An interesting problem in linear programming is the group assembly problem which is mathematically equivalent to the general transportation problem of economics. Computer programs designed for the determination of exact and approximate optimal group assemblies have been available for some time. This paper presents formulas for the mean and squared standard deviation of the distribution of all possible group assembly sums. Computational techniques are presented and the results are related to those of the analysis of variance of a k-factor problem with n levels of each factor.

Suppose that a group, or team, or crew, consists of k men each of whom is trained for only one of the k different positions of the group. Suppose further that there are n men available for each of the k positions. Then one set of men \{i_1, i_2, \ldots, i_k, \ldots, i_k\}, where 1 \leq i_j \leq n, constitutes one of the groups under consideration and there are n groups. In the personnel classification problem [5] for instance, where a modified interpretation identifies the n men (i_1) with the n jobs (i_2), the number of assignments of men to jobs is n^2.

The score which a group makes in attaining its objective is a group score, g_1, \ldots, g. It must be measured in units which indicate the effectiveness of the group such as points, percentage of targets hit, etc. The ideal group score may be a high one, as in bowling, or a low one, as in golf. In the following development we assume that the group score is known. It is sometimes difficult to satisfy this assumption by empirical determination of the group score exactly (i) because of the sampling error of such results and (ii) because of the practical impossibility of determining the group score for all the n^k possible groups with n and/or k large. In such cases approximate, or plausible, or calculated, or hypothetical group scores must be provided before the solution can be made. This aspect of the problem is much less serious in an economic version of the problem, the general transportation problem [4], where the transportation costs from origins to destinations through
intermediate points are presumably known or can be approximated. The results of this paper are applicable to the interpreted variations of the general problem.

A group assembly is a collection of groups such that each of the available \( nk \) men is assigned to one and only one group. Combinatorial argument shows that there are \( (n!)^{k-1} \) group assemblies since, for each \( i_1 \), the \( k - 1 \) sets of \( i_2, \cdots, i_k \) may be assigned in \( n! \) different ways.

In studying the measure of effectiveness of the assembly, we assume additivity and define the group assembly sum to be the sum of the group scores for the \( n \) groups in the assembly. Thus the group assembly sum is given by

\[
T = \sum_{i_1, \ldots, i_k} g_{i_1, \ldots, i_k},
\]

where \( \sum \) indicates the summation for the \( n \) terms.

**The Mean, \( \bar{T} \)**

Since there are \( (n!)^{k-1} \) assemblies, the total number of group scores in \( \sum T \) is \( n(n!)^{k-1} \). Now from symmetry,

\[
\sum_{i_1} \cdots \sum_{i_k} g_{i_1, \ldots, i_k} = g_{\ast \cdots \ast}
\]

must factor this result. Since \( g_{\ast \cdots \ast} \) is the sum of the \( n^k \) group scores,

\[
\sum T = \frac{n(n!)^{k-1}}{n^k} g_{\ast \cdots \ast} = [(n - 1)!]^{k-1} g_{\ast \cdots \ast}.
\]

Then the mean of the distribution of all possible assembly sums is

\[
\bar{T} = \frac{1}{(n!)^{k-1}} T = \frac{1}{n^{k-1}} g_{\ast \cdots \ast}.
\]

**The Value of \( T^2 \)**

In deriving the value of the standard deviation of \( T \), we first obtain the second moment, \( T^2 \). Now \( T^2 \), for a given assembly, consists of

\[
T^2 = \sum_{i_1, \ldots, i_k} g_{i_1, \ldots, i_k}^2 + \sum_{i_1^\prime \neq i_1^\prime} g_{i_1, \ldots, i_k} g_{i_1^\prime, \ldots, i_k}.
\]

The first term on the right consists of the sum of the squares of the \( n \) terms of (1) while the second term on the right consists of the \( n(n - 1) \) paired products of the \( n \) terms of (1). Then the value of \( \sum T g_{i_1, \ldots, i_k}^2 \) is

\[
\frac{n(n!)^{k-1}}{n^k} \sum_{i_1, \ldots, i_k} g_{i_1, \ldots, i_k}^2 = [(n - 1)!]^{k-1} \sum_{i_1, \ldots, i_k} g_{i_1, \ldots, i_k}^2,
\]

and the value of \( \sum T g_{i_1, \ldots, i_k} g_{i_1^\prime, \ldots, i_k} \) is

\[
\frac{n(n - 1)(n!)^{k-1}}{[n(n - 1)]^2} \sum_{i_1^\prime \neq i_1^\prime} g_{i_1, \ldots, i_k} g_{i_1^\prime, \ldots, i_k} = [(n - 2)!]^{k-1} \sum_{i_1^\prime \neq i_1^\prime} g_{i_1, \ldots, i_k} g_{i_1^\prime, \ldots, i_k}.
\]
Then dividing by \((n!)^{k-1}\) we get
\[
(4) \quad T^2 = \frac{1}{n^{k-1}} \sum_{i_j} g_{i_1} \cdots i_k + \frac{1}{n^{k-1}(n-1)^{k-1}} \sum_{i_j \neq h_j} g_{i_1, \ldots, i_k} g_{h_1, \ldots, h_k}.
\]

The next step is the evaluation of the last term of (4) in terms of sums of squares. We use the symbolism of restriction equations in order to accomplish this.

**Application of Restriction Equations**

We note that
\[
\sum_{i \neq h} f_i f_h = \sum_{i} \sum_{h} f_i f_h - \sum_{i = h} f_i f_h
\]
can be written symbolically as the restriction equation
\[
(i \neq h) = 0 - (i = h),
\]
where 0 indicates that there is no restriction on the values of \(i\) and \(h\) and where \((i = h)\) indicates that the restriction is only that \(i\) must equal \(h\). More generally for \(f_{i_1, i_2, h_{i_1, i_2, h}}\) and \(f_{i_1, i_2, h_{i_1, i_2, h}}\), respectively we have
\[
(i_1 \neq h_1)(i_2 \neq h_2) = 0 - [(i_1 = h_1) + (i_2 = h_2)] + (i_1 = h_1)(i_2 = h_2),
\]
\[
(i_1 \neq h_1)(i_2 \neq h_2)(i_3 \neq h_3) = 0 - [(i_1 = h_1) + (i_2 = h_2) + (i_3 = h_3)]
\]
\[
+ [(i_1 = h_1)(i_2 = h_2) + (i_1 = h_1)(i_3 = h_3)
\]
\[
+ (i_2 = h_2)(i_3 = h_3)] - (i_1 = h_1)(i_2 = h_2)(i_3 = h_3),
\]
where \((i_1 = h_1)(i_2 = h_2)\) means the double restriction \((i_1 = h_1)\) and \((i_2 = h_2)\). These can be written more symbolically using product \(\Pi (i_j = h_j)\), and product type \(T_v \Pi (i_j = h_j)\) which sums all products having \(r\) factors. Then the restriction equations appear as
\[
\Pi (i_1 \neq h_1) = 0 - T_v (i_1 = h_1) + \Pi (i_1 = h_1),
\]
\[
\Pi (i_1 \neq h_2)(i_2 \neq h_3) = 0 - T_v (i_1 = h_2) + T_v \Pi (i_1 = h_2) - \Pi (i_1 = h_2),
\]
and in general
\[
(5) \quad \Pi (i_i \neq h_i) = 0 - T_v (i_i = h_i)
\]
\[
+ \sum_{r=2}^{k-1} (-1)^r T_v \Pi (i_i = h_i) + (-1)^k \Pi (i_i = h_i).
\]

Application of restriction equations to the summation part of the last term of (4) gives
\[
(6) \quad \sum_{i_j \neq h_j} g_{i_1, i_2, h_{i_1, i_2, h}} = g^2_{**} - \sum_{i_1} g^2_{i_1, *} - \sum_{i_2} g^2_{*, i_2} + \sum_{i_1, i_2} g^2_{i_1, i_2},
\]
\[
\sum_{i_1 \neq h_1} g_{i_1, y_1, z_1} g_{h_1, y_1, z_1} = g_{**, **} - \sum_{i_1} g_{i_1, **} - \sum_{i_2} g_{**, i_2} - \sum_{i_3} g_{i_3, **} + \sum_{i_1, i_2} g_{i_1, i_2, **} + \sum_{i_1, i_3} g_{i_1, i_3, **} + \sum_{i_2, i_3} g_{i_2, i_3, **} - \sum_{i_1, i_2, i_3} g_{i_1, i_2, i_3, **},
\]

and in general
\[
\sum_{i_1 \neq h_1} g_{i_1, \ldots, i_k, \ldots, h_k} = g_{**, \ldots, **} - T_{v} \sum g_{i_1, **} + \sum_{r=2}^{k-1} (-1)^r T_{v} \sum g_{i_1, \ldots, i_r, **} + (1)^k \sum g_{i_1, \ldots, i_k}.
\]

The Value of \( \sigma^2_{T} \)

Defining \( \sigma^2_{T} = \bar{T}^2 - (\bar{T})^2 \) we obtain from (2), (4), (6), (7), (8) for \( k = 2, 3, 4, 5, k \)
\[
\sigma^2_{T} = \frac{1}{n-1} \sum g_{i_1, i_2} - \frac{1}{n(n-1)} \left( \sum g_{i_1, **} + \sum g_{**, i_2} \right) + \frac{1}{n^2(n-1)} g_{**, **},
\]
\[
\sigma^2_{T} = \frac{n^2 - 3n + 3}{n^2(n-1)^3} \sum g_{i_1, i_2, i_3} + \frac{1}{n^3(n-1)^3} \left[ -T_{v} \sum g_{i_1, **} \right] + 3n^2 - 3n + 1 \frac{1}{n^5(n-1)^5} g_{**, **},
\]
\[
\sigma^2_{T} = \frac{(n-2)(n^2 - 2n + 2)}{n^6(n-1)^6} \sum g_{i_1, i_2, i_3, i_4} + \frac{1}{n^7(n-1)^7} \left[ -T_{v} g_{i_1, **} \right] + 3n^2 - 3n + 1 \frac{1}{n^9(n-1)^9} g_{**, **},
\]

and in general
\[
\sigma^2_{T} = \frac{(n-1)^{k-1} - (-1)^k}{n^{k-1}(n-1)^{k-1}} \sum g_{i_1, \ldots, i_k} + \frac{1}{n^{k-1}(n-1)^{k-1}} \left[ \sum_{r=1}^{k-1} (-1)^r T_{v} \sum g_{i_1, \ldots, i_r, **} \right] + \frac{n^{k-1} - (n-1)^{k-1}}{n^{2k-2}(n-1)^{k-1}} g_{**, \ldots, **}.
\]

Direct Calculation of $\sigma_\tau^2$

The formulas are suitable for direct calculation. It is necessary to calculate the quantities $g_{i_1 \ldots i_k}**$. Then the squares can be calculated and accumulated. This process is identical with the preliminary calculations for a nonreplicated factorial analysis of variance with $k$ factors and $n$ levels for each factor. Machine programs may be used in obtaining the values of $g_{i_1 \ldots i_k}**$ or they may be obtained quite easily with desk calculator if $n$ and $k$ are not too large.

Calculation Using Deviates

The formulas are simplified with the use of a preliminary transformation to deviates. This transformation is very feasible with automatic calculation and might be considered by users of desk calculators in situations, such as this, where the main effects may be ignored. The technique is first applied to the $k = 2$ case. We define

$$d_{i_1 i_2} = g_{i_1 i_2} - \bar{g}_{i_1} - \bar{g}_{i_2} + \bar{g}** = g_{i_1 i_2} - (g_{i_1}/n) - (g_{i_2}/n) + (g**/n^2)$$

and note that $\sigma_\tau^2$ is not changed by replacing $g_{i_1 i_2}$ by $d_{i_1 i_2}$ since each assembly sum is decreased by the constant

$$\sum_{i_1} \bar{g}_{i_1} + \sum \bar{g}_{i_2} - n\bar{g}**.$$

Furthermore we note that

$$\sum_{i_2} d_{i_1 i_2} = 0 = \sum_{i_2} d_{i_1 i_2} = \sum_{i_2} \sum_{i_2} d_{i_1 i_2}$$

since

$$\sum_{i_2} d_{i_1 i_2} = g_{i_2} - (g**/n) - g_{i_2} + (g**/n).$$

Then (9) becomes the simple

$$\sigma_\tau^2 = \frac{1}{n-1} \sum d_{i_1 i_2}^2.$$

Similar treatment when $k = 3$ with

$$d_{i_1 i_2 i_3} = g_{i_1 i_2 i_3} - \bar{g}_{i_1 i_2} - \bar{g}_{i_1 i_3} - \bar{g}_{i_2 i_3} + 2\bar{g}***$$

leads to

$$\sigma_\tau^2 = \frac{n-2}{n(n-1)^2} \sum d_{i_1 i_2 i_3}^2,$$

$$+ \frac{1}{n(n-1)^2} \left[ \sum d_{i_1 i_2 i_3}^2 + \sum d_{i_2 i_3 i_4}^2 + \sum d_{i_3 i_4 i_5}^2 \right].$$
In the general case with
\[ d_{i_1 \ldots i_k} = g_{i_1 \ldots i_k} - T_v g_{i_1 \ldots i_k} + (k - 1) g_{i_1 \ldots i_k} \]
we have
\[ \sigma_v^2 = \frac{(n - 1)^{k-1} + (-1)^k}{n(n - 1)^{k-1}} \sum d_{i_1 \ldots i_k}^2 + \frac{1}{n^{k-1}(n - 1)^{k-1}} \sum_{r=2}^{k-1} (-1)^r T_v \sum d_{i_1 \ldots i_r}^2. \] (16)

**Calculation Using Large Variances**

Applying the concept of the large variance ([1], p. 302) to the d's and
noting that the number of \( d_{i_1} \), \( d_{i_1, i_2} \), \( d_{i_1, i_2, i_3} \), \ldots is \( n \), the number of \( d_{i_1, i_2} \),
\( d_{i_1, i_2, i_3} \), \( d_{i_1, i_2, i_3, i_4} \), \ldots is \( n^2 \), the number of \( d_{i_1, i_2, i_3} \), \ldots is \( n^3 \), etc. and that
\( d_{i_1, i_2, \ldots} = 0 \), we define
\[ L'(1*) = nT_v \sum d_{i_1}^2 = 0, \]
\[ L'(11*) = n^2T_v \sum d_{i_1, i_2}^2, \]
\[ L'(111*) = n^3T_v \sum d_{i_1, i_2, i_3}^2, \]
\[ \text{etc.} \] (17)

Then for \( k = 2 \) and \( k = 3 \) we have
\[ \sigma_v^2 = \frac{1}{n^2(n - 1)} L'(11), \] (18)
\[ \sigma_v^2 = \frac{1}{n^4(n - 1)^2} [(n - 2)L'(111) + L'(11*)], \] (19)

and for general \( k \)
\[ \sigma_v^2 = \frac{1}{n^{2(k-1)}(n - 1)^{k-1}} \left[ \frac{(n - 1)^{k-1} + (-1)^k}{n} L'(1^k) \right. \]
\[ + (-1)^{k-r} n^{r-1} L'(1^{r-1}*) \left. \right]. \] (20)

Thus for \( k = 4 \) and \( k = 5 \) we have respectively
\[ \sigma_v^2 = \frac{1}{n^6(n - 1)^3} [(n^2 - 3n + 3)L'(1111) + nL'(11***) - L'(111*)]. \] (21)
\[ \sigma_v^2 = \frac{1}{n^8(n - 1)^4} [(n^2 - 4n^2 + 6n - 3)L'(11111) + n^2L'(11***) \]
\[ - nL'(11***) + L'(1111*)]. \] (22)

The values of \( L' \) are easily calculated from the sums of the squares of the
d's as is illustrated below for the case with \( k = 3 \) and \( n = 3 \).

**Calculation Using I Terms**

The values of the \( L' \) are the values of the nondeviate \( L \) with
\[ L_{i_1 \cdots i_k} = L_{*i_1 \cdots i_k} = \cdots = 0. \]

Since the conventional relation of the sum of squares expressed in terms of \( L_s \) is

\[
\begin{align*}
L_{i_1i_2} &= L_{*i_1i_2} + I_{i_1i_2}, \\
L_{i_1i_2i_3} &= L_{*i_1i_2i_3} + L_{**i_2i_3} + I_{i_1i_2i_3} + I_{*i_1i_2} + I_{*i_2i_3} + I_{*i_1i_3},
\end{align*}
\]

where the value of \( I \) is \( n^k \) times the corresponding conventional sum of squares, we have

\[
\begin{align*}
L'(11) &= I_{i_1i_2} = I(11), \\
L'(11*) &= I_{i_1i_2} + L'_{*i_1i_2} + L'_{**i_2i_3} \\
&= I_{i_1i_2} + I_{i_1i_2} + I_{*i_2i_3} = I(11*), \\
L'(111) &= L'_{i_1i_2i_3} = I(111) + I(111).
\end{align*}
\]

Then the formulas for \( k = 2 \) and \( k = 3 \) become respectively

\[
\begin{align*}
\sigma_r^2 &= \frac{1}{n(n-1)} I(11), \\
\sigma_r^2 &= \frac{1}{n(n-1)^2} [(n-2)I(111) + (n-1)I(11*)].
\end{align*}
\]

In particular when \( n = 3 \), (25) becomes

\[
\sigma_r^2 = \frac{1}{324} [I(111) + 2I(11*)].
\]

When \( k = 4 \) we find that

\[
\begin{align*}
L'(1111) &= I(1111) + I(111*) + I(11**), \\
L'(111*) &= I(111*) + 2I(11**) \\
L'(11**) &= I(11**)
\end{align*}
\]

so that

\[
\sigma_r^2 = \frac{1}{n^3(n-1)^3} [(n^2 - 3n + 3)I(1111) \\
+ (n-2)(n-1)I(111*) + (n-1)^2I(11**)].
\]

For \( k = 5 \) we find

\[
\begin{align*}
L'(11111) &= I(11111) + I(1111*) + I(111**) + I(11***) \\
L'(1111*) &= I(1111*) + 2I(111**) + 3I(11***) \\
L'(111**) &= I(111**) + 3I(11***) \\
L'(11***) &= I(11***)
\end{align*}
\]
so that

\[ \sigma_T^2 = \frac{1}{n^2(n - 1)^2} \left[ (n^3 - 4n^2 + 6n - 4)I(11111) + (n^2 - 3n + 3)(n - 1) \right] 
\cdot I(1111\ast) + (n - 2)(n - 1)^2 I(111\ast\ast) + (n - 1)^3 I(11\ast\ast\ast) \].

We note that for \( k = 2, 3, 4, 5 \) at least, the coefficient of \( (1^k - r^s) \) in the bracket is \( (n - 1)^r \) times the coefficients of \( I(1^k - r) \) in the bracket. We prove that this is true in general and that the coefficient in the bracket of

\[ I(1^k - r^s) = \frac{\left( \begin{array}{c} k \\ r \end{array} \right) \left( \begin{array}{c} k - r \\ s \end{array} \right) + \left( \begin{array}{c} k \\ r + s \end{array} \right) - \left( \begin{array}{c} k - r \\ s \end{array} \right)} \]

The coefficient of \( I(1^k - r^s - r^t) \) in the expansion of \( L'(1^k - r^s - r^t) \) is

\[ \left( \begin{array}{c} k \\ r \\ s \end{array} \right) \cdot \left( \begin{array}{c} k - r \\ s \end{array} \right) + \left( \begin{array}{c} k \\ r + s \end{array} \right) - \left( \begin{array}{c} k - r \\ s \end{array} \right) \]

When \( r' = 0 \), this coefficient is \( \left( \begin{array}{c} s \\ s \end{array} \right) = 1 \). Then for a fixed \( r = r' + s \) the coefficient of \( I(1^k - r^s) \) in the bracket, using (20) with \( r \) replaced by \( r' = r - s \), is

\[ \frac{(n - 1)^{k-1} + (-1)^k}{n} \cdot \sum_{s=0}^{r-1} (-1)^{k-r+s} n^{r-s-1} \left( \begin{array}{c} s \\ s \end{array} \right) n^s \]

\[ = \frac{(n - 1)^{k-1} + (-1)^k}{n} + \frac{(-1)^{k-r}}{n} \sum_{s=0}^{r-1} (-1)^{s} \left( \begin{array}{c} r \\ s \end{array} \right) n^{r-s} \]

\[ = \frac{(n - 1)^{k-1} + (-1)^k}{n} + \frac{(-1)^{k-r}}{n} \cdot (n - 1)^r + (-1)^{r+1} \]

\[ = \frac{(n - 1)^{k-r-1} + (-1)^{k-r}}{n} [n - 1]^r. \]

We can then write, \( k \geq 2 \),

\[ \sigma_T^2 = \frac{1}{n^{2(k-1)}(n - 1)^k} \sum_{r=0}^{k-2} \left[ \frac{(n - 1)^{k-r-1} + (-1)^{k-r}}{n} \right] (n - 1)^r I(1^k - r^s). \]

The value

\[ \frac{(n - 1)^{k-r-1} + (-1)^{k-r}}{n} \]

is simply a polynomial of degree \( k - r - 2 \) with binomial coefficients of degree \( k - r - 1 \) and alternating sign.

**Calculation Using Sums of Squares**

Now

\[ I(1^k - r^s) = n^k S(1^k - r^s), \]
where $S(1^{k-r}r^*)$ is the sum of the conventional sum of squares which contribute to the interaction term of order $r$. Thus

$$S(1^{**}) = S(i_1, i_2) + S(i_1, i_3) + S(i_1, i_4) + S(i_2, i_3) + S(i_2, i_4) + S(i_3, i_4)$$

features the second-order sums of squares appearing in a conventional factorial analysis with $n = k = 4$ and no replication. The general formula then is

$$\sigma_T^2 = \frac{1}{n^{k-2}(n - 1)^{k-1}} \sum_{r=0}^{k-2} \left[ \frac{(n - 1)^{k-r-1} + (-1)^{k-r}}{n} \right] (n - 1)^r S(1^{k-r}r^*).$$

Special cases for $k = 2, 3, 4$ are

$$\sigma_T^2 = S(11)/(n - 1),$$

$$\sigma_T^2 = [(n - 2)S(111) + (n - 1)S(11*)]/n(n - 1)^2,$$

$$\sigma_T^2 = [(n^2 - 3n + 3)S(1111) + (n - 2)(n - 1)S(111*) + (n - 1)^2S(11**)]/n^2(n - 1)^3,$$

etc.

This formula is suitable for the application of the results of a computing program for analysis of variance of an unreplicated factorial experiment. It is only necessary to obtain the sum of squares for the various sources by machine and apply the multipliers indicated in (31).

The formula (31) has more than computational importance since it shows which interaction terms are responsible for the magnitude of $\sigma_T^2$. It is worthy of note that, in every case, the coefficient of $S(1^{k-r}r^*)$ in the bracket is of order $n^{k-2}$. Hence, unless $n$ is small, the weighting of every individual sum of squares, no matter what the order, is approximately the same. This makes possible the direct comparison of the different sums of squares so that one can see which sources are responsible for the size of $\sigma_T^2$.

It is also worthy of note that $\sigma_T^2 = 0$ if and only if all sums of squares for interaction terms are zero. This situation is satisfied when the group score is a linear function of the scores of the individuals composing the group and is not a function of interaction of individuals. Then all group assembly sums are the same and the groups can be assembled on any convenient basis without reference to the group score. The group assembly problem is designed to handle situations in which there are interactions in the group.
Illustration with \( k = 3 \) and \( n = 3 \)

Computation with the various formulas is illustrated in Table 1 and Table 2. Table 1 treats original \( g_{i_1i_2i_3} \) and illustrates the formulas using \( \sum g^2, I( ), \) and \( S( ). \) Table 2 treats \( d_{i_1i_2i_3} \) and illustrates the formulas for \( d^2, L( ), I( ), \) and \( S( ). \)

### Table 1

**Value of \( g_{i_1i_2i_3} \) with Sums and Analysis**

<table>
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<tr>
<th>( i_3 )</th>
<th>( i_2 )</th>
<th>( i_1 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( \bar{g}_{i_1i_2i_3} )</th>
</tr>
</thead>
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<td></td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>12</td>
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<td>6</td>
<td>4</td>
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<tr>
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<td></td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>( \bar{g}_{i_1i_2} )</td>
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<td>15</td>
<td>16</td>
<td>44</td>
<td>( \bar{g}_{i_1i_2} )</td>
</tr>
<tr>
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<td>7</td>
<td>2</td>
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<tr>
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| \( * \) | \( i_1i_2 \) | \( \bar{g}_{i_1i_2} \) | 34 | 48 | 44 | 126 | \( \bar{g}_{i_1i_2} \) |

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Examination of the I (or S) column in Table 1 and Table 2 shows that the larger contribution to $\sigma_T^2$ comes from the $i_1 i_3$ and $i_1 i_2 i_3$ interaction terms. Actually in this case with $n - 1 = 2$ and $n - 2 = 1$, the $i_1 i_3$ term, weighted twice, makes the largest contribution. If this term were zero (the other terms remaining the same) we would have $\sigma_T^2 = 6.74$ units$^2$ which is less than 50 per cent of the original $\sigma_T^2$. This illustrates the use of the results of a conventional analysis of variance in determining the major contribution to the value of $\sigma_T^2$.

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### Table 2

Value of $d_{i_1 i_2 i_3}$ with Sums and Analysis

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In Table 1 the multipliers of the $\sum g^2$ obtained from (10) are the numerators of the fractions having $n^4(n - 1)^2$ as the denominator with $n = 3$. Decimal equivalents of the coefficients in (10) could be used if preferred. In Table 2 the computed values of $L'$ are rounded to the nearest integer since it is known that each $L'$ is an integer. In most cases the methods lead to the exact answer, 14 units$^2$, though in some cases a decimal approximation results.

The various formulas are provided so that the worker may select the one which seems most suitable when considering other desired treatment of the data, the values of $k$ and $n$, and available resources such as computational equipment and assistants.

REFERENCES


Manuscript received 6/8/64.