Admittance of a Wedge Excited Co-axial Antenna With a Plasma Sheath

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ABSTRACT

Cylinder with a wedge and a coaxial shell with an axial slot make up the antenna. A stationary expression for the admittance is obtained when the antenna is enclosed by a plasma sheath. The basis of the admittance calculation is the electric field of the wedge aperture derived from a solution of two coupled integral equations. The calculations are carried out for the parameter ranges: The radii of the cylinder and shell are, in wavelength, from $0.05/\pi$ to $2/\pi$ and from $0.055/\pi$ to $2.2/\pi$, respectively. The plasma sheath thickness is from 0 to $2.5/\pi$. The plasma frequency to signal frequency ratio, ω_p/ω, is from 0 to 5. Collision frequency to signal frequency ratio, ν/ω, is 0; 0.01; 0.1, and 0.5. The angular width of the wedge slot and the shell slot are the same and equal to 0.06 radians. The results indicate: For $\omega_p/\omega > 1$, conductance and susceptance depend weakly on the plasma sheath thickness. For $\omega_p/\omega > 1$ and $\nu/\omega = 0$, conductance decreases exponentially when either the sheath thickness or ω_p/ω increases. Susceptance depends primarily on ω_p/ω and inappreciably on the sheath thickness. An increase of ν/ω increases the conductance but modifies the susceptance only slightly. The coaxial slotted shell behaves as an ideal voltage transformer in the equivalent antenna circuit.

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CHAPTER I

INTRODUCTION

1. Survey of Previous Work:

In the course of re-entry, a space vehicle travels through the upper atmosphere with hypersonic speed, thus a highly ionized non-uniform plasma layer is generated. This plasma layer encloses the body of the vehicle, therefore it tends to block the radio contact between the vehicle and the outside stations. In the last few years, this problem has attracted the attention of a number of investigators.

Hodara (1963) calculated the radiation pattern of a slot on an infinitely conducting plane covered with a homogeneous, but anisotropic plasma layer. In his approach, he first assumed a reasonable aperture field and then obtained the far field. He did not consider the slot admittance. To obtain the slot admittance, one has to know the field in the slot much more accurately than is required for the far field calculations. Galejs (1963) considered a slot on an infinitely conducting plane, backed by a rectangular cavity and excited by a current generator. He formulated a stationary expression for the slot admittance. In his more recent papers (1964, 1965a, 1965b), he applied the same technique to evaluate the slot admittance when the conducting plane is covered with a homogeneous plasma layer. A. T. Villeneuve (1965) considered a problem which involves a rectangular waveguide terminated on an infinitely conducting plane coated with a plasma layer. He employed the reaction concept to derive a stationary form for the terminal admittance of the waveguide. Both Galejs and Villeneuve limited their calculation to the case of signal frequency ω greater than plasma frequency ω_p.

Unless the aperture size and free space wavelength are much smaller than the size of the vehicle, we could not use the plane geometry to approximate the surface of the vehicle; otherwise a geometry closer to reality should be considered. Some authors choose to consider the circular cylindrical geometry. A typical geometrical configuration is shown in Fig. 1-1

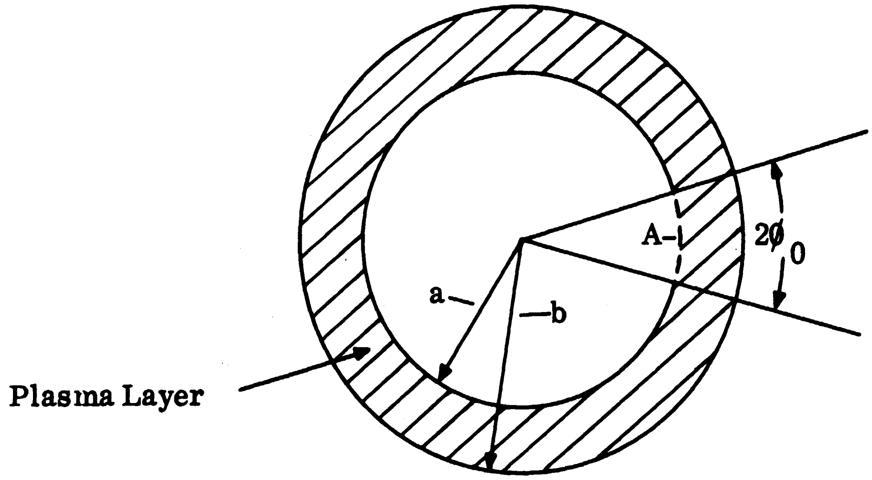

FIG. 1-1: CROSS-SECTION OF A TYPICAL SLOTTED CYLINDER WITH A PLASMA SHEATH.

where the slot A may be either axial slot or circumferencial slot. The existing work for the above configuration almost entirely is concerned with evaluating the radiation pattern of the slot when the plasma layer is assumed to be of the following:

(a) homogeneous and isotropic (Knop 1961, Sengupta 1964)

(b) homogeneous and anisotropic (Chen and Cheng, 1965)

(c) isotropic but inhomogeneous (Rusch 1964, Swift 1964, Taylor 1961)

The last case is of particular interest to us. Rusch and Swift assumed that the density of plasma varies continuously according to a specified function of the radial variable r. Taylor (Rotman and Meltz, 1961) considered the plasma sheath to be stepwise inhomogeneous. The inner step is very thin and highly overdense in comparison with the wavelength. Therefore he regarded this sublayer as metal-like sheet. This metal-like sheet is then followed by a comparatively thick dielectric-like sublayer. To prevent a short circuit on the antenna, a dielectric layer is placed between the metal surface of the vehicle and the metal-like sublayer. He also pointed out that the radio communication blackout is due to the metal-like sublayer.

Olte (1965) in a recent paper considered a conducting cylinder enclosed by a slotted coaxial metal shell with an axial slot which represents the metal-like plasma sheath. The electromagnetic field is excited by an axial magnetic line source on the cylinder (Fig. 1-2). He calculated the power radiated through the shell for different combinations of the cylinder size, shell size, and the separation angle θ between the line source A and the shell slot.

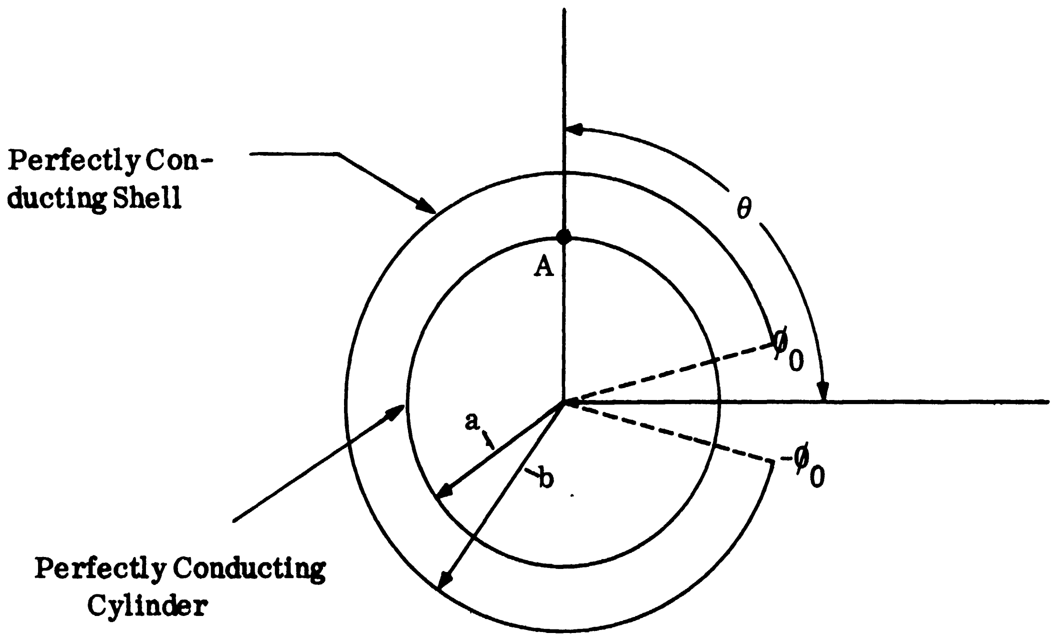

FIG. 1-2: RELATIVE POSITION OF SHELL SLOT AND MAGNETIC LINE SOURCE.

2. Problem to be Investigated

Although the radiation problem of a slot on a cylinder with a plasma sheath has been treated by many authors, few of them have been concerned with the admittance. The prime purpose of this report is to partly fill this gap. The geometrical configuration we consider is shown below:

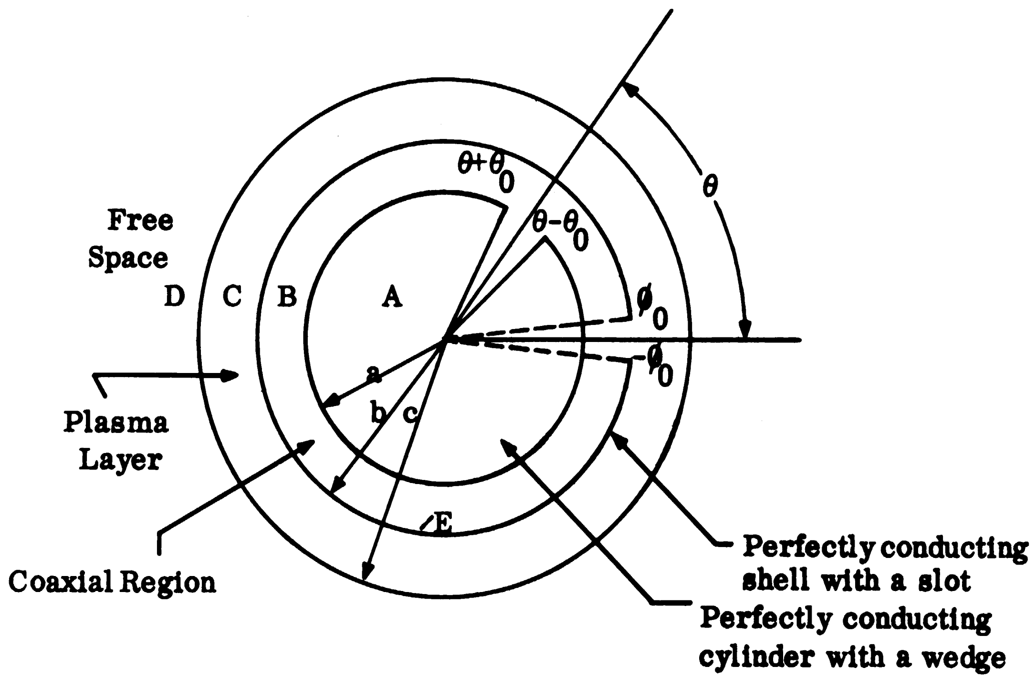

FIG. 1-3: CROSS-SECTION OF A WEDGED CYLINDER, SLOTTED SHELL, AND PLASMA SHEATH.

where A is a circular cylinder with a wedge of width $2\theta_0$, B is a dielectric with $\mu_r = 1$, $\epsilon_r = 1$, C is a uniform dielectric-like plasma layer, D is the free space region, E is a circular conducting shell with an axial slot, a, b, and c are the radii of the cylinder, circular shell and the outer boundary of the plasma layer respectively, θ represents the center to center angle between the shell slot and the wedge slot. If we assume a magnetic line source at the apex of the wedge, then the electromagnetic energy radiated from the line source is guided by the wedge to the coaxial region and then through the shell slot and the plasma sheath to the free space. Therefore we may regard the wedge as the antenna feeding line and we proceed to calculate the terminal admittance of the wedge waveguide.

3. Outline of the Report

In the next chapter, we first assume the source strength to be V_o volts and then write down the fields in the form of infinite series for the wedge guide, the coaxial region, the plasma, and the free space. From the continuity of the tangential electromagnetic fields in the two apertures, we formulate two coupled integral equations with ϕ-directed electric field in the wedge aperture and shell slot as the unknown functions. From these expressions we formulate the terminal admittance of the wedge waveguide which is proved to be stationary with respect to the variations of the wedge aperture field. In chapter III and chapter IV, we present the methods and the solutions of the coupled integral equations. Upon employing these solutions, we obtain in chapter V the explicit expressions for the voltages of the two slots and the terminal admittance of the wedge waveguide when both slots are narrow. Parallel to the stationary formulation of the terminal admittance of the wedge waveguide, in chapter VI, we formulate this admittance in an alternate form. This new formulation is not stationary, but provides a basis for the discussion of the contribution of different regions to the terminal admittance. From this formulation, we construct an equivalent circuit. In chapter VII, we present the numerical values of the terminal conductance and terminal susceptances computed from the expression of the admittance obtained in chapter V. Finally, we draw some brief conclusions for this report.

In order to maintain the main sequence of thought, we leave some of the detailed derivations to the apendices A-1 through A-10.

CHAPTER II

INTEGRAL EQUATIONS AND THE TERMINAL ADMITTANCE OF THE WEDGE WAVEGUIDE

1. Introduction:

The geometrical configuration which we choose to consider suggests us to employ the cylindrical coordinates, of which the z-axis is aligned with the axis of the cylinder and ϕ is measured in a counter-clockwise direction from the center of the slot of the shell. Because the antenna is excited by an axial magnetic line source, only the following field components exist:

H_z axial magnetic field,

E_ϕ circumferencial electric field,

E_r radial electric field.

By superscripts I, II, III and IV, we will denote the wedge waveguide, the coaxial region, the plasma sheath, and the free space, respectively. Since tangential electromagnetic field must be continuous across the wedge-guide aperture, the shell slot, and the tangential electric field must vanish on the perfectly conducting cylindrical walls one obtains at $r = a$:

$$E_\phi^{I} = E_\phi^{II} \equiv \hat{E}(\phi) \; ; \; \theta - \theta_0 < \phi < \theta + \theta_0 \qquad (2\text{-}1\text{-}1)$$

$$E_\phi^{II} = 0 \; ; \quad \theta + \theta_0 \leqslant \phi \leqslant (2\pi + \theta - \theta_0) \qquad (2\text{-}1\text{-}2)$$

$$H_z^{I} = H_z^{II} \; ; \; \theta - \theta_0 < \phi < \theta + \theta_0 \, , \qquad (2\text{-}1\text{-}3)$$

at r = b :

$$E_{\phi}^{II} = E_{\phi}^{III} \equiv E(\phi) \; ; \; -\phi_0 < \phi < \phi_0 \tag{2-1-4}$$

$$= 0 \; ; \; |\phi| \geqslant \phi_0 \tag{2-1-5}$$

$$H_z^{II} = H_z^{III} \; ; \qquad -\phi_0 < \phi < \phi_0 \, . \tag{2-1-6}$$

at r = c :

$$E_{\phi}^{III} = E_{\phi}^{IV} \quad ; \quad -\pi \leqslant \phi \leqslant \pi \tag{2-1-7}$$

$$H_z^{III} = H_z^{IV} \quad ; \quad -\pi \leqslant \phi \leqslant \pi \; . \tag{2-1-8}$$

In the last part of this chapter we use the forgoing relations to formulate two coupled integral equations with the wedge guide aperture field $\hat{E}(\phi)$ and the shell slot field $E(\phi)$ as the unknown functions.

One may consider the wedge guide as a transmission line with TEM wave as the transmission mode. We consider a section of wedged-waveguide of length L , in which the transmission line voltage and current are governed by the following equations (Montgomery, 1948)

$$\frac{d\,V(r)}{d\,r} = -j\,k_0\,Z_0(r)\,I(r) \tag{2-1-9}$$

$$\frac{d\,I(r)}{d\,r} = -j\,k_0\,Y_0(r)\,V_0(r) \tag{2-1-10}$$

where

$$Y_0(r) = \frac{1}{Z_0(r)} = \frac{L}{2\theta_0\,r}\sqrt{\frac{\epsilon_0}{\mu_0}} \tag{2-1-11}$$

The positive directions for the current and voltage are shown in Fig. 2-1

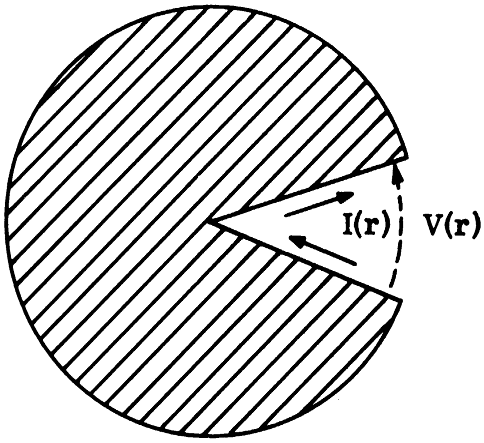

FIG. 2-1: POSITIVE VOLTAGE AND CURRENT OF WEDGE GUIDE

The solution of differential equations (2-1-9) and (2-1-10) are easily obtained as

$$I(r) = -\left[A'H_0^{(2)}(k_0 r) + B'H_0^{(1)}(k_0 r)\right] \tag{2-1-12}$$

$$V(r) = \frac{1}{j\,Y_0(r)}\left[A'H_0^{(2)'}(k_0 r) + B'H_0^{(1)'}(k_0 r)\right] \tag{2-1-13}$$

where the primes indicate the derivatives of the Hankel function. If one defines the normalized admittance at a cross-section r as

$$y(r) = \frac{I(r)}{V(r)}\,\frac{1}{Y_0(r)} \tag{2-1-14}$$

then from previous two equations one obtains

$$y(r) = j\frac{A'H_0^{(2)}(k_0r) + B'H_0^{(1)}(k_0r)}{A'H_0^{(2)'}(k_0r) + B'H_0^{(1)'}(k_0r)} \tag{2-1-15}$$

The normalized terminal admittance of the wedge waveguide is y(a). Thus our problem is to find the constants A' and B'.

2. Field Expressions:

The field expressions (Stratton, 1941) in the wedge waveguide are

$$H_z^I = k_0^2\left\{A_0'H_0^{(2)}(k_0r) + B_0'H_0^{(1)}(k_0r) + \sum_{n=1}^{\infty} A_n' J_{\frac{n\pi}{2\theta_0}}(k_0r)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\right\}, \tag{2-2-1}$$

$$E_\phi^I = j\omega\mu_0 k_0\left\{A_0'H_0^{(2)'}(k_0r) + B_0'H_0^{(1)'}(k_0r) + \sum_{n=1}^{\infty} A_n' J'_{\frac{n\pi}{2\theta_0}}(k_0r)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\right\}, \tag{2-2-2}$$

$$E_r^I = \frac{j\omega\mu_0}{r}\sum_{n=1}^{\infty}\frac{n\pi}{2\theta_0}A_n' J_{\frac{n\pi}{2\theta_0}}(k_0r)\sin\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0) . \tag{2-2-3}$$

If we let V(r) be the voltage between the walls of the wedge, then

$$V(r) = -\int_{\theta-\theta_0}^{\theta+\theta_0} r E_\phi^I \, d\phi . \tag{2-2-4}$$

Upon substituting (2-2-2) in (2-2-4), we obtain

$$V(r) = -j\omega\mu_0 k_0 r 2\theta_0\left[A_0'H_0^{(2)'}(k_0r) + B_0'H_0^{(1)'}(k_0r)\right] . \tag{2-2-5}$$

One may specify the value of A_0', or B_0', or their linear combination. We choose to specify the voltage at the apex of wedge, i.e. to fix the strength of the magnetic line source. Let

$$\lim_{r \to 0} \left\{ -j\omega\mu_0 k_0 r\, 2\theta_0 \left[A_0' H_0^{(2)'}(k_0 r) + B_0' H_o^{(1)'}(k_0 r) \right] \right\} = V_0 . \tag{2-2-6}$$

After taking the limit, the result is

$$A_0' - B_0' = - \frac{\pi V_0}{4\theta_0 \omega \mu_0} \tag{2-2-7}$$

In the coaxial region:

$$H_z^{II} = k_0^2 \sum_{n=-\infty}^{n=\infty} \left[A_n J_n(k_0 r) + B_n N_n(k_0 r) \right] e^{-jn\phi} , \tag{2-2-8}$$

$$E_\phi^{II} = j\omega\mu_0 k_0 \sum_{n=-\infty}^{n=\infty} \left[A_n J_n'(k_0 r) + B_n N_n'(k_0 r) \right] e^{-jn\phi} , \tag{2-2-9}$$

$$E_r^{II} = - \frac{\omega\mu_0}{r} \sum_{n=-\infty}^{n=\infty} n \left[A_n J_n(k_0 r) + B_n N_n(k_0 r) \right] e^{-jn\phi} . \tag{2-2-10}$$

In the plasma sheath:

$$H_z^{III} = k_1^2 \sum_{n=-\infty}^{n=\infty} \left[D_n J_n(k_1 r) + E_n N_n(k_1 r) \right] e^{-jn\phi} \tag{2-2-11}$$

$$E_{\phi}^{III} = j\omega\mu_0 k_1 \sum_{n=-\infty}^{n=\infty} \left[D_n J_n'(k_1 r) + E_n N_n'(k_1 r)\right] e^{-jn\phi} \tag{2-2-12}$$

$$E_r^{III} = -\frac{\omega\mu_0}{r} \sum_{n=-\infty}^{n=\infty} n\left[D_n J_n(k_1 r) + E_n N_n'(k_1 r)\right] e^{-jn\phi} \tag{2-2-13}$$

where

$$k_1 = k_0 \sqrt{1 - \frac{(\frac{\omega_p}{\omega})^2}{1 + j\frac{\nu}{\omega}}} = \bar{k}\, k_0 \ . \tag{2-2-14}$$

If we let

$$\bar{k} = k_r - j\, k_i \tag{2-2-15}$$

then

$$k_r = \left\{ \frac{1}{2}\left(1 - \frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\right) + \frac{1}{2}\left[\left(1 - \frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\right)^2 + \left(\frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\ \frac{\nu}{\omega}\right)^2\right]^{1/2} \right\}^{1/2} \tag{2-2-16}$$

$$k_i = \left\{ \frac{1}{2}\left[\left(1 - \frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\right)^2 + \left(\frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\ \frac{\nu}{\omega}\right)^2\right]^{1/2} - \frac{1}{2}\left(1 - \frac{(\frac{\omega_p}{\omega})^2}{1+(\frac{\nu}{\omega})^2}\right) \right\}^{1/2} \tag{2-2-17}$$

In the free space:

$$H_z^{IV} = k_0^2 \sum_{n=-\infty}^{n=\infty} C_n H_n^{(2)}(k_0 r)\, e^{-jn\phi} \ , \tag{2-2-18}$$

$$E^{IV}_{\phi} = j\omega\mu_0 k_0 \sum_{n=-\infty}^{n=\infty} C_n H_n^{(2)'}(k_0 r)\, e^{-jn\phi} \quad , \qquad (2\text{-}2\text{-}19)$$

$$E^{IV}_{r} = -\frac{\omega\mu_0}{r} \sum_{n=-\infty}^{n=\infty} n\, C_n H_n^{(2)}(k_0 r)\, e^{-jn\phi} \quad . \qquad (2\text{-}2\text{-}20)$$

3. Formulation of the integral equations

Upon substituting the required E_{ϕ} expansions in the boundary conditions at $r = a$, $r = b$ and $r = c$, i.e. in (2-1-1), (2-1-4) and (2-1-7) respectively, one obtains

$$j\omega\mu_0 k_0 \left\{ A_0^{'} H_0^{(2)'}(k_0 a) + B_0^{'} H_0^{(1)'}(k_0 a) + \sum_{n=1}^{\infty} A_n^{'} J^{'}_{\frac{n\pi}{2\theta_0}}(k_0 a) \cos\frac{n\pi}{2\theta_0}(\phi - \theta + \theta_0) \right\} = \hat{E}(\phi), \qquad (2\text{-}3\text{-}1)$$

$$j\omega\mu_0 k_0 \sum_{n=-\infty}^{n=\infty} \left(A_n J_n^{'}(k_0 a) + B_n N_n^{'}(k_0 a)\right) e^{-jn\phi} = \hat{E}(\phi) \qquad (2\text{-}3\text{-}2)$$

$$j\omega\mu_0 k_0 \sum_{n=-\infty}^{n=\infty} (A_n J_n^{'}(k_0 b) + B_n N_n^{'}(k_0 b)\, e^{-jn\phi} = E(\phi) \qquad (2\text{-}3\text{-}3)$$

$$j\omega\mu_0 k_1 \sum_{n=-\infty}^{n=\infty} \left(D_n J_n^{'}(k_1 b) + E_n N_n^{'}(k_1 b)\right) e^{-jn\phi} = E(\phi) \; , \qquad (2\text{-}3\text{-}4)$$

and

$$j\omega\mu_0 k_1 \sum_{n=-\infty}^{n=\infty} (D_n J_n'(k_1 c) + E_n N_n'(k_1 c)) e^{-jn\phi}$$

$$= j\omega\mu_0 k_0 \sum_{n=-\infty}^{n=\infty} C_n H_n^{(2)'}(k_0 c) \, e^{-jn\phi} \quad . \qquad (2\text{-}3\text{-}5)$$

Applying the orthogonality of the circular functions to eqns. (2-3-1) through (2-3-5), there results:

$$A_0' H_0^{(2)'}(k_0 a) + B_0' H_0^{(1)'}(k_0 a) = \frac{1}{2j\theta_0 \omega\mu_0 k_0} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \, d\phi' \qquad (2\text{-}3\text{-}6)$$

$$A_n' = \frac{1}{j\theta_0 \omega\mu_0 k_0 J'_{\frac{n\pi}{2\theta_0}}(k_0 a)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \cos\frac{n\pi}{2\theta_0} (\phi' - \theta + \theta_0) \, d\phi' \; ;$$

$$n \geqslant 1 \, , \qquad (2\text{-}3\text{-}7)$$

$$A_n J_n'(k_0 a) + B_n N_n'(k_0 a) = \frac{1}{j 2\pi\omega\mu_0 k_0} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \, e^{jn\phi'} \, d\phi' \, , \qquad (2\text{-}3\text{-}8)$$

$$A_n J_n'(k_0 n) + B_n N_n'(k_0 b) = \frac{1}{j 2\pi\omega\mu_0 k_0} \int_{-\phi_0}^{\phi_0} E(\phi') \, e^{jn\phi'} \, d\phi' \, , \qquad (2\text{-}3\text{-}9)$$

$$D_n J_n'(k_1 b) + E_n N_n'(k_1 b) = \frac{1}{j 2\pi\omega\mu_0 k_1} \int_{-\phi_0}^{\phi_0} E(\phi') \, e^{jn\phi'} \, d\phi' \, , \qquad (2\text{-}3\text{-}10)$$

and

$$\bar{k}\left(D_n J_n'(k_1c) + E_n N_n'(k_1c)\right) = C_n H_n^{(2)'}(k_0c) \qquad (2\text{-}3\text{-}11)$$

Combining Eqs. (2-2-7) with (2-3-6), (2-3-8) with (2-3-9), and (2-3-10) with (2-3-11), one has three sets of two variable simultaneous algebraic equations. Their solutions are

$$A_0' = -\frac{\pi V_0}{8\theta_0 \omega \mu_0} \frac{H^{(1)'}(k_0a)}{J_0'(k_0a)} + \frac{1}{4j\theta_0 \omega \mu_0 k_0} \frac{1}{J_0'(k_0a)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\, d\phi' , \qquad (2\text{-}3\text{-}12)$$

$$B_0' = \frac{\pi V_0}{8\theta_0 \omega \mu_0} \frac{H_0^{(2)'}(k_0a)}{J_0'(k_0a)} + \frac{1}{4j\theta_0 \omega \mu_0 k_0} \frac{1}{J_0'(k_0a)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\, d\phi' , \qquad (2\text{-}3\text{-}13)$$

$$A_n = \frac{1}{J_n'(k_0b) N_n'(k_0a) - J_n'(k_0a) N_n'(k_0b)} \left(N_n'(k_0a)\, p_n - N_n'(k_0b)\, q_n\right) , \qquad (2\text{-}3\text{-}14)$$

$$B_n = \frac{1}{J_n'(k_0b) N_n'(k_0a) - J_n'(k_0a) N_n'(k_0b)} \left(-J_n'(k_0a)\, p_n + J_n'(k_0b)\, q_n\right) , \qquad (2\text{-}3\text{-}15)$$

$$D_n = \frac{1}{\bar{k}\left[J_n'(k_1c) N_n'(k_1b) - J_n'(k_1b) N_n'(k_0c)\right]} \left[C_n H_n^{(2)'}(k_0c) N_n'(k_1b) - p_n N_n'(k_1c)\right] \qquad (2\text{-}3\text{-}16)$$

$$E_n = \frac{1}{\bar{k}\left[J_n'(k_1c)\,N_n'(k_1b) - J_n'(k_1b)\,N_n'(k_1c)\right]}\left[-C_n H_n^{(2)'}(k_0c)\,J_n'(k_1b) + p_n J_n'(k_1c)\right] \qquad (2\text{-}3\text{-}17)$$

where

$$p_n = \frac{1}{2\pi j\omega\mu_0 k_0}\int_{-\phi_0}^{\phi_0} E(\phi')\, e^{jn\phi'}\, d\phi' \quad , \qquad (2\text{-}3\text{-}18)$$

$$q_n = \frac{1}{2\pi j\omega\mu_0 k_0}\int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\, e^{jn\phi'}\, d\phi' \quad . \qquad (2\text{-}3\text{-}19)$$

We substitute the required H_z expansions in (2-1-3), (2-1-6), and (2-1-8), the magnetic boundary conditions at $r = a$, $r = b$, and $r = c$ respectively, and obtain

$$A_0' H_0^{(2)}(k_0a) + B_0' H_0^{(1)}(k_0a) + \sum_{n=1}^{\infty} A_n' J_{\frac{n\pi}{2\theta_0}}(k_0a)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)$$

$$= \sum_{n=-\infty}^{n=\infty} \left(A_n J_n(k_0a) + B_n N_n(k_0a)\right) e^{-jn\phi} \quad ,$$

$$\theta-\theta_0 \leqslant \phi \leqslant \theta+\theta_0 \quad , \qquad (2\text{-}3\text{-}20)$$

$$\sum_{n=-\infty}^{n=\infty} \left(A_n J_n(k_0b) + B_n N_n(k_0b)\right) e^{-jn\phi} = \bar{k}^{-2} \sum_{n=-\infty}^{n=\infty} \left(D_n J_n(k_1b) + E_n N_n(k_1b)\right) e^{-jn\phi} ,$$

$$-\phi_0 \leqslant \phi \leqslant \phi_0 \quad , \qquad (2\text{-}3\text{-}21)$$

and

$$\bar{k}^2 \sum_{n=-\infty}^{n=\infty} (D_n J_n(k_1 c) + E_n N_n(k_1 c))\, e^{-jn\phi} = \sum_{n=-\infty}^{n=\infty} C_n H_n^{(2)}(k_0 c)\, e^{-jn\phi} ,$$

$$-\pi \leq \phi \leq \pi \quad . \qquad (2\text{-}3\text{-}22)$$

Upon using (2-3-22) we eliminate C_n in the expressions for D_n and E_n, and obtain

$$D_n = \frac{\frac{1}{\bar{k}}\left[N_n'(k_1 c) H_n^{(2)}(k_0 c) - \bar{k} H_n^{(2)'}(k_0 c) N_n(k_1 c)\right] p_n}{\bar{k} H_n^{(2)'}(k_0 c)\left[J_n(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n(k_1 c)\right] - H_n^{(2)}(k_0 c)\left[J_n'(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n'(k_1 b)\right]}$$

(2-3-23)

$$E_n = -\frac{\frac{1}{\bar{k}}\left[J_n'(k_1 c) H_n^{(2)}(k_0 c) - \bar{k} H_n^{(2)'}(k_0 c) J_n(k_1 c)\right] p_n}{\bar{k} H_n^{(2)'}(k_0 c)\left[J_n(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n(k_1 c)\right] - H_n^{(2)}(k_0 c)\left[J_n'(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n'(k_1 c)\right]}$$

(2-3-24)

If we substitute (2-3-14), (2-3-15), (2-3-23) and (2-3-24) in (2-3-21), we have

$$\sum_{n=-\infty}^{n=\infty} \frac{J_n(k_0 b) N_n'(k_0 a) - N_n(k_0 b) J_n'(k_0 a)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)}\, p_n\, e^{-jn\phi}$$

$$- \sum_{n=-\infty}^{n=\infty} \frac{J_n(k_0 b) N_n'(k_0 b) - J_n'(k_0 b) N_n(k_0 b)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_o a) N_n'(k_0 b)}\, q_n\, e^{-jn\phi} = \sum_{n=-\infty}^{\infty} \pi_n p_n\, e^{-jn\phi}$$

(2-3-25)

where

$$\pi_n \equiv \frac{\left[J_n(k_1b)N_n'(k_1c) - J_n'(k_1c)N_n(k_1b)\right] - \bar{k}\dfrac{H_n^{(2)'}(k_0c)}{H_n^{(2)}(k_0c)}\left[J_n(k_0b)N_n(k_1c) - J_n(k_1c)N_n(k_1b)\right]}{\bar{k}\left[J_n(k_1c)N_n'(k_1b) - J_n'(k_1b)N_n(k_1c)\right] - \dfrac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)}\left[J_n'(k_1c)N_n'(k_1b) - J_n'(k_1b)N_n'(k_1c)\right]}\,\bar{k} \quad .$$

(2-3-26)

Substituting (2-3-18), (2-3-19), and the identity

$$J_n(k_0b)N_n'(k_0b) - J_n'(k_0b)N_n(k_0b) = \frac{2}{\pi k_0 b} \qquad \text{(2-3-27)}$$

in (2-3-25), we obtain

$$\sum_{n=-\infty}^{n=\infty}\left(\pi_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)N_n'(k_0a) - N_n(k_0b)J_n'(k_0a)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\right)\int_{-\phi_0}^{\phi_0} E(\phi')\, e^{-jn(\phi-\phi')}\, d\phi'$$

$$= -\frac{2}{\pi k_0 b}\sum_{n=-\infty}^{\infty}\frac{1}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\, e^{-jn(\phi-\phi')}\, d\phi' \quad ,$$

$$-\phi_0 \leqslant \phi \leqslant \phi_0 \, .$$

(2-3-28)

Upon interchanging the summation with integration on the left-hand side, one obtains

$$\int_{-\phi_0}^{\phi_0} E(\phi') \sum_{n=-\infty}^{n=\infty} \left(\pi_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \right) e^{-jn(\phi-\phi')} d\phi'$$

$$= -\frac{2}{\pi k_0 b} \sum_{n=-\infty}^{n=\infty} \frac{1}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') e^{-jn(\phi-\phi')} d\phi' ,$$

$$-\phi_0 < \phi < \phi_0 . \qquad (2\text{-}3\text{-}29)$$

We then substitute (2-3-7) and (2-3-12) through (2-3-15) in (2-3-20), rearrange terms, and using the Wronskian of the Hankel functions

$$H_n^{(1)'}(k_0a)\, H_n^{(2)}(k_0a) - H_n^{(1)}(k_0a)\, H_n^{(2)'}(k_0a) = \frac{4j}{\pi k_0 a} ,$$

arrive at

$$\frac{1}{2j\theta_0\omega\mu_0 k_0} \cdot \frac{V_0}{a J_0'(k_0 a)}$$

$$+ \frac{1}{2j\theta_0\omega\mu_0 k_0} \sum_{n=0}^{\infty} \frac{J_{\frac{n\pi}{2\theta_0}}(k_0 a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0 a)} \cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0) \int_{\theta-\theta_0}^{\theta+\theta_0} E(\phi')\cos\frac{n\pi}{2\theta_0}(\phi'-\theta+\theta_0)\,d\phi'$$

$$= -\frac{1}{2\pi j\omega\mu_0 k_0} \sum_{n=-\infty}^{n=\infty} \frac{N_n'(k_0 b)J_n(k_0 a) - J_n'(k_0 b)N_n(k_0 a)}{J_n'(k_0 b)N_n'(k_0 a) - J_n'(k_0 a)N_n'(k_0 b)} \int_{\theta-\theta_0}^{\theta+\theta_0} E(\phi')\, e^{-jn(\phi-\phi')}\, d\phi'$$

$$+ \frac{2}{\pi k_0 a}\, \frac{1}{2\pi j\omega\mu_0 k_0} \sum_{n=-\infty}^{n=\infty} \frac{1}{J_n'(k_0 b)N_n'(k_0 a) - J_n'(k_0 a)N_n'(k_0 b)} \int_{-\phi_0}^{\phi_0} E(\phi')\, e^{-jn(\phi-\phi')}\, d\phi' ,$$

$$\theta-\theta_0 < \phi < \theta+\theta_0 , \qquad (2\text{-}3\text{-}30)$$

where $\epsilon_n = 1$ for $n = 0$ and 2 for $n \neq 0$. We note that in (2-3-29) and on the right handside of (2-3-30) the series are summed on n from $-\infty$ to ∞. If we employ the relations

$$Z_n(r) = (-1)^n Z_{-n}(r)$$

$$Z_n'(r) = (-1)^n Z'_{-n}(r)$$

where $Z_n(r)$ and $Z_n'(r)$ denote the cylindrical function and its derivative;

one may simplify (2-3-29) and (2-3-30) respectively to

$$\int_{-\phi_0}^{\phi_0} E(\phi')\sum_{n=0}^{\infty} \epsilon_n \left(\pi_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)\,N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)\,N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\right)\cos n(\phi-\phi')\,d\phi'$$

$$= -\frac{2}{\pi k_0 b}\sum_{n=0}^{\infty} \frac{\epsilon_n}{J_n'(k_0b)\,N_n'(k_0a) - J_n'(k_0a)\,N_n'(k_0b)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\cos n(\phi-\phi')\,d\phi' \ ,$$

$$-\phi_0 < \phi \leqslant \phi_0 \ , \qquad \text{(2-3-31)}$$

and

$$\sum_{n=0}^{\infty} \epsilon_n \frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} \cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\cos\frac{n\pi}{2\theta_0}(\phi'-\theta+\theta_0)\,d\phi'$$

$$+\frac{\theta_0}{\pi}\sum_{n=0}^{\infty} \epsilon_n \frac{J_n(k_0a)N_n'(k_0b) - J_n'(k_0b)N_n(k_0a)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\cos n(\phi-\phi')\,d\phi'$$

$$= -\frac{V_0}{a}\,\frac{1}{J_0'(k_0a)} + \frac{2}{\pi k_0 a}\,\frac{\theta_0}{\pi}\sum_{n=0}^{\infty} \frac{\epsilon_n}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \int_{-\phi_0}^{\phi_0} E(\phi')\cos n(\phi-\phi')\,d\phi' \ ,$$

$$\theta-\theta_0 \leqslant \phi \leqslant \theta+\theta_0 \ . \qquad \text{(2-3-32)}$$

$$\theta-\theta_0 \leqslant \phi \leqslant \theta+\theta_0 \ . \qquad \text{(2-3-32)}$$

Equations (2-3-31) and (2-3-32) are dual integral equations. Thus the boundary value problem is reduced to the dual integral equations. In chapter III, we regard $\hat{E}(\phi')$ as a known quantity and solve integral equation (2-3-31). Then we insert this result in (2-3-32) to eliminate $E(\phi')$. An approximate solution for $\hat{E}(\phi')$ is obtained for narrow wedge and narrow shell slot. The kernels of the last two integral equations will be studied respectively in chapter III and chapter IV; it is shown that these kernels have a logarithmic singularity when $\phi' \to \phi$.

4. Terminal Admittance of the Wedge Waveguide

Using (2-3-6), the first two terms on the left hand side of (2-3-20) can be written as

$$A_0' H_0^{(2)}(k_0a) + B_0' H_0^{(1)}(k_0a) = \frac{1}{2j\theta_0 \omega\mu_0 k_0} \frac{A_0' H_0^{(2)}(k_0a) + B_0' H_0^{(1)}(k_0a)}{A_0' H_0^{(2)\prime}(k_0a) + B_0' H_0^{(2)\prime}(k_0a)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\, d\phi' ,$$

but

$$j \frac{A_0' H_0^{(2)}(k_0a) + B_0' H_0^{(1)}(k_0a)}{A_0' H_0^{(2)\prime}(k_0a) + B_0' H_0^{(1)\prime}(k_0a)} = y(a)$$

and therefore

$$A_0' H_0^{(2)}(k_0a) + B_0' H_0^{(1)}(k_0a) = -\frac{1}{2\theta_0 \omega\mu_0 k_0}\, y(a) \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\, d\phi' . \qquad (2\text{-}4\text{-}1)$$

Upon substituting (2-4-1) in (2-3-20), we have

$$-\frac{1}{2\theta_0\omega\mu_0 k_0}\, y(a)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\,d\phi + \sum_{n=1}^{\infty} A_n' J_{\frac{n\pi}{2\theta_0}}(k_0 a)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)$$

$$=\sum_{n=-\infty}^{n=\infty}(A_n J_n(k_0 a)+B_n N_n(k_0 b))e^{-jn\phi} \quad . \qquad (2\text{-}4\text{-}2)$$

If one inserts (2-3-7), (2-3-14) and (2-3-15) into the last expression, one obtains

$$y(a)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\,d\phi' = 2\sum_{n=1}^{\infty}\frac{\dfrac{J_{\frac{n\pi}{2\theta_0}}(k_0 a)}{2\theta_0}}{J'_{\frac{n\pi}{2\theta_0}}(k_0 a)}\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos\frac{n\pi}{2\theta_0}(\phi-\theta-\theta_0)\,d\phi'$$

$$+\frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{N_n'(k_0 b)J_n(k_0 a)-J_n'(k_0 b)N_n(k_0 a)}{J_n'(k_0 b)N_n'(k_0 a)-J_n'(k_0 a)N_n'(k_0 b)}\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos n(\phi-\phi')\,d\phi'$$

$$-\frac{\theta_0}{\pi}\frac{2}{\pi k_0 a}\sum_{n=0}^{\infty}\frac{\epsilon_n}{J_n'(k_0 b)N_n'(k_0 a)-J_n'(k_0 a)N_n'(k_0 b)}\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi' \; .$$

(2-4-3)

Multiplying (2-4-3) by $\hat{E}(\phi)\,d\phi$ and integrating from $\theta-\theta_0$ to $\theta+\theta_0$, we have

$$y(a)\left(\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)d\phi\right)^2 = 2j\sum_{n=1}^{\infty}\left[J_{\frac{n\pi}{2\theta_0}}(k_0a)\Big/J'_{\frac{n\pi}{2\theta_0}}(k_0a)\right]\left[\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)d\phi\right]^2$$

$$+j\frac{\theta_0}{\pi}\sum_{n=o}^{\infty}\epsilon_n\frac{J_n(k_0a)N_n'(k_0b)-J_n'(k_0b)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a)-J_n'(k_0a)N_n'(k_0b)}\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}(\phi)\int_{\theta+\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos n(\phi-\phi')d\phi'$$

$$-j\frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\frac{\epsilon_n}{J_n'(k_0b)N_n'(k_0a)-J_n'(k_0a)N_n'(k_0b)}\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}(\phi)\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi'\,.$$

(2-4-4)

One may note from (2-1-14) and (2-1-15) that y(a) is a dimensionless quantity. If the value of the terminal admittance Y(a) of the wedge waveguide is of interest, then by virtue of (2-1-11) and (2-1-14), we have

$$Y(a) = \frac{L}{2\theta_0 a}\sqrt{\frac{\epsilon_0}{\mu_0}}\;y(a)\quad\cdot \qquad (2\text{-}4\text{-}5)$$

In A-1, assuming the solution of integral equation (2-3-31) is obtainable, the stationary property of (2-4-4) with respect to a small variation of $\hat{E}(\phi)$ is established. Thus in order to use (2-4-5) to calculate the terminal admittance Y(a), one needs to solve first the integral equation (2-3-31).

CHAPTER III

SOLUTION OF INTEGRAL EQUATIONS (2-3-31)

1. Introduction

The ϕ-directed electric field in the shell slot $E(\phi)$ may be considered as the sum of $E_e(\phi)$ and $E_0(\phi)$, respectively, symmetric and antisymmetric part with regard to $\phi = 0$ (Olte, 1965).

Since

$$\int_{-\phi_0}^{\phi_0} E_e(\phi) \sin n\phi \, d\phi = 0$$

and

$$\int_{-\phi_0}^{\phi_0} E_0(\phi) \cos n\phi \, d\phi = 0$$

One may split (2-3-31) into two integral equations:

$$\int_{-\phi_0}^{\phi_0} E_e(\phi') \sum_{n=0}^{\infty} \epsilon_n \left[\pi_n \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)} - \frac{J_n(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n(k_0 b)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \right] \cos n\phi \cos n\phi' \, d\phi'$$

$$= -\frac{2}{\pi k_0 b} \sum_{n=0}^{\infty} \epsilon_n \frac{\cos(n\phi)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \int_{\theta - \theta_0}^{\theta + \theta_0} \hat{E}(\phi') \cos n\phi' \, d\phi'$$

(3-1-1)

$$\int_{-\phi_0}^{\phi_0} E_0(\phi') \sum_{n=0}^{\infty} \pi_n \left[\frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)} - \frac{J_n(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n(k_0 b)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \right] \sin n\phi \sin n\phi' \, d\phi'$$

$$= - \frac{2}{\pi k_0 b} \sum_{n=1}^{\infty} \frac{\sin n\phi}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \int_{\theta - \theta_0}^{\theta + \theta_0} \hat{E}(\phi') \sin n\phi' \, d\phi' \tag{3-1-2}$$

For large value of n, one finds

$$\pi_n \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)} \approx - \frac{k_0 b}{n} \frac{1 + \dfrac{1-(\frac{b}{c})^{2n}}{1+(\frac{b}{c})^{2n}} \bar{k}^2}{\bar{k}^2 + \dfrac{1-(\frac{b}{c})^{2n}}{1+(\frac{b}{c})^{2n}}} \bar{k}^2 \tag{3-1-3}$$

and

$$\frac{J_n(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n(k_0 b)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \approx \frac{k_0 b}{n} \frac{1+(\frac{a}{b})^{2n}}{1-(\frac{a}{b})^{2n}} \quad . \tag{3-1-4}$$

As $n \to \infty$

$$\pi_n \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)} - \frac{J_n(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n(k_0 b)}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \longrightarrow - \frac{k_0 b}{n} (1 + \bar{k}^2) \quad .$$

Thus if we define

$$\tau_0 \equiv \pi_0 \frac{H_0^{(2)}(k_0c)}{H_0^{(2)'}(k_0c)} - \frac{J_0(k_0b)N_0'(k_0a) - J_0'(k_0a)N_0(k_0b)}{J_0'(k_0b)N_0'(k_0a) - J_0'(k_0a)N_0'(k_0b)} \tag{3-1-5}$$

$$\tau_n \equiv \pi_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} + (1+\bar{k}^2)\frac{k_0b}{n}$$

$$= \frac{k_0b}{n}\left\{\bar{k}^2 \frac{\left[J_{n-1}(k_1b)N_n'(k_1c) - J_n'(k_1c)N_{n-1}(k_1b)\right]\cdot H_n^{(2)}(k_0c) - \bar{k}H_n^{(2)'}(k_0c)\left[J_{n-1}(k_1b)N_n(k_1c) - N_{n-1}(k_1b)N_n(k_1c)\right]}{\bar{k}H_n^{(2)'}(k_0c)\left[J_n(k_1c)N_n'(k_1b) - J_n'(k_1b)N_n(k_1c)\right] - H_n^{(2)}(k_0c)\left[J_n'(k_1c)N_n'(k_1b) - J_n'(k_1b)N_n'(k_1c)\right]}\right.$$

$$\left. - \frac{J_{n+1}(k_0b)N_n'(k_0a) - J_n'(k_0a)N_{n+1}(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\right\} \tag{3-1-6}$$

then the series which represents the kernels of integral equations (3-1-1) and (3-1-2) respectively become

$$\sum_{n=0}^{\infty} \epsilon_n \left(\pi_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\right)\cos n\phi \cos n\phi'$$

$$= \sum_{n=0}^{\infty} \epsilon_n \tau_n \cos n\phi \cos n\phi' - 2(1+\bar{k}^2)k_0b \sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n} \tag{3-1-7}$$

$$\sum_{n=1}^{\infty}\left(\tau_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} - \frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)}\right) \sin n\phi \sin n\phi'$$

$$= 2\sum_{n=1}^{\infty}\tau_n \sin n\phi \sin n\phi' - 2(1+\bar{k}^2)\, k_0 b \sum_{n=1}^{\infty} \frac{\sin n\phi \sin n\phi'}{n} \quad . \qquad (3\text{-}1\text{-}8)$$

From (3-1-3) and (3-1-4), it is easily found that as $n \to \infty$

$$\tau_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} + \bar{k}^2 \frac{k_0 b}{n} \longrightarrow 2\,\bar{k}^2\, k_0 b \frac{(\frac{b}{c})^{2n}}{n}$$

and

$$\frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \longrightarrow 2\, k_0 b \frac{(\frac{a}{b})^{2n}}{n} \quad .$$

Therefore the series

$$\sum_{n=1}^{\infty} |\tau_n|$$

converges, and thus

$$\sum_{n=0}^{\infty} \epsilon_n\, \tau_n \cos n\phi \cos n\phi'$$

$$\sum_{n=1}^{\infty} \tau_n \sin n\phi \sin n\phi'$$

converge uniformly in the region $-\phi_0 \leqslant \phi\, ,\ \phi' \leqslant \phi_0$.

Hence if one replaces the infinite series with the finite sums

$$\sum_{n=0}^{N} \epsilon_n \tau_n \cos n\phi \cos n\phi' \quad ,$$

$$\sum_{n=1}^{N} \tau_n \sin n\phi \sin n\phi'$$

then the error over the square region $-\phi_0 \leqslant \phi\, ,\ \phi' \leqslant \phi_0$ is less than a constant, independent of $(\phi\, ,\ \phi')$.

Upon substituting (3-1-7) and (3-1-8) in (3-1-1) and (3-1-2) respectively and replacing the uniformly convergent series by their finite sum, we obtain

$$\int_{-\phi_0}^{\phi_0} E_e(\phi')\left\{\sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n} - \frac{1}{2(1+\bar{k}^2)\, k_0 b} \sum_{n=0}^{N} \epsilon_n \tau_n \cos n\phi \cos n\phi'\right\} d\phi'$$

$$= \frac{(k_0 b)^{-2}}{\pi(1+\bar{k}^2)} \sum_{n=0}^{\infty} \epsilon_n \frac{\cos n\phi}{J_n'(k_0 b)\, N_n'(k_0 a) - J_n'(k_0 a)\, N_n'(k_0 b)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \cos n\phi'\, d\phi' \quad , \tag{3-1-9}$$

$$\int_{-\phi_0}^{\phi_0} E_0(\phi')\left\{\sum_{n=1}^{\infty} \frac{\sin n\phi \sin n\phi'}{n} - \frac{1}{(1+\bar{k}^2)\, k_0 b} \sum_{n=1}^{N} \tau_n \sin n\phi \sin n\phi'\right\} d\phi'$$

$$= \frac{2}{\pi(1+\bar{k}^2)\,(k_0 b)^2} \sum_{n=1}^{\infty} \frac{\sin n\phi}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \sin n\phi'\, d\phi' \tag{3-1-10}$$

These are the two integral equations to be solved in this chapter.

2. Noble's Scheme

In this section, we reproduce a scheme due to Noble (Langer, 1962) to solve an integral equation of the form

$$\int_A^B E(\phi')\left[K(\phi,\phi') + \sum_{n=0}^{N} \psi_n(\phi)\,\Phi(\phi')\right] d\phi' = G(\phi) \quad . \tag{3-2-1}$$

If one knew the solutions of the auxiliary integral equations

$$\int_A^B F(\phi')\,K(\phi,\phi')\,d\phi' = G(\phi) \tag{3-2-2}$$

and

$$\int_A^B f_n(\phi')\,K(\phi,\phi')\,d\phi' = \psi_n(\phi) \; , \; n = 0, 1, 2 \dots N \; , \tag{3-2-3}$$

then upon substituting (3-2-3) in (3-2-1), we have

$$\int_A^B E(\phi')\,K(\phi,\phi')\,d\phi' + \int_A^B E(\phi')\left[\sum_{n=0}^{N} \Phi_n(\phi') \int_A^B f_n(\phi'')\,K(\phi,\phi'')\,d\phi''\right] d\phi' = G(\phi) \quad . \tag{3-2-4}$$

Let

$$\sigma_n \equiv \int_A^B E(\phi')\,\Phi_n(\phi')\,d\phi' \quad . \tag{3-2-5}$$

Then (3-2-4) reduced to

$$\int_A^B \left[E(\phi') + \sum_{n=0}^{N} \sigma_n f_n(\phi')\right] K(\phi,\phi')\,d\phi' = G(\phi) \tag{3-2-6}$$

Upon comparing this equation with (3-2-2) we have

$$E(\phi') = F(\phi') - \sum_{n=0}^{N} \sigma_n f_n(\phi') \quad . \qquad (3\text{-}2\text{-}7)$$

Multiplying both side of (3-2-7) by $\bar{\Phi}_m(\phi')$ and integrating over the interval $A \leqslant \phi' \leqslant B$, we arrive at

$$\int_A^B \bar{\Phi}_m(\phi') E(\phi') d\phi' + \sum_{n=0}^{N} \sigma_n \int_A^B f_n(\phi') \bar{\Phi}_m(\phi') d\phi' = \int_A^B F(\phi') \bar{\Phi}_m(\phi') d\phi' \quad ,$$

$m = 0, 1, 2, \ldots N$.

We let

$$A_{mn} = \int_A^B f_n(\phi') \bar{\Phi}_m(\phi') d\phi' \qquad (3\text{-}2\text{-}8)$$

$$B_n = \int_A^B F(\phi') \bar{\Phi}_m(\phi') d\phi' \qquad (3\text{-}2\text{-}9)$$

then

$$\sigma_m + \sum_{n=0}^{N} A_{mn} \sigma_n = B_m \quad , \quad m = 0, 1, 2, \ldots N \quad . \qquad (3\text{-}2\text{-}10)$$

From (3-2-10), we are able to determine σ_n .

3. Solution of (3-1-9)

Comparing (3-1-9) with (3-2-1), we observe that

$$E(\phi') = E_e(\phi')$$

$$K(\phi, \phi') = \sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n}$$

$$\Phi_n(\phi') = \epsilon_n \tau_n \cos n\phi' \qquad (3\text{-}3\text{-}1)$$

$$\psi_n(\phi) = -\frac{\cos n\phi}{2(1+\bar{k}^2)\, k_0 b}$$

$$G(\phi) = \frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\phi)\Gamma_n^{(e)}}{J_n'(k_0 b)N_n'(k_0 a) - J_n'(k_0 a)N_n'(k_0 b)}$$

where

$$\Gamma_n^{(e)} = \int_{\theta-\theta_0}^{\theta+\theta_0} E(\phi') \cos n\phi' \, d\phi' \quad ,$$

and the auxiliary integral equations are

$$\int_{-\phi_0}^{\phi_0} F_0^{(e)}(\phi') \sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n} d\phi' = \frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(n\phi)\Gamma_n^{(e)}}{J_n'(k_0 b)N_n'(k_0 a) - J_n'(k_0 a)N_n'(k_0 b)} ,$$

$$-\phi_0 \leqslant \phi \leqslant \phi_0 , \qquad (3\text{-}3\text{-}2)$$

and

$$\int_{-\phi_0}^{\phi_0} f_n^{(e)}(\phi') \sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n} d\phi' = \frac{1}{2(1+\bar{k}^2)\, k_0 b} \cos n\phi , \quad n = 0, 1, 2 \ldots N ,$$

$$-\phi_0 \leqslant \phi < \phi_0 \quad . \qquad (3\text{-}3\text{-}3)$$

These two integral equations can be solved by employing Schwinger's transformation (Lewin, 1951) which is defined in (A-2-4). For detailed derivations of the solutions, the reader may refer to A-2; here we only state the results:

$$F_0^{(e)}(\phi) = \frac{\sqrt{2}}{\pi(1+\bar{k}^2)(k_0 b)^2} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \frac{S_0^{(e)}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} + \sum_{m=1}^{\infty} \frac{m S_m^{(e)}}{\pi^2} \cos\left[m\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right] \right\} ,$$

$$-\phi_0 < \phi < \phi_0 , \qquad (3\text{-}3\text{-}4)$$

$$f_n^{(e)}(\phi) = -\frac{\sqrt{2}}{2(1+\bar{k}^2)k_0 b} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \frac{X_{0n}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} + \sum_{m=1}^{n} \frac{m X_{mn}}{\pi^2} \cos\left[m\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right] \right\} ,$$

$$-\phi_0 < \phi < \phi_0 , \qquad (3\text{-}3\text{-}5)$$

where

$$S_0^{(e)} = \sum_{p=0}^{\infty} \frac{\epsilon_n \Gamma_p^{(e)} X_{op}}{J_p'(k_0 b)N_p'(k_0 a) - J_p'(k_0 a)N_p'(k_0 b)} , \qquad (3\text{-}3\text{-}6)$$

$$S_m^{(e)} = 2\sum_{p=m}^{\infty} \frac{\Gamma_p^{(e)} X_{mp}}{J_p'(k_0 b)N_p'(k_0 a) - J_p'(k_0 a)N_p'(k_0 b)} . \qquad (3\text{-}3\text{-}7)$$

and

$$X_{mp} = \int_{-\pi}^{\pi} \cos(ms)\cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s\right)\right] ds . \qquad (3\text{-}3\text{-}8)$$

Therefore the solution of equation (3-1-9) is:

$$E_e(\phi)=\frac{\sqrt{2}}{\pi(1+\bar{k}^2)(k_0b)^2}\frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi-\cos\phi_0}}\left\{\frac{S_0^{(e)}}{4\pi^2\ln\csc\frac{\phi_0}{2}}+\sum_{m=1}^{\infty}\frac{mS_m^{(e)}}{\pi^2}\cos\left[m\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi-\cot^2\frac{\phi_0}{2}\right)\right]\right.$$

$$\left.+\frac{\pi k_0b}{2}\sum_{n=0}^{N}\sigma_n^{(e)}\left[\frac{X_{on}}{4\pi^2\ln\csc\frac{\phi_0}{2}}+\sum_{m=1}^{n}\frac{m}{\pi^2}X_{mn}\cos\left[m\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi-\cot^2\frac{\phi_0}{2}\right)\right]\right]\right\},$$

$$-\phi_0<\phi<\phi_0\ . \qquad (3\text{-}3\text{-}9)$$

In the last equation, the coefficients $\sigma_n^{(e)}$ can be determined by solving the simultaneous equations

$$\sigma_m^{(e)}+\sum_{n=0}^{N}\sigma_n^{(e)}A_{mn}^{(e)}=B_m^{(e)},\quad m=0,\ 1,\ 2,\ \ldots\ N\ , \qquad (3\text{-}3\text{-}10)$$

where

$$A_{mn}^{(e)}=\epsilon_m\tau_m\int_{-\phi_0}^{\phi_0}f_n^{(e)}(\phi)\cos m\phi\, d\phi\ , \qquad (3\text{-}3\text{-}11)$$

$$B_m^{(e)}=\epsilon_m\tau_m\int_{-\phi_0}^{\phi_0}F_0^{(e)}\cos m\phi\, d\phi\ , \qquad (3\text{-}3\text{-}12)$$

To express $A_{mn}^{(e)}$ and $B_m^{(e)}$ explicitly, we substitute (3-3-5) in (3-3-11) and (3-3-4) in (3-3-12) and obtain

$$A_{mn}^{(e)}=-\frac{\epsilon_m\tau_m}{2(1+\bar{k}^2)k_0b}\left\{\frac{X_{on}X_{om}}{4\pi^2\ln\csc\frac{\phi_0}{2}}+\sum_{q=1}^{n}\frac{q}{\pi^2}X_{qn}X_{qm}\right\}, \qquad (3\text{-}3\text{-}13)$$

$$B_m^{(e)} = \frac{\epsilon_m \tau_m}{\pi(1+\bar{k}^2)(k_0 b)^2} \left\{ \frac{S_0^{(e)} X_{om}}{4\pi^2 \ln \csc \frac{\phi_0}{2}} + \sum_{q=1}^{m} \frac{q S_q^{(e)}}{\pi^2} X_{qm} \right\} . \tag{3-3-14}$$

4. Solution of (3-1-10)

To start this section, we introduce a transformation

$$W(\phi) = \int_{-\phi_0}^{\phi_0} E_0(\phi')\, d\phi' \tag{3-4-1}$$

for the odd part of the unknown slot field, $E_0(\phi)$. Since for $\phi' < \phi_0$, $E(\phi')$ behaves as

$$\frac{1}{\sqrt{\phi_0 - \phi'}} \text{ when } |\phi'| \longrightarrow \phi_0 ,$$

but is otherwise continuous, therefore the integral $\int_0^{\pm\phi_0} E_0(\phi')\, d\phi$ exists and the function $W(\phi)$ is defined at every point inside the closed interval $-\phi_0 \leq \phi \leq \phi_0$ while its first derivative exists in the corresponding open interval. Thus one has

$$E_0(\phi) = \frac{dW(\phi)}{d\phi} , \quad -\phi_0 < \phi < \phi_0 .$$

Also because $E_0(\phi)$ is an odd function of ϕ, $W(\phi)$ is symmetric with respect to ϕ. One may set

$$W(-\phi_0) = 0$$

and it then follows that

$$W(\phi_0) = 0 .$$

Thus integrating by parts

$$\int_{-\phi_0}^{\phi_0} E_0(\phi') \sin n\phi' \, d\phi' = \sin n\phi' \, W(\phi') \Big|_{-\phi_0}^{\phi_0} - n \int_{-\phi_0}^{\phi_0} W(\phi') \cos n\phi' \, d\phi'$$

$$= - n \int_{-\phi_0}^{\phi_0} W(\phi') \cos n\phi' \, d\phi' \quad . \tag{3-4-2}$$

Let

$$\Gamma_n^{(0)} = \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \sin n\phi' \, d\phi' \, . \tag{3-4-3}$$

Applying (3-4-2) and (3-4-3) to the integral equation (3-1-10), we obtain

$$\int_{-\phi_0}^{\phi_0} W(\phi') \left\{ \sum_{n=1}^{\infty} \sin n\phi \cos n\phi' - \frac{1}{(1+\bar{k}^2) k_0 b} \sum_{n=1}^{N} n \tau_n \sin n\phi \cos n\phi' \right\} d\phi'$$

$$= - \frac{2}{\pi (1+\bar{k}^2)(k_0 b)^2} \sum_{n=1}^{\infty} \frac{\Gamma_n^{(0)} \sin n\phi}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \quad . \tag{3-4-4}$$

Last equation has the same form as (3-2-1), i.e.,

$$E(\phi') = W(\phi')$$

$$K(\phi, \phi') = \sum_{n=1}^{\infty} \sin n\phi \cos n\phi'$$

$$\Phi_n(\phi') = n \tau_n \cos n\phi'$$

$$\psi_n(\phi) = - \frac{\sin n\phi}{(1+\bar{k}^2)(k_0 b)}$$

$$G(\phi) = - \frac{2}{\pi (1+\bar{k}^2)(k_0 b)^2} \sum_{n=1}^{\infty} \frac{\Gamma_n^{(0)} \sin n\phi}{J_n'(k_0 b) N_n'(k_0 a) - J_n'(k_0 a) N_n'(k_0 b)} \quad .$$

The auxiliary integral equations for (3-4-4) are:

$$\int_{-\phi_0}^{\phi_0} F_0^{(0)}(\phi')\sum_{n=1}^{\infty} \sin n\phi \cos n\phi' \, d\phi' = -\frac{2}{\pi(1+\bar{k}^2)(k_0b)^2}\sum_{n=1}^{\infty}\frac{\Gamma_n^{(0)}\sin n\phi}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0a)} ,$$

$$-\phi_0 < \phi < \phi_0 , \qquad (3\text{-}4\text{-}5)$$

$$\int_{-\phi_0}^{\phi_0} f_n^{(0)}(\phi')\sum_{n=1}^{\infty} \sin n\phi \cos n\phi' \, d\phi' = -\frac{1}{(1+\bar{k}^2)(k_0b)}\sin n\phi ,$$

$$-\phi_0 < \phi < \phi_0 . \qquad (3\text{-}4\text{-}6)$$

We may again apply the Schwinger's transformation to the last two integral equations and obtain the solutions for $F_0^{(0)}(\phi)$ and $f_n^{(0)}(\phi)$. Similar to last section, we leave the detailed derivations to A-3 and state the results as:

$$F_0^{(0)}(\phi) = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)^2}\cos\frac{\phi}{2}\sqrt{\cos\phi - \cos\phi_0}\sum_{m=1}^{\infty} a_m^{(0)} U_m(\phi) , \quad -\phi_0 < \phi < \phi_0 ,$$

(3-4-7)

$$f_n^{(0)}(\phi) = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)}\cos\frac{\phi}{2}\sqrt{\cos\phi - \cos\phi_0}\sum_{m=1}^{\infty} b_{nm}^{(0)} U_m(\phi) , \quad -\phi_0 < \phi < \phi_0$$

(3-4-8)

where

$$a_m^{(0)} = \frac{1}{2\pi}\left(\sum_{p=m-1}^{\infty}\epsilon_p L_p X_{m-1,p} - \sum_{p=m+1}^{\infty} 2 L_p X_{m+1,p}\right) , \qquad (3\text{-}4\text{-}9)$$

$$b_{nm}^{(0)} = \frac{1}{2\pi}\left(\sum_{p=0,2,4,}^{n-1}\epsilon_p X_{m-1,p} - \sum_{p=2,4,}^{n-1} 2 X_{m+1,p}\right) , \quad n = \text{odd} , \qquad (3\text{-}4\text{-}10a)$$

$$b_{nm}^{(0)} = \frac{1}{\pi} \sum_{p=1,3,5,}^{n-1} (X_{m-1,p} - X_{m+1,p}) \quad , \quad n = \text{even} \quad , \qquad (3\text{-}4\text{-}10b)$$

$$U_m(\phi) = \sum_{p=0}^{m-1} \epsilon_p (-1)^{m-p} (m-p) \cos\left[p \cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right] ,$$

$$-\phi_0 \le \phi \le \phi_0 \quad , \qquad (3\text{-}4\text{-}11)$$

and

$$L_p = \sum_{m=p+1,p+3,..}^{\infty} \frac{\Gamma_m^{(0)}}{J'_m(k_0 b) N'_m(k_0 a) - J'_m(k_0 a) N'_m(k_0 b)} \quad . \qquad (3\text{-}4\text{-}12)$$

Thus the solution of integral equation (3-4-4) is

$$W(\phi) = \frac{\sqrt{2}\cos\frac{\phi}{2}}{(1+\bar{k}^2)(\pi k_0 b)^2} \sqrt{\cos\phi - \cos\phi_0} \left\{ \sum_{m=1}^{\infty} a_m^{(0)} U_m(\phi) \right.$$

$$\left. + \pi k_0 b \sum_{n=1}^{N} \sigma_n^{(0)} \sum_{m=1}^{n} b_{nm}^{(0)} U_m(\phi) \right\} \quad . \qquad (3\text{-}4\text{-}13)$$

In the above expression, the coefficients $\sigma_n^{(0)}$ can be found by solving the simultaneous equations

$$\sigma_m^{(0)} + \sum_{n=1}^{N} A_{mn}^{(0)} \sigma_n^{(0)} = B_m^{(0)} \quad , \qquad m = 1, 2, \ldots N \quad , \qquad (3\text{-}4\text{-}14)$$

where

$$A_{mn}^{(0)} \equiv \int_{-\phi_0}^{\phi_0} m \tau_m \cos(m\phi') f_n^{(0)}(\phi')\, d\phi' \qquad (3\text{-}4\text{-}15)$$

$$B_m^{(0)} \equiv \int_{-\phi_0}^{\phi_0} m\tau_m \cos(m\phi') F_0^{(0)}(\phi') d\phi' \quad . \tag{3-4-16}$$

We perform the integrations in Appendix 5, and obtain

$$A_{mn}^{(0)} = \frac{\sin^2\left(\frac{\phi_0}{2}\right) m\tau_m}{(1+\bar{k}^2)(\pi k_0 b)} \sum_{p=1}^{n} b_{np}^{(0)} (\cos(p\pi) X_{0m} - X_{pm}) \quad , \tag{3-4-17}$$

$$B_m^{(0)} = \frac{\sin^2\left(\frac{\phi_0}{2}\right) m\tau_m}{(1+\bar{k}^2)(\pi k_0 b)^2} \sum_{p=1}^{\infty} a_p^{(0)} (\cos(p\pi) X_{0m} - X_{pm}) \quad . \tag{3-4-18}$$

The antisymmetric part of the ϕ-directed electric field in the shell slot can be obtained by differentiating $W(\phi)$, i.e.,

$$E_0(\phi) = \frac{dW(\phi)}{d\phi} \quad , \qquad -\phi_0 < \phi < \phi_0 \quad .$$

We carry out the differentiation in Appendix 6, and state that

$$E_0^{(0)} = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)^2} \frac{\sin\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \quad .$$

$$\left\{ \cos^2\left(\frac{\phi_0}{2}\right) \left[\sum_{m=1}^{\infty} a_m^{(0)} U_m(\phi) + \pi k_0 b \sum_{n=1}^{N} \sigma_n^{(0)} \sum_{m=1}^{n} b_{nm}^{(0)} U_m(\phi) \right] \right.$$

$$\left. + (1+\cos\phi) \left[\sum_{m=1}^{\infty} a_m^{(0)} m V_m(\phi) + \pi k_0 b \sum_{n=1}^{N} \sigma_n^{(0)} \sum_{m=1}^{n} b_{nm}^{(0)} m V_m(\phi) \right] \right\} ,$$

$$-\phi_0 < \phi < \phi_0 \quad , \tag{3-4-19}$$

where

$$V_m(\phi) = 1 + 2\sum_{p=1}^{m-1} \cos\left[p\cos^{-1}(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2})\right] , \quad m \text{ odd} ,$$

$$= 2\sum_{p=1}^{m-1} \cos\left[p\cos^{-1}(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2})\right] , \quad m \text{ even} . \qquad (3\text{-}4\text{-}20)$$

5. Discussion

In the first section of this chapter, we obtained a pair of integral equations (3-1-9) and (3-1-10) from (3-1-1) and (3-1-2) respectively by truncating the uniformly convergent series. Therefore (3-3-9) and (3-4-19) are the approximate solutions of (3-1-1) and (3-1-2), respectively. The accuracy of these approximate solutions depend largely on the value of N. But $N + 1$ and N, respectively, are the degrees of freedom of the simultaneous systems (3-3-10) and (3-4-14). We may encounter the usual difficulties of solving a large simultaneous system of algebraic equations. We attempt to reduce this difficulty here.

Expression (3-3-9) suggests a transformation

$$z_m^{(e)} = \sum_{n=m}^{N} \sigma_n^{(e)} X_{mn} . \qquad (3\text{-}5\text{-}1)$$

Upon substituting this transformation in (3-3-9), one has

$$E_e(\phi) = \frac{\sqrt{2}}{\pi(1+\bar{k}^2)(k_0 b)^2} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \frac{S_0^{(e)}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} \right.$$

$$\left. + \sum_{m=1}^{\infty} \frac{mS_m^{(e)}}{\pi} \cos\left[m \cos^{-1}(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2})\right] \right.$$

$$+\frac{\pi k_0 b}{2}\left[\frac{1}{4\pi^2 \ln\csc\frac{\phi_0}{2}} z_0^{(e)} + \sum_{m=1}^{N} \frac{m}{\pi^2} z_m^{(e)} \cos\left(m \cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right)\right]\Bigg\} ,$$

$$-\phi_0 < \phi < \phi_0 . \qquad (3\text{-}5\text{-}2)$$

If we multiply (3-3-10) by X_{pm}, and sum on m, with the knowledge of (3-5-1), we have

$$z_p^{(e)} - \frac{\sum_{m=o}^{N} \epsilon_m \tau_m X_{0m} X_{pm}}{8\pi^2(1+\bar{k}^2)(k_0 b)\ln\csc\frac{\phi_0}{2}} z_0^{(e)} + \frac{1}{\pi^2(1+\bar{k}^2)k_0 b} \sum_{q=1}^{N} q\left(\sum_{m=p}^{N} \tau_m X_{qm} X_{pm}\right) z_q^{(e)}$$

$$= \frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2}\left[\frac{S_0^{(e)}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} \sum_{m=p}^{N} \epsilon_m \tau_m X_{0m} X_{pm}\right.$$

$$\left. + \frac{2}{\pi^2} \sum_{q=1}^{N} q S_q^{(e)} \sum_{m=p}^{N} \tau_m X_{qm} X_{pm}\right] . \qquad (3\text{-}5\text{-}3)$$

Upon multiplying (3-5-3) with the factor R defined as

$$R \equiv \frac{4\pi^2(1+\bar{k}^2)k_0 b}{\sum_{m=0}^{N} \epsilon_m \tau_m X_{0m}^2} , \qquad (3\text{-}5\text{-}4)$$

one obtains a new system of equations

$$R z_p^{(e)} - \frac{S_{0p}}{2\ln\csc\frac{\phi_0}{2}} z_0^{(e)} - 4\sum_{q=1}^{N} q S_{qp} z_q^{(e)}$$

$$= \frac{S_0^{(e)}}{\pi k_0 b \ln\csc\frac{\phi_0}{2}} S_{0p} + 8\sum_{m=1}^{\infty} \frac{m S_m^{(e)}}{\pi k_0 b} S_{mp} ,$$

$$p = 0, 1, 2, \ldots N \qquad (3\text{-}5\text{-}5)$$

where

$$S_{qp} = \frac{\sum_{m=p}^{N} \epsilon_m \tau_m X_{qm} X_{pm}}{\sum_{m=0}^{N} \epsilon_m \tau_m X_{0m}^2} \quad . \tag{3-5-6}$$

If we express the simultaneous system in a form of a matrix equation, we have

$$\left[a_{qp}\right] \cdot z_p^{(e)}\Big] = b_p\Big] \ , \quad q, p = 0, 1, \ldots N \quad . \tag{3-5-7}$$

where $\left[a_{qp}\right]$ denotes a square matrix of order $N + 1$, while $z_p^{(e)}\Big]$ and $b_p\Big]$ denote the column matrix of the same order. Comparing (3-5-7) with (3-5-5), we obtain

$$a_{0p} = R\delta_{0p} - \frac{S_{0p}}{2 \ln \csc \frac{\phi_0}{2}} \ ,$$

$$a_{qp} = R\,\delta_{qp} - 4q\,S_{qp} \ , \quad q \neq 0 \tag{3-5-8}$$

where

$$\delta_{qp} = 1 \ , \ \text{if } q = p \ ,$$

$$= 0 \ , \ \text{if } q \neq p \ ,$$

and

$$b_p = \frac{S_0^{(e)}}{\pi k_0 b \ln \csc \frac{\phi_0}{2}} S_{0p} + 8 \sum_{m=1}^{\infty} \frac{m S_m^{(e)}}{\pi k_0 b} S_{mp} \quad . \tag{3-5-9}$$

S_{qp} plays an important role in further reducing the matrix equation (3-5-7). In the following paragraph, we state some of the properties of S_{qp}.

In Appendix 4, we show that for any ϕ_0, $X_{qm} = 0$ when $q > m$. Upon employing this property of X_{qm}, we may conclude that S_{qp} is symmetric with respect to its subscript index q and p, i.e.,

$$S_{qp} = S_{pq} \quad . \tag{3-5-10}$$

We recall that τ_m is defined by (3-1-6). It depends on k_0a, k_0b, k_0c and $\bar{k}$. For $m > m_0$ ($m_0 \sim 2k_0b < N$), τ_m behaves as $\frac{1}{m}\left[(\frac{b}{c})^{2m} + (\frac{a}{b})^{2m}\right]$ for $\frac{\omega_p}{\omega} \neq 0$ and as $\frac{1}{m}\left[\frac{1}{m(m-1)} + (\frac{a}{b})^{2m}\right]$ for $\frac{\omega_p}{\omega} = 0$. For $m < k_0b$, the τ_m values are large and may be oscillating in sign. From the (A-4-5) property of X_{jk} which is also explicitly accounted for as far as the p subscript is concerned in the definition of S_{qp} in (3-5-6), we see that as p increases, the sum making up S_{qp} consists of terms involving τ_m for which $m > p$. But the τ_m terms decrease rapidly once $m > m_0$ and thus since $|X_{jk}| < 2\pi$ we see that S_{qp} will decrease rapidly once $p > m_0$. Because of (3-5-10), the same behavior is exhibited also on the q subscript of S_{qp}. The properties of S_{qp} are further modified if we consider the angular width of the slot $2\phi_0$. From the discussion in Appendix 4, , it is clear that for ϕ_0 sufficiently small, there is a number j such that

$$X_{jk} \sim 0 \quad , \quad k < m_0 \quad .$$

The net effect of this is that the magnitude of S_{qp} is further reduced as either q-subscript or p-subscript increases.

In view of this discussion as can be seen from Eq. (3-5-8), the matrix $\left[a_{qp}\right]$ can be reduced in size. We indicate the size of this reduced matrix by N'. In fact, for a very narrow slot, we only need to consider in the matrix the first element a_{00}, i.e., N' = 0.

If the slot is very wide, ($\phi_0 = \pi - \Delta$, Δ is very small) then $\epsilon_j X_{jk} \simeq 2\pi\delta_{jk}$ and $S_{qp} \simeq (\tau_q/\tau_0)\delta_{qp}$, q, p = 0, 1, 2, ... N. Therefore the matrix $\left[a_{qp}\right]$ becomes a diagonal one, i.e., the problem becomes a separable one.

The fact that S_{qp} in (3-5-8) is multiplied by q does not change the order of magnitude of our arguments.

For the odd part of the ϕ-directed electric field $E_0(\phi)$ in the slot, given by (3-4-20), we introduce a transformation

$$z_p^{(0)} = \sum_{m=p}^{N} \sigma_m^{(0)} b_{mp}^{(0)} \quad . \tag{3-5-11}$$

Then

$$E_0(\phi) = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)} \frac{\sin\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \sum_{m=1}^{\infty} a_m^{(0)} \left[\cos^2\frac{\phi_0}{2} U_m(\phi) + m(1+\cos\phi) V_m(\phi)\right] \right.$$

$$\left. + \pi k_0 b \sum_{m=1}^{N} z_m^{(0)} \left[\cos^2\frac{\phi_0}{2} U_m(\phi) + m(1+\cos\phi) V_m(\phi)\right] \right\} \tag{3-5-12}$$

where $U_m(\phi)$ and $V_m(\phi)$ are given by (3-4-11) and (3-4-20), respectively. Upon using the transformation (3-5-11), (3-4-14) can be reduced to a new simultaneous system of algebraic equations. We express this new system in a matrix form

$$\left[a'_{qp}\right] \cdot \left[z_p^{(0)}\right] = \left[b'_p\right] \quad , \quad q, p = 1, 2, \dots N \tag{3-5-13}$$

where $\left[a'_{qp}\right]$ is a square matrix of degree N, $\left[z_p^{(0)}\right]$ and $\left[b'_p\right]$ are the column matrix of the same degree. The elements of $\left[a'_{qp}\right]$ and $\left[b'_p\right]$ are, respectively,

$$a'_{qq} = R' + \sin^2\left(\frac{\phi_0}{2}\right)\left[T_{0q}\cos q\pi - T_{qq}\right]$$

$$a'_{qp} = \sin^2\left(\frac{\phi_0}{2}\right)\left(T_{op}\cos q\pi - T_{qp}\right) \quad , \quad q \neq p \tag{3-5-14}$$

and

$$b'_p = \sin^2 \frac{\phi_0}{2} \sum_{m=1}^{\infty} a_m^{(0)} \left[T_{op} \cos m\pi - T_{mp} \right] \quad . \qquad (3\text{-}5\text{-}15)$$

In the above equations, we define

$$R' \equiv \frac{(1+\bar{k}^2)(\pi k_0 b)^2}{\sum_{m=1}^{N} m \tau_m X_{1m} b_{m1}^{(0)}} \qquad (3\text{-}5\text{-}16)$$

and

$$T_{qp} = \frac{\sum_{m=p}^{N} m\tau_m X_{qm} b_{mp}^{(0)}}{\sum_{m=1}^{N} m\tau_m X_{1m} b_{m1}^{(0)}} \quad . \qquad (3\text{-}5\text{-}17)$$

We observe that in (3-5-13) T_{qp} plays the same role as S_{qp} in (3-5-7). $X_{qm} = 0$ if $q > m$, therefore T_{qp} becomes

$$T_{qp} = \frac{\sum_{m=q}^{N} m\tau_m X_{qm} b_{mp}^{(0)}}{\sum_{m=1}^{N} m\tau_m X_{1m} b_{m1}^{(0)}} \quad . \qquad (3\text{-}5\text{-}18)$$

We note that from (3-4-10), $\left| b_{mp}^{(0)} \right| < m$, thus in the numerator of (3-5-17) τ_m is at most multiplied by m^2. But since τ_m decreases rapidly for increasing m when $m > m_0$ the τ_m behavior will prevail over m^2. Therefore the magnitude of T_{qp} will decrease rapidly as either p or q exceeds m_0. The effect of the slot width enters into T_{qp} in a similar manner as for S_{qp}.

Based on the above discussion, we conclude that $\left[a'_{qp}\right]$ can be reduced in size for a narrow slot. We denote this reduced size by N''. For a very narrow slot, we may choose $N'' = 1$, i.e., we only need to consider the first element a_{11}.

For a very wide slot, because of the property of X_{jk}, $T_{qp} = \delta_{qp}$. Matrix $\left[a'_{qp}\right]$ is thus reduced to a diagonal form.

6. Solution for Narrow Slot

In this section, we shall extract the solutions for the narrow slot from the general solutions (3-5-2) and (3-5-12). This is a case of some practical importance. In the later chapters, we use these results to attain an approximate solution for the ϕ-directed electric field in the wedge slot and to have an explicit form of the terminal admittance of the wedge waveguide.

If the slot width $2\phi_0$ is so narrow that we may apply the approximate relations

$$X_{op} \simeq 1 - O(N\phi_0^2)$$

$$X_{1p} \simeq O(N\phi_0^2)$$

to (3-5-2) and (3-5-12), then we may neglect all terms of order $O(N\phi_0^2)$. The results are:

$$E_e(\phi) \simeq \frac{\sqrt{2}}{4\pi^3(1+\bar{k}^2)(k_0 b)^2 \ln\csc\frac{\phi_0}{2}} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ S_o^{(e)} + \frac{\pi k_0 b}{2} z_0^{(e)} \right\} ,$$

$$-\phi_0 < \phi < \phi_0 \quad . \qquad (3\text{-}6\text{-}1)$$

and

$$E_0(\phi) \simeq \frac{\sqrt{2}\,(a_1^{(0)} + \pi k_0 b\, z_1^{(0)})}{(1+\bar{k}^2)(\pi k_0 b)^2} \frac{\sin\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}}$$

$$\cdot \left\{ \cos^2\!\left(\frac{\phi_0}{2}\right) U_1(\phi) + (1+\cos\phi)\, V_1(\phi) \right\} , \quad -\phi_0 < \phi < \phi_0 \quad . \qquad (3\text{-}6\text{-}2)$$

Since $U_1(\phi) = -1$, $V_1(\phi) = 1$, (Eqs. (3-4-11) and (3-4-20)) (3-6-2) can be further reduced to

$$E_0(\phi) \simeq \frac{\sqrt{2}\,(a_1^{(0)} + \pi k_0 b\, z_1^{(0)})}{(1+\bar{k}^2)\,(\pi k_0 b)^2} \; \frac{\sin\frac{\phi}{2}\cos\phi}{\sqrt{\cos\phi - \cos\phi_0}} \;, \quad -\phi_0 < \phi < \phi_0 \;. \tag{3-6-3}$$

The unknown factors $z_0^{(e)}$ and $z_1^{(0)}$ in the last two equations can be obtained by choosing $N = 0$ in (3-5-7) and $N = 1$ in (3-5-13), i.e.,

$$\left(R - \frac{S_{00}}{2\ln\csc\frac{\phi_0}{2}}\right) z_0^{(e)} = \frac{S_0^{(e)}}{\pi k_0 b\,\ln\csc\frac{\phi_0}{2}}\, S_{00} \tag{3-6-4}$$

and

$$R' z_1^{(0)} \simeq \frac{\sin^2\frac{\phi_0}{2}}{(1+\bar{k}^2)(\pi k_0 b)^2}\, a_1^{(0)}\, T_{01} \;. \tag{3-6-5}$$

From (3-6-5), it is seen that $z_1^{(0)}$ is of $O(\phi_0^2)$, therefore we may neglect $z_1^{(0)}$ in comparison with $a_1^{(0)}$ in (3-6-3). Upon substituting (3-6-4) in (3-6-1) and reorganizing terms, we have

$$E_e(\phi) \simeq \frac{2\sqrt{2}}{\pi k_0 b}\; \frac{S_0^{(e)}}{8\pi^2(1+\bar{k}^2)(k_0 b)\ln\csc\frac{\phi_0}{2} - \sum_{m=0}^{N} \epsilon_m \tau_m X_{0m}^2}\; \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \;,$$

$$-\phi_0 < \phi < \phi_0 \,. \tag{3-6-6}$$

Employing the approximate formulas

$$\cos y \simeq 1 - \frac{y^2}{2} ,$$

$$\sin y \simeq y , \quad y << 1 ,$$

the even part and odd part of the slot electric field can be further simplified to

$$E_e(\phi) \simeq \frac{4S_0^{(e)}}{\pi k_0 b \left[8\pi^2(1+\bar{k}^2) k_0 b \ln \frac{2}{\phi_0} - \sum_{m=0}^{N} \epsilon_m \tau_m X_{0m}^2 \right]} \cdot \frac{1}{\sqrt{\phi_0^2 - \phi^2}} ,$$

$$-\phi_0 < \phi < \phi_0 . \qquad (3\text{-}6\text{-}7)$$

and

$$E_0(\phi) \simeq \frac{a_1^{(0)}}{\pi^2 (1 + \bar{k}^2)(k_0 b)^2} \frac{\phi}{\sqrt{\phi_0^2 - \phi^2}} , \quad -\phi_0 < \phi < \phi_0 . \qquad (3\text{-}6\text{-}8)$$

CHAPTER IV

APPROXIMATE SOLUTION OF INTEGRAL EQUATION (2-3-32)

1. Introduction:

If we interchange summation with integration and reorganize the terms on the left hand side of (2-3-32), we have

$$\int_{\theta+\theta_0}^{\theta+\theta_0} \hat{E}(\phi') \Bigg\{ 2k_0 a \frac{\theta_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0) \cos\frac{n\pi}{2\theta_0}(\phi'-\theta+\theta_0) + \cos n(\phi-\phi') \right]$$

$$+ \frac{J_0(k_0a)}{J_0'(k_0a)} + \frac{\theta_0}{\pi} \frac{J_0(k_0a)N_0'(k_0b) - J_0'(k_0b)N_0(k_0a)}{J_0'(k_0b)N_0'(k_0a) - J_0'(k_0a)N_0'(k_0b)} + \sum_{n=1}^{\infty} \left(\frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} - \frac{k_0 a}{\frac{n\pi}{2\theta_0}} \right) .$$

$$\cos\frac{n\pi}{2\theta_0}(\phi-\theta_0+\theta_0)\cos\frac{n\pi}{2\theta_0}(\phi'+\theta-\theta_0) + \frac{\theta_0}{\pi} \cdot 2\sum_{n=1}^{\infty} \left(\frac{J_n(k_0a)N_n'(k_0b) - (J_n'(k_0b)N_n(k_0a)}{J_n'(k_0b)N_n'(k_0a) - J'_0(k_0a)N_n'(k_0b)} - \frac{k_0 a}{n} \right) \cos n(\phi-\phi') \Bigg\} d\phi'$$

$$= -\frac{V_0}{a} \frac{1}{J_0'(k_0a)} + \frac{2}{\pi k_0 a} \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \frac{\epsilon_n}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} \int_{-\phi_0}^{\phi_0} E(\phi') \cos n(\phi-\phi') d\phi' ,$$

$$\theta - \theta_0 \le \phi \le \theta + \theta_0 \quad . \tag{4-1-1}$$

From the recurrence relation of cylindrical functions

$$z Z_{p+1}(z) = p Z_p'(z) - z Z_p(z) \quad , \tag{4-1-2}$$

one may easily show that

$$\frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} - \frac{k_0 a}{\frac{n\pi}{2\theta_0}} = \frac{2k_0 a \theta_0}{\pi} \frac{J_{(\frac{n\pi}{2\theta_0}+1)}(k_0a)}{n J'_{\frac{n\pi}{2\theta_0}}(k_0a)} \tag{4-1-3}$$

and

$$\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}-\frac{k_0a}{n}=\frac{k_0a}{n}\,\frac{J_{n-1}(k_0a)N'_n(k_0b)-J'_n(k_0b)N_{n-1}(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)} \quad . \tag{4-1-4}$$

For $\frac{n\pi}{2\theta_0} >> k_0 a$ in (4-1-3) and $n >> k_0a$, k_0b in (4-1-4) the last two equations behave, respectively, as $\frac{1}{n^3}$ and $\frac{1}{n^3} + \frac{1}{n}(\frac{a}{b})^{2n}$. Therefore the second and the third series of the kernel of integral Eq. (4-1-1) are uniformly convergent on a square interval

$$\theta - \theta_0 \leq \phi, \phi' \leq \theta + \theta_0 \quad ,$$

while the first series has a logarithmic singularity when $\phi' \longrightarrow \phi$. Thus the chance of solving (4-1-1) depends largely on whether or not one can solve the integral equation

$$\int_{-\theta_0}^{\theta_0} F(\eta') \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\frac{n\pi}{\theta_0}\eta \cos\frac{n\pi}{\theta_0}\eta' + \cos n\eta \cos n\eta'\right] d\eta' = G(\eta) \quad . \tag{4-1-5}$$

Unfortunately Schwinger's transformation is not applicable to this integral equation. Therefore, to solve (4-1-1), a new transformation of some form is required. If both the wedge slot width $2\theta_0$ and shell slot width $2\phi_0$ are much smaller than unity, we may substitute (3-6-7) and (3-6-8) in (2-3-32) and then employ Galekin's method (Kantorovich, 1958) to obtain an approximate solution for the integral equation (2-3-32).

2. Reduction of Integral Eq. (2-3-32):

In Eq. (2-3-32), the variables ϕ and ϕ' are referred to the center of the shell slot, while the unknown function is the ϕ-directed electric field in the wedge aperture. It is more convenient to express (2-3-32) as function of a new set of variables η and η' defined as

$$\eta = \phi - \theta \ ,$$

$$\eta' = \phi' - \theta \ . \qquad (4\text{-}2\text{-}1)$$

It is seen that from (4-2-1), η and η' are referred to the center of the wedge aperture.

Upon substituting (4-2-1) in (2-3-32), we obtain

$$\sum_{n=0}^{\infty} \epsilon_n \frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} \cos \frac{n\pi}{2\theta_0}(\eta+\theta_0) \int_{-\theta_0}^{\theta_0} \hat{E}(\eta')\cos \frac{n\pi}{2\theta_0}(\eta'+\theta_0)\, d\eta'$$

$$+ \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{J_n(k_0a)N'_n(k_0b) - J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)} \int_{-\theta_0}^{\theta_0} \hat{E}(\eta')\cos n(\eta-\eta')\, d\eta'$$

$$= -\frac{V_0}{a}\frac{1}{J'_0(k_0a)} + \frac{1}{\pi k_0 a}\frac{\theta_0}{\pi}\sum_{n=0}^{\infty} \frac{\epsilon_n}{J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)} \int_{-\phi_0-\theta}^{\phi_0-\theta} \hat{E}(\eta')\cos n(\eta-\eta')\, d\eta' \ .$$

(4-2-2)

In (4-2-2) we may regard $\hat{E}(\eta)$ as the sum of a symmetric part $\hat{E}_e(\eta)$ and antisymmetric part $\hat{E}_0(\eta)$; thus

$$\hat{E}(\eta) = \hat{E}_e(\eta) + \hat{E}_0(\eta) \ . \qquad (4\text{-}2\text{-}3)$$

Since $\cos \frac{n\pi}{2\theta_0}(\eta+\theta_0)$ is an even function of η when n is even, and an odd function of η when n is odd, we have

$$\int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta)\cos \frac{n\pi}{2\theta_0}(\eta+\theta_0)\, d\eta = 0 \ , \quad n = 1, 3, 5, \ldots \qquad (4\text{-}2\text{-}4)$$

$$\int_{-\theta_0}^{\theta_0} \hat{E}_o(\eta) \cos \frac{n\pi}{2\theta_0} (\eta + \theta_0) d\eta = 0 \quad , \qquad n = 0, 2, 4, \ldots \quad . \tag{4-2-5}$$

We also know that

$$\int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta) \sin n\eta \, d\eta = 0 \quad , \tag{4-2-6}$$

$$\int_{-\theta_0}^{\theta_0} \hat{E}_o(\eta) \cos n\eta \, d\eta = 0 \quad . \tag{4-2-7}$$

If we substitute (4-2-3) into (4-2-2), and use the relations (4-2-4) through (4-2-7) we may obtain two equations:

$$\sum_{n=0}^{\infty} \epsilon_n \frac{J_{\frac{n\pi}{\theta_0}}(k_0 a)}{J'_{\frac{n\pi}{\theta_0}}(k_0 a)} \cos \frac{n\pi}{\theta_0}(\eta + \theta_0) \int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta') \cos \frac{n\pi}{\theta_0}(\eta' + \theta_0) d\eta'$$

$$+ \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \frac{J_n(k_0 a) N'_n(k_0 b) - J_n(k_0 b) N'_n(k_0 a)}{J'_n(k_0 b) N'_n(k_0 a) - J'_n(k_0 a) N'_n(k_0 b)} \cos n\eta \int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta') \cos n\eta' \, d\eta$$

$$= -\frac{V_0}{a} \frac{1}{J'_0(k_0 a)} + \frac{2}{\pi k_0 a} \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \frac{\epsilon_n \gamma_n^{(e)} \cos n\eta}{J'_n(k_0 b) N'_n(k_0 a) - J'_n(k_0 a) N'_n(k_0 b)} \quad . \tag{4-2-8}$$

and

$$\sum_{n=1}^{\infty} \frac{J_{\frac{2n-1}{2\theta_0}\pi}(k_0 a)}{J'_{\frac{2n-1}{2\theta_0}\pi}(k_0 a)} \cos \frac{2n-1}{2\theta_0}\pi(\eta+\theta_0) \int_{-\theta_0}^{\theta_0} \hat{E}_o(\eta')\cos\frac{2n-1}{2\theta_0}\pi(\eta'+\theta_0)\,d\eta'$$

$$+ \frac{\theta_0}{\pi}\sum_{n=1}^{\infty} \frac{J_n(k_0a)N'_n(k_0b) - J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)} \sin n\eta \int_{-\theta_0}^{\theta_0} \hat{E}_o(\eta')\sin n\eta'\,d\eta$$

$$= \frac{2}{\pi k_0 a}\,\frac{\theta_0}{\pi}\sum_{n=1}^{\infty} \frac{\gamma_n^{(0)}\sin n\eta}{J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)} \quad . \tag{4-2-9}$$

where

$$\gamma_n^{(e)} = \int_{-\phi_0-\theta}^{\phi_0-\theta} E(\eta)\cos n\eta\,d\eta \quad , \tag{4-2-10}$$

and

$$\gamma_n^{(0)} = \int_{-\phi_0-\theta}^{\phi_0-\theta} E(\eta)\sin n\eta\,d\eta \quad . \tag{4-2-11}$$

In the preceeding chapter we expressed the shell slot field as function of ϕ, therefore to perform the last two integrals, it is more convenient to go back to the ϕ variable. If we employ (4-2-1), then (4-2-10) and (4-2-11) respectively, become

$$\gamma_n^{(e)} = \cos n\theta \int_{-\phi_0}^{\phi_0} E_e(\phi')\cos n\phi'\,d\phi'$$

$$+ \sin n\theta \int_{-\phi_0}^{\phi_0} E_o(\phi')\sin n\phi'\,d\phi' \quad , \tag{4-2-12}$$

and

$$\gamma_n^{(0)} = \cos n\theta \int_{-\phi_0}^{\phi_0} E_0(\phi')\sin n\phi' \, d\phi'$$

$$- \sin n\theta \int_{-\phi_0}^{\phi_0} E_e(\phi')\cos n\phi \, d\phi' \quad . \tag{4-2-13}$$

If we limit ourselves to the case that the angular width of the shell slot $2\phi_0$ is much smaller than unity, then substituting (3-6-7) and (3-6-8) in (4-2-12) and (4-2-13), respectively, and neglecting the term of $O(\phi_0^{\,2})$, we obtain

$$\gamma_n^{(e)} \simeq \frac{4 S_e^{(e)} J_0(n\phi_0)\cos n\theta}{k_0 b\left[8\pi^2(1+\bar{k}^{-2})k_0 b\, \ell n \frac{2}{\phi_0} - \sum_{n=0}^{\infty} \epsilon_n \tau_n X_{0m}^{\;2}\right]} \quad , \tag{4-2-14}$$

and

$$\gamma_n^{(0)} \simeq \frac{4 S_0^{(e)} J_0(n\phi_0)\sin n\theta}{k_0 b\left[8\pi^2(1+\bar{k}^{-2})k_0 b\, \ell n \frac{2}{\phi_0} - \sum_{n=0}^{N} \epsilon_n \tau_n X_{0m}^{\;2}\right]} \quad . \tag{4-2-15}$$

As was stated in the introduction of the present chapter, we confine ourselves to the case that the angular width of the wedge slot is much smaller than unity. Therefore $\frac{\pi}{\theta_0} >> 1$, and the first series on the left hand side of (4-2-8) and (4-2-9), respectively, can be approximated by

$$2\sum_{n=1}^{\infty}\frac{J_{\frac{n\pi}{\theta_0}}(k_0a)}{J'_{\frac{n\pi}{\theta_0}}(k_0a)}\cos\frac{n\pi}{\theta_0}(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta')\cos\frac{n\pi}{\theta_0}(\eta'+\theta_0)\,d\eta$$

$$\simeq\frac{\theta_0}{\pi}\cdot\sum_{n=1}^{\infty}\frac{2k_0a}{n}\cos\frac{n\pi}{\theta_0}(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta')\cos\frac{n\pi}{\theta_0}(\eta'+\theta_0)\,d\eta'+O(\theta_0^{\,3})\quad,$$

(4-2-16)

and

$$2\sum_{n=1}^{\infty}\frac{J_{\frac{2n-1}{2\theta_0}\pi}(k_0a)}{J'_{\frac{2n-1}{2\theta_0}\pi}(k_0a)}\cos\frac{2n-1}{2\theta_0}\pi(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}_0(\eta')\cos\frac{2n-1}{2\theta_0}\pi(\eta'+\theta_0)\,d\eta'$$

$$\simeq\frac{\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{2k_0a}{(2n-1)}\cos\frac{2n-1}{2\theta_0}\pi(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}(\eta')\cos\frac{2n-1}{2\theta_0}\pi(\eta'+\theta_0)\,d\eta'+O(\theta_0^{\,3})$$

(4-2-17)

If one inserts (4-2-14) and (4-2-15) into (4-2-8) and (4-2-9), respectively, and introduces the notations

$$v_n^{(1)}=\frac{1}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\quad,\qquad(4\text{-}2\text{-}18)$$

$$v_0^{(2)}=\frac{J_0(k_0a)N'_0(k_0b)-J'_0(k_0b)N_0(k_0a)}{J'_0(k_0b)N'_0(k_0a)-J'_0(k_0a)N'_0(k_0b)}\quad,$$

$$v_n^{(2)}=\frac{k_0a}{n}\,\frac{J_{n-1}(k_0a)N'_n(k_0b)-J'_n(k_0b)N_{n-1}(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\quad,\quad n\neq0$$

(4-2-19)

one arrives at

$$\frac{J_0(k_0a)}{J_0'(k_0a)}\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta)\,d\eta' + \frac{\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{2k_0a}{n}\cos\frac{n\pi}{\theta_0}(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta')\cos\frac{n\pi}{\theta_0}(\eta'+\theta_0)\,d\eta'$$

$$+\frac{\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{2k_0a}{n}\cos n\eta\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta')\cos n\eta\,d\eta + \frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}\cos n\eta\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta)\cos n\eta\,d\eta$$

$$-\frac{2}{\pi k_0a}\frac{\theta_0}{\pi}\frac{4S_0^{(e)}}{k_0b\left[8\pi^2(1+\bar{k}^{-2})k_0b\,\ell n\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_nX_{0n}^{\;2}\right]}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}J_0(n\phi_0)\cos n\theta\cos n\eta$$

$$= -\frac{V_0}{a}\frac{1}{J_0'(k_0a)}\quad, \tag{4-2-20}$$

and

$$\sum_{n=1}^{\infty}\frac{2k_0a}{(2n-1)}\cos\frac{2n-1}{2\theta_0}\pi(\eta+\theta_0)\int_{-\theta_0}^{\theta_0}\hat{E}_0(\eta')\cos\frac{2n-1}{2\theta_0}\pi(\eta'+\theta_0)\,d\eta'$$

$$+\sum_{n=1}^{\infty}\frac{2k_0a}{n}\sin n\eta\int_{-\theta_0}^{\theta_0}\hat{E}_0(\eta')\sin n\eta'\,d\eta' + \sum_{n=1}^{\infty}v_n^{(2)}\sin n\eta\int_{-\theta_0}^{\theta_0}\hat{E}_0(\eta')\sin n\eta'\,d\eta'$$

$$+\frac{2}{\pi k_0a}\frac{4S_0^{(e)}}{k_0b\left[8\pi^2(1+\bar{k}^{-2})k_0b\,\ell n\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_nX_{0n}^{\;2}\right]}\sum_{n=1}^{\infty}v_n^{(1)}J_0(n\phi_0)\sin n\theta\sin n\eta = 0$$

$$\tag{4-2-21}$$

where $S_0^{(e)}$ is given by (3-3-6) and can be rewritten as

$$S_0^{(e)} = \sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} \cos(n\theta) K_{0n} \int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta') \cos n\eta' \, d\eta'$$

$$-2\sum_{n=1}^{\infty} v_n^{(1)} \sin(n\theta) K_{0n} \int_{-\theta_0}^{\theta_0} \hat{E}_0(\eta') \sin n\eta' \, d\eta' \quad . \tag{4-2-22}$$

In the next section, from the last three equations, we will find the solutions for $\hat{E}_e(\eta)$ and $\hat{E}_0(\eta)$.

3. Approximate Solutions for $\hat{E}_e(\eta)$ and $\hat{E}_0(\eta)$:

We will apply Çalerkin's method (Kantorovich, 1958) to find the approximate solutions for $\hat{E}_e(\eta)$ and $\hat{E}_0(\eta)$. This method requires us to choose the forms of $\hat{E}_e(\eta)$ and $\hat{E}_0(\eta)$ in advance and then to determine the arbitrary constant for each field by substituting back in the integral equations. Since the electromagnetic fields in the vicinity of a perfectly conducting right angle edge behave as $r^{\frac{1}{3}}$ (R. E. Collin, 1960) where r is the distance from the field point to the edge. Thus $\hat{E}_e(\eta)$ and $\hat{E}_0(\eta)$ may take the forms

$$\hat{E}_e(\eta) = \frac{A_0^{(e)}}{\sqrt[3]{\theta_0^2 - \eta^2}} \quad , \tag{4-3-1}$$

and

$$\hat{E}_0(\eta) = \frac{A_0^{(0)} \eta}{\sqrt[3]{\theta_0^2 - \eta^2}} \quad . \tag{4-3-2}$$

The remaining problem is to determine the constants $A_0^{(e)}$ and $A_0^{(0)}$.

We first substitute (4-3-1) and (4-3-2) in (4-2-20) and (4-2-21). Then multplying (4-2-20) with $1/(\theta_0^2-\eta^2)^{1/3}$ and (4-2-21) with $\eta/(\theta_0^2-\eta^2)^{1/3}$ and integrating with respect to η from $-\theta_0$ to θ_0, we have

$$\left\{\frac{J_0(k_0a)}{J_0'(k_0a)}(Q_0^{(e)})^2+\frac{\theta_0}{\pi}\left[\sum_{n=1}^{\infty}\frac{2k_0a}{n}(P_n^{(e)})^2+\sum_{n=1}^{\infty}\frac{2k_0a}{n}(Q_n^{(e)})^2+\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)})^2\right]\right\}A_0^{(e)}$$

$$-\frac{2}{\pi k_0a}\frac{\theta_0}{\pi}\frac{4\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}J_0(n\phi_0)\cos(n\theta)Q_n^{(e)}}{k_0b\left[8\pi^2(1+\bar{k}^2)k_0b\,\ell n\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_nX_{0n}^{\;2}\right]}S_0^{(e)}=-\frac{V_0}{a}\frac{1}{J_0(k_0a)}Q_0^{(e)},\tag{4-3-3}$$

$$\left\{\sum_{n=1}^{\infty}\frac{2k_0a}{2n-1}(P_n^{(0)})^2+\sum_{n=1}^{\infty}\frac{2k_0a}{n}(Q_n^{(0)})^2+\sum_{n=1}^{\infty}v_n^{(2)}(Q_n^{(0)})^2\right\}A_0^{(0)}$$

$$+\frac{2}{\pi k_0a}\frac{4\sum_{n=1}^{\infty}v_n^{(1)}J_0(n\phi_0)\sin(n\theta)Q_n^{(0)}}{k_0b\left[8\pi^2(1+\bar{k}^2)k_0b\,\ell n\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_nX_{0n}^{\;2}\right]}S_0^{(e)}=0\ ,\tag{4-3-4}$$

and

$$S_0^{(e)}=A_0^{(e)}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}\cos(n\theta)X_{0n}Q_n^{(e)}-2A_0^{(0)}\sum_{n=1}^{\infty}v_n^{(1)}\sin(n\theta)X_{0n}Q_n^{(0)}\tag{4-3-5}$$

where

$$P_n^{(e)}=\int_{-\theta_0}^{\theta_0}\frac{\cos\frac{n\pi}{\theta_0}\eta\,d\eta}{\sqrt[3]{\theta_0^{\,2}-\eta^2}}\tag{4-3-6}$$

$$P_n^{(0)} = \int_{-\theta_0}^{\theta_0} \frac{\eta \sin \frac{2n-1}{2\theta_0} \eta}{\sqrt[3]{\theta_0^2 - \eta^2}} \, d\eta \tag{4-3-7}$$

$$Q_n^{(e)} = \int_{-\theta_0}^{\theta_0} \frac{\cos n\eta \, d\eta}{\sqrt[3]{\theta_0^2 - \eta^2}} \tag{4-3-8}$$

$$Q_n^{(0)} = \int_{-\theta_0}^{\theta_0} \frac{\eta \sin n\eta \, d\eta}{\sqrt[3]{\theta_0^2 - \eta^2}} \tag{4-3-9}$$

Integrals (4-3-6) through (4-3-9) are discussed in A-7. It is shown in A-7 that

$$\left.\begin{aligned} P_n^{(e)} &= \theta_0^{\frac{1}{3}} \sqrt{\pi}\,\Gamma(\tfrac{2}{3})(\tfrac{2}{n\pi})^{\frac{1}{6}} \, J_{1/6}(n\pi) \\ Q_n^{(e)} &= \theta_0^{\frac{1}{3}} \sqrt{\pi}\,\Gamma(\tfrac{2}{3})(\tfrac{2}{n\theta_0})^{\frac{1}{6}} \, J_{1/6}(n\theta_0) \end{aligned}\right\} \tag{4-3-10}$$

while $P_n^{(0)}/P_n^{(e)}$ and $Q_n^{(0)}/Q_n^{(e)}$ are at least of $O(\theta_0)$ and $O(\theta_0^2)$, respectively.

From (4-3-3) through (4-3-5), one can easily obtain that

$$A_0^{(e)} \simeq -\frac{V_0}{a} \cdot \frac{Q_n^{(e)}}{\Lambda} \tag{4-3-11}$$

and

$$A_0^{(0)} \simeq \frac{V}{a} \cdot \frac{Q_n^{(e)}}{\Lambda} \cdot \frac{\left[\frac{2}{\pi k_0 a} \sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} J_0(n\phi_0) Q_n^{(e)} \cos n\theta\right]\left[\frac{2}{\pi k_0 a} \sum_{n=1}^{\infty} v_n^{(1)} J_0(n\phi_0) Q_n^{(0)} \sin n\theta\right]}{(k_0 b)\left[\sum_{n=1}^{\infty} (P_n^{(0)})^2/(2n-1)\right]\left[2(1+\bar{k}^2) k_0 b \,\ell n \frac{2}{\phi_0} - \sum_{n=0}^{N} \epsilon_n \tau_n J_0^2(n\phi_0)\right]}$$

(4-3-12)

where

$$\Lambda = J_0(k_0a)(Q_0^{(e)})^2 - \frac{\theta_0}{\pi} J_1(k_0a) \Bigg\{ 2k_0a \sum_{n=1}^{\infty} (P_n^{(e)})^2/n + 2k_0a \sum_{n=1}^{\infty} (Q_n^{(e)})^2/n$$

$$+ \sum_{n=0}^{\infty} \epsilon_n v_n^{(2)} (Q_n^{(e)})^2 - \frac{4}{\pi k_0^2 ab} \frac{\left(\sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} J_0(n\phi_0) Q_n^{(e)} \cos n\theta\right)^2}{2(1+\bar{k}^2)k_0 b \, \ell n(2/\phi_0) - \sum_{n=0}^{N} \epsilon_n \tau_n J_0^2(n\phi_0)} \Bigg\} .$$

(4-3-13)

In Eq. (4-3-11) and (4-3-12) we use the approximately equal sign because in (4-3-13) we have neglected the terms of $O(\theta_0^{8/3})$ and replaced X_{0n} with $2\pi J_0(n\phi_0)$ on account of (A-4-16) and for the convenience of computation. One also notices from (4-3-10) that the series $\sum_{n=1}^{\infty} (Q_n^{(e)})^2/n$ in (4-3-13) converges very slowly when $\theta_0 << 1$. Fortunately, under this condition, it is found in A-8 that

$$\sum_{n=1}^{\infty} (Q_n^{(e)})^2/n = (Q_0^{(e)})^2 \, \ell n \, (2/\theta_0) + 0.05053 \, \theta_0^{2/3} \quad . \qquad (4\text{-}3\text{-}14)$$

V

SLOT VOLTAGES AND TERMINAL ADMITTANCE OF THE WEDGE WAVEGUIDE FOR NARROW SLOTS

1. Introduction

In this chapter we will obtain explicitly the three important physical quantities: the wedge slot voltage, the shell slot voltage and the terminal admittance of the wedge waveguide when the angular width of the shell slot and the wedge aperture are very small in comparison with unity. The voltages of the wedge aperture and of the shell slot are defined, respectively, as

$$V_w = -\int_{-\theta_0}^{\theta_0} a\hat{E}(\eta)\,d\eta \quad , \tag{5-1-1}$$

and

$$V_s = -\int_{-\phi_0}^{\phi_0} b\,E(\phi)\,d\phi \quad . \tag{5-1-2}$$

Since

$$\int_{-\theta_0}^{\theta_0} a\hat{E}_0(\eta)\,d\eta = 0$$

and

$$\int_{-\phi_0}^{\phi_0} b\,E_0(\phi)\,d\phi = 0 \quad ,$$

consequently, we arrive at

$$V_w = -\int_{-\theta_0}^{\theta_0} a\hat{E}_e(\eta)\, d\eta \qquad (5\text{-}1\text{-}3)$$

$$V_s = -\int_{-\phi_0}^{\phi_0} b E_e(\phi)\, d\phi \quad . \qquad (5\text{-}1\text{-}4)$$

In deriving the explicit form of the terminal admittance, we neglect all terms whose magnitude are of $O(\phi_0^{\,2})$, or $O(\theta_0^{\,2})$, or less in comparison with the magnitude of $S_0^{(e)}$.

2. Voltages of the Wedge Aperture and the Shell Slot

From (5-1-3), (5-1-4), (3-6-7) and (4-3-1), it is obvious that

$$V_w = V_0 \left[Q_0^{(e)}\right]^2 / \Lambda \quad , \qquad (5\text{-}2\text{-}1)$$

and

$$\frac{V_s}{V_w} \simeq \frac{2}{\pi k_0 a Q_0^{(e)}} \; \frac{\sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} \cos(n\theta) J_0(n\phi_0) Q_n^{(e)}}{2(1+\bar{k}^2) k_0 b \,\ell n \frac{2}{\phi_0} - \sum_{n=0}^{N} \epsilon_n \tau_n J_0^2(n\phi_0)} \qquad (5\text{-}2\text{-}2)$$

where Λ is given by (4-3-13). It is seen from (4-3-13) that V_w is only weakly dependent on the slot separation angle θ, plasma sheath and the coaxial spacing, except when $J_0(k_0 a)$ is close to a zero. For this exception one can show for $\theta_0 << 1$ that

$$V_w \simeq V_0 / J_0(k_0 a) \quad . \qquad (5\text{-}2\text{-}3)$$

Thus for the radiation problem, one may even regard the narrow wedge aperture as a constant voltage source. On the other hand the voltage of the shell slot depends not only on V_w, but also on θ, $(k_0b - k_0a)$, and plasma sheath.

3. Terminal Admittance of the Wedge Waveguide When $\theta_0, \phi_0 << 1$

The terminal admittance shown in (2-4-4) consists of three series. We will consider these series in the next few paragraphs.

The first series is

$$I_1 = 2j \sum_{n=1}^{\infty} \frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi')\cos\frac{n\pi}{2\theta_0}(\phi' - \theta + \theta_0)\, d\phi'$$

which can be separated into two series, i.e.,

$$I_1 = 2j \left\{ \sum_{n=1}^{\infty} \frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)} \left(\int_{-\theta_0}^{\theta_0} \hat{E}_e(\eta')\cos\frac{n\pi}{\theta_0}(\eta' + \theta_0)\, d\eta' \right)^2 \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{J_{\frac{2n-1}{2\theta_0}\pi}(k_0a)}{J'_{\frac{2n-1}{2\theta_0}\pi}(k_0a)} \left(\int_{-\theta_0}^{\theta_0} \hat{E}_0(\eta')\sin\frac{2n-1}{2\theta_0}(\eta' + \theta_0)\, d\eta' \right)^2 \right\}$$

since ϕ' and η' are related by (4-2-1). Since $\theta_0 << 1$, similar to (4-2-16) and (4-2-17), we employ the approximate formulas

$$\frac{J_{\frac{n\pi}{\theta_0}}(k_0a)}{J'_{\frac{n\pi}{\theta_0}}(k_0a)} \simeq \frac{\theta_0}{\pi} \frac{k_0a}{n}$$

$$\frac{\dfrac{J_{\frac{2n-1}{2\theta_0}}(k_0a)}{}}{\dfrac{J'_{\frac{2n-1}{2\theta_0}}(k_0a)}{}} \simeq \frac{\theta_0}{\pi}\,\frac{2k_0a}{2n-1} ,$$

then using (4-3-1) and (4-3-2), we have

$$I_1 \simeq 2jk_0a\cdot\frac{\theta_0}{\pi}\left\{(A_0^{(e)})^2\sum_{n=1}^{\infty}(P_n^{(e)})^2/n+(A_0^{(e)})^2\sum_{n=1}^{\infty}2(P_n^{(e)})^2/(2n-1)\right\} .$$

As we indicate in A-7, the magnitude of $P_n^{(e)}/P_n^{(e)}$ is at least of $O(\theta_0)$, therefore one may neglect the second term and arrive at

$$I_1 \simeq j\frac{\theta_0}{\pi}(A_0^{(e)})^2\sum_{n=1}^{\infty}\frac{2k_0a}{n}(P_n^{(e)})^2 . \tag{5-3-1}$$

The second series of (2-4-4) by change of variable (4-2-1) and using (4-2-3) can be written in the form

$$I_2 = j\frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\left[\left(\int_{-\theta_0}^{\theta_0}\hat{E}_e(\eta)\cos n\eta\, d\eta\right)^2\right.$$

$$\left.+\left(\int_{-\theta_0}^{\theta_0}\hat{E}(\eta)\sin n\eta\, d\eta\right)^2\right] .$$

If one substitutes (4-3-1) and (4-3-2) in last equation, and neglects the terms of $O(\theta_0^{\,2})$, one has

$$I_2 \simeq j\frac{\theta_0}{\pi}(A_0^{(e)})^2\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}(Q_n^{(e)})^2 .$$

Using (4-1-4), we can reduce I_2 in a more convenient form, i.e.,

$$I_2 = j\frac{\theta_0}{\pi}(A_0^{(e)})^2\left\{\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)})^2 + 2k_0 a\sum_{n=1}^{\infty}(Q_n^{(e)})^2/n\right\} . \tag{5-3-2}$$

The last series of (2-4-4) is

$$I_3 = j\frac{\theta_0}{\pi}\frac{2}{\pi k_0 a}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}\left[\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos n\phi\, d\phi\int_{-\phi_0}^{\phi_0}E_e(\phi')\cos n\phi'\, d\phi'\right.$$

$$\left.+\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\sin n\phi\, d\phi\int_{-\phi_0}^{\phi_0}E_0(\phi)\sin n\phi'\, d\phi'\right] \tag{5-3-3}$$

where $v_n^{(1)}$ is given by (4-2-18). Again if we make use of (4-2-1) and (4-2-3) and consider $\hat{E}(\eta)$ as the sum of the even function $A_0^{(e)}/\sqrt[3]{\theta_0^2-\eta^2}$ and the odd function $A_0^{(0)}\eta/(\theta_0^2-\eta^2)^{1/3}$, it is obvious that

$$\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos n\phi\, d\phi = A_0^{(e)}Q_n^{(e)}\cos n\theta - A_0^{(0)}Q_n^{(0)}\sin n\theta \tag{5-3-4}$$

and

$$\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\sin n\phi\, d\phi = A_0^{(0)}Q_n^{(0)}\cos n\theta + A_0^{(e)}Q_n^{(e)}\sin n\theta . \tag{5-3-5}$$

From (3-6-7) and (3-6-8), respectively,

$$\int_{-\phi_0}^{\phi_0}E_e(\phi)\cos n\phi\, d\phi = \frac{4S_0^{(e)}J_0(n\phi_0)}{k_0 b\left[8\pi^2(1+\bar{k}^2)k_0 b\,\ell n(2/\phi_0) - \sum_{n=0}^{N}\epsilon_n\tau_n X_{0n}^2\right]} \tag{5-3-6}$$

and

$$\int_{-\phi_0}^{\phi_0} E_0(\phi)\sin n\phi \, d\phi = -\frac{a_1^{(0)}\phi_0}{\pi(1+\bar{k}^2)(k_0 b)^2} J_1(n\phi_0) \, . \tag{5-3-7}$$

Upon substituting (5-3-4) through (5-3-7) in (5-3-3) and neglecting the terms of $O(\phi_0^2)$, $O(\theta_0^2)$, and $O(\theta_0\phi_0)$

$$I_3 \simeq -j\frac{2\theta_0}{\pi}\frac{A_0^{(e)}S_0^{(e)}}{\pi k_0 a}\frac{\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}\cos(n\theta)J_0(n\phi_0)Q_n^{(e)}}{\pi^2 k_0 b \quad 2(1+\bar{k}^2)(k_0 b)\ln\frac{2}{\phi_0} - \sum_{n=0}^{N}\epsilon_n \tau_n J_0^2(n\phi_0)} \tag{5-3-8}$$

where $S_0^{(e)}$ is given by (4-3-5). Therefore, inserting (4-3-5) in the last equation, one obtains

$$I_3 \simeq -j\frac{\theta_0}{\pi}\frac{2}{\pi k_0 a}\frac{2\pi(A_0^{(e)})^2\left(\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}J_0(n\phi_0)Q_n^{(e)}\cos n\theta\right)^2}{\pi^2 k_0 b\left[2(1+\bar{k}^2)k_0 b\ln\frac{2}{\phi_0} - \sum_{n=0}^{N}\epsilon_n\tau_n J_0^2(n\phi_0)\right]} \, . \tag{5-3-9}$$

Now we add (5-3-1), (5-3-2), and (5-3-9) together and obtain

$$y(a) \simeq j\frac{\theta_0}{\pi}\frac{1}{(Q_0^{(e)})^2}\left\{2k_0 a\left[\sum_{n=1}^{\infty}\frac{1}{n}(P_n^{(e)})^2 + \sum_{n=1}^{\infty}\frac{1}{n}(Q_n^{(e)})^2\right]\right.$$

$$\left. + \sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)})^2 - \frac{\frac{4}{\pi^2 k_0^2 ab}\left(\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}J_0(n\phi_0)\cos(n\theta)Q_n^{(e)}\right)^2}{2(1+\bar{k}^2)k_0 b\ln\frac{2}{\phi_0} - \sum_{n=0}^{N}\epsilon_n\tau_n J_0^2(n\phi_0)}\right\} \, . \tag{5-3-10}$$

Thus substituting (5-3-10) in (2-4-5), we obtain an explicit formula for the terminal admittance Y(a) of the wedge waveguide. For the convenience in presenting data, in Eq. (2-4-5), we choose L = a and attain the resulting form of

$$Y(a) \approx j\frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\left\{2k_0a\left[\sum_{n=1}^{\infty}(P_n^{(e)}/Q_0^{(e)})^2/n+\sum_{n=1}^{\infty}(Q_n^{(e)}/Q_0^{(e)})^2/n\right]\right.$$
$$\left.+\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)})^2-\frac{a}{b}\frac{\left[\frac{2}{\pi k_0 a}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}J_0(n\phi_0)(Q_n^{(e)}/Q_0^{(e)})\cos n\theta\right]^2}{2(1+\bar{k}^2)k_0 b\,\ln\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_n J_0^2(n\phi_0)}\right\}.$$

(5-3-11)

The last term inside the brace can be written as

$$-\frac{a}{b}\left|\frac{V_s}{V_w}\right|^2\cdot\left\{2\left[1+(\bar{k}^2)^*\right]k_0 b\,\ln\frac{2}{\phi_0}-\sum_{n=0}^{N}\epsilon_n\tau_n^* J_0^2(n\phi_0)\right\}$$

where the voltage ratio V_s/V_w is given by (5-2-2). In view of (3-1-6), we further introduce the notations

$$v_0^{(3)}=\frac{J_0(k_0b)N_0'(k_0a)-J_0'(k_0a)N_0(k_0b)}{J_0'(k_0b)N_0'(k_0a)-J_0'(k_0a)N_0'(k_0b)},$$

$$v_n^{(3)}=\frac{k_0b}{n}\,\frac{J_{n+1}(k_0b)N_n'(k_0a)-J_n'(k_0a)N_{n+1}(k_0b)}{J_n'(k_0b)N_n'(k_0a)-J_n'(k_0a)N_n'(k_0b)}\quad;\ n\geq 1,\qquad (5\text{-}3\text{-}12)$$

and

$$\pi'_0 = \pi_0 \quad ,$$

$$\pi'_n = \bar{k}^{-2} \frac{k_0 b}{n} \cdot \frac{\left[J_{n-1}(k_1 b) N'_n(k_1 c) - J'_n(k_1 c) N_{n-1}(k_1 b) \right] - \bar{k} \cdot \frac{H_n^{(2)'}(k_0 c)}{H_n^{(2)}(k_0 c)} \left[J_{n-1}(k_1 b) N_n(k_1 c) - N_{n-1}(k_1 b) J_n(k_1 c) \right]}{\bar{k} \left[J_n(k_1 c) N'_n(k_1 b) - J'_n(k_1 b) N_n(k_1 c) \right] - \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)}(k_0 c)} \cdot \left[J'_n(k_1 c) N'_n(k_1 b) - J'_n(k_1 b) N'_n(k_1 c) \right]} \cdot \tag{5-3-13}$$

Then τ_0 and τ_n become

$$\tau_0 = \pi'_0 \frac{H_0^{(2)}(k_0 c)}{H_0^{(2)'}(k_0 c)} - v_0^{(3)} \quad ,$$

$$\tau_n = \pi'_n \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)} - v_n^{(3)} \quad , \qquad n \geq 1 \quad . \tag{5-3-14}$$

Thus if G and B respectively, denote the terminal conductance and susceptance, then from (5-3-11) through (5-3-14) one arrives at

$$G \simeq -\frac{1}{2\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{a}{b} \frac{|V_s|^2}{|V_w|^2} \cdot \mathrm{Im} \left[\sum_{n=0}^{N} \epsilon_n (\pi'_n)^* \frac{H_n^{(1)}(k_0 c)}{H_n^{(1)'}(k_0 c)} J_0^2(n\phi_0) - 2(\bar{k}^{-2})^* k_0 b \, \ell n(2/\phi_0) \right] \quad , \tag{5-3-15}$$

$$B \simeq \frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\left\{2k_0 a\left[\sum_{n=1}^{\infty}(P_n^{(e)}/Q_0^{(e)})^2/n+\sum_{n=1}^{\infty}(Q_n^{(e)}/Q_0^{(e)})^2/n\right]\right.$$

$$+\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)}/Q_0^{(e)})^2-\frac{a}{b}\frac{|V_s|^2}{|V_w|^2}\left[\sum_{n=0}^{N}\epsilon_n v_n^{(3)}J_0^2(n\phi_0)+2k_0 b\,\ell n(2/\phi_0)\right]$$

$$\left.+\frac{a}{b}\frac{|V_s|^2}{|V_w|^2}\mathrm{Re}\left[\sum_{n=0}^{N}\epsilon_n(\pi_n')^*\frac{H_n^{(1)}(k_0c)}{H_n^{(1)'}(k_0c)}J_0^2(n\phi_0)-2(\bar{k}^{-2})^*k_0 b\,\ell n(2/\phi_0)\right]\right\}$$

(5-3-16)

where Im and Re are the abreviations of "real part" and "imaginary part" respectively. Since (5-3-15) and (5-3-16) are so complicated that in general one can hardly obtain any information before actually performing the numerical computations. However, in a certain special cases, some properties of the conductance G and susceptance B can be read from the expressions.

Case a: In this case, we assume no plasma sheath, i.e. we let $c \rightarrow b$ and $\bar{k} \rightarrow 1$. From (2-3-26), it can be shown that $\pi_n \rightarrow 1$. Since

$$\mathrm{Im}\left(\frac{H_n^{(1)}(k_0b)}{H_n^{(1)'}(k_0b)}\right) = \frac{2}{\pi k_0 b}$$

then (5-3-15) is easily reduced to a form

$$G \simeq \frac{|V_s|^2}{|V_w|^2}\frac{a}{b}\frac{1}{\pi^2 b\omega\mu_0}\sum_{n=0}^{N}\frac{\epsilon_n J_0^2(n\phi_0)}{|H_n^{(1)'}(k_0b)|^2} \quad . \qquad (5\text{-}3\text{-}17)$$

If we divide (5-3-17) by (A-9-4), then we obtain a formula of similar form as Eq. (38) in one of Olte's recent papers (1965). This coincidence is a physical consequence because from the circuit point of view, if V is the terminal voltage, G is input conductance then it is well known that the power $P = GV^2$. It is interesting to note that as $\phi_0 \to 0$, G decreases as $-1/\ln\phi_0$ and B tends to an expression independent of ϕ_0 and θ,

$$B \simeq \frac{k_0 a}{\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\left[\sum_{n=1}^{\infty}(P_n^{(e)}/Q_0^{(e)})^2/n + \sum_{n=1}^{\infty}(Q_n^{(e)}/Q_0^{(e)})^2/n\right.$$

$$\left.+ \frac{1}{2k_0 a}\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)}/Q_0^{(e)})^2\right]. \qquad (5\text{-}3\text{-}18)$$

The susceptance given by last equation is the terminal susceptance of the wedge waveguide for the case of no shell slot, i.e. a continuous shell shrouds the cylindrical antenna.

Case b: In the present case, we assume that $|\bar{k}| < 1.0$, $k_0 b - k_0 a \ll k_0 a, k_0 b$ and $k_0 a = m$, a positive integer. In this report, we limit ourselves to $k_0 a < 5$. Since the series $\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)}/Q_0^{(e)})^2$ and $\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)} J_n(n\phi_0)(Q_n^{(e)}/Q_0^{(e)})\cos n\theta$

(the numerator of V_s/V_w, (5-2-2)) converge absolutely, we may truncate them at Mth term and then these truncated series as well as the $\sum_{n=0}^{N}\epsilon_n v_n^{(3)} J_0^2(n\phi_0)$

will be proved to be dominated by their respective mth terms.

If $k_0 b - k_0 a \ll k_0 a$, $k_0 b$, it is found that

$$J_{n-1}(k_0 a)N_n'(k_0 b) - J_n'(k_0 b)N_{n-1}(k_0 b)$$

$$= \frac{n}{k_0 a}\,\frac{2}{\pi k_0 a} + k_0(b-a)\left[J_{n-1}(k_0 a)N_n''(k_0 a) - J_n''(k_0 a)N_{n-1}(k_0 a)\right] + \dots ,$$

$$J_{n+1}(k_0b)N'_n(k_0a) - J'_n(k_0a)N_{n+1}(k_0b)$$

$$= \frac{n}{k_0b}\frac{2}{\pi k_0b} - k_0(b-a)\left[J_{n+1}(k_0b)N''_n(k_0b) - J''_n(k_0b)N_{n+1}(k_0b)\right] + \ldots ,$$

$$J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)$$

$$= \frac{2k_0(b-a)}{\pi k_0a}\left[\left(\frac{n}{k_0a}\right)^2 - 1\right] + \frac{k_0^{\,2}(b-a)^2}{2}\left[J'''_n(k_0a)N'_n(k_0a) - J'_n(k_0a)N'''_n(k_0a)\right]$$

$$+ \ldots ,$$

where $n \geq 1$. Thus if one made use of the above results, it is clear that

$$\sum_{n=0}^{M}\epsilon_n v_n^{(2)}(Q_n^{(e)}/Q_0^{(e)})^2 \simeq \frac{4}{k_0^2(b-a)^2}\cdot\frac{2}{\pi k_0a}\left\{\frac{1}{F_m}\left(\frac{Q_m^{(e)}}{Q_0^{(e)}}\right)^2 + O\left[k_0b - k_0a\right]\right\} , \tag{5-3-19}$$

$$\sum_{n=0}^{M}\epsilon_n v_n^{(1)} J_0(n\phi_0)(Q_n^{(e)}/Q_0^{(e)})\cos n\theta \simeq \frac{4}{k_0^2(b-a)^2}\left\{\frac{J_0(m\phi_0)Q_m^{(e)}\cos m\theta}{Q_0^{(e)}F_m} + O\left[k_0b - k_0a\right]\right\}, \tag{5-3-20}$$

and

$$\sum_{n=0}^{N}\epsilon_n v_n^{(3)} J_0^2(n\phi_0) \simeq \frac{4}{k_0^2(b-a)^2}\,\frac{2}{\pi k_0b}\left\{\frac{J_0^2(m\phi_0)}{F_m} + O(k_0b - k_0a)\right\} , \tag{5-3-21}$$

where

$$F_m = J'''_m(k_0a)N'_m(k_0a) - J'_m(k_0a)N'''_m(k_0a) \tag{5-3-22}$$

If one substitutes (5-3-19) through (5-3-22) in (5-3-16) and (5-2-2), one obtains

$$B \simeq \frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{4}{k_0^2(b-a)^2}\left\{\frac{2}{\pi F_m k_0 a}\left(\frac{Q_m^{(e)}}{Q_0^{(e)}}\right)^2 + \frac{k_0^2(b-a)^2}{4}\sum_{n=1}^{\infty}\frac{2k_0 a}{n}\left(\frac{Q_n^{(e)}}{Q_0^{(e)}}\right)^2\right.$$

$$-\frac{|V_s|^2}{|V_w|^2}\left[\frac{2}{\pi F_m k_0 a} J_0^2(m\phi_0) + \frac{k_0^2(b-a)^2}{4}\,2k_0 a\,\ell n(2/\phi_0)\right]$$

$$+\frac{a}{b}\left|\frac{V_s}{V_w}\right|^2\frac{k_0^2(b-a)^2}{4}\,R_e\left[\sum_{n=0}^{\infty}\epsilon_n(\tau_n')^*\,\frac{H_n^{(1)}(k_0c)}{H_n^{(1)'}(k_0c)}\,J_0^2(n\phi_0)\right.$$

$$\left.\left.-2(\bar{k}^{-2})^* k_0 b\,\ell n\,(2/\phi_0)\right]\right\},$$

$$k_0 a = m \qquad (5\text{-}3\text{-}23)$$

and

$$\frac{V_s}{V_w} \simeq \frac{2}{\pi k_0 a}\,\frac{\left[J_0(m\phi_0)Q_m^{(e)}\cos m\theta/(Q_0^{(e)}F_m)\right]}{\dfrac{2J_0^2(m\phi_0)}{\pi k_0 b F_m} + \dfrac{k_0^2(b-a)^2}{4}\left[2(1+\bar{k}^{-2})k_0 b\,\ell n(2/\phi_0) - \sum_{n=0}^{N}\epsilon_n\tau_n\dfrac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)}J_0^2(n\phi_0)\right]},$$

$$k_0 a = m \quad . \qquad (5\text{-}3\text{-}24)$$

Since $|\bar{k}| < 1$ and the angular width of the wedge aperture and shell slot in practical case are small but finite, Eqs. (5-3-23) and (5-3-24) can be further reduced to

$$\frac{V_s}{V_w} \simeq \frac{b\,Q_m^{(e)}}{a\,Q_0^{(e)}} \, \frac{\cos m\theta}{J_0(m\phi_0)} + O\left[k_0(b-a)\right] \quad , \qquad k_0 a = m \quad , \tag{5-3-25}$$

$$B \simeq \frac{4}{k_0 a F_m} \left[\frac{Q_m^{(e)}}{Q_0^{(e)} k_0(b-a)} \right]^2 \left\{ 1 - \frac{b}{a} \cos^2 m\theta + O\left[k_0(b-a)\right] \right\} ,$$

$$k_0 a = m \quad . \tag{5-3-26}$$

If we substitute (5-3-25) in (5-3-15), we obtain

$$G \simeq -\frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \; \frac{b}{a} \left(\frac{Q_m^{(e)}}{Q_0^{(e)} J_0(m\phi_0)} \right)^2 \cos^2 m\theta$$

$$\cdot \, I_m \left\{ \sum_{n=0}^{N} \epsilon_n (\pi'_n)^* \, \frac{H_n^{(1)}(k_0 c)}{H_n^{(1)'}(k_0 c)} \; J_0^2(n\phi_0) - 2(\bar{k}^{-2})^* k_0 b \,\ell n\,(2/\phi_0) \right\}$$

$$+ \; O\left[k_0(b-a)\right] \quad , \qquad k_0 a = m \quad . \tag{5-3-27}$$

It is interesting to note that in the present case, both G and B depends strongly on $k_0 a$, $k_0(b-a)$ and θ but only G depends on the plasma sheath and shell slot width $2\phi_0$.

Case C: If we keep the radii $k_0 a$ and $k_0 b$ constant, ϕ_0 and θ_0 small but finite, $\theta = 0$, $v/\omega = 0$ and $\omega_p/\omega >> 1$, then

$$\bar{k} = j\sqrt{\left(\frac{\omega_p}{\omega}\right)^2 - 1} \simeq j\,\frac{\omega_p}{\omega}$$

and

$$\pi'_n \frac{H_n^{(2)}(k_0c)}{H_n^{(2)'}(k_0c)} \simeq -\left(\frac{\omega_p}{\omega}\right)^2 \frac{k_0 b}{n}$$

$$-\frac{\omega_p}{\omega} \frac{H_n^{(2)}(k_0c)\cosh\left[\frac{\omega_p}{\omega}k_0(c-b)\right] - \frac{\omega_p}{\omega}H_n^{(2)'}(k_0c)\sinh\left[\frac{\omega_p}{\omega}k_0(c-b)\right]}{\frac{\omega_p}{\omega}H_n^{(2)'}(k_0c)\cosh\left[\frac{\omega_p}{\omega}k_0(c-b)\right] + H_n^{(2)}(k_0c)\sinh\left[\frac{\omega_p}{\omega}k_0(c-b)\right]} \quad . \tag{5-3-28}$$

Therefore

$$\mathrm{Im}\left[\sum_{n=0}^{N} \epsilon_n (\pi'_n)^* \frac{H_n^{(1)}(k_0c)}{H_n^{(1)}(k_0c)} J_0^2(n\phi_0)\right]$$

$$= -\frac{2}{\pi k_0 c}\, e^{-\frac{2\omega_p}{\omega}k_0(c-b)} \sum_{n=0}^{N} \epsilon_n \frac{J_0^2(n\phi_0)}{\left|H_n^{(1)'}(k_0c)\right|^2} \quad . \tag{5-3-29}$$

$$\mathrm{Re}\left[\sum_{n=0}^{\infty} \epsilon_n (\pi'_n)^* \frac{H_n^{(1)}(k_0c)}{H_n^{(1)'}(k_0c)} J_0^2(n\phi_0)\right]$$

$$= -2\left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=0}^{N} \frac{k_0 b}{n} J_0^2(n\phi_0) - \frac{\omega_p}{\omega} \sum_{n=0}^{N} \epsilon_n J_0^2(n\phi_0) \quad . \tag{5-3-30}$$

On the right hand-side of (5-3-30), one notes that from (A-8-12), the first term is approximately equal to $-2(\frac{\omega_p}{\omega})^2 k_0 b \ell n(2/\phi_0)$. Hence, upon employing (5-3-29) and (5-3-30), $|V_s/V_w|^2$, G and B respectively become

$$\left|\frac{V_s}{V_w}\right|^2 \simeq (\frac{\omega}{\omega_p})^2 (C_s/C_t)^2 \quad , \tag{5-3-31}$$

$$C_s = \frac{2}{\pi k_0 a} \sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} (Q_n^{(e)}/Q_0^{(e)})\, J_0(n\phi_0) \cos n\theta \quad , \tag{5-3-32}$$

$$C_t = \sum_{n=0}^{N} \epsilon_n J_0^2(n\phi_0) \quad , \tag{5-3-33}$$

$$G \simeq \frac{k_0 a}{\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \left[\frac{1}{\pi k_0^2 bc} \sum_{n=0}^{N} \epsilon_n \frac{J_0^2(n\phi_0)}{\left|H_n^{(1)'}(k_0 c)\right|^2} \right] (\frac{C_s}{C_t})^2 (\frac{\omega}{\omega_p})^2 e^{-2\frac{\omega_p}{\omega} k_0 (c-b)} \tag{5-3-34}$$

$$B \simeq \frac{k_0 a}{\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \left\{ \sum_{n=1}^{\infty} (P_n^{(e)}/Q_0^{(e)})^2/n + \sum_{n=1}^{\infty} (Q_n^{(e)}/Q_0^{(e)})^2/n + \frac{1}{2k_0 a} \sum_{n=0}^{\infty} \epsilon_n v_n^{(2)} (Q_n^{(e)}/Q_0^{(e)})^2 \right.$$
$$\left. - (C_s^2/2k_0 b) \left[\frac{1}{C_t} \frac{\omega}{\omega_p} + \frac{\sum_{n=0}^{N} \epsilon_n v_n^{(3)} J_0^2(n\phi_0) + 2k_0 b \ell n(2/\phi_0)}{C_t^2} (\frac{\omega}{\omega_p})^2 \right] \right\} . \tag{5-3-35}$$

Equation (5-3-34) and (5-3-35) show that for an overdense plasma sheath, if we ignore the collision effects, the terminal conductance G decreases with a factor $(\omega/\omega_p)^2 e^{-2k_0(c-b)\omega_p/\omega}$ and the terminal susceptance B approaches to the case of no shell slot as shown in (5-3-18).

VI

EQUIVALENT CIRCUIT FOR A COAXIAL ANTENNA WITH A PLASMA SHEATH

1. Introduction:

In chapter II, we derived the general stationary form for the terminal admittance of the wedge waveguide. Upon using the results from chapter III and chapter IV for $\theta_0 << 1$ we finally arrived at an explicit formula for this admittance in chapter V. It is clear that this admittance is a function of the following factors: k_0a, k_0b, k_0c, θ, ω_p/ω, ν/ω, θ_0, and ϕ_0. If one can find some explicit expressions to indicate the individual role of each of the above factors in $Y(a)$, then one knows all details of the coaxial antenna. Unfortunately, this is practically impossible. However, it is also valuable to know the individual influence of the wedge region, the coaxial region, plasma sheath and free space on $Y(a)$, respectively. If we refer to the normalized stationary form of the terminal admittance $y(a)$, (2-4-4), it is found that one can hardly identify the individual influence of each of the above four regions on $y(a)$. Thus we turn to seek some other way to formulate the normalized terminal admittance of the wedge waveguide so that the effects of the above four regions can be discussed. In section 2 of this chapter, we furnish a new formulation of $y(a)$ which allows one to propose an equivalent circuit for the antenna. In section 3, we also discuss the physical significance of each circuit component of the equivalent circuit. However, the new formulation of $y(a)$ is not stationary with respect to the functional variation of the wedge aperture field and therefore, as long as the exact solution for the wedge aperture field is not found, the stationary formulation of $y(a)$, (2-4-4), is still important in producing the numerical results.

2. An Equivalent Circuit of the Coaxial Antenna:

Upon multiplying (2-4-3) with $\frac{1}{2\theta_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\hat{E}^*(\phi)\,d\phi$ and integrating from $\theta-\theta_0$ to $\theta+\theta_0$, we have

$$\frac{1}{2\theta_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\,y(a)\left|\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\,d\phi\right|^2$$

$$=\frac{1}{2\theta_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\left\{2j\sum_{n=1}^{\infty}\left(J_{\frac{n\pi}{2\theta_0}}(k_0a)\Big/J'_{\frac{n\pi}{2\theta_0}}(k_0a)\right)\left|\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\,d\phi\right|^2\right.$$

$$+j\frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}^*(\phi)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos n(\phi-\phi')\,d\phi'$$

$$\left.-j\frac{\theta_0}{\pi}\frac{2}{\pi k_0a}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}^*(\phi)\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi'\right\}\quad. \tag{6-2-1}$$

Following the same procedures as in A-1, one may find that the above equation is not stationary. From the definition (2-4-5) of Y(a), taking $L = a$ and by virtue of integral Eq. (2-3-31), we write (6-2-1) in the form

$$Y(a) = jB_w + j(B_{c1} - \ell^2 B_{c2}) - \ell^2 Y_{pf} \tag{6-2-2}$$

where

$$B_w = \frac{1}{\theta_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=1}^{\infty}\left(J_{\frac{n\pi}{2\theta_0}}(k_0a)\Big/J'_{\frac{n\pi}{2\theta_0}}(k_0a)\right)\frac{\left|\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\,d\phi\right|^2}{|T_w|^2}, \tag{6-2-3}$$

$$B_{c1} = \frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\frac{\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}^*(\phi)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos n(\phi-\phi')\,d\phi'}{|T_w|^2}, \tag{6-2-4}$$

$$B_{c2} = \frac{b}{a}\frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N_n(k_0b)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0a)N'_n(k_0b)}\frac{\int_{-\phi_0}^{\phi_0}d\phi\,E^*(\phi)\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi'}{|T_s|^2}, \tag{6-2-5}$$

$$Y_{pf} = -j\frac{b}{a}\frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=0}^{\infty}\epsilon_n\pi_n^*\frac{H_n^{(1)}(k_0c)}{H_n^{(1)'}(k_0c)}\frac{\int_{-\phi_0}^{\phi_0}d\phi\,E^*(\phi)\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi'}{|T_s|^2}, \tag{6-2-6}$$

$$T_w = \int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\,d\phi, \tag{6-2-7}$$

$$T_s = \int_{-\phi_0}^{\phi_0} E(\phi)\,d\phi \quad , \tag{6-2-8}$$

and

$$\ell = \frac{|T_s|}{|T_w|} \quad . \tag{6-2-9}$$

Equation (6-2-2) suggests an equivalent terminal circuit of the wedge transmission line as shown in Fig. 6-1.

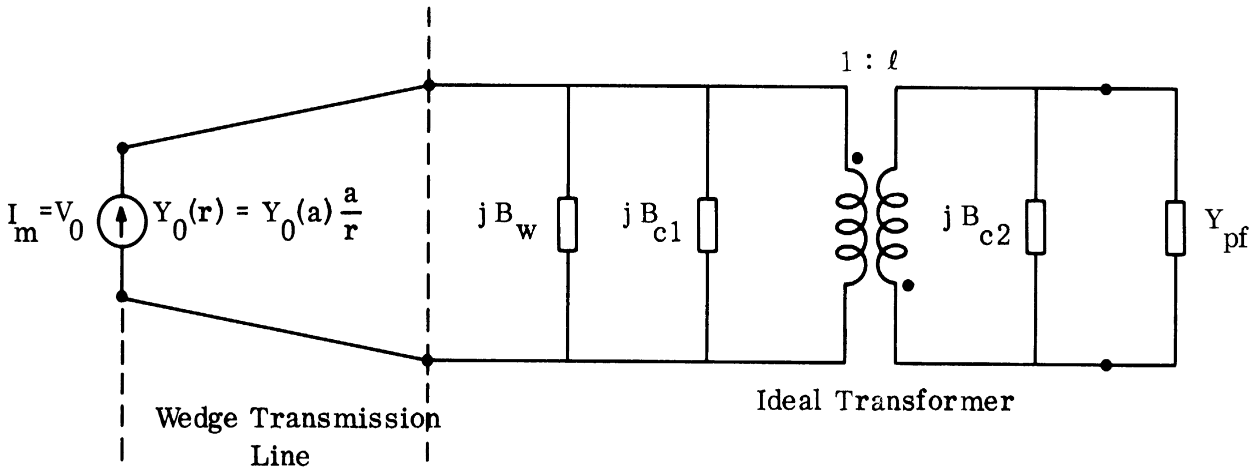

FIG. 6-1: EQUIVALENT CIRCUIT FOR THE COAXIAL ANTENNA

It is seen from (6-2-3) through (6-2-9) that B_w, B_{c1}, B_{c2} are real quantities while Y_{pf} is a complex quantity, therefore the first three circuit components are susceptances and last one is an admittance. One may also note that the only θ dependent circuit component is the transformer turn ratio ℓ. In the next section, we will discuss the physical significance of these circuit components in detail.

3. Physical Significance of the Circuit Components:

In (6-2-3), we note that B_w depends only on k_0a, θ_0 and $\hat{E}(\phi)$, the wedge aperture field. Furthermore, if we let E_r^h, E_ϕ^h and H_z^h represent the electric field and magnetic field in the wedge region minus the TEM mode field, then the integral

$$\frac{\omega}{2|T_w|^2}\int_0^a dz\int_0^a dr\int_{\theta-\theta_0}^{\theta+\theta_0}\left[\epsilon_0(E_r^h E_r^{h*} + E_\phi^h E_\phi^{h*}) - \mu_0 H_z^h H_z^{h*}\right] r\, d\phi$$

in view of Eqs. (2-2-1) through (2-2-3), can be reduced to a form exactly the same as shown on the right hand-side of (6-2-3) which defines the susceptance B_w. Therefore, we may regard B_w as the susceptance due to the higher order mode fields in the wedge region. For narrow angular width of the wedge or small k_0a, B_w can be reduced to

$$B_w = \omega\left[C_w + O(k_0a\theta_0)^2\right] \qquad (6\text{-}3\text{-}1)$$

where

$$C_w = \frac{\epsilon_0 a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\,d\phi\right|^2\Big/\left|T_w\right|^2 \ . \qquad (6\text{-}3\text{-}2)$$

It is seen from (6-3-1) that the dominant part in square bracket is the capacitor C_w which from (6-3-2), depends upon the radius of the cylinder and the angular width of the wedge aperture.

In Eq. (6-2-4), if we make use of (4-2-19), we may separate the right hand-side into two series and obtain

$$B_{c1} = \omega C_{c1} + \frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{n=0}^{\infty} \epsilon_n v_n^{(2)} \frac{\int_{\theta-\theta_0}^{\theta+\theta_0} d\phi\, \hat{E}^*(\phi) \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi) \cos n(\phi - \phi')\, d\phi'}{\left|T_w\right|^2} , \tag{6-3-3}$$

where

$$C_{c1} = \frac{\epsilon_0 a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\int_{\theta-\theta_0}^{\theta+\theta_0} d\phi\, \hat{E}^*(\phi) \int_{\theta-\theta_0}^{\theta+\theta_0} \hat{E}(\phi) \cos n(\phi - \phi')\, d\phi'}{\left|T_w\right|^2} . \tag{6-3-4}$$

When the angular width of the wedge is very small, we may employ the wedge aperture field (4-3-1) and then because of (A-8-12), we have

$$C_{c1} \simeq \frac{\epsilon_0 a}{\pi} \ln(2/\theta_0) . \tag{6-3-5}$$

The second term of (6-3-3), because of $v_n^{(2)}$, (4-2-19), converges rapidly. C_{c1} may be considered as the capacitance due to the fringe fields of the wedge aperture. The same fringe capacitance can be found when the circular shell and plasma sheath are not present. Schelkunoff (1952) in deriving the terminal admittance of the biconical antenna also found a capacitance which has a logarithmic singularity as the cone angle $\theta \rightarrow \frac{\pi}{2}$.

The next circuit component to be discussed is B_{c2}. Since

$$\frac{J_n(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n(k_0b)}{J_n'(k_0b)N_n'(k_0a) - J_n'(k_0a)N_n'(k_0b)} = \frac{k_0b}{n} + v_n^{(3)} \quad , \quad n \geq 1$$

where $v_n^{(3)}$ is given by (5-3-12), we may write (6-2-5) in a form

$$B_{c2} = \omega C_{c2} + \frac{b}{a}\frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=0}^{\infty}\epsilon_n v_n^{(3)} \frac{\int_{-\phi_0}^{\phi_0} d\phi E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi')\cos n(\phi - \phi')d\phi'}{|T_s|^2} \tag{6-3-6}$$

where

$$C_{c2} = \frac{b}{a}\frac{\epsilon_0 b}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} \frac{\int_{-\phi_0}^{\phi_0} d\phi E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi')\cos n(\phi - \phi')d\phi'}{|T_s|^2} \quad . \tag{6-3-7}$$

Similar to C_{c1}, we regard C_{c2} as the capacitance due to the fringing field of the shell slot in the coaxial region. For narrow shell slot, upon substituting (3-6-7) in (6-3-7) and making use of (A-8-12), we obtain

$$C_{c2} \simeq \frac{b}{a}\frac{\epsilon_0 b}{\pi} \ln (2/\phi_0) \quad .$$

Hence, as the angular width of the shell slot approaches zero, the capacitance C_{c2} also has a logarithmic singularity. The series in (6-3-6) converges rapidly and for narrow shell slot $2\phi_0$, it is weakly dependent on ϕ_0.

Since in the plasma sheath, there is also a fringe field neighboring to the shell slot, we expect that this fringe field will contribute to a capacitance. To investigate the nature of this capacitance, we turn our attention to Y_{pf}, (6-2-6). Comparing (5-3-14) with (3-1-6), we observe that $\pi_n = \pi'_n - \frac{\bar{k}^2 k_1 b}{n}$ for $n \neq 0$ where π'_n is given by (5-3-13). Therefore Y_{pf} can be reduced to

$$Y_{pf} = j \frac{\omega \epsilon_0 a}{\pi} (\bar{k}^2)^* A_{fr} - j \frac{b}{a} \frac{1}{2\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{n=0}^{\infty} \epsilon_n (\pi'_n)^* \frac{H_n^{(2)}(k_0 c)}{H_n^{(2)'}(k_0 c)}$$

$$\cdot \frac{\int_{-\phi_0}^{\phi_0} d\phi \, E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi) \cos n(\phi - \phi') \, d\phi'}{|T_s|^2} , \qquad (6\text{-}3\text{-}8)$$

and

$$A_{fr} = \left(\frac{b}{a}\right)^2 \sum_{n=1}^{\infty} \frac{\int_{-\phi_0}^{\phi_0} d\phi \, E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi) \cos n(\phi - \phi') \, d\phi'}{n \, |T_s|^2} . \qquad (6\text{-}3\text{-}9)$$

We see from (6-3-9) that for $2\phi_0 << 1$, we find

$$A_{fr} \simeq \left(\frac{b}{a}\right)^2 \ln\left(\frac{2}{\phi_0}\right) . \qquad (6\text{-}3\text{-}10)$$

In (6-3-8), we may regard the first term on the right side as the admittance associated with shell slot fringe field in the plasma sheath. From the defining equation of the dielectric constant $\bar{k}$ of the plasma, (2-2-14), we can show that the above fringe admittance is composed of three parallel branches and can be written as

$$j \frac{\omega \epsilon_0 a}{\pi} (\bar{k}^2)^* A_{fr} = j\omega \left(C_{c3} - \frac{1}{\omega^2 L_{c3}}\right) + G_{c3} \qquad (6\text{-}3\text{-}11)$$

where

$$C_{c3} = \frac{\epsilon_0 a}{\pi} A_{fr} \quad , \tag{6-3-12}$$

$$L_{c3} = \frac{\mu_0 a}{\pi} \left(\frac{\pi}{k_0 a}\right)^2 \frac{1+(\nu/\omega)^2}{(\omega_p/\omega)^2 A_{fr}} \quad , \tag{6-3-13}$$

$$G_{c3} = \frac{\nu \epsilon_0 a}{\pi} \frac{(\omega_p/\omega)^2 A_{fr}}{1+(\nu/\omega)^2} \quad . \tag{6-3-14}$$

C_{c3} is a capacitance due to fringing fields, L_{c3} is an inductance due to the plasma, and G_{c3} is a conductance which accounts for the power dissipatedby the shell slot fringe field in the plasma sheath. As $\phi_0 \longrightarrow 0$, it is seen from (6-3-12) through (6-3-14) that C_{c3} and G_{c3} have logarithmic singularity while L_{c3} approaches zero. Since A_{fr} is only weakly dependent on the plasma sheath, thus C_{c3} is also weakly dependent on the plasma constants. L_{c3} is inversely proportional to $(\omega_p/\omega)^2$ and G_{c3} increases as $(\omega_p/\omega)^2$. Thus increasing the plasma density tends to short out the shell slot. The real part of the second term on the right hand-side of (6-3- 8) may be associated with the power radiated into the free space and the power loss in the plasma sheath by other than the fringe field of the slot. The imaginary part of this term may be related to the stored energy in the plasma sheath and the free space with the slot fringe field excluded. To investigate the connections between Y_{pf} and the stored energies, the power loss in the plasma sheath, and the power radiated into the free space, it is more convenient to start with (6-2-6). From (A-10-9), it can be easily shown that

$$2P^{III} + j4\omega(W_H^{III} - W_E^{III}) + 2P_r + j4\omega(W_H^{IV} - W_E^{IV}) = b^2 |T_s|^2 Y_{pf} \quad . \tag{6-3-15}$$

where

P_r = time averaged power radiated into the free space

P^{III} = time averaged power loss in the plasma sheath

W_H = time averaged energy stored in the magnetic field

W_E = time averaged energy stored in the electric field

and the superscripts III and IV denote the plasma sheath and the free space, respectively. If we define the radiation conductance G_r and plasma conductance G_p, respectively, as

$$G_r = \frac{2P_r}{b^2 |T_s|^2} \quad , \tag{6-3-16}$$

$$G_p = \frac{2P^{III}}{b^2 |T_s|^2} \quad , \tag{6-3-17}$$

then

$$G_r + G_p = \mathrm{Re}\ Y_{pf} \quad . \tag{6-3-18}$$

The imaginary part of Y_{pf}, from (6-3-15) is a susceptance which accounts for the difference of the time averaged stored energies in the magnetic field and electric field exterior to the conducting shell. It can be visualized that $\omega \left[C_{c3} - 1/(\omega^2 L_{c3})\right]$ and G_{c3} are a part of the susceptance and conductance represented by $\mathrm{Im}\ Y_{pf}$ and G_p respectively. In (A-10-10) we derived the expression for P_r. From this equation and (6-3-16), when $\omega_p >> \omega$, one can show that G_r

decreases as the factor $e^{-x_p k_0 (c-b)}$ if $x_p k_0 (c-b) >> 1$ where

$$x_p = \frac{\omega_p}{\omega} \left\{ \frac{1}{2} + \frac{1}{2} \left[1 + 4 \left(\frac{\nu}{\omega} \right)^2 \right]^{1/2} \right\}^{1/2}$$

The coaxial region not only behaves as a reactive element, but also couples the two slots. In the equivalent circuit, we indicate this coupling effect by a transformer of turn ratio ℓ as defined in (6-2-9). Since these two slots are separated by an angle θ, ℓ will be a circuit constant in Fig. 6-1 that depends upon the separation angle θ. Apparently, ℓ also depends on the radii $k_0 a$ and $k_0 b$. However, since the shell slot opens into the plasma sheath, ℓ is also modified by the plasma constants. To have an idea as to how ℓ depends upon these factors, the reader may refer to (5-2-2) in which the angular width of wedge aperture and shell slot are assumed very narrow.

VII

NUMERICAL RESULTS AND CONCLUSIONS

1. Introduction:

In this chapter, we present the numerical results based on computations from (5-3-11). From this equation we note that Y(a) is a function of k_0a, k_0b, k_0c, θ, ω_p/ω and ν/ω; thus in presenting data, we successively choose θ, $k_0(b-a)$, $k_0(c-b)$, k_0a and ω_p/ω as the abscissas. The computations were performed on a digital computer 7090 for θ_0 and ϕ_0 equal to 0.03 radians. Since the method of solution of the integral Eqs. (2-3-31) and (2-3-32) given in chapter III and chapter IV, respectively, is primarily a low frequency approximation, we limit k_0a in the computations to the interval $0.1 \leq k_0a \leq 4.3$. In (5-3-11), we sum the series $\sum_{n=1}^{\infty}(P_n^{(e)}/Q_0^{(e)})^2/n$ to 250 terms and the series $\sum_{n=1}^{\infty}(Q_n^{(e)}/Q_0^{(e)})^2/n$ by the method shown in A-8. The factor $v_n^{(1)}$ enters into the series defining the numerator of the last term inside the brace of (5-3-11); this series we sum to M terms. The number M is determined by two conditions: a) in the last terms preceeding the Mth term, $\left|v_n^{(1)}\right|$ decreases monotonically, b) $\left|v_M^{(1)}\right| / \left|v_p^{(1)}\right| \leq 10^{-6}$ where $\left|v_p^{(1)}\right|$ is the largest among $\left|v_n^{(1)}\right|$. Since $v_n^{(2)}$, τ_n decrease faster than $v_n^{(1)}$ as n becomes large, we sum the series $\sum_{n=0}^{\infty}\epsilon_n v_n^{(2)}(Q_n^{(e)}/Q_0^{(e)})^2$ to Mth term and for the finite sum $\sum_{n=0}^{N}\epsilon_n\tau_n J_0^2(n\phi_0)$ we set N = M.

In the following sections, we first present and discuss the numerical results, then we summarize what we have done in this report. Finally, we make some brief conclusions based on the theoretical discussions and numerical results.

2. Numerical Results:

In Fig. 7-1 (a) and (b), we plot the normalized conductance G/G' and the normalized susceptance B/B' as a function of θ for the no plasma sheath case. In this figure, the radii k_0a and k_0b are the parameters; G' and B', respectively, are the terminal conductance and susceptance of the wedge waveguide without the conducting shell and plasma sheath. Their formulas are (A-9-4) and (A-9-5), respectively. G' and B' depend only upon the radius k_0a and the wedge width $2\theta_0$. In Fig. 7-1 (a) and (b), four different values of the radius k_0a are used. We tabulate the corresponding values of G' and B' in Table VII-1 for reference.

k_0a	G', mhos	B', mhos
0.2	1.33×10^{-4}	8.72×10^{-4}
1.0	1.04×10^{-3}	3.59×10^{-3}
1.8	2.08×10^{-3}	5.69×10^{-3}
4.3	5.35×10^{-3}	10.5×10^{-3}

TABLE VII-1: TERMINAL CONDUCTANCE AND SUSCEPTANCE WITHOUT CONDUCTING SHELL. WEDGE WIDTH 0.06 RADIANS.

In Fig. 7-2(a) and (b), we plot the terminal conductance and the terminal susceptance versus separation angle θ, with ω_p/ω as the parameter for the collision-free plasma sheath. The radii k_0a, k_0b, and k_0c are kept at 1.0, 1.1, and 1.2, respectively. In Fig. 7-3(a) and (b), we repeat Fig. 7-2, except ν/ω is used as the parameter and $\omega_p/\omega = 1.5$. It is seen

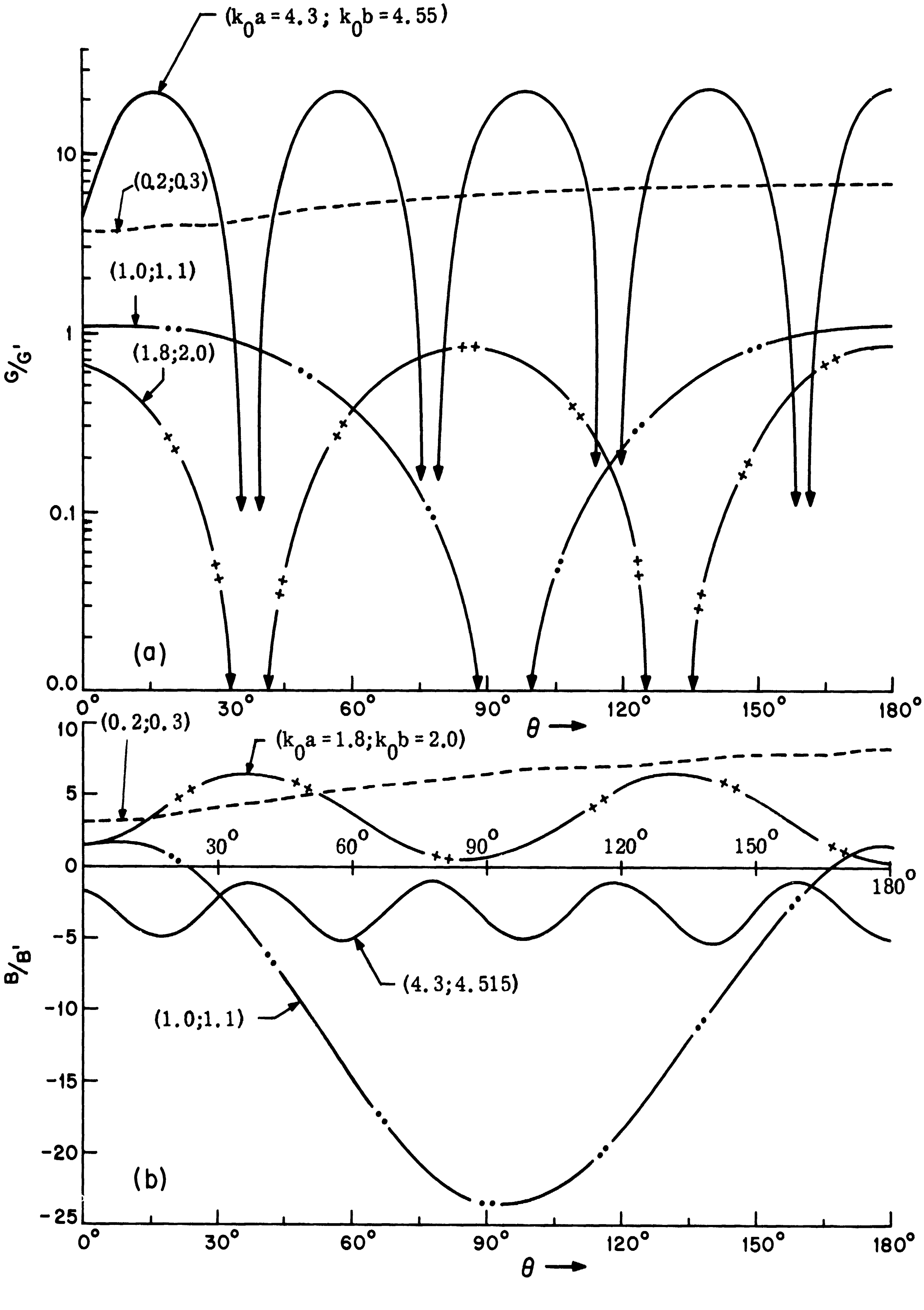

FIG. 7-1: (a) NORMALIZED CONDUCTANCE AND (b) SUSCEPTANCE VERSUS θ WITH NO PLASMA SHEATH AND $(k_0a\,;\,k_0b)$ AS THE PARAMETER

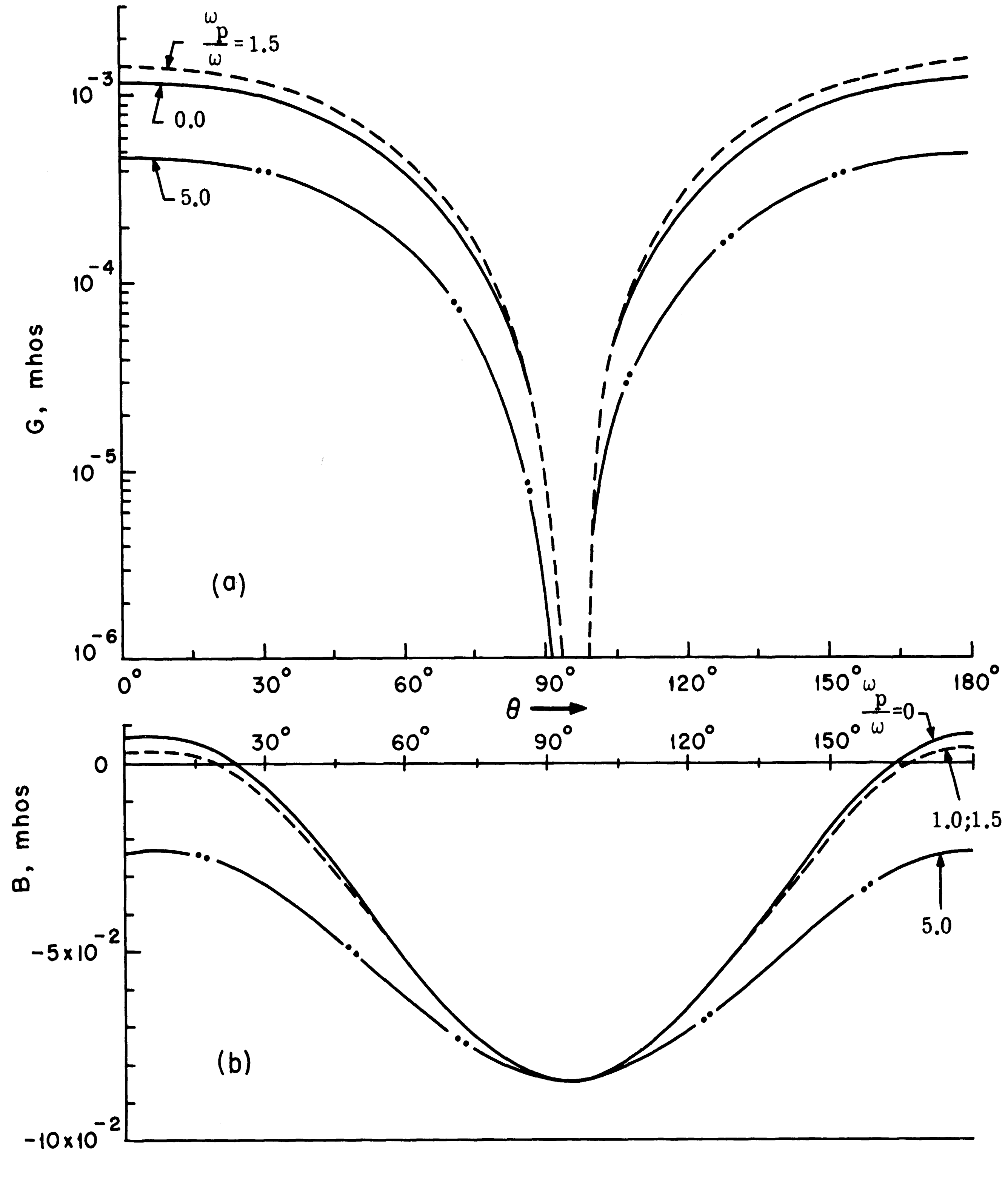

FIG. 7-2: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH A COLLISIONLESS PLASMA SHEATH VERSUS θ FOR $k_0a = 1.0$, $k_0b = 1.1$, $k_0c = 1.2$ AND ω_p/ω AS THE PARAMETER

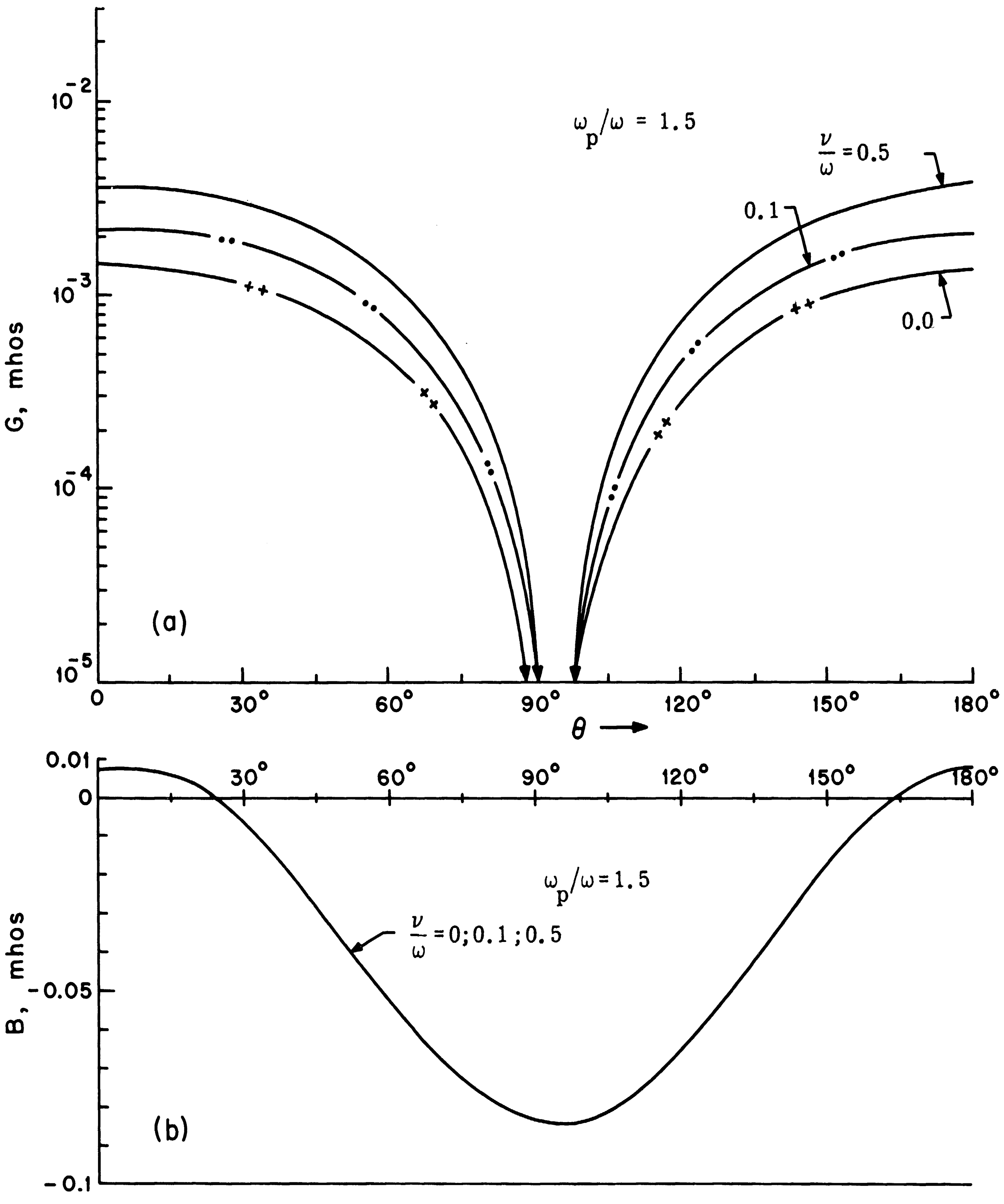

FIG. 7-3: (a) CONDUCTANCE AND (b) SUSCEPTANCE VERSUS θ WITH $k_0a = 1.0$, $k_0b = 1.1$, $k_0c = 1.2$, $\omega_p/\omega = 1.5$ AND ν/ω AS THE PARAMETER

from the last three figures that a) conductance versus θ curve and susceptance versus θ curve for the case $k_0a = 1.0$ and $k_0b = 1.1$ are neatly checked with Eqs. (5-3-26) and (5-3-27) except $\omega_p/\omega = 5$; b) when ν/ω increases, the conductance also increases while the susceptance is practically not affected.

In Fig. 7-4(a) and (b) we plot the terminal conductance and the terminal susceptance against the width of coaxial region, $k_0(b-a)$, for the no plasma sheath case, the separation angle θ as the parameter, and $k_0a = 1.0$. One may observe that for small $k_0(b-a)$, when $\theta = 0^\circ$ and 180° the conductance and the susceptance are approximately equal to G' and B' for $k_0a = 1.0$; when $\theta = 90^\circ$, G becomes very small while B becomes a large inductive susceptance. If we refer back to (5-3-26) and (5-3-27), a similar result can be observed. For a large value of $k_0(b-a)$, the conductance is small while the susceptance approaches a positive constant, i.e., a capacitive susceptance. Furthermore, one may note that the conductance versus $k_0(b-a)$ curves shown in Fig. 7-4(a) are maximum when $k_0(b-a) \simeq 0.4$.

In Fig. 7-5(a) and (b), we plot G and B versus the plasma sheath thickness $k_0(b-c)$ with ω_p/ω as the parameter and $k_0a = 1.0$, $k_0b = 1.1$, $\theta = 0^\circ$, $\nu/\omega = 0$. In Fig. 7-6(a) and (b), we repeat the last figure except for $\nu/\omega = 0.1$. From the last two figures one may observe that: a) When $\omega_p/\omega = 0.5$, the conductance and the susceptance are weakly dependent on the sheath thickness for the cases $\nu/\omega = 0$ and $\nu/\omega = 0.1$. Galejs (1964) in a paper on the admittance of a slot in a perfectly conducting plate covered with a plasma sheath showed that the slot conductance and susceptance are practically independent of the thickness of the plasma sheath when $\omega_p/\omega < 1$. His numerical results are not accurate for a thin sheath. b) When $\omega_p/\omega = 1.5$ and the sheath thickness $k_0(c-b)$ exceeds 1.5, further increasing the sheath thickness will decrease the conductance exponentially for $\nu/\omega = 0$, but makes it approach a constant for $\nu/\omega = 0.1$. For $k_0(c-b) > 1.5$ the susceptance is essentially independent of $k_0(c-b)$ and the collision frequency. c) When $k_0(c-b)$

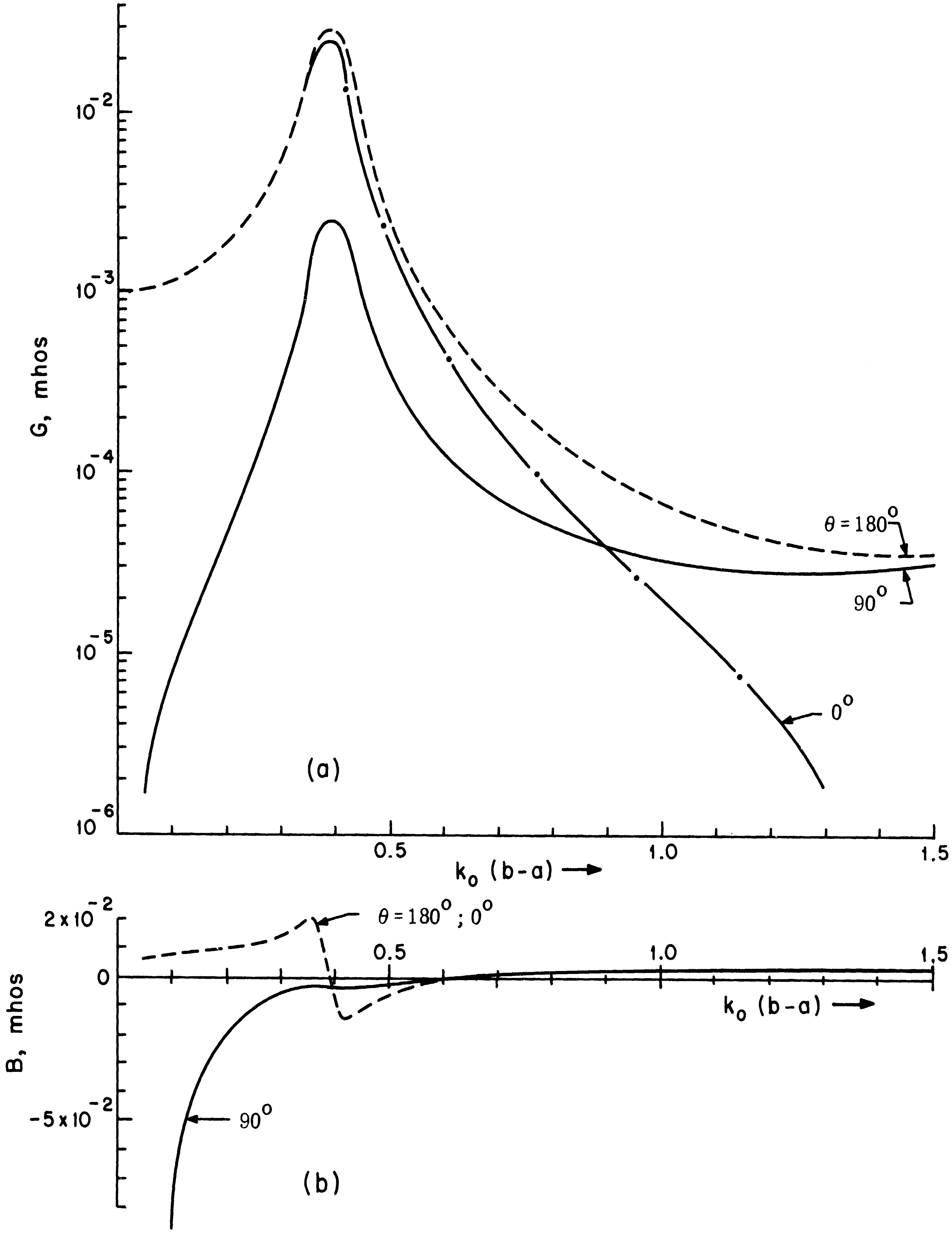

FIG. 7-4: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH NO PLASMA SHEATH VERSUS $k_0(b-a)$ FOR $k_0a = 1.0$, AND θ AS THE PARAMETER

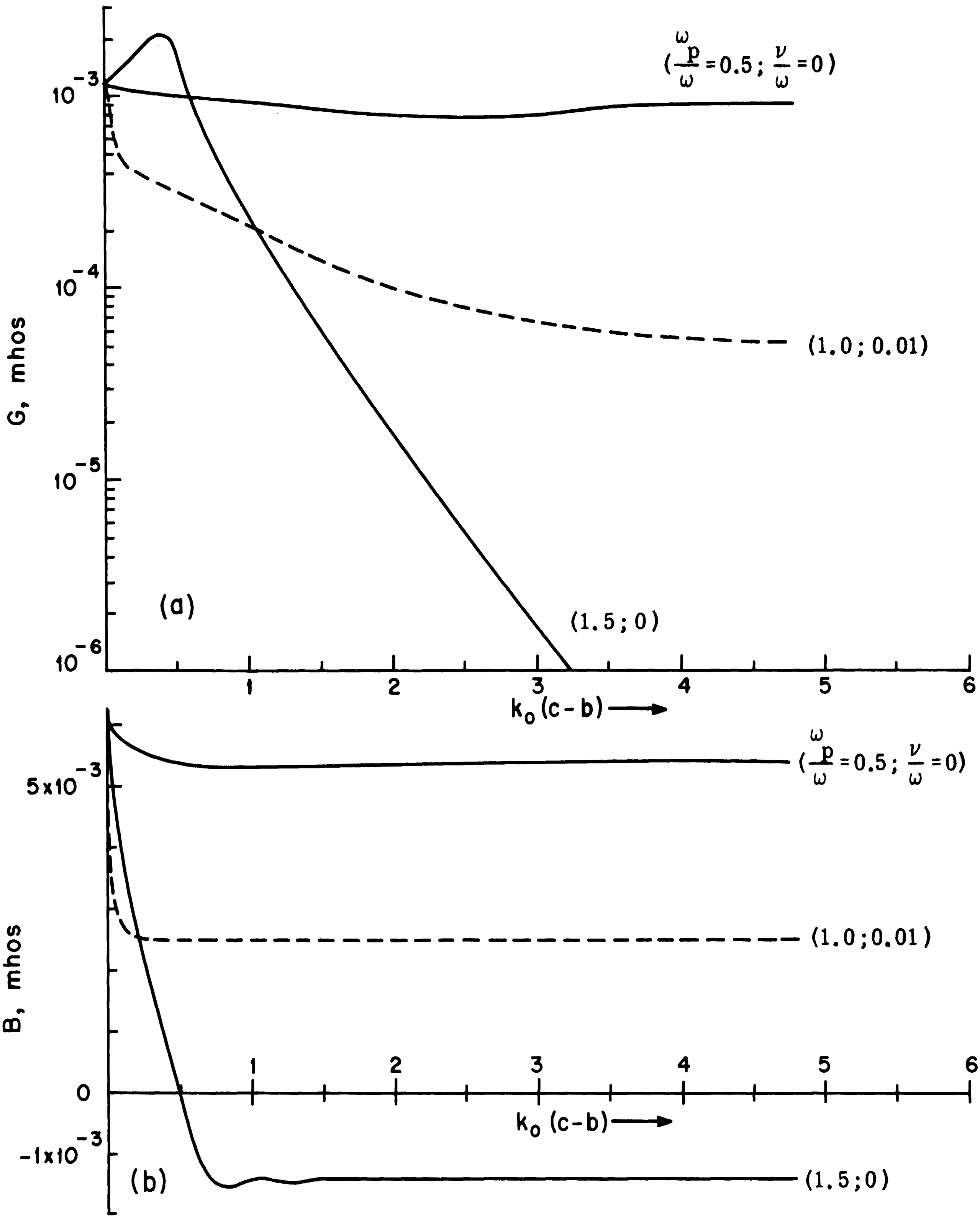

FIG. 7-5: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH COLLISIONLESS PLASMA SHEATH VERSUS $k_0(c-b)$ FOR $k_0a = 1.0$, $k_0b = 1.1$, $\theta = 0^o$, AND ω_p/ω AS THE PARAMETER

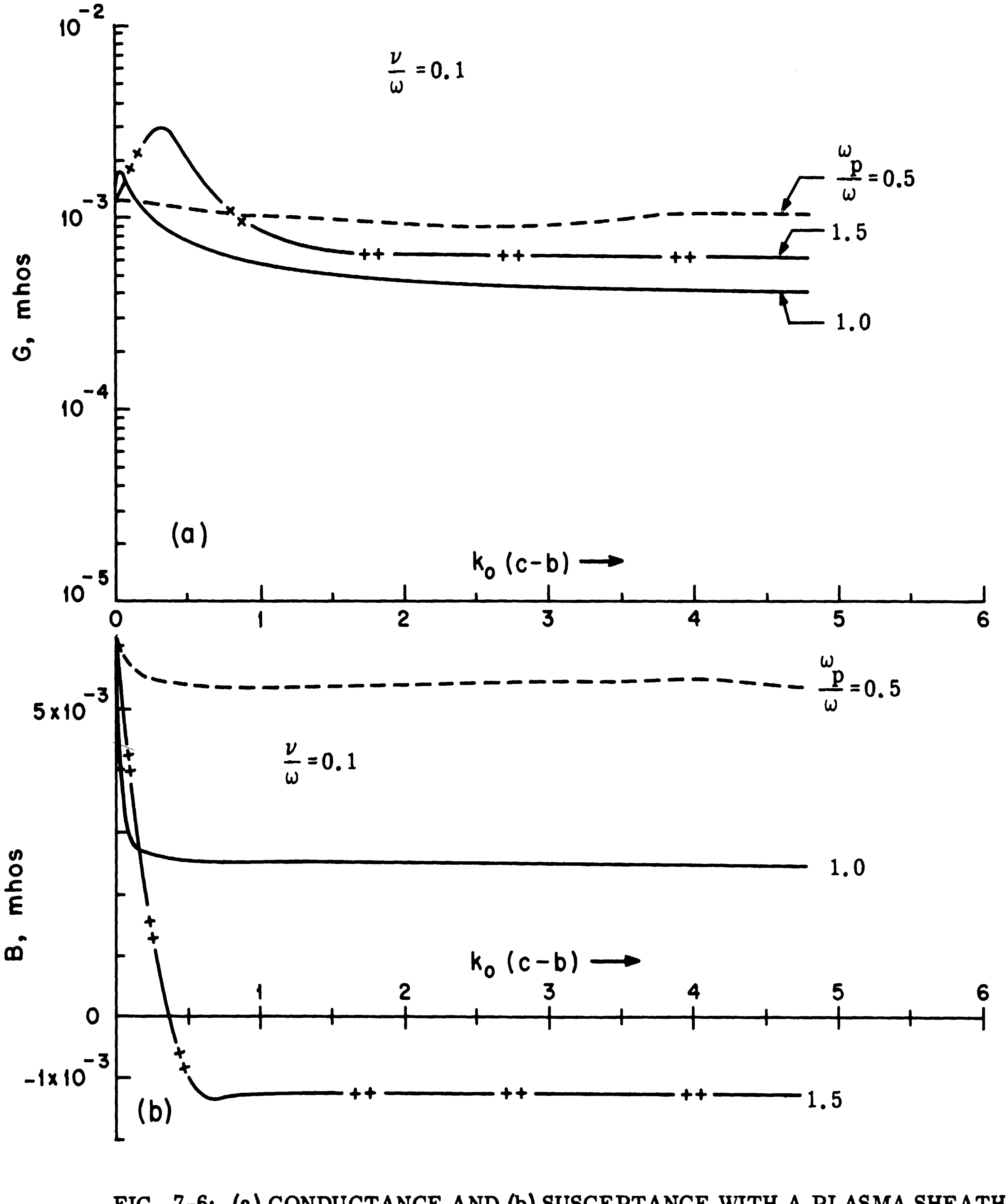

FIG. 7-6: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH A PLASMA SHEATH VERSUS $k_0(c-b)$ FOR $k_0a = 1.0$, $k_0b = 1.1$, $\theta = 0^o$, $\nu/\omega = 0.1$, AND ω_p/ω AS THE PARAMETER

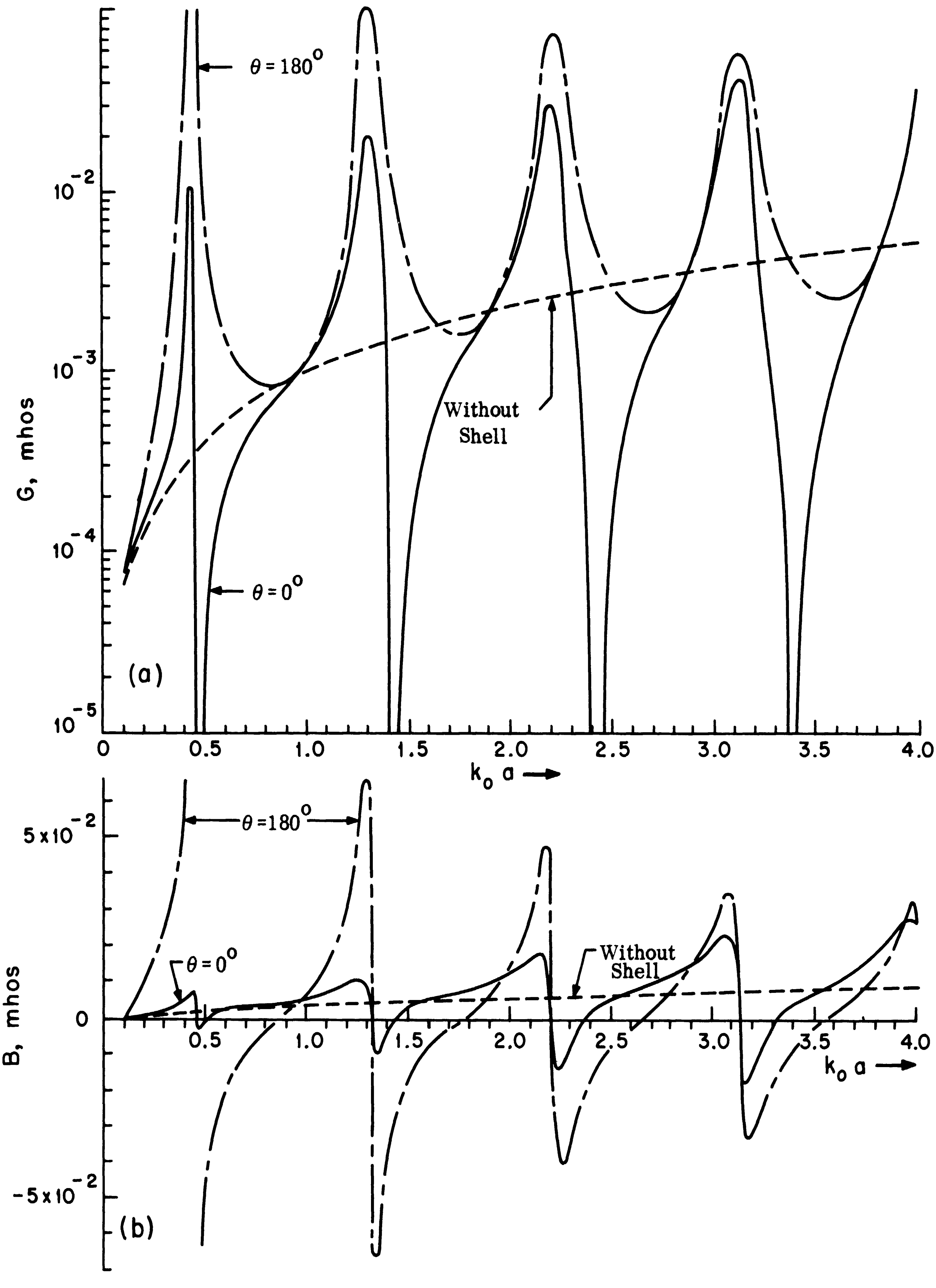

FIG. 7-7: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH NO PLASMA SHEATH VERSUS k_0a, $k_0b/k_0a = 1.1$ AND θ AS THE PARAMETER.

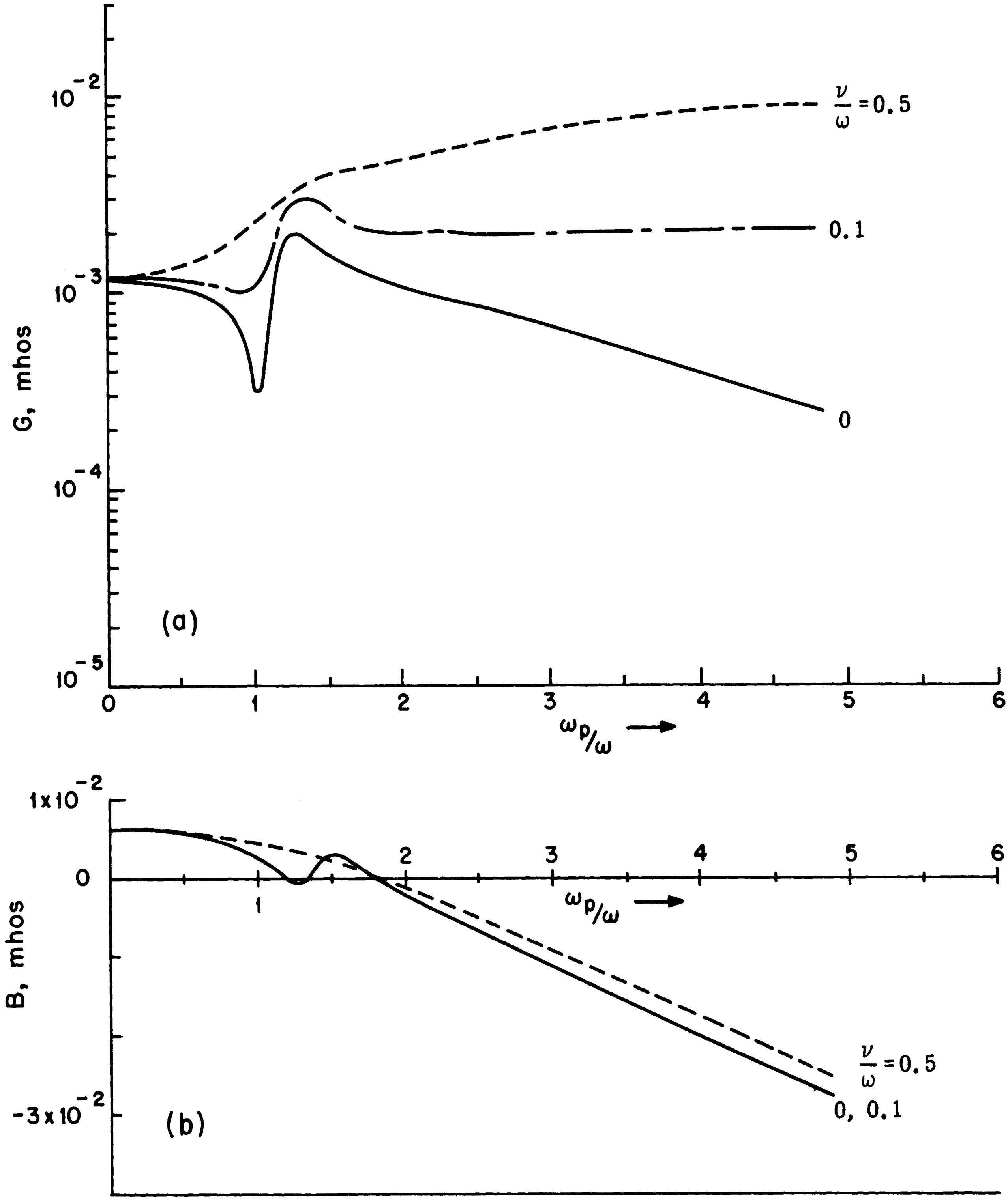

FIG. 7-8: (a) CONDUCTANCE AND (b) SUSCEPTANCE WITH PLASMA SHEATH VERSUS ω_p/ω FOR $k_0a = 1.0$, $k_0b = 1.1$, $k_0c = 1.3$, AND ν/ω AS THE PARAMETER

approaches zero, the conductance and the susceptance for all cases approach respectively to 1.18×10^{-3} mhos and 6.26×10^{-3} mhos, the magnitude of the terminal conductance and the terminal susceptance for the case without plasma sheath. We did not plot the case $\omega_p/\omega = 1.0$, $\nu/\omega = 0$ because when the plasma frequency is equal to the radio frequency, the effect of the collision can not be neglected.

In Fig. 7-7(a) and (b), we keep $k_0b/k_0a = 1.1$ and have no plasma sheath. G and B are plotted against k_0a with θ as the parameter. The primary purpose of this figure is to show the effect of the radio frequency on G and B for a constant cylinder radius a. We note that as k_0a increases, the conductance peaks at $k_0a \simeq 0.43$, 1.3, 2.2, 3.13, etc. The susceptance peaks almost at the same values of k_0a as G.

In Fig. 7-8(a) and (b), we plot G and B as function of ω_p/ω with ν/ω as the parameter . The values of k_0a, k_0b and k_0c are chosen as 1, 1.1, and 1.3, respectively. One notes that for large values of ω_p/ω, G decreases exponentially with further increasing of ω_p/ω when $\nu/\omega = 0$ and approaches to a constant value when $\nu/\omega \neq 0$. The susceptance, on the other hand, for large ω_p/ω, is approximately a straight line with a negative slope. The effect of ν/ω is to shift the straight line upward. The susceptance in this region of ω_p/ω is inductive. In Fig. 7-8(a) we plotted G versus ω_p/ω for the cases $\nu/\omega = 0$, 0.1, and 0.5 and in Fig. 7-8(b), B versus ω_p/ω for the same parameters. Notice that $\nu/\omega = 0$, and 0.1 curves for B are not distinguishable on the graph.

3. Conclusion:

The antenna problem encountered in this report is basically a boundary value problem. To attack such a problem, we first express the electromagnetic fields in the wedge region, the coaxial region, the plasma sheath and the free space in a series whose coefficients are in terms of the ϕ-directed electric field, $\hat{E}(\phi)$ and $E(\phi)$, in the wedge aperture and the shell slot, respectively. Then upon applying the boundary conditions, we formulate two coupled integral equations in which $\hat{E}(\phi)$ and $E(\phi)$ are the unknown functions. Both integral equations are of the first kind of the Fredholm type if one of the slots fields is assumed known. Only in one, however, the magnetic current source is present. This we will call the inhomogeneous equation; the other one — the homogeneous equation, for the purpose of present discussion. Thus the boundary value problem is reduced to the problem of solving these two coupled integral equations. However, for practical purposes, we may regard the wedge region as a transmission line loaded at the cylinder surface by a terminal admittance. The knowledge of the terminal admittance is fundamentally important in studying the behavior of an antenna. For this purpose, from the inhomogeneous integral equation, we formulated two different expressions for the terminal admittance. On the assumption that the solution of the homogeneous integral equation mentioned above is obtainable, one of the above two expressions for the terminal admittance is proved to be stationary with respect to the functional variation of $\hat{E}(\phi)$. An analytical solution of the homogeneous integral equation in a series form has been found for the low frequency region. This solution depends on the radii k_0a, k_0b, k_0c; ω_p/ω, ν/ω, and the angular width of the shell slot $2\phi_0$. For narrow shell slot, the series which represents the solution converges rapidly. The other form of the terminal admittance of the wedge waveguide is not found stationary with respect to the functional variation of $\hat{E}(\phi)$. However, this new form of the terminal admittance gives us some physical insight about the antenna via an equivalent circuit.

When the angular width of the wedge aperture and shell slot are very narrow, from the stationary form of the terminal admittance, we obtained an explicit expression for the terminal admittance. Based on this explicit form, in some special cases, we were able to discuss the behavior of the terminal admittance theoretically. From the above discussions and the numerical results presented in the preceeding section of this chapter we may briefly conclude:

a) The slotted circular shell functions as a tuning element and a matching transformer. Therefore a suitable choice of the width of the coaxial region and the slot separation angle θ will result in more power radiated into the free space than by the wedge cylinder alone.

b) The frequency response of the conductance and susceptance of the coaxial antenna peak repeatedly at different frequencies, with narrow bandwidth in comparison with the wedge cylinder.

c) When $\omega_p/\omega < 1$, the plasma sheath thickness $k_0(c-b)$ has little effect on the conductance and susceptance. When $\omega_p/\omega > 1$, and plasma collisions are neglected, for large sheath thickness, the conductance decreases exponentially while the susceptance approaches to a constant which depends on the ratio ω_p/ω. If the collisions are not negligible, we observe that the behavior of the susceptance is not changed but the conductance approaches to a constant depending upon ν/ω.

d) For a fixed operating frequency and plasma sheath thickness, when $\omega_p/\omega < 1$, the collision term ν/ω has little effect on the susceptance, but increases the magnitude of the conductance. For large ω_p/ω, further increasing the plasma density will have the same effect on the conductance as the increasing of plasma sheath thickness, but will make the terminal susceptance decreases continuously to the case of unslotted conducting shell.

APPENDIX
A-1
PROOF OF THE STATIONARY PROPERTY OF y(a)

To start the proof we take the first variation of Eq. (2-4-4); the result is

$$\delta y(a)\left[\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi)\,d\phi\right]^2 + 2y(a)\int_{\theta-\theta_0}^{\theta+\theta_0}\delta\hat{E}(\phi)\,d\phi\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\,d\phi'$$

$$= j4\sum_{n=1}^{\infty}\frac{J_{\frac{n\pi}{2\theta_0}}(k_0a)}{J'_{\frac{n\pi}{2\theta_0}}(k_0a)}\int_{\theta-\theta_0}^{\theta+\theta_0}\delta\hat{E}(\phi)\cos\frac{n\pi}{2\theta_0}(\phi-\theta+\theta_0)\,d\phi\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos\frac{n\pi}{2\theta_0}(\phi'-\theta-\theta_0)\,d\phi'$$

$$+ j\frac{2\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(k_0a)N'_n(k_0b)-J'_n(k_0b)N_n(k_0a)}{J'_n(k_0b)N'_n(k_0a)-J'_n(k_0b)N'_n(k_0b)}\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\delta\hat{E}(\phi)\int_{\theta-\theta_0}^{\theta+\theta_0}\hat{E}(\phi')\cos n(\phi-\phi')\,d\phi'$$

$$-\frac{2\theta_0}{\pi}\frac{2}{\pi k_0a}\sum_{n=0}^{\infty}\epsilon_n v_n^{(1)}\left[\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\hat{E}\phi\int_{-\phi_0}^{\phi_0}\delta E(\phi')\cos n(\phi-\phi')\,d\phi'\right.$$

$$\left.+\int_{\theta-\theta_0}^{\theta+\theta_0}d\phi\,\delta\hat{E}(\phi)\int_{-\phi_0}^{\phi_0}E(\phi')\cos n(\phi-\phi')\,d\phi'\right]\quad . \qquad \text{(A-1-1)}$$

From integral Eq. (2-3-31) one can show that

$$\frac{2}{\pi k_0 a} \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} \int_{\theta-\theta_0}^{\theta+\theta_0} d\phi \hat{E}(\phi) \int_{-\phi_0}^{\phi_0} \delta E(\phi') \cos n(\phi-\phi') d\phi'$$

$$= \frac{2}{\pi k_0 a} \frac{\theta_0}{\pi} \sum_{n=0}^{\infty} \epsilon_n v_n^{(1)} \int_{\theta-\theta_0}^{\theta+\theta_0} d\phi \, \delta \hat{E}(\phi) \int_{-\phi_0}^{\phi_0} E(\phi') \cos n(\phi-\phi') d\phi' \quad . \qquad \text{(A-1-2)}$$

Upon applying (A-1-2) to (A-1-1) and moving the second term on the left hand side of (A-1-1) to the right hand side it is seen from (2-4-3) that the right hand side is zero, i.e.

$$\delta y(a) = 0 \quad .$$

One thus concludes that a first variation in the aperture field of the wedge gives a second variation of the terminal admittance of the wedge waveguide.

APPENDIX
A-2
SOLUTION OF INTEGRAL EQS. (3-3-2) and (3-3-3)

In this appendix, we will employ the Schwinger transformation (Lewin, 1951) and use the trigonometric series (Schmeidler, 1955) to solve integral Eqs. (3-3-2) and (3-3-3).

Since

$$\sum_{n=1}^{\infty} \frac{\cos n\phi \cos n\phi'}{n} = -\frac{1}{2} \ln 2\left|\cos\phi' - \cos\phi\right| \tag{A-2-1}$$

(3-3-2) and (3-3-3) become, respectively,

$$-\frac{1}{2}\int_{-\phi_0}^{\phi_0} F_0^{(e)}(\phi') \ln 2\left|\cos\phi' - \cos\phi\right| d\phi'$$

$$= \frac{1}{\pi(1+\bar{k}^{2})(k_0 b)^2} \sum_{n=0}^{\infty} \frac{\epsilon_n \Gamma_n^{(e)} \cos n\phi}{J_n'(k_0 b)N_n'(k_0 a) - J_n'(k_0 a)N_n'(k_0 b)} \tag{A-2-2}$$

and

$$-\frac{1}{2}\int_{-\phi_0}^{\phi_0} f_n^{(e)}(\phi) \ln 2\left|\cos\phi' - \cos\phi\right| d\phi'$$

$$= \frac{1}{(1+\bar{k}^{2})(k_0 b)} \cos n\phi \quad . \tag{A-2-3}$$

We then introduce a transformation due to Schwinger, i.e.,

$$\cos\phi = \cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s \quad . \tag{A-2-4}$$

It is obvious that one may map the region $-\phi_0 \leq \phi \leq \phi_0$ into the region $-\pi \leq s \leq \pi$ with the transformation (A-2-4) but not in one to one correspondence. Thus we further introduce the restrictions

$-\pi \leq s \leq 0$ corresponding to $-\phi_0 < \phi \leq 0$

$0 \leq s \leq \pi$ corresponding to $0 \leq \phi \leq \phi_0$

to the transformation (A-2-4). In this report, whenever the Schwinger transformation is mentioned, these two restrictions as well as (A-2-4) are implied.

Upon applying the Schwinger transformation to (A-2-2) and (A-2-3), one obtains

$$\ln\csc\frac{\phi_0}{2}\int_{-\pi}^{\pi}\bar{F}_0^{(e)}(t)\frac{d\phi'}{dt}\,dt + \int_{-\pi}^{\pi}\bar{F}_0^{(e)}(t)\frac{d\phi'}{dt}\sum_{m=1}^{\infty}\frac{\cos ms\cos mt}{m}\,dt$$

$$= \frac{1}{\pi(1+\bar{k}^{-2})(k_0b)^2}\sum_{n=0}^{\infty}\frac{\epsilon_n\Gamma_n^{(e)}\cos\left[n\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right]}{J_n'(k_0b)N_n'(k_0a)-J_n'(k_0a)N_n'(k_0b)} \tag{A-2-5}$$

and

$$\ln\csc\frac{\phi_0}{2}\int_{-\pi}^{\pi}\bar{f}_n^{(e)}(t)\frac{d\phi'}{dt}\,dt + \int_{-\pi}^{\pi}\bar{f}_n^{(e)}(t)\frac{d\phi'}{dt}\sum_{m=1}^{\infty}\frac{\cos ms\cos mt}{m}\,dt$$

$$= -\frac{1}{2(1+\bar{k}^{-2})k_0b}\cdot\cos\left[n\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right] \tag{A-2-6}$$

where

$$\bar{F}_0^{(e)}(s) \equiv F_0^{(e)}\left[\phi(s)\right] \tag{A-2-7}$$

$$\bar{f}_n^{(e)}(s) \equiv f_n^{(e)}\left[\phi(s)\right] \tag{A-2-8}$$

for $-\phi_0 < \phi < \phi_0$ and $-\pi < s < \pi$.

The free terms of Eqs. (A-2-5) and (A-2-6), respectively, can be expanded into Fourier series; thus,

$$\sum_{m=0}^{\infty} \frac{\epsilon_m \Gamma_m^{(e)} \cos\left[m\cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s\right)\right]}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} = \sum_{p=0}^{\infty} a_p^{(e)} \cos p\,s \tag{A-2-9}$$

$$\cos\left[n\cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s\right)\right] = \sum_{p=0}^{\infty} b_{np}^{(e)} \cos p\,s \tag{A-2-10}$$

with

$$a_0^{(e)} \equiv \frac{1}{2\pi}\sum_{m=0}^{\infty} \frac{\epsilon_m \Gamma_m^{(e)} X_{0m}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \tag{A-2-11}$$

$$a_p^{(e)} \equiv \frac{2}{\pi}\sum_{m=p}^{\infty} \frac{\Gamma_m^{(e)} X_{pm}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \tag{A-2-12}$$

$$b_{n0}^{(e)} = \frac{1}{2\pi} X_{0n} \tag{A-2-13}$$

$$b_{np}^{(e)} = \frac{1}{\pi} X_{pn} \tag{A-2-14}$$

where

$$X_{pm} = \int_{-\pi}^{\pi} \cos ps \cos\left[m \cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s\right)\right] ds \qquad \text{(A-2-15)}$$

The properties of X_{pm} will be investigated in A-4, however for the present discussion, we note that

$$X_{pm} = 0 \quad \text{for} \quad p > m \quad . \qquad \text{(A-2-16)}$$

If we let

$$\bar{F}_0^{(e)}(t)\frac{d\phi'}{dt} = \alpha_0^{(e)} + \sum_{p=1}^{\infty} \alpha_p^{(e)} \cos pt \quad , \qquad \text{(A-2-17)}$$

$$\bar{f}_n^{(e)}(t)\frac{d\phi'}{dt} = t_{n0}^{(e)} + \sum_{p=1}^{\infty} t_{np}^{(e)} \cos pt \quad , \qquad \text{(A-2-18)}$$

for $-\pi \leq t \leq \pi$;

upon substituting (A-2-9), (A-2-10), (A-2-17) and (A-2-18) in (A-2-5) and (A-2-6), respectively, and employing the expressions (A-2-11) through (A-2-14), we obtain

$$\alpha_0^{(e)} = \frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2} \frac{1}{4\pi^2 \ell n \csc\frac{\phi_0}{2}} \cdot \sum_{m=0}^{\infty} \frac{\epsilon_m \Gamma_m^{(e)} X_{0m}}{J'_m(k_0 b)N'_m(k_0 a) - J'_m(k_0 a)N'_m(k_0 b)} \; ,$$

(A-2-19)

$$\alpha_p^{(e)} = \frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2} \frac{2p}{\pi} \sum_{m=p}^{\infty} \frac{\Gamma_m^{(e)} X_{pm}}{J'_m(k_0 b)N'_m(k_0 a) - J'_m(k_0 a)N'_m(k_0 b)} , \quad \text{(A-2-20)}$$

$$\beta_{n0}^{(e)} = -\frac{1}{2(1+\bar{k}^2)(k_0 b)} \frac{X_{0n}}{4\pi^2 \ln \csc \frac{\phi_0}{2}} , \quad \text{(A-2-21)}$$

$$\beta_{np}^{(e)} = -\frac{1}{2(1+\bar{k}^2)(k_0 b)} \frac{p}{\pi^2} X_{pn} . \quad \text{(A-2-22)}$$

If one differentiates the Schwinger transformation with respect to ϕ, one has

$$\frac{dt}{d\phi} = \frac{\sqrt{2}\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} , \quad -\phi_0 < \phi < \phi_0 . \quad \text{(A-2-23)}$$

Thus from (A-2-7) and (A-2-17), we obtain

$$F_0^{(e)}(\phi) = \frac{\sqrt{2}\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \alpha_0^{(e)} + \sum_{p=1}^{\infty} \alpha_p^{(e)} \cos\left[p \cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right]\right\} ,$$

$$-\phi_0 < \phi < \phi_0 . \quad \text{(A-2-24)}$$

and from (A-2-8) and (A-2-18),

$$f_n^{(e)}(\phi) = \frac{\sqrt{2}\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \beta_{n0}^{(e)} + \sum_{p=1}^{n} \beta_{np}^{(e)} \cos\left[p \cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right]\right\} ,$$

$$-\phi_0 < \phi < \phi_0 , \quad n = 0, 1, 2, \ldots N , \quad \text{(A-2-25)}$$

If one defines

$$S_0^{(e)} = \sum_{m=0}^{\infty} \frac{\epsilon_m \Gamma_m^{(e)} X_{0m}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \quad , \tag{A-2-26}$$

$$S_p^{(e)} = 2 \sum_{m=p}^{\infty} \frac{\Gamma_m^{(e)} X_{pm}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \tag{A-2-27}$$

then $F_0^{(e)}(\phi)$ and $f_n^{(e)}(\phi)$ becomes

$$F_0^{(e)}(\phi) = \frac{\sqrt{2}}{\pi(1+\bar{k}^2)(k_0b)^2} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \frac{S_0^{(e)}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} + \sum_{p=1}^{\infty} \frac{pS_p^{(e)}}{\pi^2} \cdot \cos\left[p\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right] \right\} ,$$

$$-\phi_0 < \phi < \phi_0 \tag{A-2-28}$$

$$f_n^{(e)}(\phi) = -\frac{\sqrt{2}}{2(1+\bar{k}^2)(k_0b)} \frac{\cos\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \frac{X_{0n}}{4\pi^2 \ln\csc\frac{\phi_0}{2}} + \sum_{p=1}^{n} \frac{pX_{pn}}{\pi^2} \cdot \cos\left[p\cos^{-1}\left(\csc^2\frac{\phi_0}{2}\cos\phi - \cot^2\frac{\phi_0}{2}\right)\right] \right\} ,$$

$$-\phi_0 < \phi < \phi_0 \quad . \tag{A-2-29}$$

APPENDIX
A-3
SOLUTION OF INTEGRAL EQ. (3-4-5) and (3-4-6)

Same as in A-2 we employ the Schwinger transformation and the trigonometrical series method to solve integral Eqs. (3-4-5) and (3-4-6).

If we differentiate the well known formula (A-2-1) with respect to ϕ for all ϕ, $\phi' < \phi_0$ except $\phi = \phi'$, we have

$$\sum_{m=1}^{\infty} \sin m\phi \cos m\phi' = \frac{1}{2} \frac{\sin\phi}{\cos\phi' - \cos\phi} \quad . \tag{A-3-1}$$

Upon substituting (A-3-1) into (3-4-5) and (3-4-6), one obtains, respectively,

$$\int_{-\phi_0}^{\phi_0} \frac{1}{2} \frac{F_0^{(0)}(\phi')\,d\phi'}{\cos\phi' - \cos\phi} = -\frac{1}{\pi(1+\bar{k}^2)(k_0 b)^2} \sum_{m=1}^{\infty} \frac{\Gamma_m^{(0)}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \frac{\sin m\phi}{\sin\phi} \tag{A-3-2}$$

and

$$\int_{-\phi_0}^{\phi_0} \frac{1}{2} \frac{f_n^{(0)}(\phi')\,d\phi'}{\cos\phi' - \cos\phi} = -\frac{1}{(1+\bar{k}^2)(k_0 b)} \frac{\sin n\phi}{\sin\phi} \quad , \quad n = 1, 2, \ldots N \quad . \tag{A-3-3}$$

Since

$$\frac{\sin n\phi}{\sin\phi} = 2 \sum_{m=1,3,5\ldots}^{n-1} \cos m\phi \qquad \text{for } n \text{ even}$$

$$= \sum_{m=0,2,4,\ldots}^{n-1} \epsilon_m \cos m\phi \quad \text{for } n \text{ odd} \quad , \tag{A-3-4}$$

the free term of (A-3-2) can be written as

$$\sum_{m=1}^{\infty} \frac{\Gamma_m^{(0)}}{J'_m(k_0b)N'_m(k_0a) - J'_m(k_0a)N'_m(k_0b)} \frac{\sin n\phi}{\sin\phi} = \sum_{p=0}^{\infty} \epsilon_p L_p \cos p\phi \qquad \text{(A-3-5)}$$

where

$$L_p = \sum_{m=1}^{\infty} v_{p+2m-1}^{(1)} \Gamma_{p+2m-1}^{(1)} , \qquad \text{(A-3-6)}$$

$$v_n^{(1)} = \frac{1}{J'_n(k_0b)N'_n(k_0a) - J'_n(k_0a)N'_n(k_0b)} . \qquad \text{(A-3-7)}$$

Upon substituting (A-3-5) in (A-3-2) and (A-3-4) in (A-3-3), we have

$$\int_{-\phi_0}^{\phi_0} \frac{1}{2} \frac{F_0^{(0)}(\phi')\,d\phi'}{\cos\phi' - \cos\phi} = -\frac{1}{\pi(1+\bar{k}^2)(k_0b)^2} \sum_{p=0}^{\infty} \epsilon_p L_p \cos p\phi \; ; \qquad \text{(A-3-8)}$$

$$\int_{-\phi_0}^{\phi_0} \frac{1}{2} \frac{f_n^{(0)}(\phi')\,d\phi'}{\cos\phi' - \cos\phi} = -\frac{2}{(1+\bar{k}^2)(k_0b)} \sum_{p=1,3,5\ldots}^{n-1} \cos p\phi, \quad n = \text{even} ,$$

$$= -\frac{1}{(1+\bar{k}^2)(k_0b)} \sum_{p=0,2,4\ldots}^{n-1} \epsilon_p \cos p\phi , \quad n = \text{odd} . \qquad \text{(A-3-9)}$$

We apply Schwinger's transformation (A-2-4) to (A-3-8) and (A-3-9) and let

$$\bar{F}_0^{(0)}(t) \equiv F_0^{(0)}[\phi(t)] \qquad \text{(A-3-10)}$$

$$\bar{f}_n^{(0)}(t) \equiv f_n^{(0)}[\phi(t)] \tag{A-3-11}$$

for $-\phi_0 \leq \phi \leq \phi_0$ and $-\pi \leq t \leq \pi$, and obtain respectively

$$\int_{-\phi_0}^{\phi_0} \frac{\frac{1}{2}\,\bar{F}_0^{(0)}(t)\frac{d\phi'}{dt}dt}{\sin^2\frac{\phi_0}{2}(\cos t-\cos s)} = -\frac{1}{\pi(1+\bar{k}^2)(k_0b)^2}\sum_{p=0}^{\infty}\epsilon_p L_p \cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right]$$

(A-3-12)

$$\int_{-\phi_0}^{\phi_0} \frac{\frac{1}{2}\,\bar{f}_n^{(0)}(t)\frac{d\phi'}{dt}dt}{\sin^2\frac{\phi_0}{2}(\cos t-\cos s)} = -\frac{2}{(1+\bar{k}^2)(k_0b)}\sum_{p=1,3,5\ldots}^{n-1}\cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right],$$

$n = \text{even}$,

$$= -\frac{1}{(1+\bar{k}^2)(k_0b)}\sum_{p=0,2,4\ldots}^{n-1}\epsilon_p\cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right],$$

$n = \text{odd}$. (A-3-13)

In order to generate convenient expansions we multiply both sides of the last three integral equations by $\sin s$. The free terms of these new integral equations can be expanded by Fourier series, i.e.,

$$\sum_{p=0}^{n-1}\epsilon_p L_p\cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right]\sin s = \sum_{m=1}^{\infty}a_m^{(0)}\sin ms \tag{A-3-14}$$

$$\sum_{p=1,3,\ldots}^{n-1} 2\cos\left[p\cos^{-1}\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s\right)\right]\sin s = \sum_{m=1}^{\infty}b_{nm}^{(0)}\sin ms ,$$

$n = \text{even}$, (A-3-15)

and

$$\sum_{p=0,2,\ldots}^{n-1} \epsilon_p \cos\left[p\cos^{-1}(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos s)\right]\sin s = \sum_{m=1}^{\infty} b_{nm}^{(0)} \sin m s ,$$

$$n = \text{odd} . \qquad \text{(A-3-16)}$$

Since

$$\int_{-\pi}^{\pi} \sin ms \sin s \cos\left[p\cos^{-1}(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos s)\right] ds$$

$$= \frac{1}{2}(X_{m-1,p} - X_{m+1,p})$$

we obtain

$$a_m^{(0)} = \frac{1}{2\pi}\left(\sum_{p=m-1}^{\infty} \epsilon_p L_p X_{m-1,p} - 2\sum_{p=m+1}^{\infty} L_p X_{m+1,p}\right) , \qquad \text{(A-3-17)}$$

$$b_{nm}^{(0)} = \frac{1}{2\pi}\left(\sum_{p=0,2,4,\ldots}^{n-1} \epsilon_p X_{m-1,p} - 2\sum_{p=2,4,\ldots}^{n-1} X_{m+1,p}\right) , \quad n=\text{odd}, \qquad \text{(A-3-18)}$$

$$b_{nm}^{(0)} = \frac{1}{\pi}\sum_{p=1,3,\ldots}^{n-1} (X_{m-1,p} - X_{m+1,p}) , \quad n = \text{even} . \qquad \text{(A-3-19)}$$

We note that $b_{nm}^{(0)} = 0$ when $m > n$. Because

$$\frac{\sin s}{2(\cos t - \cos s)} = \sum_{m=1}^{\infty} \sin m s \cos m t ,$$

(A-3-12) and (A-3-13) become

$$\int_{-\pi}^{\pi} \bar{F}_0^{(0)}(t)\,\frac{d\phi'}{dt} \sum_{m=1}^{\infty} \sin ms \cos mt\, dt = -\frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^2)(k_0 b)^2} \sum_{m=1}^{\infty} a_m^{(0)} \sin ms\,, \tag{A-3-20}$$

$$\int_{-\pi}^{\pi} \bar{f}_n^{(0)}(t)\,\frac{d\phi'}{dt} \sum_{m=1}^{\infty} \sin ms \cos mt\, dt = -\frac{\sin^2\frac{\phi_0}{2}}{(1+\bar{k}^2)(k_0 b)} \sum_{m=1}^{n} b_{nm}^{(0)} \sin ms\,,$$

$$n = 1, 2, \ldots N\,. \tag{A-3-21}$$

Now we let

$$\bar{F}_0^{(0)}(t)\,\frac{d\phi'}{dt} = C_0 + C_1 \cos t + C_2 \cos 2t + \ldots\,, \tag{A-3-22}$$

$$\bar{f}_n^{(0)}(t)\,\frac{d\phi'}{dt} = d_{n0} + d_{n1} \cos t + d_{n2} \cos 2t + \ldots\,. \tag{A-3-23}$$

Upon substituting (A-3-22) and (A-3-23) in (A-3-20) and (A-3-21), the Fourier coefficients C_m and d_{nm} are found as

$$C_m = -\frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^2)(k_0 b)^2}\,\frac{a_m^{(0)}}{\pi} \tag{A-3-24}$$

$$d_{nm} = -\frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^2)(k_0 b)}\, b_{nm}^{(0)}\,, \qquad m \geq 1\,. \tag{A-3-25}$$

Hence

$$\bar{F}_0^{(0)}(t)\,\frac{d\phi'}{dt} = C_0 - \frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^{-2})(k_0 b)^2}\sum_{m=1}^{\infty}\frac{a_m^{(0)}}{\pi}\cos mt \;,$$

$$-\pi \le t \le \pi \qquad \text{(A-3-26)}$$

$$\bar{f}_n^{(0)}(t)\,\frac{d\phi'}{dt} = d_{n0} - \frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^{-2})(k_0 b)}\sum_{m=1}^{n} b_{nm}^{(0)}\cos mt \;,$$

$$-\pi \le t \le \pi \;. \qquad \text{(A-3-27)}$$

We recall that

$$\cos\phi' = \cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t$$

and therefore

$$\frac{d\phi'}{dt} = \frac{\sin^2\frac{\phi_0}{2}\sin t}{\sqrt{1-\left(\cos^2\frac{\phi_0}{2}+\sin^2\frac{\phi_0}{2}\cos t\right)^2}} \;.$$

It is obvious

$$\frac{d\phi'}{dt} \longrightarrow 0 \qquad \text{as} \qquad t \longrightarrow \pm\pi \;.$$

Therefore,

$$\left.\begin{array}{l}\bar{F}_0^{(0)}(t)\,\dfrac{d\phi'}{dt}\\[2ex]\bar{f}_0^{(0)}(t)\,\dfrac{d\phi'}{dt}\end{array}\right\}\longrightarrow 0 \qquad \text{as} \quad t \rightarrow \pm\pi$$

and hence from (A-3-26) and (A-3-27), respectively, we have

$$C_0 = \frac{\sin^2\frac{\phi_0}{2}}{\pi^2(1+\bar{k}^2)(k_0 b)^2}\sum_{m=1}^{\infty} a_m^{(0)} \cos m\pi \quad , \tag{A-3-28}$$

$$d_{n0} = \frac{\sin^2\frac{\phi_0}{2}}{\pi^2(1+\bar{k}^2)\, k_0 b}\sum_{m=1}^{n} b_{nm}^{(0)} \cos m\pi \quad . \tag{A-3-29}$$

Thus $F_0^{(0)}(t)$ and $f_n^{(0)}(t)$ are

$$\bar{F}_0^{(0)}(t) = \frac{\sin^2\frac{\phi_0}{2}}{\pi^2(1+\bar{k}^2)(k_0 b)^2}\,\frac{dt}{d\phi'}\sum_{m=1}^{\infty} a_m^{(0)}(\cos m\pi - \cos mt) \quad ,$$

$$-\pi \leq t \leq \pi \quad , \tag{A-3-30}$$

$$\bar{f}_n^{(0)}(t) = \frac{\sin^2\frac{\phi_0}{2}}{\pi(1+\bar{k}^2)\, k_0 b}\,\frac{dt}{d\phi'}\sum_{m=1}^{n} b_{nm}^{(0)}(\cos m\pi - \cos mt) \quad ,$$

$$-\pi \leq t \leq \pi \quad . \tag{A-3-31}$$

For the convenience of further investigation we introduce a new function

$$\bar{U}_m(t) = \frac{\cos m\pi - \cos mt}{1 + \cos t} = \sum_{p=0}^{m-1} \epsilon_p (-1)^{m-p} (m-p) \cos pt \quad . \tag{A-3-32}$$

If we let $U_m(\phi)$ denote the function $U_m(t)$ in ϕ interval, then

$$U_m(\phi) = \sum_{p=0}^{m-1} \epsilon_p (-1)^{m-p} (m-p) \cos\left[p \cos^{-1}\left(\csc^2 \frac{\phi_0}{2} \cos \phi - \cot^2 \frac{\phi_0}{2}\right)\right] \tag{A-3-33}$$

Thus Eqs. (A-3-30) and (A-3-31) become

$$\bar{F}_0^{(0)}(t) = \frac{\sin^2 \frac{\phi_0}{2}}{\pi^2 (1+\bar{k}^2)(k_0 b)^2} \frac{dt}{d\phi'} (1+\cos t) \sum_{m=1}^{\infty} a_m^{(0)} \bar{U}_m(t) \quad ,$$

$$-\pi \leq t \leq \pi \quad , \tag{A-3-34}$$

and

$$\bar{f}_n^{(0)}(t) = \frac{\sin^2 \frac{\phi_0}{2}}{(1+\bar{k}^2)\pi k_0 b} \frac{dt}{d\phi'} (1+\cos t) \sum_{m=1}^{n} b_{nm}^{(0)} \bar{U}_m(t) \quad ,$$

$$-\pi \leq t \leq \pi \quad . \tag{A-3-35}$$

From the Schwinger's transformation, we obtain

$$\sin^2 \frac{\phi_0}{2} \frac{dt}{d\phi'} (1+\cos t) = 2 \cos \frac{\phi}{2} \sqrt{\cos\phi - \cos\phi_0} \tag{A-3-36}$$

Therefore,

$$F_0^{(0)}(\phi) = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)^2} \cdot \cos\left(\frac{\phi}{2}\right)\sqrt{\cos\phi - \cos\phi_0} \sum_{m=1}^{\infty} a_m^{(0)} U_m(\phi) ,$$

$$-\phi_0 \leq \phi \leq \phi_0 \tag{A-3-37}$$

$$f_n^{(0)}(\phi) = \frac{\sqrt{2}}{(1+\bar{k}^2)(\pi k_0 b)} \cos\left(\frac{\phi}{2}\right)\sqrt{\cos\phi - \cos\phi_0} \sum_{m=1}^{n} b_{nm}^{(0)} U_m(\phi) ,$$

$$-\phi_0 \leq \phi \leq \phi_0 \quad . \tag{A-3-38}$$

APPENDIX
A-4

PROPERTIES OF X_{qp}

In Eq. (3-3-8) we put

$$\cos s = x \quad , \quad \cos^2 \frac{\phi_0}{2} = b \quad \text{and} \quad \sin^2 \frac{\phi_0}{2} = a \tag{A-4-1}$$

then X_{qp} becomes

$$X_{qp} = 2 \int_{-1}^{1} \frac{\cos(q \cos^{-1} x) \cos\left[p \cos^{-1}(b + ax)\right]}{\sqrt{1 - x^2}} dx \quad . \tag{A-4-2}$$

Tchebychev polynomial is defined as

$$T_q(x) = \cos\left[q \cos^{-1} x\right] ,$$

$$T_0(x) = 1 ,$$

therefore one may rewrite (A-4-2) in the form

$$X_{qp} = 2 \int_{-1}^{1} \frac{T_q(x) T_p(b + ax)}{\sqrt{1 - x^2}} dx \quad . \tag{A-4-3}$$

It is obvious that for any ϕ_0

$$X_{00} = 2\pi \quad . \tag{A-4-4}$$

$T_p(b+ax)$ is a p-th order polynomial of $(b+ax)$, while $T_m(x)$ is a m-th order polynomial of x, therefore

$$T_p(b+ax) = \sum_{m=0}^{p} \alpha_m T_m(x) \quad .$$

If we multiply both sides of that equation with $2T_q(x) \Big/ \sqrt{1-x^2}$, and integrate from $x=-1$ to $x=1$, we find

$$X_{qp} = 2 \sum_{m=0}^{p} \alpha_m \int_{-1}^{1} \frac{T_m(x) T_q(x)}{\sqrt{1-x^2}} dx \quad .$$

But

$$\int_{-1}^{1} \frac{T_m(x) T_q(x) dx}{\sqrt{1-x^2}} = \begin{cases} 0 \ , & m \neq q \ , \\ \frac{\pi}{2} \ , & m = q \neq 0 \ , \\ \pi \ , & m = q = 0 \ , \end{cases}$$

and therefore we conclude that for any ϕ_0

$$X_{qp} = 0 \text{ if } q > p \quad . \qquad \text{(A-4-5)}$$

DuHamel (1953), Salzer (1956), Brown (1957) and others in their works on radiation pattern of antenna arrays also studied integral (A-4-3). By different approaches, they carried out the integration and arrived at a tedious formula

$$\epsilon_q X_{q,\,q+n} = 2\pi(2b)^n a^q \sum_{r=0}^{[n/2]} \Bigg\{ \left(-\frac{1}{4b^2}\right)^r \left[2\binom{q+n-r}{r} - \binom{q+n-r-1}{r} \right]$$

$$\cdot \sum_{j=0}^{[n/2]-r} \left[\binom{q-2j}{j} \binom{q+n-2r}{q+2j} \left(\frac{a^2}{4b^2}\right)^j \right] \Bigg\} \qquad \text{(A-4-6)}$$

where $\binom{q}{r}$ denotes the binomial coefficient $\frac{q!}{r!\,(q-r)!}$, and $[y]$ denotes the largest integer not exceeding y. For the convenience of further discussion, we list $X_{q,\,q+n}$ for $n=0$ to $n=4$:

$$\epsilon_q X_{q,\,q} = 2\pi a^q \,,$$

$$\epsilon_q X_{q,\,q+1} = 2\pi (q+1) a^q (2b) \,,$$

$$\epsilon_q X_{q,\,q+2} = 2\pi (q+2) a^q \left[\left(\frac{q+1}{2!} (2b)^2 - 1 \right) + a^2 \right] ,$$

$$\epsilon_q X_{q,\,q+3} = 2\pi (q+3) a^q (2b) \left[(q+1) \left(\frac{q+2}{3!} (2b)^2 - 1 \right) + (q+2) a^2 \right] ,$$

$$\epsilon_q X_{q,\,q+4} = 2\pi (q+4) a^q \left\{ \left[\frac{(q+1)(q+2)(q+3)}{4!} - \frac{(q+1)(q+2)}{2!} \right] (2b)^4 \right. ,$$

$$\left. + \left[\frac{(q+2)(q+3)}{2!} - (q+2) \right] a^2 (2b)^2 + \frac{q+3}{2!} a^4 \right\} . \qquad \text{(A-4-7)}$$

For $n \longrightarrow \infty$, one may evaluate the integral (A-4-2) by the method of stationary phase:

$$X_{q,\,q+n} \simeq 2\pi \cos q\pi \, \frac{\cos (n\phi_0 + q\phi_0 - \frac{\pi}{4})}{\sqrt{2\pi n \tan \frac{\phi_0}{2}}} . \qquad \text{(A-4-8)}$$

Combining the informations given by (A-4-7) and (A-4-8), we may state the following behavior of $X_{q,\,q+n}$: For any $q \neq 0$, as n increases from 0, $X_{q,\,q+n}$ increases gradually from πa^q to its first maximum and then repeatedly swings from negative maximum to positive maximum with a decreasing amplitude. For $q=0$, as n increases from 0, X_{0n} decreases gradually from 2π to a negative maximum, then swings up and down with a gradually reducing amplitude. In the case $q \neq 0$, suppose the first maximum of $X_{q,q+n}$ occurs at $n = M$, then it is seen that the increasing of q reduces the value of $X_{q,q}$ as well as increases the value of M.

Now we consider the special case of narrow slot. The necessary p values for the narrow slot satisfy the condition $p\phi_0^2 << 1$. If in

$$\cos[p\cos^{-1} z] = \sum_{r=0}^{[p/2]} (-1)^r 2^{p-2r-1}\left[2\binom{p-r}{r} - \binom{p-r-1}{r}\right] z^{p-2r} \qquad \text{(A-4-9)}$$

we replace z by $\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t$ and make use of

$$(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t)^{p-2r} \simeq 1 - \frac{\phi_0^2}{4}(p-2r)(1-\cos t) \qquad \text{(A-4-10)}$$

we have

$$\cos\left[p\cos^{-1}(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t)\right]$$

$$\simeq \sum_{r=0}^{[p/2]} (-1)^r 2^{p-2r-1}\left[2\binom{p-r}{r} - \binom{p-r-1}{r}\right]$$

$$- \frac{\phi_0^2}{4} g(p) + \frac{\phi_0^2}{4} g(p)\cos t \qquad \text{(A-4-11)}$$

where

$$g(p) = \sum_{r=0}^{[p/2]} (-1)^r 2^{p-2r-1}\left[2\binom{p-r}{r} - \binom{p-r-1}{r}\right]\cdot(p-2r) \quad . \qquad \text{(A-4-12)}$$

We note that $g(p) < p$. Upon substituting (A-4-11) in (A-4-2) and employing the identity

$$\sum_{r=0}^{[p/2]} (-1)^r 2^{p-2r-1} \left[2\binom{p-r}{r} - \binom{p-r-1}{r} \right] = 1 \quad ,$$

we attain

$$X_{0p} \simeq 2\pi \left[1 - \frac{\phi_0^{\,2}}{4} g(p) \right] \quad , \tag{A-4-13}$$

$$X_{1p} \simeq \pi \frac{\phi_0^{\,2}}{4} g(p) \quad , \tag{A-4-14}$$

and

$$X_{qp} \approx 0 \qquad q \geq 2 \quad . \tag{A-4-15}$$

There is an alternate approach to find X_{0p} for $\phi_0 << 1$. From Schwinger's transformation, for $q = 0$ we may rewrite (A-4-2) as

$$X_{0p} = \int_{-\phi_0}^{\phi_0} \frac{2 \cos \frac{\phi}{2} \cos p\phi}{\sqrt{\cos\phi - \cos\phi_0}} \, d\phi$$

$$\simeq 2 \int_{-\phi_0}^{\phi_0} \frac{\cos p\phi}{\sqrt{\phi_0^{\,2} - \phi_0}} \, d\phi = 2\pi J_0(p\phi_0) \tag{A-4-16}$$

If the angular width of the shell slot is very wide, we may let $\phi_0 = \pi - \Delta$, where Δ is much smaller than unity, then we obtain

$$\cos^2 \frac{\phi_0}{2} + \sin^2 \frac{\phi_0}{2} \cos t = \sin^2 \frac{\Delta}{2} + \cos^2 \frac{\Delta}{2} \cos t \quad .$$

Now we let

$$b = \sin^2 \frac{\Delta}{2} \quad , \quad a = \cos^2 \frac{\Delta}{2} \quad .$$

Salzer (1956) showed that

$$T_p(ax+b) = \sum_{q=0}^{p} \left\{ \sum_{r=0}^{[(p-q)/2]} (-1)^r 2^{p-2r} \left[2\binom{n-r}{r} - \binom{n-r-1}{r} \right] \right.$$

$$\left. \cdot \sum^{[(p-q)/2]-r} \binom{q+2j}{j} \binom{p-2r}{q-2j} \cdot \left(\frac{a}{2}\right)^{q+2j} b^{p-q-2r-2j} \right\}$$

$$\cdot \; T_q(x) \quad . \qquad \text{(A-4-17)}$$

If $p\Delta^2 << 1$, (A-4-17) can be reduced to

$$T_p(ax+b) \simeq T_p(x) + \frac{p\Delta^2}{2} T_{p-1}(x) \quad . \qquad \text{(A-4-18)}$$

Therefore

$$X_{p,p} \simeq \pi \quad , \quad p \neq 0$$

$$X_{p,p+1} \simeq \frac{(p+1)\Delta^2}{2} \pi$$

and

$$X_{p,p+n} \simeq 0 \quad , \quad \text{for } n \geq 2$$

APPENDIX
A-5
INTEGRATION OF $A_{mn}^{(0)}$ AND $B_{m}^{(0)}$

Using the Schwinger transformation we obtain from (3-4-15) and (3-4-16), respectively,

$$A_{mn}^{(0)} = \int_{-\pi}^{\pi} m\tau_m \cos\left[m\cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t\right)\right] \cdot \bar{f}_n^{(0)}(t)\,\frac{d\phi'}{dt}\,dt \qquad \text{(A-5-1)}$$

and

$$B_{m}^{(0)} = \int_{-\pi}^{\pi} m\tau_m \cos\left[m\cos^{-1}\left(\cos^2\frac{\phi_0}{2} + \sin^2\frac{\phi_0}{2}\cos t\right)\right] \cdot \bar{F}_0^{(0)}(t)\,\frac{d\phi'}{dt}\,dt \ . \qquad \text{(A-5-2)}$$

Upon substituting (A-3-34) and (A-3-35) in (A-5-1) and (A-5-2), we have, respectively

$$A_{mn}^{(0)} = \frac{m\tau_m \sin^2\frac{\phi_0}{2}}{(1+\bar{k}^2)\pi k_0 b} \sum_{p=1}^{n} b_{np}^{(0)} (\cos p\pi X_{0m} - X_{pm}) \qquad \text{(A-5-3)}$$

$$B_{m}^{(0)} = \frac{m\tau_m \sin^2\frac{\phi_0}{2}}{(1+\bar{k}^2)\pi k_0 b} \sum_{p=1}^{\infty} a_{p}^{(0)} (\cos p\pi X_{0m} - X_{pm}) \ . \qquad \text{(A-5-4)}$$

APPENDIX

A-6

DIFFERENTIATION OF $W(\phi)$

Since

$$\sqrt{2}\,\frac{d}{d\phi}\left[\cos\frac{\phi}{2}\sqrt{\cos\phi-\cos\phi_0}\right]=-\frac{1}{\sqrt{2}}\;\frac{(2\cos\phi+1-\cos\phi_0)\sin\frac{1}{2}\phi}{\sqrt{\cos\phi-\cos\phi_0}} \tag{A-6-1}$$

we find from (3-4-13)

$$(1+\bar{k}^2)(\pi k_0 b)\,\frac{dW(\phi)}{d\phi}=-\frac{1}{\sqrt{2}}\;\frac{(2\cos\phi+1-\cos\phi_0)\sin\frac{1}{2}\phi}{\sqrt{\cos\phi-\cos\phi_0}}$$

$$\cdot\left\{\sum_{m=1}^{\infty}a_m^{(0)}U_m(\phi)+\pi k_0 b\sum_{n=1}^{N}\sigma_n^{(0)}\sum_{m=1}^{n}b_{nm}^{(0)}U_m(\phi)\right\}$$

$$+\sqrt{2}\cos\frac{\phi}{2}\sqrt{\cos\phi-\cos\phi_0}\left\{\sum_{m=1}^{\infty}a_m^{(0)}\frac{dU_m(\phi)}{d\phi}+\pi k_0 b\sum_{n=1}^{N}\sigma_n^{(0)}\sum_{m=1}^{n}b_{nm}^{(0)}\frac{dU_m(\phi)}{d\phi}\right\}. \tag{A-6-2}$$

But

$$\frac{dU_m(\phi)}{d\phi}=\frac{d\bar{U}_m(t)}{dt}\,\frac{dt}{d\phi}, \tag{A-6-3}$$

and

$$\frac{d\bar{U}_m(t)}{dt}=\frac{\sin t}{1+\cos t}\left[\frac{m\sin mt}{\sin t}+\frac{\cos m\pi-\cos mt}{1+\cos t}\right]. \tag{A-6-4}$$

If we define

$$\bar{V}_m(t) = \frac{\sin mt}{\sin t} \quad , \qquad -\pi \le t \le \pi \quad ,$$

$$= 1 + 2 \sum_{p=1}^{m-1} \cos pt \quad , \qquad m = \text{odd} \quad ,$$

$$= 2 \sum_{p=1}^{m-1} \cos pt \quad , \qquad m = \text{even} \qquad \text{(A-6-5)}$$

and

$$V_m(\phi) = 1 + 2 \sum_{p=1}^{m-1} \cos p\left[\cos^{-1}(\csc^2 \frac{\phi_0}{2} \cos\phi - \cot^2 \frac{\phi_0}{2})\right] \quad , \quad m = \text{odd} \quad ,$$

$$= 2 \sum_{p=1}^{m-1} \cos p\left[\cos^{-1}(\csc^2 \frac{\phi_0}{2} \cos\phi - \cot^2 \frac{\phi_0}{2})\right] \quad , \qquad m = \text{even} \quad ,$$

(A-6-6)

then we have

$$\frac{d\bar{U}_m(t)}{dt} = \frac{\sin t}{1 + \cos t}\left[m\bar{V}_m(t) + \bar{U}_m(t)\right] \quad . \qquad \text{(A-6-7)}$$

From the Schwinger's transformation, one can show that

$$\frac{dt}{d\phi} = \frac{1}{\sin^2 \frac{\phi_0}{2}} \; \frac{\sin\phi}{\sin t} \quad . \qquad \text{(A-6-8)}$$

Thus it is seen from (A-6-3), (A-6-7) and (A-6-8) that

$$\frac{dU_m(\phi)}{d\phi} = \frac{\sin\phi}{\cos\phi - \cos\phi_0} \cdot \left[m V_m(\phi) - U_m(\phi)\right] \quad . \qquad \text{(A-6-9)}$$

Hence (A-6-2) can be written as

$$(1+k^{-2})(\pi k_0 b)^2 \frac{dW(\phi)}{d\phi} = \frac{\sqrt{2}\,\sin\frac{\phi}{2}}{\sqrt{\cos\phi - \cos\phi_0}} \left\{ \cos^2\frac{\phi_0}{2} \left[\sum_{m=1}^{\infty} a_m^{(0)} U_m(\phi) + \pi k_0 b \sum_{n=1}^{N} \sigma_n^{(0)} \sum_{m=1}^{n} b_{nm}^{(0)} U_m(\phi) \right] \right.$$

$$\left. + (1+\cos\phi) \left[\sum_{m=1}^{\infty} m\, a_m^{(0)} V_m(\phi) + \pi k_0 b \sum_{n=1}^{N} \sigma_n^{(0)} \sum_{m=1}^{n} m\, b_{nm}^{(0)} V_m(\phi) \right] \right\} ,$$

$$-\phi < \phi < \phi_0 \quad . \qquad \text{(A-6-10)}$$

APPENDIX
A-7
PROPERTIES OF THE INTEGRALS (4-3-6) TO (4-3-9)

If we let $x = \eta/\theta_0$ then from (4-3-6) to (4-3-9) we obtain, respectively,

$$P_n^{(e)} = \theta_0^{\frac{1}{3}} \int_{-1}^{1} \frac{\cos n\pi x \, dx}{\sqrt[3]{1 - x^2}} , \tag{A-7-1}$$

$$Q_n^{(e)} = \theta_0^{\frac{1}{3}} \int_{-1}^{1} \frac{\cos n\theta_0 x \, dx}{\sqrt[3]{1 - x^2}} , \tag{A-7-2}$$

$$P_n^{(0)} = \theta_0^{\frac{4}{3}} \int_{-1}^{1} \frac{\sin \frac{2n-1}{2} n\pi \, dx}{\sqrt[3]{1 - x^2}} , \tag{A-7-3}$$

$$Q_n^{(0)} = \theta_0^{\frac{4}{3}} \int_{-1}^{1} \frac{\sin n\theta_0 x}{\sqrt[3]{1 - x^2}} dx . \tag{A-7-4}$$

It is well known that

$$\int_0^1 (1 - x^2)^{\nu - \frac{1}{2}} \cos zx \, dx = \frac{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})}{2} \left(\frac{2}{z}\right)^{\nu} J_{\nu}(z) \tag{A-7-5}$$

where $\Gamma(\nu + \frac{1}{2})$ is the gamma function with $\nu + \frac{1}{2}$ as its argument. Therefore for $n \geq 1$, $P_n^{(e)}$ and $Q_n^{(e)}$, respectively, can be written as

$$P_n^{(e)} = \theta_0^{\frac{1}{3}} \sqrt{\pi}\, \Gamma(\tfrac{2}{3}) (\frac{2}{n\pi})^{\frac{1}{6}} J_{\frac{1}{6}}(n\pi) \quad , \tag{A-7-6}$$

$$Q_n^{(e)} = \theta_0^{\frac{1}{3}} \sqrt{\pi}\, \Gamma(\tfrac{2}{3}) (\frac{2}{n\pi})^{\frac{1}{6}} J_{\frac{1}{6}}(n\pi) \quad . \tag{A-7-7}$$

It is difficult to express $P_n^{(0)}$ and $Q_n^{(0)}$ in terms of any classical functions. However, it is rather obvious that $P_n^{(0)}/P_0^{(e)}$ is at least of $O(\theta_0)$ and $Q_n^{(0)}/Q_0^{(e)}$ is at least of the order $\theta_0{}^2$ when $\theta_0 << 1$.

APPENDIX
A-8

SUMMATION OF THE SERIES

$$\sum_{n=1}^{\infty} (Q_n^{(e)}/Q_0^{(e)})^2/n \text{ AND } \sum_{n=1}^{\infty} J_0^2(n\phi_0)/n$$

Letting $\eta = \theta_0 \cos \alpha$ we transform (4-3-8) into

$$Q_n^{(e)} = \theta_0^{\frac{1}{3}} \int_0^{\pi} \sin^{\frac{1}{3}} \alpha \cos\left(n\theta_0 \cos \alpha\right) d\alpha \qquad \text{(A-8-1)}$$

and hence

$$\sum_{n=1}^{\infty} (Q_n^{(e)})^2/n = Q_0^{\frac{2}{3}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi} \sin^{\frac{1}{3}} \alpha \cos\left(n\theta_0 \cos \alpha\right) d\alpha$$

$$\cdot \int_0^{\pi} \sin^{\frac{1}{3}} \beta \cos\left(n\theta_0 \cos \beta\right) d\beta \quad . \qquad \text{(A-8-2)}$$

Upon interchanging the summation and integration, we arrive at

$$\sum_{n=1}^{\infty} (Q_n^{(e)})^2/n = \theta_0^{\frac{2}{3}} \int_0^{\pi} d\alpha \sin^{\frac{1}{3}} \alpha \int_0^{\pi} \sin^{\frac{1}{3}} \beta \quad .$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta_0 \cos \alpha) \cos(n\theta_0 \cos \beta)}{n} \, d\beta \quad . \qquad \text{(A-8-3)}$$

But

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta\cos\alpha)\cos(n\theta\cos\beta)}{n} = -\frac{1}{2}\ln 2\left|\cos(\theta_0\cos\alpha) - \cos(\theta_0\cos\beta)\right| \tag{A-8-4}$$

For $\theta_0 << 1$, we have

$$\cos\theta_0\cos\alpha \simeq 1 - \frac{\theta_0^2}{4}(1+\cos 2\alpha)$$

$$\cos\theta_0\cos\beta \simeq 1 - \frac{\theta_0^2}{4}(1+\cos 2\beta) \tag{A-8-5}$$

and (A-8-4) becomes

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta_0\cos\alpha)\cos(n\theta_0\cos\beta)}{n} \simeq \ln\frac{2}{\theta_0} - \frac{1}{2}\ln 2|\cos 2\beta - \cos 2\alpha|$$

$$= \ln\frac{2}{\theta_0} + \sum_{n=1}^{\infty}\frac{\cos 2\alpha\cos 2\beta}{n} \quad . \tag{A-8-6}$$

Upon substituting (A-8-6) in (A-8-3) one has

$$\sum_{n=1}^{\infty}\frac{1}{n}(Q_n^{(e)})^2 \simeq \theta_0^{\frac{2}{3}}\left\{\ln\left(\frac{2}{\theta_0}\right)\left(\int_0^{\pi}\sin^{\frac{1}{3}}\alpha\, d\alpha\right)^2 \right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{\pi}\sin^{\frac{1}{3}}\alpha\cos 2n\alpha\, d\alpha\right)^2\right\} \quad . \tag{A-8-7}$$

One can see from (A-8-1) that

$$Q_0^{(e)} = \theta_0^{\frac{1}{3}} \int_0^{\pi} \sin^{\frac{1}{3}} \alpha \, d\alpha \tag{A-8-8}$$

and because the series on the right-hand side of (A-8-7) converges very fast to $0.05053 \ \theta_0^{2/3}$, the series

$$\sum_{n=1}^{\infty} (Q_n^{(e)}/Q_0^{(e)})^2/n = \ell n(\frac{2}{\theta_0}) + 0.0195 \quad . \tag{A-8-9}$$

Since

$$J_0(n\phi_0) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi_0 \cos\theta)\,d\theta \quad , \tag{A-8-10}$$

we have

$$\sum_{n=1}^{\infty} J_0^2(n\phi_0)/n = \sum_{n=1}^{\infty} \frac{1}{n\pi^2} \int_0^{\pi} \cos(n\phi_0 \cos\alpha)\,d\alpha \int_0^{\pi} \cos(n\phi_0 \cos\beta)\,d\beta \quad . \tag{A-8-11}$$

Following the same steps from (A-8-3) to (A-8-9), we obtain

$$\sum_{n=1}^{\infty} J_0^2(n\phi_0)/n \simeq \ell n\,(\frac{2}{\phi_0}) \tag{A-8-12}$$

APPENDIX
A-9

TERMINAL ADMITTANCE OF A WEDGE WAVEGUIDE IN A PERFECTLY CONDUCTING CYLINDER

The geometry is as shown in Fig. A-9-1.

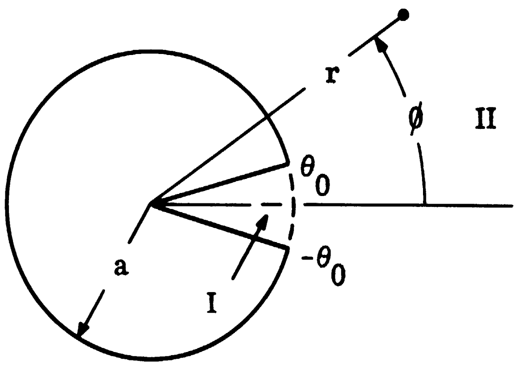

FIG. A-9-1: PERFECTLY CONDUCTING CYLINDER SLOTTED BY A WEDGE.

The perfectly conducting circular cylinder body is of radius a. The width of the wedge is $2\theta_0$. If we put a magnetic line source at the apex of the wedge, then the source excites EM fields in the free space (region II) as well as inside the wedge (region I). In Fig. 1-3 if we let $\theta = 0$, $\phi_0 \geq \theta_0$ and $c \to b \to a$, then we obtain the same geometry as shown Fig. A-9-1. If $y'(a)$ is the normalized wedge terminal admittance defined by (2-1-15) where $r = a$, it can be shown by a similar procedure as in chapter II that

$$y'(a) = j\,2 \sum_{n=1}^{\infty} \frac{J_{\frac{n\pi}{\theta_0}}(k_0 a)}{J'_{\frac{n\pi}{\theta_0}}(k_0 a)} \left(\frac{\int_{-\theta_0}^{\theta_0} \hat{E}(\phi') \cos \frac{n\pi}{\theta_0}\phi' \, d\phi}{\int_{-\theta_0}^{\theta_0} \hat{E}(\phi')\, d\phi'} \right)^2$$

$$-j\,\frac{\theta_0}{\pi}\sum_{n=0}^{\infty}\epsilon_n\,\frac{H_n^{(2)}(k_0a)}{H_n^{(2)'}(k_0a)}\left(\frac{\int_{-\theta_0}^{\theta_0}\hat{E}(\phi')\cos n\phi'\,d\phi'}{\int_{-\theta_0}^{\theta_0}\hat{E}(\phi')\,d\phi'}\right)^2 \qquad \text{(A-9-1)}$$

where $\hat{E}(\phi')$ is the tangential electric field in the wedge aperture. This is a stationary expression with respect to $\hat{E}(\phi)$. Hence, when $\theta_0 << 1$, one may let

$$\hat{E}(\phi') = \frac{A_w}{\sqrt[3]{\theta_0^{\,2} - \phi^2}}$$

and since

$$\frac{J_{\frac{n\pi}{\theta_0}}(k_0a)}{J'_{\frac{n\pi}{\theta_0}}(k_0a)} \simeq \frac{\theta_0}{\pi}\,\frac{k_0a}{n} \qquad \text{for} \quad \theta_0 << 1, \quad n > 0$$

Eq. (A-9-1) becomes

$$y'(a) = j\,\frac{\theta_0}{\pi}\left[\frac{2k_0a}{(Q_0^{(e)})^2}\sum_{n=1}^{\infty}(P_n^{(e)})^2/n - \sum_{n=0}^{\infty}\epsilon_n\,\frac{H_n^{(2)}(k_0a)}{H_n^{(2)'}(k_0a)}\,(Q_n^{(e)})^2\right] . \qquad \text{(A-9-2)}$$

If $Y'(a)$ is the terminal admittance of a section of the wedge waveguide of length a meters, then

$$Y'(a) = \frac{1}{2\theta_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\; y'(a) \quad . \qquad \text{(A-9-3)}$$

It is straight forward to write down the conductance G' and susceptance B' from (A-9-3) as

$$G' = \frac{1}{\pi^2 k_0 a}\sqrt{\frac{\epsilon_0}{\mu_0}}\sum_{n=0}^{\infty}\frac{\epsilon_n}{1+x_n^2}\left(\frac{Q_n^{(e)}}{Q_0^{(e)} N_n'(k_0 a)}\right)^2 \quad , \quad \text{mhos} \qquad \text{(A-9-4)}$$

$$B' = \frac{1}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\left\{2k_0 a\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{P_n^{(e)}}{Q_0^{(e)}}\right)^2 - \sum_{n=0}^{\infty}\epsilon_n\frac{1+x_n y_n}{1+x_n^2}\left(\frac{Q_n^{(e)}}{Q_0^{(e)}}\right)^2\frac{N_n(k_0 a)}{N_n'(k_0 a)}\right\} \quad , \quad \text{mhos} \qquad \text{(A-9-5)}$$

where

$$x_n = J_n'(k_0 a)/N_n'(k_0 a)$$

$$y_n = J_n(k_0 a)/N_n(k_0 a) \quad .$$

APPENDIX
A-10

POYNTING'S ENERGY THEOREM IN THE PLASMA SHEATH AND THE FREE SPACE

We consider a volume V enclosed by a surface S in which the electromagnetic fields are of periodic time variation. The Poynting's theorem for this volume then is

$$-\int_S (\vec{E} \times \vec{H}^*) \cdot \vec{n}\, ds = 4j\omega(W_H - W_E) + 2P \qquad \text{(A-10-1)}$$

where $\vec{n}$ is the outward normal of S and

W_H = time-averaged stored magnetic energy in V

$$= \frac{1}{4}\mu \int_V \vec{H}^* \cdot \vec{H}\, dv \ , \qquad \text{(A-10-2)}$$

W_E = time-averaged stored electric energy in V

$$= \frac{1}{4}\epsilon \int_V \vec{E}^* \cdot \vec{E}\, dv \ , \qquad \text{(A-10-3)}$$

P = time-averaged dissipated power in V

$$= \frac{1}{2}\sigma \int_V \vec{E}^* \cdot \vec{E}\, dv \ . \qquad \text{(A-10-4)}$$

As in the main text, we choose to consider a section of the coaxial antenna of length a meters in z-direction and apply (A-10-1) to the plasma sheath and free space of this section respectively; we have

$$2P^{III} + j4\omega(W_H^{III} - W_E^{III}) = a\int_{-\pi}^{\pi} E_\phi^{III} H_z^{III*}\, b\, d\phi\Big|_{r=b} - a\int_{-\pi}^{\pi} E_\phi^{III} H_z^{III*}\, c\, d\phi\Big|_{r=c} , \quad \text{(A-10-5)}$$

$$2P^{IV} + j4\omega(W_H^{IV} - W_E^{IV}) = a\int_{-\pi}^{\pi} E_\phi^{IV} H_z^{IV*}\, c\, d\phi\Big|_{r=c} - a\int_{-\pi}^{\pi} E_\phi^{IV} H_z^{IV*}\, r\, d\phi\Big|_{r\to\infty} . \quad \text{(A-10-6)}$$

It is obvious that there is no dissipated power in the free space, thus $P^{IV} = 0$. Furthermore, if P_r denotes the power radiated by this section of the antenna, then

$$P_r = \frac{a}{2}\int_{-\pi}^{\pi} E_\phi^{IV} H_z^{IV*}\, r\, d\phi\Big|_{r\to\infty} . \quad \text{(A-10-7)}$$

Since at $r = c$, $E_\phi^{III} = E_\phi^{IV}$ and $H_z^{III} = H_z^{IV}$, one may combine (A-10-5) through (A-10-7) together and obtain

$$2P^{III} + j4\omega(W_H^{III} - W_E^{III}) + j4\omega(W_H^{IV} - W_E^{IV}) + 2P_r = ab\int_{-\pi}^{\pi} E_\phi^{III} H_z^{III*}\, d\phi\Big|_{r=b} . \quad \text{(A-10-8)}$$

Upon substituting (2-2-11) and (2-2-12) in the right hand-side of (A-10-8) and carrying out the integration, we have

$$2P^{III} + 2P_r + j4\omega(W_H^{IV} - W_E^{III}) + j4\omega(W_H^{IV} - W_E^{IV})$$

$$= -j \frac{ab}{2\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{n=0}^{\infty} \epsilon_n \pi_n^* \frac{H_n^{(1)}(k_0 c)}{H_n^{(1)'}(k_0 c)} \;.$$

$$\cdot \int_{-\phi_0}^{\phi_0} d\phi E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi) \cos n(\phi - \phi') d\phi' \;. \qquad \text{(A-10-9)}$$

Similarly, if we substitute (2-2-18) and (2-2-19) in (A-10-7), we attain a formula as shown,

$$P_r = \frac{ac}{4\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \left(\frac{2}{\pi k_0 c}\right)^3 \sum_{n=0}^{\infty} \frac{\epsilon_n}{\Delta_n \Delta_n^*} \int_{-\phi_0}^{\phi_0} d\phi E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi') \cos n(\phi - \phi') d\phi'$$

(A-10-10)

where

$$\Delta_n = \bar{k} H_n^{(0)'}(k_0 c) \left[J_n(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n(k_1 c)\right]$$

$$- H_n^{(0)}(k_0 c) \left[J_n'(k_1 c) N_n'(k_1 b) - J_n'(k_1 b) N_n'(k_1 c)\right] \;. \qquad \text{(A-10-11)}$$

For no plasma sheath case, we let $c \rightarrow b$, $\bar{k} \rightarrow 1$ and obtain straight forwardly from (A-10-10) that

$$P_r = \frac{ab}{4\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \left(\frac{2}{\pi k_0 b}\right) \sum_{n=0}^{\infty} \frac{\epsilon_n}{\left|H_n^{(2)'}(k_0 b)\right|^2} \int_{-\phi_0}^{\phi_0} d\phi E^*(\phi) \int_{-\phi_0}^{\phi_0} E(\phi') \cos n(\phi - \phi') d\phi'$$

(A-10-12)

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