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Achievement and Avoidance Games for Generating Abelian Groups

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Abstract: For any finite group G, the DO GENERATE game is played by two players Alpha and Beta as follows. Alpha moves first and chooses $x_1 \in G$. The k-th play consists of a choice of $x_k \in G - S_{k-1}$ where $S_n = \{x_1, \ldots, x_n\}$. Let $G_n = \langle S_n \rangle$. The game ends when $G_n = G$. The player who moves x_n wins. In the corresponding avoidance game, DON'T GENERATE, the last player to move loses. Of course neither game can end in a draw. For an arbitrary group, it is an unsolved problem to determine whether Alpha or Beta wins either game. However these two questions are answered here for abelian groups.

1 Introduction

These are examples of 2-person games with perfect information in which one of the players has a strategy which guarantees victory regardless of the moves made by his opponent. The strategies derived below, dealing with the theory of groups, constitute a class of combinatorial games involving finite sets and are a part of discrete mathematics. Our games, being derived from a theorem, cannot end in a draw. The theorem involved here is trivial, but the games are not. Here the "theorem" is simply that a group G generates itself. The referee, to whom we are grateful for his helpful comments, kindly noted that "The games ... of the paper are new in the game theoretic literature".

In the graph game, DO CONNECT, the two players Ms. Alpha and Mr. Beta begin with n isolated vertices and Alpha makes the first move by joining some vertex pair by an edge. This has been called [3] a "shrewd move" as the choice of two ver-

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tices to join has absolutely no effect on the resulting configuration and hence has no impact on the outcome of the game. Beta then joints a different pair of vertices, and so on. The player who first forms a connected graph on all n vertices wins. In the corresponding avoidance game, DON'T CONNECT, this player loses. The outcome of these two games was determined in [4] by specifying the values of n for which each player wins with rational moves.

These games on graphs suggest corresponding games on groups in which the object is to generate the group as the analog to connecting the graph. Given a finite group G, Alpha picks $x_1 \in G$, Beta chooses x_2, and so forth. Let set $S_n = \{x_1, \ldots, x_n\}$ and let subgroup $G_n = \langle S_n \rangle$. The game ends when $G_n = G$. In the DO GENERATE game, the player who moves x_n wins; in DON'T GENERATE he or she loses.

Our object is to determine the winner in each of these two games when G is abelian.

2 Do Generate

We first consider the achievement game, DO GENERATE, by handling six separate cases. Let 1 be the identity element. We use the well known fact that G is the direct sum of cyclic groups Z_n of prime power order n. We write $G = G'$ to indicate group isomorphism.

Case 1: G is cyclic.

Here *Alpha obviously wins* because she can choose a generator for x_1.

Case 2: G is not cyclic but for all $x \neq 1$ in G, there exists $y \neq x$ such that $\langle x, y \rangle = G$.

This means that for each $x \neq 1$, $G/\langle x \rangle$ is cyclic, that is, $G = Z_p + Z_p$ for some prime p. Therefore by choosing $x_1 = 1$, clearly *Alpha wins.*

Case 3: Neither Case 1 or Case 2 holds and $|G|$ is odd.

Since Case 1 does not hold, Alpha cannot win on the first move. Since Case 2 does hold, Alpha has a safe move so $G \neq G_2$. Now $|G_2|$ is odd as it divides $|G|$, but the number of choices already made after the first move by Beta is two which is even. So Alpha now wins if she can; otherwise she chooses $x_3 \in G_2$ making it impossible for Beta to win on the fourth move, x_4. This parity argument applies at every stage and hence *Alpha wins.*

Case 4: None of Cases 1, 2, 3 hold, and for all x of even order $G/\langle x \rangle$ is cyclic.

We prove that here $G = Z_2 + Z_{2n}$ for n odd. Consider $x \in G$ of order 2. Because $G/\langle x \rangle$ is cyclic, $G = Z_2 + Z_k$. Now if k is odd then G itself would be the cyclic group Z_{2k} contrary to the hypothesis of this case; thus $k = 2n$. As Z_{2n} has a unique element y of order two, let $H = \langle y \rangle$; then $G/H = Z_2 + Z_n$. Since this is a cyclic group, n must be odd.

We can now determine the outcome of the achievement game in this case. Clearly the first player to choose an element of even order loses. Since there are an odd number n of elements of odd order, including the identity, *Alpha must win.*

Case 5: Cases 1, 2, 3, 4 do not hold and there exists $x \neq 1$ of odd order such that $G/\langle x \rangle$ is not cyclic, but for all $b \in G$ of order 2, $G/\langle b, x \rangle$ is cyclic.

We prove that in this case $G = Z_2 + Z_k + Z_r$ where $k, r > 1$ are odd. Let r be the order of x and choose any b of order 2. Then for some y, $G = \langle b \rangle + \langle x \rangle + \langle y \rangle$ since $G/\langle b, x \rangle$ is cyclic. If y has odd order m, then $G/\langle x \rangle = Z_{2m}$, a cyclic group. Thus y has even order $2k$, so $G/\langle x, y^k \rangle = \langle b \rangle + Z_k$. Since this group is cyclic, k must be odd. This implies that $G = Z_2 + Z_k + Z_r$ since Case 4 does not hold.

We can now determine the outcome of this case. Alpha chooses $x_1 = x$. Clearly the first player to choose an element of even order loses. Since there is an odd number kr of elements of odd order, *Alpha wins*.

Case 6: None of cases 1–5 holds.

There are two possibilities. In the first instance, Alpha chooses an element x_1 of even order. If Beta cannot win at this point, then since x_1 is a subgroup of even order, Beta can choose $x_2 \in \langle x_1 \rangle$. This parity argument applies at each stage and so Beta wins.

The other alternative is that Alpha chooses an element x_1 of odd order where $G/\langle x_1 \rangle$ is not cyclic so that Beta cannot win on the next move. Since Case 5 does not hold, Beta can find x_2 of order 2, so that $G/\langle x_1, x_2 \rangle$ is not cyclic. But now Alpha cannot win on the third move, x_3, and since each succeeding subgroup G_k has even order, Beta can always choose some $x_k \in G_{k-1}$ if he cannot find an x_k which wins. Thus Beta wins with this possibility also, so *Beta wins* in Case 6.

Summary: In the finite abelian group achievement game, Alpha wins if any of Case 1–5 holds and Beta wins otherwise. This is now stated formally.

Theorem 1: For the DO GENERATE game played on a finite abelian group G, Alpha wins if and only if G is cyclic or $|G|$ is odd or

$$G = Z_2 + Z_{2m+1} + Z_{4k+2}, \quad \text{for } m,\ k = 0, 1, 2, \ldots;$$

Beta wins otherwise.

3 Don't Generate

Perhaps surprisingly the analysis of this game is easier. It can be handled in just four cases.

Case 1: G is the trivial group.

Alpha must pick $x_1 = 1 = G$ and obviously loses. This has been called [1] a "Banker's Game" for Beta; clearly it is a game which the polite Banker always wins by saying, "After you". *Beta wins.*

Case 2: G has odd order but is not the trivial group.

Alpha chooses x_1 which does not generate G, picking $x_1 = 1$ if necessary. At each stage x_k, Beta either generates and loses, or else G_k is a subgroup of odd order generated by an even number of choices. If every choice outside of G_k (from $G - G_k$) then generates, Alpha has room to choose inside G. If not, Alpha makes any choice she pleases which does not generate. To speed up the game she will choose outside G_k if possible. Thus *Alpha wins*.

Case 3: $G = Z_{2 \cdot k}$, k odd.

Alpha chooses some element x_1 of order k. Then Beta must choose an element x_1 of order k. Thus Beta must choose an element x_2 of $\langle x_1 \rangle$ or he will generate G. But this group $\langle x_1 \rangle = Z_k$ is of odd order so Alpha has the last choice in $\langle x_1 \rangle$ and hence *Alpha wins*.

Case 4: G is of even order but $G \neq Z_{2k}$ for k odd.

If Alpha chooses an element of even order, the same sort of parity argument as in Case 3 implies that Beta wins.

If Alpha picks x_1 of odd order, then Beta can find an element x_2 of even order so that $G_2 \neq G$ because of the hypothesis that $G \neq Z_{2k}$ for k odd. But then the order of every succeeding G_n is even, and the same parity argument again implies that *Beta wins.*

Theorem 2: For the DON'T GENERATE game played on a finite group G, Alpha wins if and only if G is of odd order but is nontrivial, or $G = Z_{2k}$ with k odd; Beta wins otherwise.

4 Unsolved Problems

A. Determine the outcome of these achievement and avoidance games for the standard families of nonabelian groups including D_n (the dihedral group), A_n (the alternating group), and S_n (the symmetric group). At present it appears quite hopeless to settle these questions for arbitrary finite groups.

B. There is a well known observation that for any finite group G and for each partition $G = X_1 \cup X_2$, at least one X_i generates G. As mentioned in [2], this suggests the "2-color" game in which Alpha builds $\{x_1, x_3, \ldots\}$ and Beta gets $\{x_2, x_4, \ldots\}$. Then in this achievement game the first player who generates G wins; in the avoidance game he loses. The outcome of these games has not yet been determined for any family of groups.

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