THE UNIVERSITY OF MICHIGAN

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APPLICATION OF ULTRASPHERICAL POLYNOMIALS TO NON-LINEAR OSCILLATIONS - FREE OSCILLATIONS

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1. Introduction. In a recent paper [1], Denman and Howard introduced a procedure for the linearization of the non-linear ordinary differential equation governing the free oscillation of the simple pendulum, by approximating the non-linear torque with an ultraspherical polynomial. This linearization yields an approximation for the period of the pendulum as a function of the amplitude of the motion. The present paper applies this procedure to the following non-linear functions: $ax + bx^3$, $\sinh x$ and $\tanh x$. Certain asymptotic results are obtained. Further, it extends the procedure to the cubic ultraspherical polynomial approximation and applied it to the non-linear functions $\sin x$ and $\sinh x$. All results are either compared with the exact expressions for the period, if they are available, or to numerical results, if they are not. This extension yields a marked improvement over the linear approximation, with little increase in complexity.

2. The general free oscillation problem. A general free oscillation problem is characterized by the differential equation

$$d^2x/dt^2 + f(x) = 0. \tag{1}$$

In this paper, $f(x)$ is assumed to be an odd (non-linear) function representing a force or torque. The general initial conditions are: $x(0) = x_0$ and $(dx/dt)_0 = v_0$. In the oscillatory case, these conditions may be replaced by $x_0 = A$, $v_0 = 0$, where A is the amplitude of motion. The first integral of (1) is

$$(dx/dt)^2/2 + V(x) = E,$$

where $V(x)$ is a potential function ($dV/dx = f(x)$), and E is proportional to the (constant) energy of the system. The turning points of the motion are $(-A, A)$.

The second integral of (1) can be written

$$\tau/4 = \int_0^A [2(E - V)]^{-1/2} dx,$$

where τ is the period of oscillation, and is in general a function of A. Since $E = V(A)$, one may write

$$\tau/2^{3/2} = \int_0^A [V(A) - V(x)]^{-1/2} dx. \tag{2}$$

3. The ultraspherical polynomials. The ultraspherical polynomials on the interval $[-1, 1]$ are the sets of polynomials orthogonal on this interval with respect to the weight factors $(1 - x^2)^{\lambda - 1/2}$, each set corresponding to a value of $\lambda > -1/2$. They may be obtained from Rodrigues' formula [2]

$$P_n^{(\lambda)}(x) = A_n^{(\lambda)}(1 - x^2)^{-\lambda+1/2}(d/dx)^n (1-x^2)^{n+\lambda-1/2},$$

where $A_n^{(\lambda)}$ is a normalization factor.

Some subsets of the ultraspherical polynominals are: $\lambda = 0$, Tchebycheff polynomials of the first kind; $\lambda = 1/2$, Legendre polynomials; $\lambda = 1$, Tchebycheff polynomials of the second kind, and $\lambda \to \infty$, the powers of x.

The ultraspherical polynomials on the interval $[-A, A]$ are defined as the sets of polynomials orthogonal on this interval with respect to the weight factors $[1 - (x/A)^2]^{\lambda - 1/2}$, $\lambda > -1/2$. For a function $f(x)$ expandable in these polynomials one obtains

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\lambda)} P_n^{(\lambda)} (x/A), \tag{3}$$

where the coefficients $a_n^{(\lambda)}$ may be written

$$a_n^{(\lambda)} = \frac{\int_{-1}^{+1} f(Ax) P_n^{(\lambda)} (x) (1 - x^2)^{\lambda - 1/2} dx}{\int_{-1}^{+1} [P_n^{(\lambda)} (x)]^2 (1 - x^2)^{\lambda - 1/2} dx} . \tag{4}$$

Since $a_n^{(\lambda)} P_n^{(\lambda)} (x/A)$ is unchanged if $P_n^{(\lambda)} (x/A)$ is multiplied by a constant, one may choose any convenient normalization factor $A_n^{(\lambda)}$.

For all odd $f(x)$, the approximate period τ^* resulting from the <u>linear</u> ultraspherical polynomial approximation in $[-A, A]$, i.e., $f^*(x) = a_1^{(\lambda)} P_1^{(\lambda)} (x/A)$, can be expressed as

$$\tau^* = 2\pi \, [\pi^{1/2} \, \Gamma(\lambda + 1/2) \, A/4\Gamma(\lambda + 2) \, S]^{1/2}, \tag{5}$$

where

$$S = \int_0^1 x \, f(Ax) \, (1 - x^2)^{\lambda - 1/2} \, dx , \tag{6}$$

for $\lambda > -1/2$. (This expression for τ^* actually converges for values of $\lambda > -2$ for many $f(x)$, but the accuracy of the approximate results for $-2 < \lambda < -1/2$ has been found to be rather poor; therefore this range is not considered in this paper).

4. Application of the linear ultraspherical approximation to some typical problems. The restoring forces or torques encountered in non-linear oscillation problems may be roughly classified as: hardening, softening, flattening, and bottoming. Examples from the first three classes will be examined here, and include: (I) (odd) cubic, $f(x) = ax + bx^3$; (II) sine, $f(x) = \sin x$, and hyperbolic sine, $f(x) = \sinh x$; and (III) hyperbolic tangent, $f(x) = \tanh x$.

I. Cubic Non-linearity.

The equation of free oscillation is taken as

$$d^2x/dt^2 + ax + bx^3 = 0. \tag{7}$$

Equation (7) can be classified according to whether a and b are positive or negative. The four combinations are shown in Table I.

Table I

Classification	Name	Sign a	Sign b	Motion
Case 1	Hardening	+	+	Oscillatory
Case 2	Softening	+	-	Conditionally Oscillatory
Case 3	Softening-Hardening	-	+	Oscillatory
Case 4	Softening-Softening	-	-	Non-oscillatory

Only the first three cases can yield bounded oscillatory motion; Case 4 is without any "restoring" force or torque regardless of the amplitude of motion. Case 3 is included both for completeness and because it will be used later (Section 6).

A. Exact Solutions.

Case 1. Hardening Cubic. The expression for the period of (7) in terms of the complete elliptic integral of the first kind is well known [3]. The exact period is

$$\tau = 4K(k_1)/(a + bA^2)^{1/2}, \tag{8}$$

or, in dimensionless form,

$$\tau/\tau' = 2K(k_1)/\pi(1 + \nu)^{1/2}, \tag{9}$$

where $\tau' = 2\pi/a^{1/2}$ is the period of the system if $b = 0$, $K(k_1)$ is the complete elliptic integral of the first kind, $\nu = bA^2/a$ is a dimensionless quantity, and $k_1^2 = \nu/2(1+\nu)$ is the modulus of the elliptic integral. The quantity ν is a measure of the non-linearity of the problem, since the maximum ratio of the non-linear to linear term in (7) is $bA^2/a = \nu$.

Case 2. Softening Cubic. If b is negative, i.e., the non-linear spring is soft, a solution similar to (9) is obtained by making the transformation

$$k_2^2 = -\nu/(2 + \nu), \quad -1 < \nu \leq 0.$$

After some simplification, the period ratio becomes

$$\tau/\tau' = 2(1 + k_2^2)^{1/2}K(k_2)/\pi. \tag{10}$$

Note, however, that the above solution is no longer bounded, since $\nu \to -1$ implies $k_2^2 \to 1$, and $K \to \infty$. This ν (or amplitude) is called the critical ν (or amplitude). Plots of τ/τ' against ν for Cases 1 and 2 are shown in Figures 1 and 2.

Case 3. Softening-Hardening Cubic. This case possesses rather interesting properties: for x small, the linear term dominates the cubic, and vice-versa for large x; thus the force is away from the center for small x, but restoring for large x. The potential curve is shown in Fig. 3. For initial conditions represented by the energy E_1, the motion is periodic, but not symmetric about the origin. The conditions represented by E_2; however, do give symmetric oscillations about the origin. The demarcation line is E_o. Only the case represented by E_2 is considered here. Then one obtains

$$\tau(-a)^{1/2} = 4(-\nu - 1)^{-1/2} K(k_3), \tag{11}$$

where $\nu = bA^2/a < 0$, and $k_3^2 = \nu/2(1 + \nu)$.

For $E > E_o$, $\nu < -2$, so that $1/2 \leq k_3^2 < 1$. For $\tau' = 2\pi/|a|^{1/2}$,

$$\tau/\tau' = 2K(k_3)/\pi(-\nu - 1)^{1/2}. \tag{12}$$

Equation (12) is plotted as curve E in Figure 4.

B. Linear Ultraspherical Polynomial Approximation.

In (7), if one expands $f(x) = ax + bx^3$ in ultraspherical polynomials in $[-A, A]$, one obtains the linear approximation

$$(ax + bx^3)^*_\lambda = [a + 3bA^2/2(\lambda + 2)]\, x,$$

where * indicates a linear approximation. This linear approximation, when substituted into (7), results in the approximate linear differential equation

$$d^2x/dt^2 + [a + 3bA^2/2(\lambda + 2)]x = 0. \tag{13}$$

Therefore, for Cases 1 and 2,

$$\tau^*/\tau' = [1 + 3\nu/2(\lambda + 2)]^{-1/2}, \ \nu \geq -1, \tag{14a}$$

and for Case 3,

$$\tau^*/\tau' = [-1 - 3\nu/2(\lambda + 2)]^{-1/2}, \ \nu < -2, \tag{14b}$$

where $\tau' = 2\pi/|a|^{1/2}$, and $\nu = bA^2/a$.

On Figs. 1, 2, and 4 are shown curves given by the above approximations for certain values of λ, for these cubic springs. (For $a = 0$, $\tau_o \to \infty$ and $\nu \to \infty$, but by using (8) and (13) one obtains the usual results.)

II. Sine and Hyperbolic Sine.

A. Exact Solutions.

The governing differential equations are written

$$d^2x/dt^2 + \omega_o^2 \sin x = 0, \tag{15}$$

$$d^2x/dt^2 + \omega_o^2 \sinh x = 0. \tag{16}$$

Solutions of (15), both exact and for the linear ultraspherical polynomial approximation, are found in [1]. Equation (16) can also be solved in terms of the complete elliptic integral of the first kind. While the solution for (15) is oscillatory only if $A < \pi$, the solution to (16) has no such limitation; the exact period is given by

$$\tau/\tau_o = (2/\pi)\mathrm{sech}(A/2)\ K[\tanh(A/2)]. \tag{17}$$

B. Linear Ultraspherical Polynomial Approximation.

If one expands sinh x in ultraspherical polynomials in [-A, A] and truncates after the linear term, one obtains

$$(\sinh x)^*_\lambda = [\Gamma(\lambda + 2)\ I_{\lambda+1}(A) / (A/2)^{\lambda+1}]\ x\ , \tag{18}$$

where $I_n(x) = (-i)^n J_n(ix)$ is a modified Bessel function. The approximate period ratio τ^*/τ_o is then

$$\tau^*/\tau_o = [(A/2)^{\lambda+1} / \Gamma(\lambda+2)\ I_{\lambda+1}(A)]^{1/2}. \tag{19}$$

The comparison between (19) and the exact solution (17) for various λ values is shown in Fig. 5.

III. Hyperbolic Tangent Nonlinearity.

The governing differential equation of motion is

$$d^2x/dt^2 + \omega_o^2 \tanh x = 0. \tag{20}$$

No exact solution of (20) in simple functions is known to exist. To provide comparison for the approximate results, numerical quadrature is used to provide the "exact" results.

From (2), the exact period in this case can be written

$$\tau/\tau_o = (2^{1/2}/\pi) \int_0^A [\ln(\cosh A/\cosh x)]^{-1/2} dx. \tag{21}$$

For numerical quadrature, to remove the singularity at the upper limit, consider the expansion ($x^2 < \pi^2/4$)

$$\ln(\cosh x) = x^2/2 - x^4/12 + \ldots\ (-1)^{n-1}\ 2^{2n-1}(2^{2n}-1)B_{2n-1}x^{2n}/n(2n)!,$$

where B_{2n-1} are the odd Bernoulli numbers [4]. Equation (21) becomes

$$\tau/\tau_o = (2^{1/2}/\pi) \int_0^A [(A^2-x^2)/2-(A^4-x^4)/12 + \ldots.]^{-1/2} dx. \quad (22)$$

Setting $x = A \cos \theta$, one obtains (for $A < \pi/2$)

$$\tau/\tau_o = (2/\pi) \int_0^{\pi/2} [1-A^2(1+\cos^2\theta)/6 + \ldots]^{-1/2} d\theta. \quad (23)$$

Equation (23) is now free of singularities and is susceptible to numerical integration. For small values of A, one can truncate after the second term and express τ/τ_o in terms of an elliptic integral

$$\tau/\tau_o = (2/\pi) [6/(6-A^2)]^{1/2} K(k_4), \quad (24)$$

where $k_4^2 = A^2/(6-A^2)$. For intermediate and large A, expand ln(cosh x) in a Taylor series about A. Then

$$\ln \cosh x = \ln \cosh A + \tanh A\ (x-A) + (1/2) \operatorname{sech}^2 A(x-A)^2 - \operatorname{sech}^2 A \tanh A\ (x-A)^3/3 + \ldots .$$

Setting $y^2 = A-x$, one obtains

$$\tau/\tau_o = (2/\pi)(6 \cosh^3 A/\sinh A)^{1/2} \int_0^{\sqrt{A}} [3 \cosh^2 A-(3/2)y^2 \operatorname{ctnh} A - y^4 \ldots]^{-1/2} dy.$$

If one truncates the series after the third term, the above integration can be carried out be means of an elliptic integral of the first kind [5]

$$\tau/\tau_o \doteq (4/\pi) g \cosh A\ F(\varphi, k_5), \quad (25)$$

where $k_5^2 = (1/2)(1-g^2)$, $g = [1+(16/3) \sinh^2 A]^{-1/4}$
and $\sin^2\varphi = 2/[\ 1+(2g^2 \sinh 2A)/A-g^2]$.

Retaining only two terms of the integrand yields the approximation

$$\tau/\tau_o \doteq (4/\pi)(\cosh A)\sin^{-1}(A/\sinh 2A)^{1/2}. \tag{26}$$

For asymptotically large A, (26) approaches

$$\tau/\tau_o = (2/\pi)(2A)^{1/2} = 0.9003\ A^{1/2}. \tag{27}$$

Equations (24), (25), (26) and (27) span the entire range of A and overlap one another as far as graphical accuracy is concerned. The "exact" curve E in Fig. 6 was constructed using them.

B. <u>Linear Ultraspherical Polynomial Approximation.</u>

For $f(x) = \omega_o^2 \tanh x$, S in (6) becomes

$$S = \omega_o^2 \int_0^1 x \tanh Ax(1-x^2)^{\lambda-1/2} dx. \tag{28a}$$

This integral does not seem to exist in closed form in terms of simple functions. For $A < \pi/2$, tanh Ax can be expanded in a Maclaurin series, and

$$\tau^*/\tau_o = \{\pi^{1/2} A/[4\Gamma(\lambda+2)\sum_{n=1}^{\infty} (-1)^{n-1} (2A)^{2n-1} (2^{2n}-1) B_{2n-1}\Gamma(n+1/2)/(2n)!]\}^{1/2} \tag{28b}$$

Alternately, an integration of (28a) by parts yields

$$S = (2\lambda+1)^{-1} \omega_o^2 A \int_0^1 (1-x^2)^{\lambda+1/2} \operatorname{sech}^2 Ax\, dx, \tag{28c}$$

which removes the singularity in the integrand of (28a) for $x = 1$, $\lambda < 1/2$. Thus τ^*/τ_o can be expressed as

$$\tau^*/\tau_o = \left\{ \frac{8\Gamma(\lambda+2)}{\sqrt{\pi}\,\Gamma(\lambda+3/2)} \int_0^1 \frac{(1-x^2)^{\lambda+1/2}}{e^{2Ax}(1+e^{-2Ax})^2} dx \right\}^{-1/2}. \tag{29}$$

For arbitrary λ, numerical quadrature was used to evaluate (29).

However, certain values of λ yield somewhat simpler results. For example, if $\lambda = 1/2$,

$$S = \omega_o^2 \int_0^1 x \tanh Ax \, dx. \tag{30}$$

Since, for $x \geq 0$, $A > 0$,

$$x \tanh Ax = x(1-2e^{-2Ax} + 2e^{-4Ax} - 2e^{-6Ax} + \ldots), \tag{31}$$

the period ratio for $\lambda = 1/2$ can be expressed as

$$(\tau^*/\tau_o)_{1/2} = \left\{\frac{3}{2A}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2nA+1)}{(nA)^2 e^{2nA}} - \frac{\pi^2}{12A^2}\right)\right\}^{-1/2}. \tag{32}$$

This expression is particularly convenient for large values of A, and for very large A, $(\tau^*/\tau_o)_{1/2}$ approaches $0.8165\, A^{1/2}$. For $\lambda \to -1/2$, (5) and (28c) yield

$$\tau^*/\tau_o)_{-1/2} \to [A/\tanh A]^{1/2}. \tag{33}$$

Figure 6 compares the "exact" solution with the approximations corresponding to λ values $-1/2$, 0, and $1/2$.

5. Asymptotic behavior of the linear ultraspherical polynomial approximation. Two special regions of interest are those for which A approaches 0 and for A large.

I. Lower Limit.

For small oscillations in the neighborhood of the origin, one can expand V(x) in (2) in a Maclaurin series

$$V(x) = V_o + a_2x^2 + a_4x^4 + \ldots,$$

or

$$V(A) - V(x) = a_2(A^2-x^2)\ [1+a_4(A^2+x^2)/a_2 + \ldots],$$

where a_2 must be positive for <u>small</u> oscillations about the origin to exist. Setting $x/A = \cos\theta$, one finds

$$\tau/2^{3/2} = a_2^{-1/2} \int_0^{\pi/2} [1+a_4A^2(1+\cos^2\theta)/a_2 + \ldots]^{-1/2} d\theta\ .$$

For sufficiently small values of A, one can expand the integrand by the Binomial Theorem, so that

$$\tau/2^{3/2} = a_2^{-1/2} \int_0^{\pi/2} [1-a_4A^2(1+\cos^2\theta)/2a_2 \ldots]d\theta\ .$$

The solution for small oscillation is, therefore,

$$\tau = 2\pi(2a_2)^{-1/2}\ [1-3a_4A^2/4a_2 + \ldots]\ . \tag{34}$$

In (6), setting $x = \cos\theta$, one obtains

$$S = \int_0^{\pi/2} f(A\cos\theta)\ \sin^{2\lambda}\theta\ \cos\theta\ d\theta. \tag{35}$$

Expanding $f(A\cos\theta)$ in its Maclaurin series,

$$f(A\cos 0) = \sum_{m=o}^{\infty} c_mA^m\cos^m\theta\ .$$

Then (5) becomes

$$\tau^* = 2\pi\ [\pi^{1/2}\ \Gamma(\lambda+1/2)A/4\Gamma(\lambda+2) \sum_{m=o}^{\infty} c_mA^m \int_0^{\pi/2} \cos^{m+1}\theta\ \sin^{2\lambda}\theta\ d\theta]^{1/2}. \tag{36}$$

For f(x) odd, (36) yields

$$\tau^* = 2\pi[c_1+3c_3A^2/2(2+\lambda) + \ldots]^{-1/2}. \tag{37}$$

Since the c_i's and a_i's are the Maclaurin series coefficients for f(x) and V(x) respectively, their relationships are

$$a_2 = c_1/2, \quad a_4 = c_3/4, \quad \ldots\ldots \quad .$$

Equation (37) becomes

$$\tau^* = 2\pi(2a\ )^{1/2}[1-3a\ A\ /2a\ (\lambda+2)\ \ldots.] \ . \qquad (37')$$

When one compares (37') with the expression for the exact period (34), one notes that the first two terms agree if and only if $\lambda = 0$. Hence, for <u>small</u> free oscillations, $\lambda = 0$, corresponding to Tchebycheff polynomials of the first kind, provides the most accurate period approximation of all the linear ultraspherical polynomials, for all odd analytic f(x). (This result can not be applied to the softening-hardening cubic; only the case $E \geq E_o$ is considered here, for which arbitrarily small oscillations about 0 do not exist.)

II. Large Asymptotic Limit.

For large non-linearities, each case was examined individually.

A. Cubic Non-linearity.

For the hardening and softening-hardening cases (Cases 1 and 3), the exact solutions are given by (9) and (12). For $|\nu| >> 1$, (9) and (12) yield

$$\lim_{|\nu|\to\infty} (\tau/\tau') = 2(1.8541)/|\nu|^{1/2}\pi,$$

while from (14a) and (14b), one finds

$$\lim_{|\nu|\to\infty} (\tau^*/\tau') = [2(2+\lambda)/3|\nu|]^{1/2} \ .$$

Hence, the optimum value of λ, for large non-linearity, is found from

$$[2(\lambda+2)/3]^{1/2} = 2(1.8541)/\pi, \tag{38}$$

which yields $\lambda = 0.0899$.

For the softening case (Case 2), the exact solution (10) and the approximation (14a) behave as follows:

$$\tau/\tau' \to \infty \qquad \text{as } \nu \to -1, \tag{39a}$$

$$\tau^*/\tau' \to \infty \qquad \text{as } 3\nu/2(2+\lambda) \to 1. \tag{39b}$$

Hence, the value of λ which yields the correct critical ν_c is $\lambda = -1/2$.

For the softening-hardening spring, the exact solution (12) becomes ∞ for $\nu = -2$; the approximation (14b) gives the same result for $\lambda=1$.

B. Sine and Hyperbolic Sine Non-linearities.

The exact solution for the free oscillation of a simple pendulum is

$$\tau/\tau_o = (2/\pi)K(k),$$

where $k = \sin(A/2)$. Thus, τ/τ_o approaches infinity as $A \to \pi$. The linear ultraspherical polynomial approximation yields [1]

$$\tau^*/\tau_o = [(A/2)^{\lambda+1}/\Gamma(\lambda+2)J_{\lambda+1}(A)]^{1/2}.$$

From a table of the first zero of $J_p(x)$ as a function of p, such as in [6], one notes that $J_{1/2}(A)$ has its first zero at $A=\pi$. Hence, the same critical amplitude occurs for the linear approximation when the index λ is $-1/2$.

The exact solution by elliptic integrals for the sinh spring is given by (17). For $A >> 1$, $k^2 \to 1$, and

$$\tau/\tau_o \to C_1 A \exp(-A/2), \tag{40}$$

where C_1 is a constant. The linear ultraspherical polynomial approximation result is given by (19). For very large A [7],

$$I_p(A) \to e^A/(2\pi A)^{1/2},$$

and

$$\tau^*/\tau_o \to C_2 A \exp(-A/2) \tag{41}$$

only if $\lambda = 1/2$.

C. The Hyberbolic Tangent Non-linearity.

The asymptotic τ/τ_o is given by (27). For very large A, $\tanh Ax \to 1$, $x > 0$, and (28a) and (5) yield

$$\tau^*/\tau_o \to \left\{A\ \pi^{1/2}\Gamma(\lambda+3/2)/2\Gamma(\lambda+2)\right\}^{1/2}. \tag{42a}$$

Equating the coefficients of $A^{1/2}$ in (27) and (42a), one obtains

$$\pi^{1/2}\Gamma(\lambda+3/2)/2\Gamma(\lambda+2) = 8/\pi^2,$$

from which

$$\lambda = -0.075. \tag{42b}$$

These asymptotic results are summarized in Table II, and are verified by the computations leading to Figs. 1, 2, 4, 5, and 6.

Table II

Type of Spring	Optimum Value of λ for Large Non-linearity or Correct Critical Value
cubic	
(1) hard	.089
(2) soft	-.5
(3) softening-hardening	.089 (large ν) 1.0 (correct ν_c)
sine	-.5
sinh	.5
tanh	-.075

6. Cubic ultraspherical polynomial approximation. If, instead of truncating the ultraspherical polynomial expansion of f(x) in [-A, A] after the linear term, one does so after the cubic, much more accurate results are to be expected. Then

$$f^{**}(x) = a_1^{(\lambda)} P_1^{(\lambda)}(x/A) + a_3^{(\lambda)} P_3^{(\lambda)}(x/A).$$

When this result is substituted in (1), the resulting approximate cubic differential equation can be either solved exactly by elliptic functions or, preferably, utilize the previously plotted "exact" results for the cubic springs to extract the approximate solution. The sin and sinh cases will be considered below (the hyperbolic tangent has no accurate cubic approximation for large amplitude, as is true for any flattening spring).

The evaluation of the ultraspherical polynomial coefficient $a_3^{(\lambda)}$ will be shown for $f(x) = \sin x$, that for $\sinh x$ being very similar. From (4),

$$a_3^{(\lambda)} = \frac{\int_0^\pi \sin(A\cos\theta)\sin^{2\lambda}\theta\, P_3^{(\lambda)}(\cos\theta)\, d\theta}{\int_0^\pi [P_3^{(\lambda)}(\cos\theta)]^2 \sin^{2\lambda}\theta\, d\theta} .$$

The denominator is a standard integral, while the numerator is an integral found in [8]. Thus it can be shown

$$(\sin x)_\lambda^{**} = \frac{\Gamma(\lambda+2)}{(A/2)^{\lambda+1}} [J_{\lambda+1}(A) + (\lambda+3)\, J_{\lambda+3}(A)]\, x$$

$$- \frac{\Gamma(\lambda+4)\, J_{\lambda+3}(A)}{6(A/2)^{\lambda+3}} x^3 , \qquad (43a)$$

$$= [\Lambda_{\lambda+1}(A) - \frac{A^2}{4(\lambda+2)} \Lambda_{\lambda+3}(A)]x - \frac{1}{6} \Lambda_{\lambda+3}(A)x^3, \qquad (43b)$$

where $\Lambda_\lambda(A) = \Gamma(\lambda+1)\, J_\lambda(A)/(A/2)^\lambda$. The function Λ_λ is plotted in [6], while (43a) agrees with the result given in [9] for $\lambda = 0$.

The cubic ultraspherical polynomial approximation for $\sinh x$ is

$$(\sinh x)_\lambda^{**} = \frac{\Gamma(\lambda+2)}{(A/2)^{\lambda+1}} \left[I_{\lambda+1} - (\lambda+3)\, I_{\lambda+3}\right] x + \frac{\Gamma(\lambda+4) I_{\lambda+3}}{6(A/2)^{\lambda+3}} x^3 . \qquad (44)$$

Consider the case $\lambda = 0$ and $f(x) = \omega_0^2 \sin x$. The differential equation (1) is replaced by

$$d^2x/dt^2 + [2\omega_0^2(J_1+3J_3)/A]x - (8\omega_0^2 J_3/A^3)x^3 = 0. \qquad (45)$$

Since J_1+3J_3 and J_3 are positive in $[0, \pi]$, (45) corresponds to the softening cubic of section 4, with

$$a = 2\omega_0^2(J_1+3J_3)/A, \qquad b = -8\omega_0^2 J_3/A^3.$$

Following solution (10), one finds

$$\frac{\tau^{**}}{2\pi}\left\{\frac{2\omega_0^2}{A}(J_1+3J_3)\right\}^{1/2} = 2/\pi\ (1+k_6^2)^{1/2}\ K(k_6), \tag{46}$$

where τ^{**} is the approximation to the period resulting from the cubic ultraspherical approximation to $f(x)$ for $\lambda = 0$,

$$k_6^2 = -\nu/2+\nu = 2J_3/J_1+J_3\ , \qquad 0 \leq k_6^2 < 1,$$

and

$$\nu = -4J_3/(J_1+3J_3), \qquad -1 < \nu \leq 0. \tag{47}$$

The critical amplitude A_c for this cubic approximation is found from

$$k_6^2 = 2J_3/(J_1+J_3) = 1, \quad \text{or} \quad J_1(A_c) = J_3(A_c).$$

The smallest solution of this equation is

$$(A_c^{**})_0 = 3.054, \tag{48}$$

compared with $A_c = \pi$ for the exact solution.

Corresponding to a given value of $A < A_c$, one can compute ν from (47). With this value of ν known, one finds the corresponding numerical value of τ/τ' on the exact solution curve (E) in Fig. 1. This numerical value is set equal to the left-hand side of (46), whereupon one can obtain the new period ratio approximation τ^{**}/τ_0. A numerical example will show the manipulations.

Example: Letting the amplitude $A = 2$, one finds $\nu = -0.535$, which corresponds to $\tau/\tau' = 1.30$ in Fig. 1 (curve E). Hence, using (46) $\tau^{**}/\tau_o = 1.32$.

Instead of using Fig. 1 to furnish the above result, one can, of course, evaluate (46) directly, or equivalently, use

$$\tau^{**}/\tau_o = (2/\pi)\ \Lambda_2^{-1/2}(A)\ K(k_6). \tag{49}$$

For comparison, using the value $A = 2$ of the above sample, one obtains $\tau^{**}/\tau_o = 1.3296$. The numerical difference between these results is due to graph-reading inaccuracy.

For $\lambda = 1/2$, the cubic Legendre polynomial approximation for sin x yields

$$(\tau^{**}/2\pi)[3\omega_o^2\ (j_1 + \tfrac{7}{2}j_3)/A]^{1/2} = 2(1+k_7^2)^{1/2}K(k_7)/\pi, \tag{50}$$

where

$$\nu = -35j_3/(6j_1+21j_3), \quad k_7^2 = 35j_3/(12j_1+7j_3),$$

and j_1 and j_3 are spherical Bessel functions, i.e., $j_n(x) = (\pi/2x)^{1/2}J_{n+\frac{1}{2}}(x)$. To establish the critical amplitude, set $k_7^2 = 1$, which results in the equation

$$3j_1(A_c) = 7j_3(A_c). \tag{51}$$

The smallest solution of (51) is

$$(A_c^{**})_{1/2} = 2.98. \tag{52}$$

Thus the critical amplitude A_c based on the cubic Legendre polynomial approximation is smaller than that from the Tchebycheff polynomials of

the first kind. This means that the Legendre approximation becomes poorer at a smaller amplitude A. This is substantiated by the curves of Fig. 7, where these results and the exact solution are plotted. Included also in Fig. 7 is the linear approximation curve from [1] in order to demonstrate the improvement brought about by the cubic approximation.

For sinh x and $\lambda = 0$, one obtains from (44)

$$(\sinh x)_o^{**} = (2/A)[I_1 - 3I_3(A)]x + [8I_3/A^3]x^3.$$

The cubic approximate differential equation of motion is

$$d^2x/dt^2 + [2\omega_o^2(I_1 - 3I_3)/A]x + [8\omega_o^2 I_3/A^3]x^3 = 0. \tag{53}$$

When compared to (7) one notes

$$a = 2\omega_o^2(I_1 - 3I_3)/A, \quad b = 8\omega_o^2 I_3/A .$$

It is interesting to observe that while b will always be positive, a becomes negative for certain values of the amplitude A, i.e., $I_1 < 3I_3$ for $A > 3.91$. When a becomes negative, one must treat (53) as a "softening-hardening" (Case 3) cubic spring. In the present case,

$$\nu = bA^2/a = 4I_3/(I_1 - 3I_3). \tag{54}$$

For a given A, one can compute ν. If ν is positive, use Fig. 1 or 2; if ν is negative, use Fig. 4. Equate the period ratio found to $\tau^{**}(a)^{1/2}/2\pi$, and solve for the cubic-approximation period ratio τ^{**}/τ_o. An example is given to illustrate the procedure:

Example: Given $A = 2$. Then $\nu = 0.8934$, and Fig. 1 gives $\tau/\tau_0 = 0.776$. Hence, $\tau^{**}/\tau_0 = 0.795$, while a solution of (53) directly in terms of elliptic integrals yields 0.7954.

Using the procedure shown for $\lambda = 1/2$ also, one obtains the results in Fig. 8. It is worthwhile to note that the approximations for $\lambda = 0$ and $\lambda = 1/2$ are so close to the exact solution that the errors are hardly discernible. The results of the cubic Maclaurin series approximation has also been plotted for comparison.

7. Connection with method of Krylov and Bogoliuboff. The first approximation of Krylov and Bogoliuboff [10] yields the period approximation for (1)

$$(\tau^*)_{K.B.} = 2\pi\left[\frac{1}{A\pi}\int_0^{2\pi} f(A\cos\theta)\cos\theta\, d\theta\right]^{-1/2}. \tag{55}$$

On the other hand, the linear ultraspherical polynomial approximation gives, from (5) and (6),

$$(\tau^*)_{U.P.} = 2\pi\left[\frac{4\Gamma(\lambda+2)}{\pi^{1/2}\Gamma(\lambda+1/2)A}\int_0^{\pi/2} f(A\cos\theta)\sin^{2\lambda}\theta\cos\theta\, d\theta\right]^{-1/2}. \tag{56}$$

We note that (55) and (56) are equivalent for odd $f(x)$ when $\lambda = 0$ in (56). Hence, a linear Tchebycheff polynomial approximation of $f(x)$ in (1) gives results identical to those of the first approximation of Krylov and Bogoliuboff.

Higher order approximations in the Krylov and Bogoliuboff method depend on an iteration procedure, whereas the cubic ultraspherical polynomial approximation yields a solution in elliptic integrals of the first kind. No general comparison was made.

8. Discussion. The characteristic feature of <u>any</u> ultraspherical polynomial approximation ($\lambda < \infty$) for arbitrary $f(x)$ in (1) is that it yields an approximate period which will in general depend on the amplitude of the motion. The exact solution of a non-linear oscillation problem also has this property, while the <u>linear</u> Maclaurin series approximation does not. It is this feature which stimulated these investigations, while the use of particular ultraspherical polynomials permitted comparison of results for various values of λ.

As was shown in section 5 and indicated in all the graphical results, for small oscillations and $f(x)$ odd, the use of Tchebycheff polynomials ($\lambda = 0$) not only gives the best results of all the ultraspherical polynomials, but also τ^* agrees with the exact result to terms of order A^2. For large amplitudes or non-linearities, no such general results could be obtained. For the hardening cubics (Cases 1 and 3) and the hard sinh force, the λ value corresponding to the correct asymptotic form for τ as A or ν become infinite was small and positive, while $\lambda = 0$ yields smaller errors out to quite large amplitudes. For the only flattening spring considered here (tanh), for large amplitudes the choice $\lambda = -0.075$ gives the correct asymptotic results.

For the softening cubic (Case 2) and the soft sine force, oscillations at unlimited amplitudes cannot occur. Instead a critical amplitude exists at which the exact period becomes infinite. It can be shown that the linear ultraspherical polynomial approximation which produces the same critical amplitude corresponds to $\lambda = -1/2$. Although the integrals for the coefficients do not exist for this value, the polynomials and the results for $\tau^*_{-1/2}$ can be obtained by a limiting process. The numerical results

were quite poor for the problems considered here, despite its yielding the correct ν_c or A_c.

The extension to cubic ultraspherical approximations yields the expected increase in the accuracy of the results τ^{**}. In comparing this extension with other higher order procedures, one notes that the cubic ultraspherical approximation method can be applied without a substantial increase in work, in contrast to higher order approximations in other methods, as the perturbation treatment. This is due to two factors. First, if the coefficient of the linear term $a_1^{(\lambda)}$ can be found, then $a_3^{(\lambda)}$ can usually be found in a similar manner. If $a_1^{(\lambda)}$ must be evaluated numerically, then $a_3^{(\lambda)}$ will probably have to be also, but with no greater difficulty. Secondly, the procedure uses the fact that not only are the exact results for any cubic known in terms of elliptic integrals, but also τ/τ' depends only on one parameter ν, so that one can obtain τ/τ' graphically or in tabular form once and use these reults for the cubic ultraspherical approximations.

Finally, at no point in these approximation procedures was it necessary to restrict the non-linearity ν or the amplitude A to a small value. In fact, <u>all</u> of ultraspherical polynomials ($\lambda < \infty$) gave reasonably good results as ν (or A) became infinite.

Given a (non-linear) f(x) other than those studied here, (and for which the amplitude does not become extremely large), then, if a single ultraspherical polynomial approximation is desired, the choice, $\lambda = 0$ is indicated by our results.

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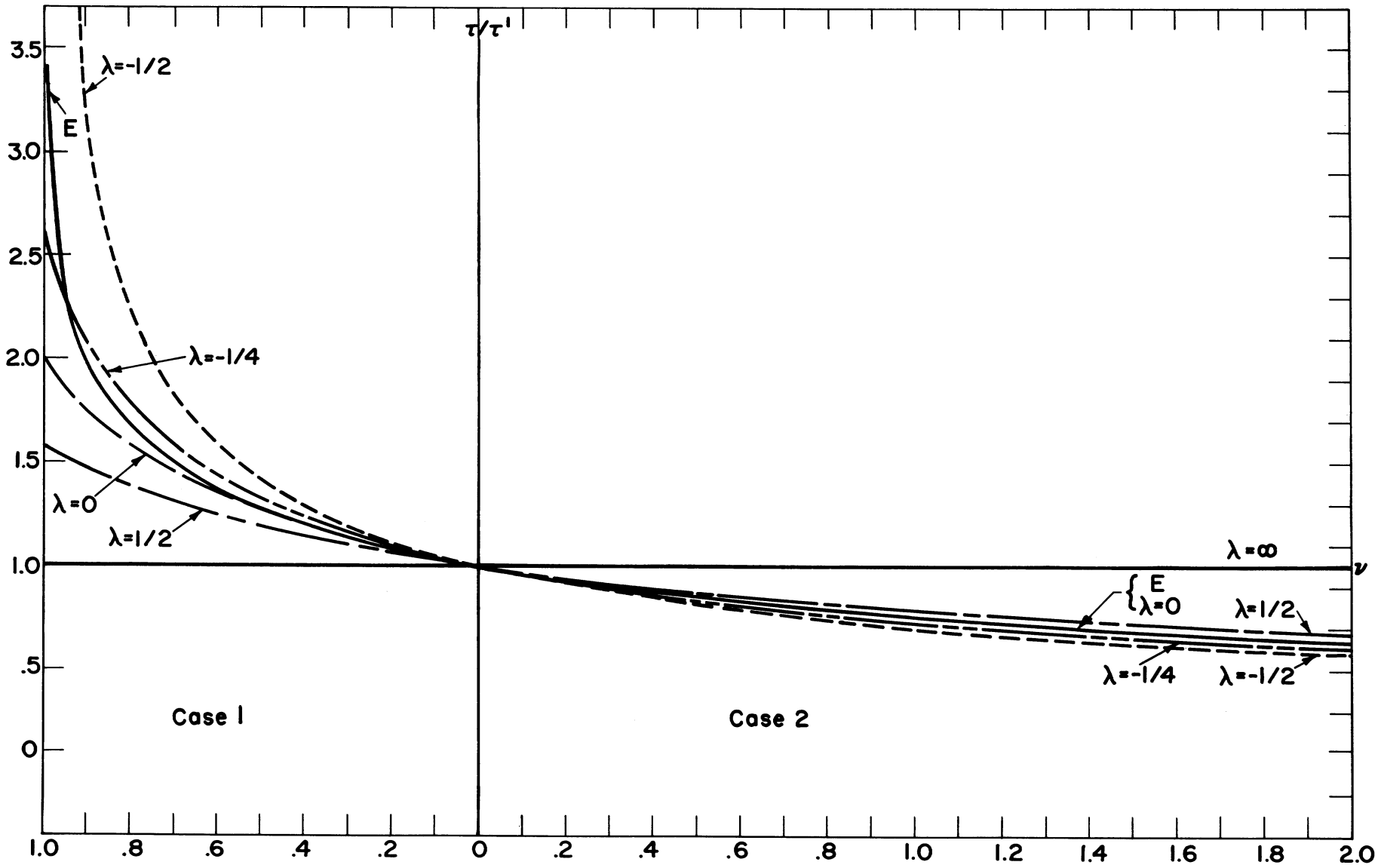

Figure 1. Period ratio (τ/τ')-non-linearity factor ν curve for cubic spring -- free oscillation.

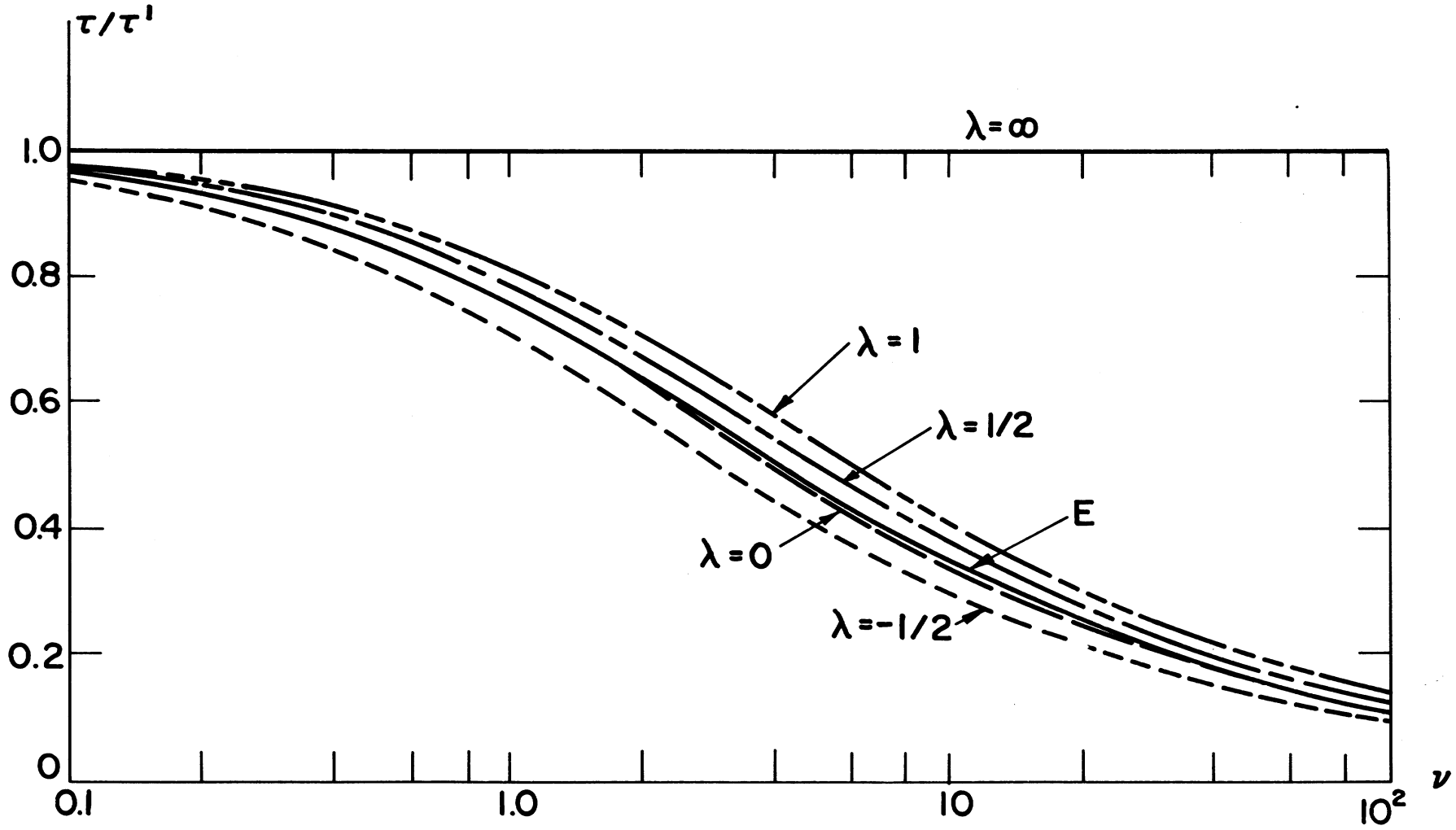

Figure 2. Period ratio (τ/τ')-non-linearity factor ν curve for hard cubic spring -- free oscillation.

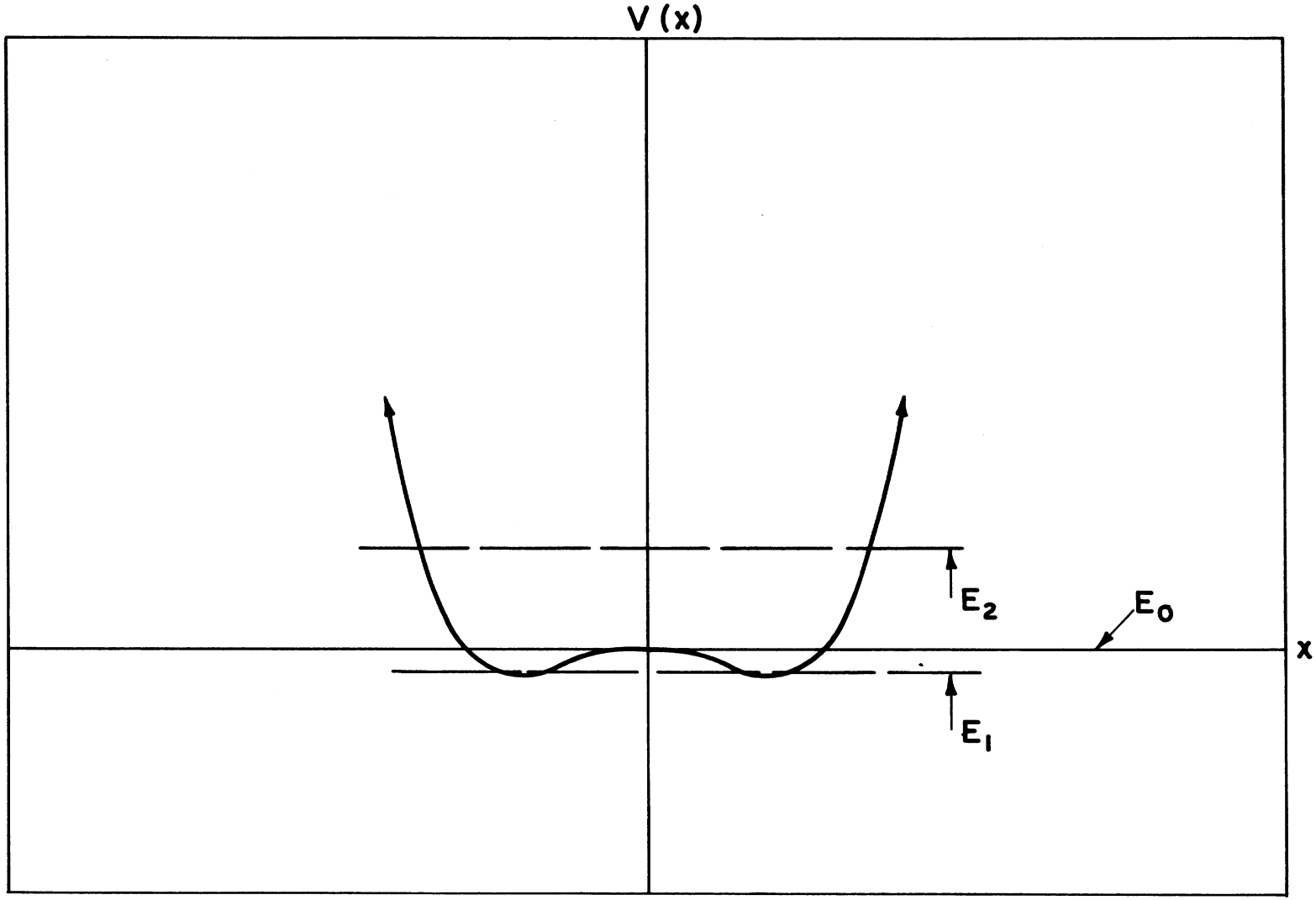

Figure 3. Potential curve for softening-hardening cubic spring -- free oscillation.

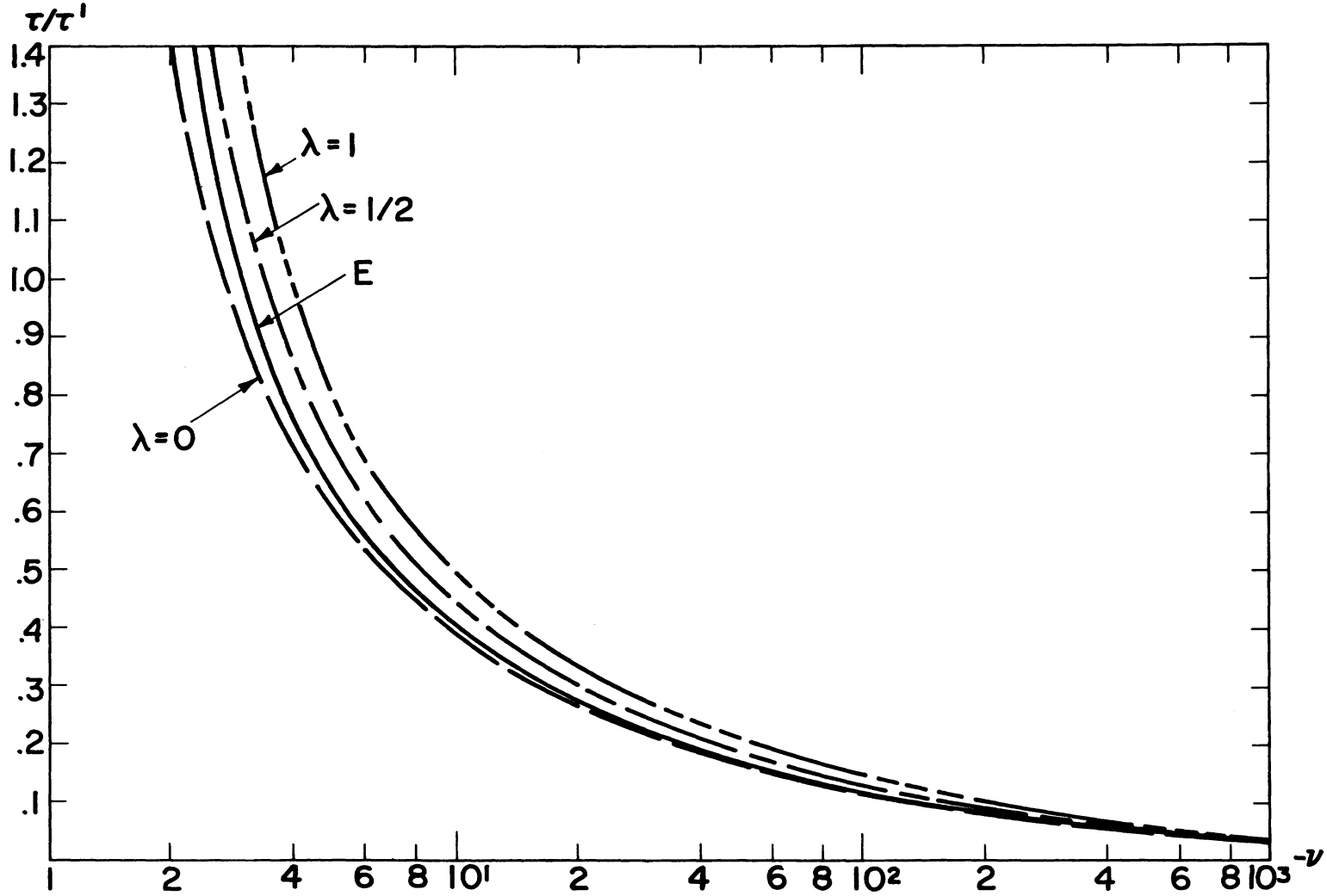

Figure 4. Period ratio (τ/τ')-non-linearity factor ν curve for softening-hardening cubic spring -- free oscillation.

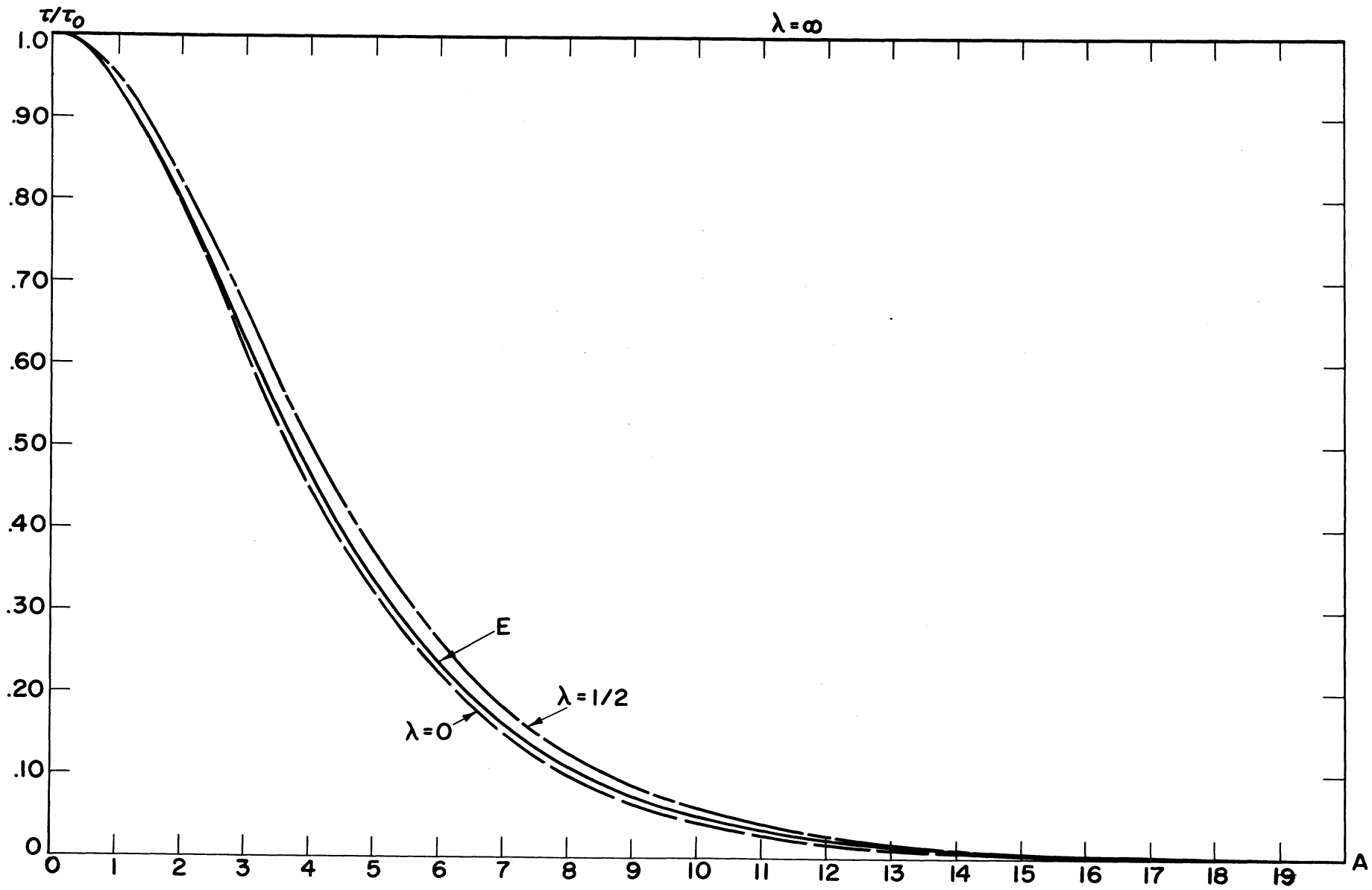

Figure 5. Period ratio (τ/τ_o)-amplitude curve for sinh spring -- free oscillation.

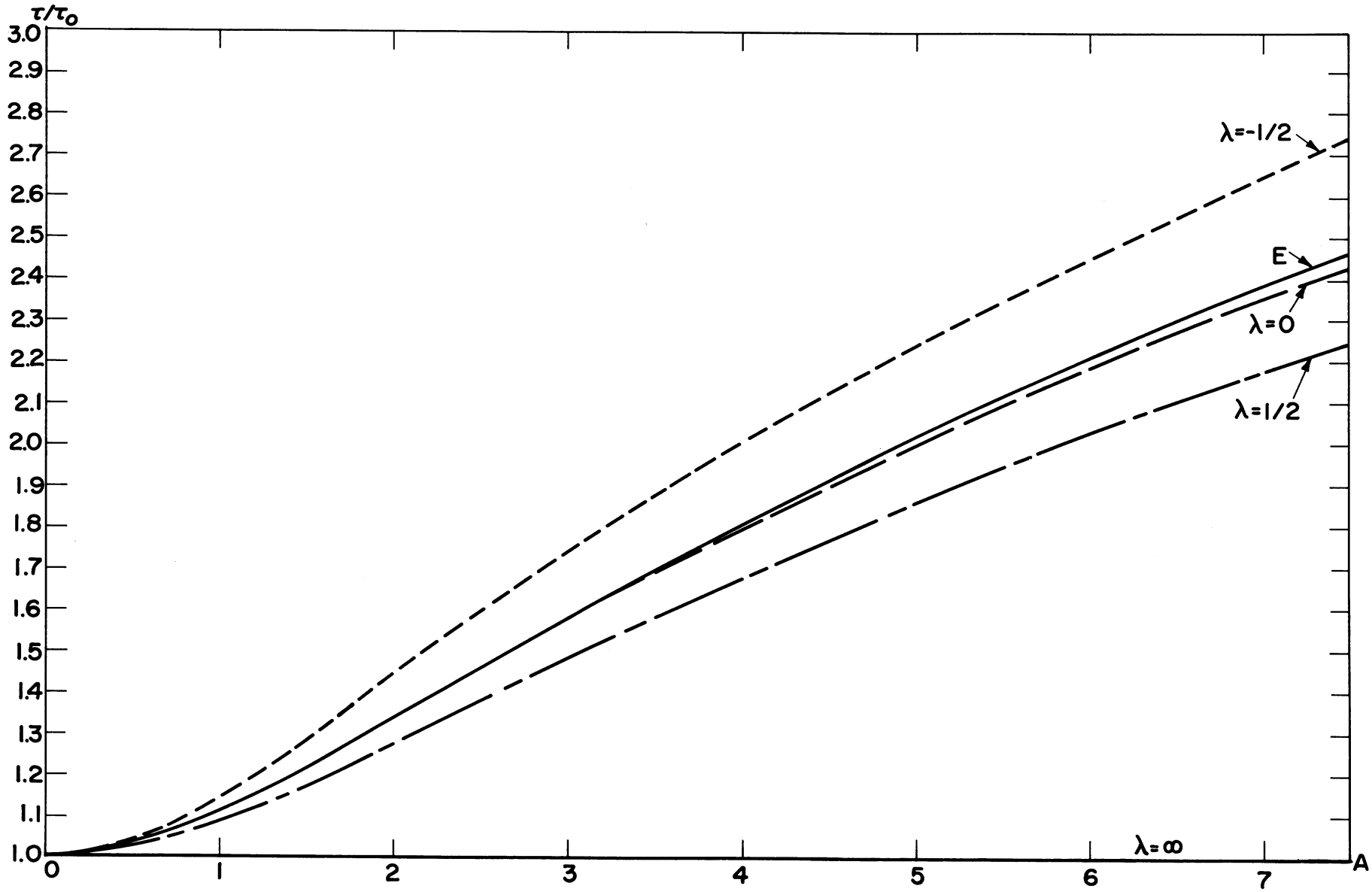

Figure 6. Period ratio (τ/τ_o)-amplitude curve for tanh spring -- free oscillation.

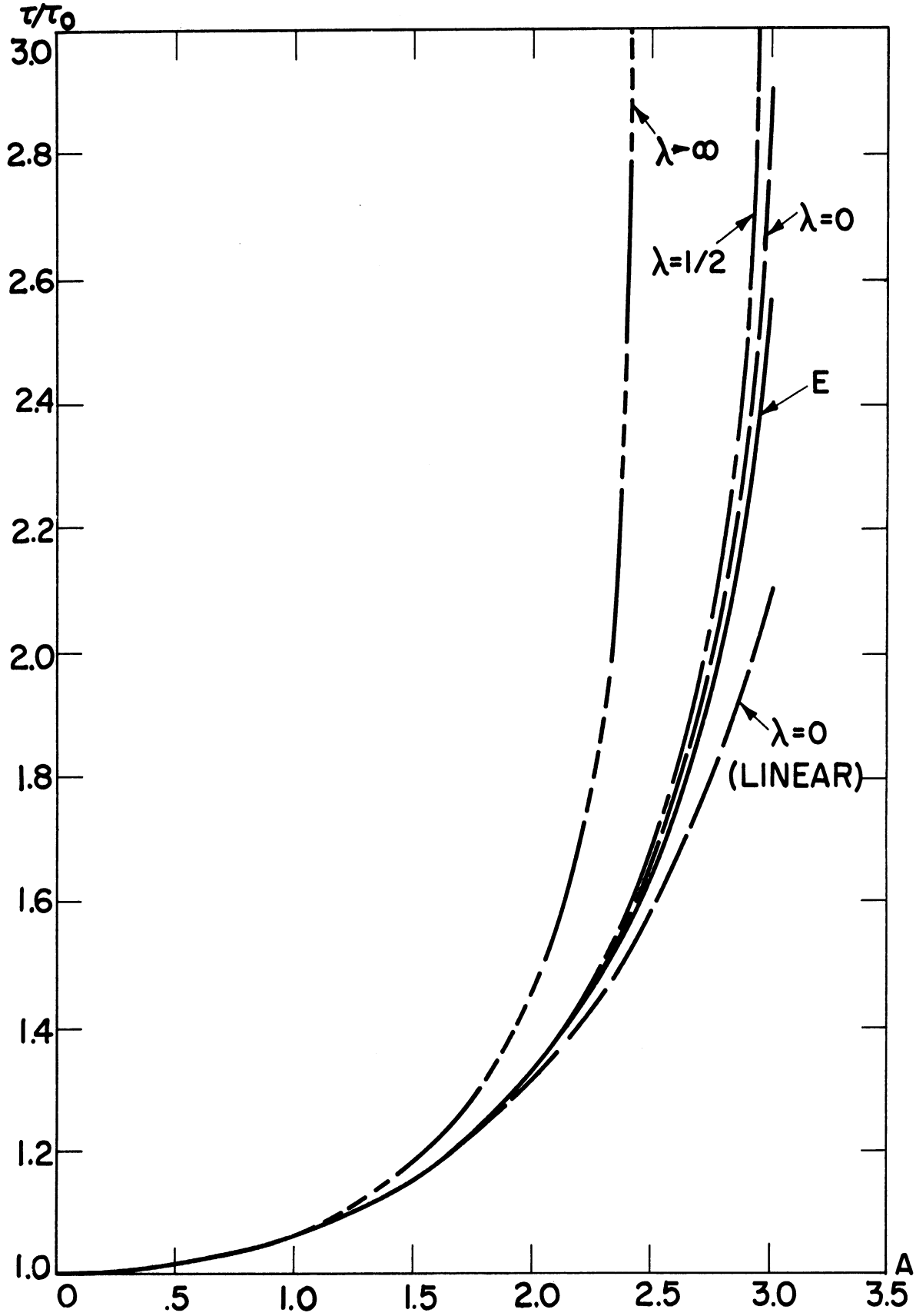

Figure 7. Period ratio (τ/τ_o)-amplitude curve for sine spring for cubic approximation -- free oscillation.

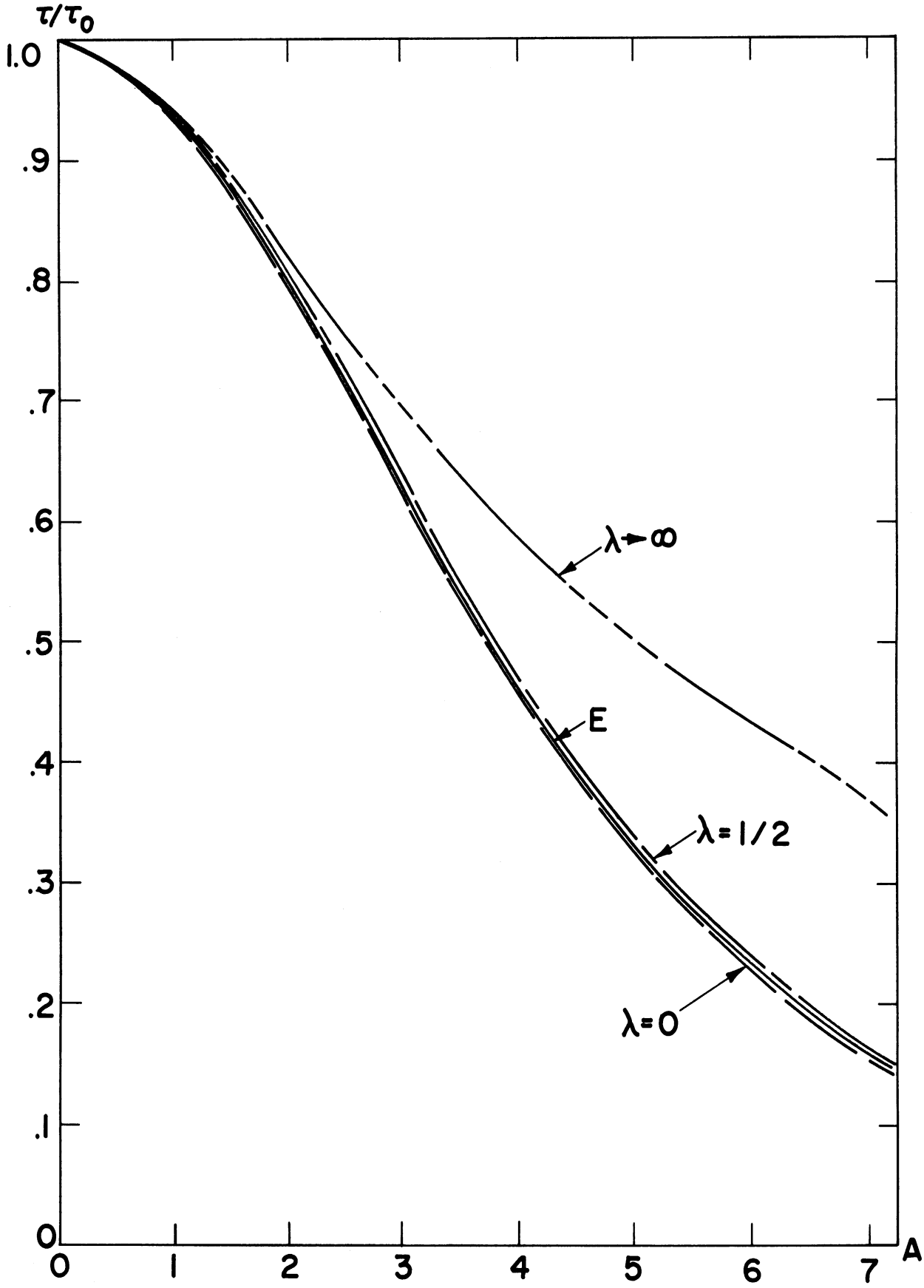

Figure 8. Period ratio (τ/τ_0)-amplitude curve for sinh spring for cubic approximation -- free oscillation.

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