THE NUMBER OF WAYS TO LABEL A STRUCTURE*

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It has been observed that the number of different ways in which a graph with \( p \) points can be labelled is \( p! \) divided by the number of symmetries, and that this holds regardless of the species of structure at hand. In this note, a simple group-theoretic proof is provided.

The article by Harary and Read [1966] concluded with a table listing the probabilities \( P(n, k) \) that a connected functional digraph with \( n \) points has a cycle of length \( k \), for \( n = 2 \) to 7. We wish to acknowledge that the entries in this table are given by the formula

\[
P(n, k) = \frac{(n-1)!}{(n-k)! (n-1)^n}
\]

in accordance with the theorem in Katz [1955]. This result was anticipated in turn by Rubin and Sitgreaves in an unpublished memorandum cited in Katz [1955].

In order to contribute something positive in this note, we now prove the theorem about graphs and groups which justifies the formula given in Harary and Read [1966] for the number of ways to label a structure. Since this is a sequel to Harary and Read [1966], its notation and terminology will be used. Thus we write \( s(G) \) for the symmetry number of graph \( G \) (the order of its automorphism group \( \Gamma(G) \)) and \( \ell(G) \) for the number of labelings of \( G \). As usual we denote the number of points of \( G \) by \( p \).

The notation used in the following proof follows that in Harary [in press] and Harary and Palmer [1965]. Accordingly, \( S_p \) is the symmetric group of degree \( p \) acting on \( X = \{1, 2, \cdots, p\} \); \( X^{(2)} \) is the set of unordered pairs of the objects in \( X \); \( S_p^{(2)} \) is the pair group acting on \( X^{(2)} \) as induced by \( S_p \); and \( E_2 \) is the identity group on \( Y = \{0, 1\} \). The power group (introduced in Harary and Palmer [1965]) \( E_2^{\times^*} \) acts on \( Y^{X^{(2)}} \) and each function \( f \) from \( X^{(2)} \) into \( Y \) represents a labeled graph with point set \( X \). Two points \( i, j \in X \) are considered adjacent in the graph of \( f \) whenever \( f([i, j]) = 1 \).

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In order to present this proof concisely, we assume the basic properties of permutation groups $A$ acting on $X$. These include the "stabilizer" of an object $x \in X$ (the subgroup of $A$ which fixes $x$), the "orbit" of $A$ which contains $x$ (the set of all objects to which $x$ can be mapped by permutations in $A$), and the "index" of a subgroup $B$ of $A$ (the ratio of the order of $A$ to that of $B$). We also recall the well known result:

Lemma. The index in the group $A$ of the stabilizer $A_x$ of an object $x \in X$ is the number of objects in the orbit of $A$ which contains $x$.

The theorem is stated for graphs, but is easily modified to apply to any type of structure, e.g., trees, directed graphs, tournaments, relations, 1-choice structures (functional digraphs), and nets.

**Theorem.** The number of different ways in which the points of $G$ can be labeled is:

$$I(G) = \frac{p!}{s(G)}.$$

**Proof.** Since the theorem is obvious for $p = 1, 2$, we assume $p \geq 3$.

Now let $G$ be the unlabeled graph on $p$ points which corresponds to the function $f$ mentioned above. It is clear that the number of ways in which $G$ can be labeled is simply the number of functions in the orbit of $f$ regarded as an element in the object set of the power group $E_2^{S_p}$. Furthermore, the stabilizer of $f$ in $E_2^{S_p}$ is obviously isomorphic to $\Gamma(G)$. Applying the lemma to this power group, we have the result that the number of ways of labeling $G$ is the order of $E_2^{S_p}$ divided by the order of $\Gamma(G)$, i.e., the index of $\Gamma(G)$ regarded as a subgroup of the power group. The proof is completed by observing that the order of this power group is $p!$ when $p \geq 3$.

**REFERENCES**


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