions to the finite-dimensional subproblems (P_N), $N = 1, 2, \ldots$, obtained by truncating (P) after N vector variables and N vector constraints. This is done in Section 2. In particular, we construct a sequence of optimal solutions to the subproblems (P_N), which when embedded in H, converges to x^*, thus yielding solution convergence. Analogously, the corresponding sequence of optimal objective values converges to the objective value of x^*, thus yielding value convergence. These are the main results of this paper. In Section 3, we apply these results to a general time-varying, infinite-horizon, positive-definite, linear-quadratic control problem. This extends to the time-varying case the work of Lee, Chou, and Barr (Ref. 2), who demonstrated weak convergence of the finite-horizon optimal controls for the time-invariant case, and that of Zabczyk (Ref. 3), who later showed that this convergence is in fact strong. See also Schochetman and Smith (Ref. 6), who demonstrated strong convergence in the presence of a bounded control space.

2. Approximation of Optimal Solutions

Our objective in this section is to approximate the optimal solution x^* of (P) by optimal solutions to finite-dimensional subproblems of (P). We will do the same for the corresponding objective function values.

We first observe however that there is no loss of optimality in requiring that the feasible region X in problem (P) satisfy $X \subseteq H$, as opposed to $X \subseteq \Pi_{j=1}^{\infty} \Re^{n_j}$ more generally. The objective value $C(x)$ for $x \in \Pi_{j=1}^{\infty} \Re^{n_j}$ is given by

$$C(x) = \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle,$$

where $0 \le C(x) \le \infty$. Since Q is positive definite, there exists $\alpha_{Q_j} > 0$, for each j, such that

$$\alpha_{Q_j} \|x_j\|_2^2 \le \langle x_j, Q_j x_j \rangle,$$

$$\inf_{1 \le j < \infty} \alpha_{Q_j} > 0.$$

Suppose that

$$x \in \Pi_{j=1}^{\infty} \Re^{n_j} \quad \text{and} \quad x \notin H,$$

i.e.,

$$\sum_{j=1}^{\infty} \|x_j\|_2^2 = \infty.$$

Then,

$$C(x) = \sum_{j=1}^{\infty} \langle x_j, Q_j x_j \rangle$$

$$\geq \sum_{j=1}^{\infty} \alpha_{Q_j} \|x_j\|_2^2$$

$$\geq \inf_{1 \leq j < \infty} \alpha_{Q_j} \sum_{j=1}^{\infty} \|x_j\|_2^2,$$

which implies that $C(x) = \infty$ for such x. Thus, if Q is positive definite, there is no loss of optimality in requiring that the feasible solutions to (P) be square summable. Moreover, since $X \neq \varnothing$, it follows that the optimal objective value $C^* = C(x^*)$ for (P) satisfies $0 \leq C^* < \infty$, because

$$C(x) = \langle x, Q(x) \rangle, \qquad x \in H.$$

Next, we construct finite-dimensional approximating subproblems of (P) corresponding to finite truncations of (P). Specifically, for each $N = 1, 2, \ldots$, let (P_N) be defined as follows:

$$(P_N) \quad \min \quad \sum_{j=1}^{N} \langle x_j, Q_j x_j \rangle,$$

$$\text{s.t.} \quad A_{i,i-1} x_{i-1} + A_{ii} x_i = b_i, \qquad i = 1, 2, \ldots, N,$$

$$x_j \in \Re^{n_j}, \qquad j = 1, 2, \ldots, N.$$

If we let Φ_N denote the feasible region to (P_N), then Φ_N is a closed, affine subset of $\Pi_{j=1}^{N} \Re^{n_j}$, which is nonempty, since it must contain $(x_1^*, \ldots, x_N^*)$. Moreover, the operator $Q_1 \times \cdots \times Q_N$ is positive definite on $\Pi_{j=1}^{N} \Re^{n_j}$. Consequently, as for (P), there exists a unique solution in Φ_N to (P_N), which we denote by $(\xi_1^N, \ldots, \xi_N^N)$. The optimal value C_N^* for (P_N) is then given by

$$C_N^* = \sum_{j=1}^{N} \langle \xi_j^N, Q_j \xi_j^N \rangle, \qquad N = 1, 2, \ldots.$$

Next, we embed (P_N) in H. Let X_N be the closed, affine space in H given by

$$X_N = \Phi_N \times (\bigoplus_{j=N+1}^{\infty} \Re^{n_j}), \qquad N = 1, 2, \ldots;$$

i.e., X_N consists of arbitrary square-summable extensions in H of the elements of Φ_N. Then,

$$X_{N+1} \subseteq X_N, \qquad N = 1, 2, \ldots,$$

$$X = \bigcap_{N=1}^{\infty} X_N.$$

In particular, $X_N \neq \varnothing$, all N.

Let $Q^N: H \to H$ be defined by

$$Q^N(x) = (Q_1 x_1, \ldots, Q_N x_N, 0, \ldots), \qquad x \in H,$$

so that Q^N is a positive-semidefinite operator on H, for all N. Define (P^N) to be the problem obtained by embedding (P_N) in H, that is,

$$(P^N) \quad \min \quad C_N(x) = \langle x, Q^N(x) \rangle = \sum_{j=1}^{N} \langle x_j, Q_j x_j \rangle,$$

$$\text{s.t.} \quad x \in X_N.$$

Clearly, each (P^N) has infinitely many optimal solutions X_N^* in X_N given by

$$X_N^* = \{ x \in H : x_j = \xi_j^N, j = 1, 2, \ldots, N; \ x_j \text{ arbitrary}, j > N \}.$$

In particular, we let ξ^N denote the element of X_N^* given by

$$\xi^N = (\xi_1^N, \ldots, \xi_N^N, 0, 0, \ldots),$$

so that $\xi^N \in H$, for all N, and

$$C_N^* = \langle \xi^N, Q^N(\xi^N) \rangle, \qquad N = 1, 2, \ldots.$$

It is important to note that although in general

$$\langle x, Q^N(x) \rangle \leq \langle x, Q(x) \rangle = \|x\|_Q^2, \qquad x \in H,$$

since $\xi_j^N = 0$, for $j > N$, we have that

$$C_N^* = \langle \xi^N, Q^N(\xi^N) \rangle = \langle \xi^N, Q(\xi^N) \rangle = \|\xi^N\|_Q^2, \qquad N = 1, 2, \ldots.$$

Lemma 2.1. The sequence $\{\xi^N\}_{N=1}^{\infty}$ is bounded in H.

Proof. Since $x^* \in X$, it follows that $x^* \in X_N$, for all N. Hence,

$$C_N^* \leq \langle x^*, Q^N(x^*) \rangle, \qquad N = 1, 2, \ldots.$$

But

$$\langle x^*, Q^N(x^*) \rangle = \sum_{j=1}^{N} \langle x_j^*, Q_j x_j^* \rangle$$

$$\leq \sum_{j=1}^{\infty} \langle x_j^*, Q_j x_j^* \rangle$$

$$= \langle x^*, Q(x^*) \rangle, \qquad N = 1, 2, \ldots,$$

so that $\{C_N^*\}_{N=1}^{\infty}$ is as bounded sequence of real numbers.

Now, suppose that $\{\xi^N\}_{N=1}^\infty$ is not bounded in H, i.e., $\{\|\xi^N\|\}_{N=1}^\infty$ is not a bounded sequence of real numbers. Then, $\{\|\xi^N\|_Q\}_{N=1}^\infty$ is not bounded either. But

$$\|\xi^N\|_Q^2 = \langle \xi^N, Q(\xi^N) \rangle = \langle \xi^N, Q^N(\xi^N) \rangle = C_N^*, \qquad N = 1, 2, \ldots,$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following is the main result of this section.

Theorem 2.1. If Q is positive definite in problem (P), then the sequence $\{\xi^N\}_{N=1}^\infty$ converges strongly to x^*. In particular, the sequence $\{\xi_j^N\}_{N=1}^\infty$ converges to x_j^*, for each $j = 1, 2, \ldots$. Moreover, the sequence $\{C_N^*\}_{N=1}^\infty$ converges to C^*.

Proof. Let $\{\xi^{N_k}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{\xi^N\}_{N=1}^\infty$, which is necessarily bounded also. By the Banach–Alaoglu theorem (Ref. 7), we may assume (passing to a subsequence if necessary) that $\{\xi^{N_k}\}_{k=1}^\infty$ converges weakly to some y in H.

The constraints of (P) have been given as a system of vector equations. However, each scalar constraint of (P) is easily seen to be of the form $\langle a, x \rangle = b$, where $x \in H$, $b \in \Re$, and a is of the form $(a_1, 0, \ldots)$, for $a_1 \in \Re^{n_1}$, or of the form $(0, \ldots, 0, a_{i-1}, a_i, 0, \ldots)$, for $a_{i-1} \in \Re^{n_{i-1}}$ and $a_i \in \Re^{n_i}$. Thus, a is in H in either case. Also, $\{\xi^{N_k}\}_{k=1}^\infty$ must eventually satisfy every such scalar constraint. Therefore, $\langle a, \xi^{N_k} \rangle = b$, for k sufficiently large. However, $\langle a, \xi^{N_k} \rangle \to \langle a, y \rangle$, as $k \to \infty$, so that $\langle a, y \rangle = b$. Thus, y must satisfy each constraint of (P), i.e., $y \in X$ necessarily.

Next we show that $\{\xi^{N_k}\}_{k=1}^\infty$ converges strongly to y. By definition of the inner product on H_Q, it follows that $\{\xi^{N_k}\}_{k=1}^\infty$ also converges weakly to y in H_Q. Since $y \in X$, it follows that $y \in X_{N_k}$, for all k, so that

$$C_{N_k}^* \leq \langle y, Q^{N_k}(y) \rangle,$$

i.e.,

$$\begin{aligned}
\|\xi^{N_k}\|_Q^2 &= \langle \xi^{N_k}, Q(\xi^{N_k}) \rangle \\
&= \langle \xi^{N_k}, Q^{N_k}(\xi^{N_k}) \rangle \\
&= C_{N_k}^* \\
&\leq \langle y, Q^{N_k}(y) \rangle \\
&\leq \langle y, Q(y) \rangle \\
&= \|y\|_Q^2, \qquad k = 1, 2, \ldots.
\end{aligned}$$

Thus,

$$\|\xi^{N_k}\|_Q \leq \|y\|_Q, \qquad k=1,2,\ldots.$$

Now, observe that a weakly convergent sequence in a Hilbert space is strongly convergent, provided the norms of the sequence elements are bounded by the norm of the weak limit (Ref. 8, p. 206). Consequently, $\{\xi^{N_k}\}_{k=1}^{\infty}$ converges strongly to y in H_Q, and hence in H.

Next, we show that $y=x^*$. If not, then

$$C(x^*) < C(y),$$

i.e.,

$$\|x^*\|_Q^2 < \|y\|_Q^2,$$

because $y \in X$. Since $\{\xi^{N_k}\}_{N=1}^{\infty}$ converges strongly to y in H, we have that

$$\lim_{k \to \infty} \|\xi^{N_k}\|^2 = \|y\|^2,$$

and hence,

$$\lim_{k \to \infty} \|\xi^{N_k}\|_Q^2 = \|y\|_Q^2.$$

Thus, for large k,

$$\|\xi^{N_k}\|_Q^2 > \|x^*\|_Q^2.$$

But since $x^* \in X_{N_k}$, we have

$$C_{N_k}(\xi^{N_k}) \leq C_{N_k}(x^*),$$

for all k. Therefore,

$$\begin{aligned}
\|\xi^{N_k}\|_Q^2 &= \langle \xi^{N_k}, Q(\xi^{N_k}) \rangle \\
&= \langle \xi^{N_k}, Q^{N_k}(\xi^{N_k}) \rangle \\
&= C_{N_k}(\xi^{N_k}) \\
&\leq C_{N_k}(x^*) \\
&= \langle x^*, Q^{N_k}(x^*) \rangle \\
&\leq \langle x^*, Q(x^*) \rangle \\
&= \|x^*\|_Q^2, \qquad k=1,2,\ldots,
\end{aligned}$$

so that

$$\|\xi^{N_k}\|_Q^2 \le \|x^*\|_Q^2, \qquad \text{all } k,$$

a contradiction. Hence, $y = x^*$ and $\xi^{N_k} \to x^*$ in H, as $k \to \infty$.

We have shown that an arbitrary subsequence of $\{\xi^N\}_{N=1}^\infty$ has a subsequence which converges to x^* in H. Thus, $\{\xi^N\}_{N=1}^\infty$ converges to x^* in H. Consequently,

$$\lim_{N \to \infty} \xi_j^N = x_j^*, \qquad j = 1, 2, \ldots.$$

Finally, since $\xi^N \to x^*$ in H_Q necessarily, we have that

$$C_N^* = \langle \xi^N, Q^N(\xi^N) \rangle$$
$$= \langle \xi^N, Q(\xi^N) \rangle$$
$$= \|\xi^N\|_Q^2$$

converges to $\|x^*\|_Q^2 = C^*$, as $N \to \infty$.   □

To summarize the main result of this section for problem (P), in the case of Q positive definite, (P) admits a unique solution x^* which may be arbitrarily well approximated by solving sufficiently large finite-dimensional versions of (P).

3. Application to Control Theory

Consider the following time-varying, infinite-horizon, linear quadratic control problem:

(C)   min $\displaystyle\sum_{j=1}^\infty [y_j^t S_j y_j + u_{j-1}^t R_{j-1} u_{j-1}]$,

   s.t. $y_i = A_{i-1} y_{i-1} + B_{i-1} u_{i-1} + d_i, \qquad i = 1, 2, \ldots,$

   $y_j \in \Re^n, \qquad u_{j-1} \in \Re^m, \qquad j = 1, 2, \ldots,$

where y_0 is given and each d_i is a known exogenous parameter (e.g., demand in production-inventory models). For the jth period, y_j is the jth state and u_j is the resulting jth control. Note that the cost of the initial state y_0 is constant and may be omitted. This problem is essentially an infinite-horizon version of the familiar LQ regulator problem (Ref. 1).

Problem (C) may, without loss of optimality, be rewritten as follows:

$$(C) \quad \min \quad \sum_{j=1}^{\infty} \left\langle \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix}, \begin{bmatrix} S_j & 0 \\ 0 & R_{j-1} \end{bmatrix} \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right\rangle$$

$$\text{s.t.} \quad [-I, B_0] \begin{bmatrix} y_1 \\ u_0 \end{bmatrix} = A_0 y_0 - d_1, \quad i=1,$$

$$[A_{i-1}, 0] \begin{bmatrix} y_{i-1} \\ u_{i-2} \end{bmatrix} + [-I, B_{i-1}] \begin{bmatrix} y_i \\ u_{i-1} \end{bmatrix}$$

$$= -d_i, \quad i=2, 3, \ldots,$$

$$\begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \in \mathfrak{R}^{n+m}, \quad j=1, 2, \ldots,$$

where, in this case,

$$x_j = \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix}, \quad j=1, 2, \ldots,$$

$$Q_j = \begin{bmatrix} S_j & 0 \\ 0 & R_{j-1} \end{bmatrix}, \quad j=1, 2, \ldots,$$

$$A_{i,i-1} = [A_{i-1}, 0], \quad i=2, 3, \ldots,$$

$$A_{ii} = [-I, B_{i-1}], \quad i=1, 2, \ldots,$$

$$b_1 = A_0 y_0 - d_1, \quad i=1,$$

$$b_i = -d_i, \quad i=2, 3, \ldots,$$

$$H = \left\{ \left(\begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right)_{j=1}^{\infty} : \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \in \mathfrak{R}^{n+m}, \right.$$

$$\left. j=1, 2, \ldots; \sum_{j=1}^{\infty} [\|y_j\|_2^2 + \|u_{j-1}\|_2^2] < \infty \right\},$$

$$Q\left(\left(\begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right)_{j=1}^{\infty} \right) = \left(\begin{bmatrix} S_j & 0 \\ 0 & R_{j-1} \end{bmatrix} \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right)_{j=1}^{\infty}, \left(\begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right)_{j=1}^{\infty} \in H.$$

Lemma 3.1.   If

$$\sup_{1 \leq j < \infty} \|S_j\|_2 < \infty \quad \text{and} \quad \sup_{1 \leq j < \infty} \|R_{j-1}\|_2 < \infty,$$

then Q is a bounded linear operator on H.

Proof.   In this case, $\sup_{1 \leq j < \infty} \|Q_j\|_2$ is bounded by the maximum of $\sup_{1 \leq j < \infty} \|S_j\|_2$ and $\sup_{1 \leq j < \infty} \|R_{j-1}\|_2$.   □

Lemma 3.2.   If the S_j and $R_{j-1}, j = 1, 2, \ldots$, are positive definite and the α_{S_j} and $\alpha_{R_{j-1}}$ can be chosen such that

$$\inf_{1 \leq j < \infty} \alpha_{S_j} > 0 \quad \text{and} \quad \inf_{1 \leq j < \infty} \alpha_{R_{j-1}} > 0,$$

then Q is positive definite on H.

Proof.   In this case, each Q_j is positive definite and $\inf_{1 \leq j < \infty} \alpha_{Q_j}$ is bounded below by the minimum of $\inf_{1 \leq j < \infty} \alpha_{S_j}$ and $\inf_{1 \leq j < \infty} \alpha_{R_{j-1}}$.   □

We assume that the matrices S_j and R_{j-1} are symmetric, so that each matrix Q_j is symmetric, $j = 1, 2, \ldots$. We also assume the hypotheses of Lemmas 3.1 and 3.2, so that Q is a bounded linear operator on H which is positive definite. Finally, we assume that the feasible region X (contained in H) is nonempty.

As in Section 2, problem (C) has a unique solution $x^* = (x_j^*)_{j=1}^{\infty}$ in X for

$$x_j^* = \begin{bmatrix} y_j^* \\ u_{j-1}^* \end{bmatrix}, \qquad j = 1, 2, \ldots,$$

where for each stage j, $y_j^* \in \mathfrak{R}^n$ is the resulting optimal state and $u_{j-1}^* \in \mathfrak{R}^m$ is the optimal control. For convenience, we let

$$y^* = (y_j^*)_{j=1}^{\infty} \quad \text{and} \quad u^* = (u_{j-1}^*)_{j=1}^{\infty}.$$

Our objective here is to approximate the vector y^* of optimal states and the vector u^* of optimal controls by corresponding optimal solutions for finite truncations of (C). We will also approximate the optimal cost,

$$C^* = \sum_{j=1}^{\infty} [(y_j^*)' S_j y_j^* + (u_{j-1}^*)' R_{j-1} u_{j-1}^*],$$

for (C) by the optimal costs to these finite truncations.

To this end, for each $N = 1, 2, \ldots$, let (C_N) denote the finite-dimensional truncation of (C) which is analogous to (P_N), i.e.,

$$(C_N) \quad \min \quad \sum_{j=1}^{N} [y_j' S_j y_j + u_{j-1}' R_{j-1} u_{j-1}],$$

$$\text{s.t.} \quad y_i = A_{i-1} y_{i-1} + B_{i-1} u_{i-1} + d_i, \qquad i = 1, 2, \ldots, N,$$

$$y_j \in \mathfrak{R}^n, \qquad u_{j-1} \in \mathfrak{R}^m, \qquad j = 1, 2, \ldots, N,$$

with y_0 given. As above, (C_N) can be rewritten as

$$(C_N) \quad \min \quad \sum_{j=1}^{N} \left\langle \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix}, \begin{bmatrix} S_j & 0 \\ 0 & R_{j-1} \end{bmatrix} \begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \right\rangle,$$

$$\text{s.t.} \quad [-I, B_0] \begin{bmatrix} y_1 \\ u_0 \end{bmatrix} = A_0 y_0 - d_1, \qquad i = 1,$$

$$[A_{i-1} 0] \begin{bmatrix} y_{i-1} \\ u_{i-2} \end{bmatrix} + [-I, B_{i-1}] \begin{bmatrix} y_i \\ u_{i-1} \end{bmatrix} = -d_i, \qquad i = 2, 3, \ldots, N,$$

$$\begin{bmatrix} y_j \\ u_{j-1} \end{bmatrix} \in \mathfrak{R}^{n+m}, \qquad j = 1, 2, \ldots, N.$$

As in Section 2, for each N, problem (C_N) has a unique solution $(\xi_1^N, \ldots, \xi_N^N)$, where this time

$$\xi_j^N = \begin{bmatrix} \eta_j^N \\ \mu_{j-1}^N \end{bmatrix},$$

for $\eta_j^N \in \mathfrak{R}^n$, the optimal state for (C_N) beginning stage j, and $\mu_{j-1}^N \in \mathfrak{R}^m$, the optimal control for (C_N) at stage j, for $j = 1, 2, \ldots, N$. As in Section 2, we define

$$\xi^N = (\xi_1^N, \ldots, \xi_N^N, 0, \ldots), \qquad N = 1, 2, \ldots.$$

Analogously, we let

$$\eta^N = (\eta_1^N, \ldots, \eta_N^N, 0, \ldots), \qquad N = 1, 2, \ldots,$$

$$\mu^N = (\mu_1^N, \ldots, \mu_N^N, 0, \ldots), \qquad N = 1, 2, \ldots.$$

The corresponding optimal cost C_N^* for (C_N) is then

$$\sum_{j=1}^{N} [(\eta_j^N)' S_j \eta_j^N + (\mu_{j-1}^N)' R_{j-1} \mu_{j-1}^N], \qquad N = 1, 2, \ldots.$$

We next apply the main results of Section 2 to problem (C).

Theorem 3.1. Suppose that, for problem (C), the matrices S_j and R_{j-1} are symmetric and positive definite, for each $j = 1, 2, \ldots$, and satisfy the hypotheses of Lemmas 3.1 and 3.2. Suppose that the feasible region for problem (C) is nonempty. Then, the state sequence $\{\eta^N\}_{N=1}^{\infty}$ converges strongly to the vector y^* of optimal states, i.e.,

$$\sum_{j=1}^{\infty} \| \eta_j^N - y_j^* \|_2^2 \to 0, \qquad N \to \infty,$$

and the control sequence $\{\mu^N\}_{N=1}^{\infty}$ converges strongly to the vector μ^* of optimal controls, i.e.,

$$\sum_{j=1}^{\infty} \| \mu_{j-1}^N - \mu_{j-1}^* \|_2^2 \to 0, \qquad N \to \infty.$$

In particular, for each $j = 1, 2, \ldots$, the sequence $\{\eta_j^N\}_{N=1}^{\infty}$ converges to y_j^* and the sequence $\{\mu_{j-1}^N\}_{N=1}^{\infty}$ converges μ_{j-1}^*. Finally, the cost sequence $\{C_N^*\}$ converges to the optimal cost C^*, i.e.,

$$\sum_{j=1}^{N} [(\eta_j^N)' S_j \eta_j^N + (\mu_{j-1}^N)' R_{j-1} \mu_{j-1}^N]$$

$$\to \sum_{j=1}^{\infty} [(y_j^*)' S_j y_j^* + (u_{j-1}^*)' R_{j-1} \mu_{j-1}^*], \qquad \text{as } N \to \infty.$$

Proof. Apply Theorem 2.2. □

References

1. LEWIS, F. L., *Optimal Control*, Wiley, New York, New York, 1986.
2. LEE, K. Y., CHOW, S., and BARR, R. O., *On the Control of Discrete-Time Distributed Parameter Systems*, SIAM Journal on Control, Vol. 10, pp. 361–376, 1972.
3. ZABCZYK, J., *Remarks on the Control of Discrete-Time Distributed Parameter Systems*, SIAM Journal on Control, Vol. 12, pp. 721–773, 1974.
4. MILNE, R. D., *Applied Functional Analysis*, Pitman, Boston, Massachusetts, 1980.
5. AUBIN, J. P., *Applied Functional Analysis*, Wiley, New York, New York, 1979.

6. SCHOCHETMAN, I. E., and SMITH, R. L., *Solution Approximation in Infinite-Horizon Linear Quadratic Control*, IEEE Transactions on Automatic Control, Vol. 39, pp. 596–601, 1994.
7. DUNFORD, N., and SCHWARTZ, J. T., *Linear Operators, Vol. 1,* Wiley-Interscience, New York, New York, 1964.
8. KOLMOGOROV, A. N., and FOMIN, S. V., *Introductory Real Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1970.