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TECHNICAL NOTE

Convergence of the Steepest Descent Method for Minimizing Quasiconvex Functions[1,2]

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Abstract. To minimize a continuously differentiable quasiconvex function $f: \mathbb{R}^n \to \mathbb{R}$, Armijo's steepest descent method generates a sequence $x^{k+1} = x^k - t_k \nabla f(x^k)$, where $t_k > 0$. We establish strong convergence properties of this classic method: either $x^k \to \bar{x}$, s.t. $\nabla f(\bar{x}) = 0$; or $\arg\min f = \emptyset$, $\|x^k\| \to \infty$, and $f(x^k) \downarrow \inf f$. We also discuss extensions to other line searches.

Key Words. Steepest descent methods, convex programming, Armijo's line search.

1. Introduction

To minimize a continuously differentiable quasiconvex function $f: \mathbb{R}^n \to \mathbb{R}$, Cauchy's steepest descent method (Ref. 1) with Armijo's stepsizes (Ref. 2) generates a sequence $\{x^k\}$ via

$$x^{k+1} = x^k - t_k g^k, \quad g^k = \nabla f(x^k), \qquad k = 0, 1, \ldots, \tag{1}$$

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where

$$t_k = \arg\max\{t : f(x^k - tg^k) \le f(x^k) - \alpha t\|g^k\|^2,\ t = 2^{-i},\ i = 0, 1, \ldots\}, \tag{2}$$

with $\alpha \in (0, 1)$. We prove in Section 2 the following strong convergence result.

Theorem 1.1. Global Convergence. Either $x^k \to \bar{x} \in X := \{x : \nabla f(x) = 0\}$, or $\check{X} := \arg\min f = \emptyset$, $\|x^k\| \to \infty$, and $f(x^k) \downarrow \inf f$.

A closely related result appeared in Ref. 3 after an earlier version of this note was accepted. The present version provides a considerably simpler convergence proof that permits generalization to the quasiconvex case. Other related results for nondifferentiable optimization methods are given in Ref. 4 and Ref. 5, Remark 3.2. These relations and extensions are discussed in Section 3.

2. Global Convergence of Steepest Descent

We make the following standing assumption that generalizes Armijo's condition (2).

Assumption 2.1. Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that:

(A1) $\exists \alpha \in (0, 1),\ \tau_\alpha > 0,\ \forall t \in (0, \tau_\alpha]$: $\phi(t) \le \alpha t$,

(A2) $\exists \beta > 0,\ \tau_\beta \in (0, \infty],\ \forall t \in (0, \tau_\beta] \cap \mathbb{R}$: $\phi(t) \ge \beta t^2$,

(A3) $\forall k, f(x^{k+1}) \le f(x^k) - \phi(t_k)\|g^k\|^2$ and $0 < t_k \le \tau_\beta$ in (1),

(A4) $\exists \gamma > 1,\ \tau_\gamma > 0,\ \forall k$: $t_k \ge \tau_\gamma$ or
$[\exists \tilde{t}_k \in [t_k, \gamma t_k] : f(x^k - \tilde{t}_k g^k) \ge f(x^k) - \phi(\tilde{t}_k)\|g^k\|^2]$.

Note that (2) corresponds to

$$\phi(t) = \alpha t, \qquad \beta = \alpha, \qquad \gamma = 2, \qquad \tau_\alpha = \tau_\beta = \tau_\gamma = 1.$$

As in Ref. 4, we start by considering the condition

$$f(x^k) \ge f(\tilde{x}), \qquad \text{for some fixed } \tilde{x} \text{ and all } k, \tag{3}$$

which holds if $\check{X} \ne \emptyset$ or $\tilde{x}$ is a cluster point of $\{x^k\}$.

Lemma 2.1. If (3) holds, then

$$\sum_{k=0}^{\infty} t_k^2 \|g^k\|^2 \le [f(x^0) - f(\tilde{x})]/\beta. \tag{4}$$

Moreover, $x^k \to \bar{x}$ for some $\bar{x}$.

Proof. By (A2)–(A3),

$$\beta t_k^2 \|g^k\|^2 \le \phi(t_k)\|g^k\|^2 \le f(x^k) - f(x^{k+1});$$

adding these inequalities yields (4). Next, since $\langle g^k, \tilde{x} - x^k \rangle \le 0$ by (3) and quasiconvexity of f [Ref. 6, Theorem 9.1.4], and since $x^k - x^{k+1} = t_k g^k$, we deduce that

$$\begin{aligned}\|\tilde{x} - x^{k+1}\|^2 &= \|\tilde{x} - x^k\|^2 + 2\langle \tilde{x} - x^k, x^k - x^{k+1}\rangle + \|x^{k+1} - x^k\|^2 \\ &\le \|\tilde{x} - x^k\|^2 + t_k^2 \|g^k\|^2,\end{aligned}$$

so that

$$\|\tilde{x} - x^l\|^2 \le \|\tilde{x} - x^k\|^2 + \sum_{j=k}^{\infty} t_j^2 \|g^j\|^2 < \infty,$$

if $l > k$. Hence, $\{x^k\}$ is bounded and has a cluster point $\bar{x}$, so we may set $\tilde{x} = \bar{x}$ above to deduce from (4) for any $\epsilon > 0$ the existence of k such that $\|\bar{x} - x^k\|^2 \le \epsilon/2$ and

$$\sum_{j=k}^{\infty} t_j^2 \|g^j\|^2 \le \epsilon/2;$$

thus, $\|\bar{x} - x^l\|^2 \le \epsilon$ for all $l > k$, i.e., $x^k \to \bar{x}$. □

Lemma 2.2. If $\bar{x}$ is a cluster point of $\{x^k\}$, then $\bar{x} \in X$, i.e., $\nabla f(\bar{x}) = 0$.

Proof. Suppose that $x^k \xrightarrow{K} \bar{x}$, but $\bar{g} := \nabla f(\bar{x}) \neq 0$. Then, $t_k \xrightarrow{K} 0$ from [cf. (A2)–(A3)]

$$0 \le \beta t_k^2 \|g^k\|^2 \le f(x^k) - f(x^{k+1}) \xrightarrow{K} 0,$$

with $g^k \xrightarrow{K} \bar{g} \neq 0$ and $f(x^k) \downarrow f(\bar{x})$ by continuity. Thus, for all large $k \in K$,

$$f(x^k - \tilde{t}_k g^k) - f(x^k) \ge -\phi(\tilde{t}_k)\|g^k\|^2 \ge -\alpha \tilde{t}_k \|g^k\|^2, \tag{5}$$

by (A4) and (A1), where the left side equals $-\tilde{t}_k \langle g^k, \nabla f(x^k - \breve{t}_k g^k)\rangle$ for some $\breve{t}_k \in [0, \tilde{t}_k]$ by the mean-value theorem, and by (A4), $0 \le \tilde{t}_k \le \gamma t_k \xrightarrow{K} 0$. Hence, dividing (5) by $\tilde{t}_k$ and letting $k \xrightarrow{K} \infty$ yields $-\|\bar{g}\|^2 \ge -\alpha \|\bar{g}\|^2$, a contradiction with $\alpha < 1$ [cf. (A1)]. □

We can now prove Theorem 1.1 under Assumption 2.1 that generalizes (2).

Proof of Theorem 1.1. If (3) holds, e.g., $\check{X} \neq \emptyset$ or $\{x^k\}$ has a cluster point, then the preceding results yield $x^k \to \bar{x} \in X$. If $\|x^k\| \not\to \infty$, then $\{x^k\}$ has a cluster point. If $\lim_{k\to\infty} f(x^k) > \inf f$, then (3) holds. □

3. Discussion of Other Line Searches

First, suppost that $\alpha \in (\frac{1}{2}, 1)$ and f is convex. Then, the proof of Lemma 2.2 simplifies, since

$$\langle g^k, x^k - \tilde{x}\rangle \geq f(x^k) - f(\tilde{x}) \geq f(x^k) - f(x^{k+1}) \geq \alpha t_k \|g^k\|^2,$$

$$\begin{aligned}\|\tilde{x} - x^{k+1}\|^2 - \|\tilde{x} - x^k\|^2 &\leq -2\alpha t_k^2 \|g^k\|^2 + t_k^2 \|g^k\|^2 \\ &= -(2\alpha - 1)\|x^{k+1} - x^k\|^2 \leq 0.\end{aligned}$$

This observation is used in Ref. 7 to prove that $x^k \to \bar{x} \in \check{X}$ if $\check{X} \neq \emptyset$ and ∇f is Lipschitz continuous; thus, our result improves that of Ref. 7.

Second, it is easy to verify Theorem 1.1 for any line search for which Lemma 2.3 holds and for all k,

$$f(x^{k+1}) \leq f(x^k) - \alpha t_k \|g^k\|^2$$

and

$$t_k \in (0, t_{\max}], \qquad \text{for some fixed } t_{\max} > 0.$$

Such stepsizes may be found by many procedures [Refs. 8–12]. Note that exact line searches are not admissible, but one may use, as in Ref. 12, Section 10.7.2,

$$t_k \approx \arg\min\{f(x^k - tg^k) : f(x^k - tg^k) \leq f(x^k) - \alpha t \|g^k\|^2, 0 < t \leq t_{\max}\}.$$

Third, under (A1)–(A2) to satisfy (A3)–(A4), one may let (cf. the proof of Lemma 2.3)

$$\begin{aligned} t_k = \arg\max\{t : f(x^k - tg^k) \leq f(x^k) - \phi(t)\|g^k\|^2, \\ t = 2^{-i}\min[\tau_\alpha, \tau_\beta], i = 0, 1, \ldots\}. \end{aligned} \tag{6}$$

We note that (6) with $\phi(t) = \alpha t^2$ was used in Ref. 13. Again, the Armijo-type search (6) may be relaxed as in the preceding paragraph. In particular, one may use

$$t_k \approx \check{t}_k := \arg\min\{f(x^k - tg^k) + \alpha t^2 \|g^k\|^2 : t > 0\}.$$

If f is pseudoconvex, then $X = \check{X}$ [Ref. 6, Theorem 9.3.3]; so if $\check{X} \neq \emptyset$ and $t_k = \check{t}_k$ for all k, then $x^k \to \bar{x} \in X$; thus, we recover the result of Ref. 14.

Fourth, one may verify Assumption 2.1 for the algorithms of Ref. 3; in their notation, let $\phi(t)=\beta t^2$ with $\beta=L\delta_2/2(1-\delta_2)$ for Algorithm A, $\phi=\psi$ for Algorithm B. Theorem 1.1 is stronger than Theorem 3 of Ref. 3, and our proof is simpler.

We note that quasiconvexity of f is necessary for Lemma 2.2, and consequently Theorem 1.1. For example, let

$$n=2, \qquad f(x)=e^{x_1}-x_2^2, \qquad x^0=(0,0)^T.$$

Each of the above methods generates

$$x^k=(x_1^k,0)^T, \text{ with } x_1^k\downarrow -\infty \text{ and } f(x^k)\downarrow 0, \text{ while } \inf f=-\infty.$$

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