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Pricing of point-to-point bandwidth contracts*

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Abstract. In this paper we consider the pricing of point-to-point bandwidth leasing contracts and options. The underlying asset of these contracts is a point-to-point telecommunications connection. Due to the network structure the network capacity prices depend nonlinearly on each other. A leasing contract on a point-to-point connection can be seen as an option because the seller of the connection selects the cheapest path between the points. Therefore, a bandwidth option is a compound option.

Key words: Bandwidth, Network, Leasing contract, Option pricing, Real option

1. Introduction

Term bandwidth corresponds to the amount of data transferred on a given transmission path within a specified block of time. Thus, bandwidth is a synonym for telecommunications capacity, and it is measured in units of bits per second (bps). For example, downloading a picture in one second requires more bandwidth than downloading a text page in one second. Large sound files, computer programs, and videos require even more bandwidth for acceptable system performance.

Managed bandwidth services are connections between two or multiple points on a certain capacity. The bandwidth seller and buyer define the start

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date and end date of the service contract as well as the quality of service. Quality can be measured by permissible errored seconds, severely errored seconds, and unavailable seconds over time [see Mayfield (2000)]. At the start date the buyer buys the connection from the seller at the price specified in the service contract, and after the end date the owner can sell the connection to the next customer. Thus, the bandwidth service contracts are actually forwards [for forward contracts see e.g. Hull (1997)]. Capacity level, connection's start and end points, and the quality of service specify the underlying asset of the forward. The start and end dates define the maturity and duration of the forward.

It is estimated that about $25 billion worth of telecommunications capacity is bought and sold worldwide annually [see Ryan (2000)]. According to Mayfield (2000) there are about 7500 market participants in US alone and roughly double the number of that internationally. Fletcher and DiClemente (2001) have grouped the bandwidth buyers into five categories: 1) large existing carriers, 2) emerging telcos, 3) service providers, e.g. ISPs, 4) dotcoms, which operate through the Internet, and 5) enterprise customers, which need huge amounts of bandwidth in their everyday operations. In the bandwidth market there are also capacity providers of several magnitudes.

The pricing of bandwidth contingent claims is similar to the pricing of corresponding electricity instruments in the sense that both these commodities are held for consumption and they cannot be stored [see e.g. Kenyon and Cheliotis (2001) for a discussion about the properties of bandwidth]. Therefore, the bandwidth contracts cannot be hedged by using the underlying asset like usual financial derivative instruments and arbitrage argument is possible only between different bandwidth contracts. However, in the present paper we do not use this kind of hedging argument because the bandwidth markets are currently illiquid. In the bandwidth markets there is a new arbitrage condition due to the structure of the network and to the market's optimal point-to-point routing selection. The buyer of a bandwidth is buying the point-to-point connection independent of the routing. Thus, the seller can provide an alternative routing in case the direct routing between the points costs more at the delivery time. Therefore, the risk of the seller is in a way bounded above and the seller's optimal routing selection leads to the network arbitrage condition, also called geographical arbitrage condition [see, e.g., Chiu and Crametz (1999, 2000) and Upton (2002)]. This condition implies that at each time a point-to-point capacity market price has to be equal to the minimum capacity price over all possible routings between these two points. That is, the bandwidth market prices are obtained by using the cheapest paths between the start and end points. This new arbitrage condition has to be considered in the pricing of bandwidth contracts and, therefore, the pricing is partly different from the pricing of corresponding electricity instruments.

The purpose of this paper is to calculate point-to-point forward (leasing contracts) and option prices under the network arbitrage condition. Note that in the bandwidth markets there are also other special characteristics [see e.g. Kenyon and Cheliotis (2001), Crametz (1999, 2000), and Upton (2002)] but here we focus on the network arbitrage condition. We firstly model the network prices without the network arbitrage condition, i.e., the initial situation is close to the electricity market. Therefore, we assume similar stochastic processes for the fixed routing prices that are used in several electricity papers [see e.g. Deng, Johnson, and Sogomonian (2001), Keppo and Räsänen (1999, 2000), Manoliu and Tompaidis (2000), and Oren (2001)]. Because the underlying bandwidth is non-storable, the forward prices are functions of the bandwidth market price of risk. Thus, unlike in a storable commodity derivative market the risk attitude of bandwidth market's participants affects the forward prices. We assume that all the bandwidth instrument prices are given by the same general pricing function. This implies that the bandwidth instruments are in equilibrium with each other and they are priced with respect to the same market price of risk, i.e., with respect to the same utility function. The risk pricing is important in bandwidth markets because the bandwidth prices are highly volatile [see e.g. Mayfield (2000)], according to Cass (2000) 20–40%. In the pricing function we model the network arbitrage condition as a feature of the forward contracts. That is, the bandwidth forward prices are solved by using the processes of fixed (straight) routing point-to-point prices and the option nature of the cheapest path selection. Fixed routing bandwidth prices are used in the pricing models because their process parameters can be assumed to be constant and, therefore, they act as the risk factors of the bandwidth market. Further, this way we can better understand the effect of the network arbitrage condition on the bandwidth market prices. Arte and Keppo (2003) illustrate parameter estimation for the present paper's forward model and show how the framework is used in practice. Due to the cheapest path selection a bandwidth forward price is equal to the forward price with fixed routing minus the option to change the point-to-point routing. This routing option is a kind of exchange option [for the pricing of exchange options see e.g. Margrabe (1978)] and by using the pricing models of exchange options we can derive analytical pricing formulas for bandwidth forwards. Therefore, we can also derive analytical formulas for the nonlinear dependencies between the market prices. According to the model, the network arbitrage condition increases positive correlations between the capacity market prices, and the probability distributions of the market prices have short upper tails. This is natural since network routing smoothens the point-to-point demands over the whole network and therefore prices move together [for routing see, e.g., Gune and Keppo (2002)].

In addition to the cheapest path selection, some bandwidth service contracts include also other option type characteristics for the parties of the contracts. For instance, the seller can have a right to disconnect the service for a predefined penalty payment. These rights can be modeled as bandwidth options and, therefore, the understanding of option pricing is important in the bandwidth markets even though there do not yet exist traded option contracts. Further, this helps the application of real option theory in telecommunications markets [see e.g. Alleman and Noam (1999)]. Bandwidth options can be modeled as options on bandwidth forwards, because at maturity the forward price equals the underlying connection price. This gives that the bandwidth options are a kind of compound options [for compound options see Geske (1979) and Rubinstein (1992)] since the underlying bandwidth forward contracts can be seen as exchange options. The option prices are described in terms of bivariate and trivariate normal distributions because we have to integrate the forward price function, which now includes cumulative normal distributions due to the routing option, over the underlying point-to-point price distribution and also over the alternative routing's price distribution. This ends up with the pricing equation that includes the trivariate and bivariate probability distributions. Because the probability distribution of the underlying forward price has a short upper tail, the bandwidth call option is cheaper than the corresponding option price implied by the Black-76 model [see Black (1976)] that is a usual commodity option pricing formula.

Bandwidth pricing is considered, e.g., in Kenyon and Cheliotis (2001) that considers a jump-diffusion model for bandwidth spot prices. Present paper uses only continuous uncertainties but can be extended, e.g., to the jump processes by using transformation analysis [for the transformation analysis see e.g. Duffie, Pan, and Singleton (2000)]. The possible drawback of our continuous uncertainty assumption is a reduced accuracy. However, the advantage is that we are able to obtain analytical pricing formulas that are easily implemented to everyday industry practice [see Arte and Keppo (2003)]. These analytical results enable the pricing and analyzing of huge portfolios within a short time period and this is important in practice since telcos' portfolios may include thousands of different instruments. There are several other papers that have considered bandwidth pricing. Rasmusson (2002) presents a new design for bandwidth markets. Reiman and Sweldens (2001) analyze the calculation of network arbitrage free bandwidth forward prices. Upton (2002) studies methods for pricing and agent behavior in bandwidth markets by using financial models and stochastic control. Gupta, Kalyanaraman, and Zhang (2003) calculate bandwidth spot prices by using a nonlinear pricing method. Courcoubetis, Kelly, Stamoulis, and Manolakis (1998) analyze bandwidth allocation and pricing model, where each user is assumed to select his/her willingness-to-pay so as to maximize his/her net benefit, i.e., the difference of the utility induced to the user by the quality of service (QoS) received minus his/her willingness-to-pay. In the present paper we do not consider QoS and assume that the QoS is the same for all the point-to-point connections. Keon and Anandalingam (2003) consider pricing of multiple services with guaranteed QoS levels and a single telecommunications network. Courcoubetis, Kelly, Siris, and Weber (2000) study simply usage-based charging schemes for bandwidth networks. Songhurst and Kelly (1997) consider the issues of network interaction that are inherent in appropriate usage-sensitive charging schemes. The stability and fairness of rate control algorithms for communication networks are studied in Kelly, Maulloo, and Tan (1998). Courcoubetis, Dimakis, and Reiman (2001) analyze pricing in a best effort network and they also consider option pricing. Our option pricing model can be seen as an extension to that since we add the network arbitrage condition to their framework. MacKie-Mason and Varian (1995a, 1995b) and Paschalidis and Tsitsiklis (2000) study congestion dependent pricing. Odlyzko (2001) discusses Internet pricing and the history of telecommunications. The financial option pricing theory under continuous uncertainties is developed, e.g., in Black and Scholes (1973), Merton (1973), and Black (1976), and the hedging of financial instruments is considered in many papers [see e.g. Bertsimas, Kogan, and Lo (2001), Duffie and Jackson (1990), Lioui and Poncet (1996), and Keppo and Peura (1999)]. General financial contingent claims pricing theory is derived in Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981), and Cox and Huang (1986). Utility function based financial asset pricing is considered, e.g., in Davis (1998, 2001) and Cochrane (2002). Our pricing function for bandwidth contracts can be viewed as an example of this kind of utility approach.

To sum up, the main objectives of our model are the following:

- Analytical pricing models for forwards and options under the network arbitrage condition.
- Nonlinear dependencies of bandwidth prices due to the network structure.

The rest of the paper is divided as follows: Section 2 introduces the bandwidth price models used in the paper. The stochastic processes for the capacity prices are defined and the processes are then applied to the forward pricing problem in Section 3. Section 4 uses the forward prices in the pricing of bandwidth options. Section 5 illustrates the derived models with numerical examples and finally Section 6 concludes.

2. Model

For simplicity, we consider a network of three point-to-point capacity prices. All the point-to-point connections have the same capacity level and quality. For example, the points are New York, Los Angeles, and Atlanta, and the capacity is OC-3 (155.52 Mbps). The extension to more general networks is discussed in Arte and Keppo (2003). Figure 1 illustrates the network structure.

The S-prices of Figure 1 are the bandwidth prices by using the capacity between the points and without the network structure, i.e., the routing is fixed. For instance, S_1 is the bandwidth price between the up and left points by using the direct routing between these two points. Price S_1 does not necessary equal the market price between the up and left points because if we have $S_2 + S_3 \leq S_1$ then the market uses the longer routing and, therefore, the market price equals $S_2 + S_3$. This cheapest path selection is called network arbitrage condition [see e.g. Upton (2002)], i.e., it can be seen as the optimal behavior of telecommunications companies.

Let us denote the market spot prices by Y_1, Y_2, and Y_3. Then we have

$$\begin{aligned} Y_1(t) &= \min[S_1(t), S_2(t) + S_3(t)] \\ Y_2(t) &= \min[S_2(t), S_1(t) + S_3(t)] \text{ for all} \in [0, \tau]. \\ Y_3(t) &= \min[S_3(t), S_1(t) + S_2(t)] \end{aligned} \tag{2.1}$$

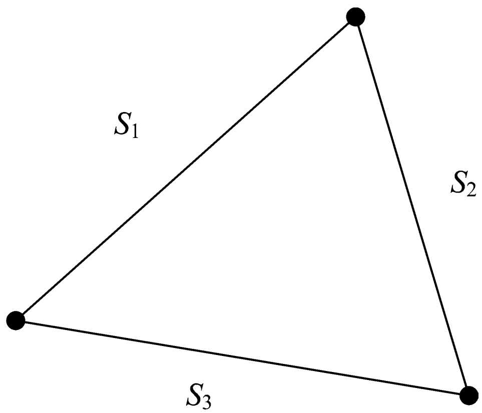

Fig. 1. Network prices

According to (2.1) spot prices are free of network arbitrage ($Y_2 + Y_3 \geq Y_1$, $Y_1 + Y_3 \geq Y_2$, $Y_1 + Y_2 \geq Y_3$). Currently, in the bandwidth commodity market there does not exist pure spot prices and all the instruments in the market are derivative instruments. Therefore, in practice the spot price refers to the forward price (leasing contract price) with shortest maturity and duration. At present, this means one-month forward price that starts after two months. From now on, we assume that the spot prices can be approximated with the forwards of shortest maturity and duration.

From (2.1) we see that even though S_1, S_2, and S_3 were independent Y_1, Y_2, and Y_3 would not be, because the market prices depend on the point-to-point routings. Also from (2.1) we see that the probability distribution of future Y_{i} is different from S_{i} for all $i \in \{1,2,3\}$, because $Y_i(t) \leq \sum_{j\in\{1,2,3\}-\{i\}} S_j(t)$ and, therefore, the upper tail of Y_{i}'s probability distribution is shorter than the S_{i}'s corresponding tail. However, all the uncertainties in the market spot prices are from the S-processes and, therefore, these processes are the risk factors of the bandwidth market. If we had more complex network structure then in equation (2.1) we would have the minimum over all possible routings.

We consider a finite time horizon $[0, \tau]$. In describing the probabilistic structure of the market, we will refer to an underlying probability space (Ω, F, P). Here Ω is a set, F is a σ-algebra of subsets of Ω, and P is a probability measure on F. We assume that $E[|S_i(t)|] < \infty$ for all $t \in [0,\tau]$ and $i \in \{1,2,3\}$ so that we can calculate the expected values and we denote the conditional expected value of the fixed routing price S_i by $S_i(t,T) = E[S_i(T)|F_t]$ for all $t \in [0,T]$ and $T \in [0,\tau]$. We model these expected prices and the following assumption characterizes their stochastic process.

Assumption 2.1 *The process of the expected fixed routing price $S_i(t,T)$ is given by the following Itô stochastic differential equation*

$$dS_i(t,T) = S_i(t,T)\sigma_i dB_i(t) \quad \text{for all} \quad t \in [0,T], i \in \{1,2,3\}, \tag{2.2}$$

where $S_i(t,T) = E[S_i(T)|F_t], S_i(\cdot,T) : [0,T] \to \mathbf{R}_+, T \in [0,\tau], \sigma_i$ is bounded and constant, and $B_{\mathrm{i}}(\cdot)$ is a standard Brownian motion corresponding to the link i on the probability space (Ω, F, P) along with the standard filtration $\{F_t : t \in [0,\tau]\}$. We will denote by $\rho_{\mathrm{i,j}}$ the correlation between the i'th and j'th Brownian motions.

According to equation (2.2) the stochastic processes for the expected fixed routing prices follow geometric Brownian motion processes where $S_i(t,T)^2\sigma_i^2$ is the rate of change of the conditional variance of $S_{\mathrm{i}}(t,T)$. The boundedness of the volatility parameter σ_{i} guarantees the existence and uniqueness of the solution to (2.2). The solution to (2.2) can be written as

$$S_i(T,T) = S_i(T) = S_i(t,T)\exp\left(-\frac{1}{2}\sigma_i^2(T-t) + \sigma_i[B_i(T) - B_i(t)]\right).$$

The extension to time dependent and deterministic volatility is straightforward since we can set σ_i^2 to equal the average of volatility square over the lifetime of the bandwidth contract under consideration. For instance, in the calculating of $S_i(T)$'s distribution with time dependent volatility $\sigma_i(t,T)$ we use $\sigma_i = \sqrt{\frac{1}{T-t}\int_t^T \sigma_i(y,T)^2 dy}$. Because in Assumption 2.1 we model the expected values, the process of the fixed routing price $S_i(t)$ can be e.g. geometric Brownian motion or mean-reverting, and we can use, for instance, the electricity price models presented in Deng, Johnson, and Sogomonian (2001), Keppo and Räsänen (1999, 2000), Manoliu and Tompaidis (2000), and Oren (2001). Thus, equation (2.2) has already been used with a non-storable underlying asset. Also note that there can be cycles in the expected fixed routing prices. For instance, in our model we can have $S_i(0,1) = 100$ and $S_i(0,1.1) = 1$

The continuous-time process in (2.2) can also be seen as a limit of the corresponding discrete-time process. Since prices are always positive let us model log-expected prices and define

$$D_i^m(k) = \log(S_i(k\tfrac{T}{m},T)) - \log(S_i((k-1)\tfrac{T}{m},T))$$
$$\text{for all } k \in \{1,\ldots,m\}, \quad m \in \{1,2\ldots\},$$

where m is the number of discrete time intervals on $[0,T]$ and k is the index for discrete times. This gives $\log\big(S_i^m(T)\big) = \log(S_i(0,T)) + \sum_{k=1}^m D_i^m(k)$. We assume that $\{D_i^m(k)\}$ are mutually independent and identically distributed random variables with mean $\beta_i \frac{T}{m}$ and standard deviation $\sigma_i\sqrt{\frac{T}{m}}$. Then we get from the central limit theorem [see e.g. Billingsley (1995)] $\lim_{m\to\infty} \sum_{k=1}^{m} \big(D_i^m(k) - \beta_i \frac{T}{m}\big) = \sigma_i B_i(T)$, i.e.,

$$\lim_{m\to\infty} \log\big(S_i^m(T)\big) = \log(S_i(0,T)) + \beta_i T + \sigma_i B_i(T),$$

where according to (2.2) $\beta_i = -\frac{1}{2}\sigma_i^2$ and therefore the limit of our discrete-time model is given by (2.2). Thus, if we assume that in discrete-time the differences of the log-expected prices are independent and identically distributed then if we speed up the arrivals of the discrete-time events we get our approximation, Assumption 2.1.

Because there are three risk factors they cause at most three independent sources of uncertainty in the market. That is, all the uncertainties in the market prices are driven by the risk factors and the correlation structure between the risk factors is free in our model. If the market capacity supply is constant then the uncertainties in the S-prices are generated from the capacity demand processes.

In order to get analytical formulas for bandwidth point-to-point contracts, we will approximate the sum of two geometric Brownian motions with a geometric Brownian motion. Thus, we make the following assumption on the expected alternative routing prices and the same assumption is done in the pricing of financial derivatives.

Assumption 2.2 *The expected alternative routing prices are given by the following Itô stochastic differential equations*

$$
\begin{aligned}
dX_1(t,T) &= dS_2(t,T)+dS_3(t,T) = X_1(t,T)\left[\omega_{1,2}\sigma_2 dB_2(t)+\omega_{1,3}\sigma_3 dB_3(t)\right] \\
dX_2(t,T) &= dS_1(t,T)+dS_3(tT) = X_2(t,T)\left[\omega_{2,1}\sigma_1 dB_1(t)+\omega_{2,3}\sigma_3 dB_3(t)\right] \\
dX_3(t,T) &= dS_1(t,T)+dS_2(t,T) = X_3(t,T)\left[\omega_{3,1}\sigma_1 dB_1(t)+\omega_{3,2}\sigma_2 dB_2(t)\right]
\end{aligned}
\tag{2.3}
$$

where $X_i(t,T) = \sum_{k\in\{1,2,3\}-\{i\}} S_k(t,T)$ and $\omega_{i,j} = \frac{S_j(t,T)}{X_i(t,T)}$ *is constant for all* $i,j \in \{1,2,3\}$.

In many practical situations, equation (2.3) is accurate enough and, therefore, the same method is used, e.g., in the pricing of basket options [see Gentle (1993)]. The correct processes of $X_1(t,T) = S_2(t,T) + S_3(t,T)$, $X_2(t,T) = S_1(t,T) + S_3(t,T)$, and $X_3(t,T) = S_1(t,T) + S_2(t,T)$ are given by

$$
\begin{aligned}
dX_1(t,T) &= dS_2(t,T) + dS_3(t,T) = S_2(t,T)\sigma_2 dB_2(t) + S_3(t,T)\sigma_3 dB_3(t) \\
dX_2(t,T) &= dS_1(t,T) + dS_3(tT) = S_1(t,T)\sigma_1 dB_1(t) + S_3(t,T)\sigma_3 dB_3(t) \\
dX_3(t,T) &= dS_1(t,T) + dS_2(t,T) = S_1(t,T)\sigma_1 dB_1(t) + S_2(t,T)\sigma_2 dB_2(t)
\end{aligned}
$$

and, therefore, combining this with equation (2.3) we get $\omega_{i,j} = \frac{S_j(t,T)}{X_i(t,T)}$. In order to justify more the X-process of Assumption 2.2, Appendix 1 analyzes the approximation error by using two first moments and compares equation (2.3) with Monte Carlo Simulation. Note that equation (2.3) is for the expected values $\sum_{k\in\{1,2,3\}-\{i\}} S_k(t,T)$ and not for $\sum_{k\in\{1,2,3\}-\{i\}} S_k(t)$.

We consider a market where telecommunications capacity and bandwidth instruments are bought and sold continuously. Because of the new exchanges (for instance Band-X and InterXion) and OTC-market this kind of international market already exists, but the market is not liquid and is in an early stage. Therefore, we assume the following general pricing function for bandwidth instruments and it is not based on continuous time hedging.

Assumption 2.3 *The price of T-maturity bandwidth contingent claim is given by*

$$
\pi(t,T) = \exp(-r(T-t))E[\phi(T)|F_t] \quad \text{for all} \quad t \in [0,T], \quad T \in [0,\tau],
\tag{2.4}
$$

where r *is a constant discount rate,* $\phi(T)$ *is the payoff at time* T*, and the expectation is with respect to probability measure* P*.*

Assumption 2.3 implies that the agents in the market price the bandwidth instruments by using the discounted expected payoff formula. The discount rate and the probability measure in equation (2.4) may depend on the agent's utility function. However, for simplicity we assume that the probability measure is equal to the objective measure P and the discount rate is constant. Note that if the market was liquid, arbitrage free, and complete then the discount rate in (2.4) would be the risk-free rate and the expectation would be under the risk-neutral pricing measure Q [see e.g. Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981), and Björk (1998)]. However, since in this paper we do not assume this kind of market we use directly

Assumption 2.3. The payoff $\phi(T)$ in equation (2.4) can be viewed as a money metric utility and, therefore, Assumption 2.3 can also be seen as a utility based pricing approach. For the general utility based pricing models see, e.g., Davis (1998, 2001) and Cochrane (2002) and for applications in telecommunications see, e.g., Stoenescu and Taneketzis (2002) and Courcoubetis, Kelly, Siris, and Weber (2000).

In bandwidth markets there are leasing contracts traded between telecommunications companies. As mentioned earlier, these leasing contracts can be modeled as forward contracts since they oblige the buyer of the contract to acquire the underlying bandwidth connection at a certain future date (maturity) for a certain payment (forward price). Therefore, in this paper we call these contracts as forwards. When the forward contract is agreed upon, no payments are made. Instead, at maturity the seller of the contract receives the forward price from the buyer. Thus, if the claim in Assumption 2.3 is a forward contract then we get from equation (2.4) that the T-maturity forward price at time t with instantaneous duration is given by

$$Y_i(t,T) = E[Y_i(T)|F_t] \tag{2.5}$$

because, as mentioned above, in this case by definition of the forward contract $\phi(T) = Y_i(T) - Y_i(t,T)$ and $\pi(t,T) = 0$, where $Y_i(T)$ is the corresponding bandwidth spot price at time T. Note that according to Assumption 2.3 the forward price with duration D can be calculated as follows

$$Y_i(t,T,D) = \frac{-r\int_T^{T+D} \exp(-r(y-t))Y_i(t,y)dy}{\exp(-r(T+D-t)) - \exp(-r(T-t))}, \tag{2.6}$$

where the contract's start date is T and the end date is $T+D$.

3. Forward pricing

From now on we only consider the pricing of contracts on Y_1 and the contract prices on Y_2 and Y_3 can be derived in the same way. Unlike in other commodity markets, the bandwidth forward contract includes optionality. That is, the seller can provide either the direct routing or the alternative one. According to Section 2, we assume that the expectation hypothesis for bandwidth forward prices holds under P and, therefore, the forward prices are equal to the expected spot prices:

$$Y_1(t,T) = E(\min[S_1(T), X_1(T)]|F_t) \quad \text{for all} \quad t \in [0,T], \quad T \in [0,\tau], \tag{3.1}$$

where $Y_1(t,T)$ is the T-maturity forward price at time t and $X_1(t) = S_2(t) + S_3(t)$.

Equation (3.1) implies that the forward contracts are a kind of combined options where the forward price is equal to the minimum value of an asset and a portfolio. Note that (3.1) is independent of the stochastic processes of S_1 and X_1. Further, because the spot prices [equation (2.1)] are free of network arbitrage, according to (3.1) also the forward prices satisfy the no-arbitrage condition. Equation (3.1) can also be written in the following way

$$Y_1(t,T) = E(S_1(T)|F_t) - E(\max[0, S_1(T) - X_1(T)]|F_t) \tag{3.2}$$

Thus, the market forward price is equal to the price with fixed routing minus the option to change the point-to-point routing. This option is due to the network arbitrage condition and it is a kind of exchange option [for the pricing of exchange options see e.g. Margrabe (1978)].

By allowing T vary from t to τ we get the whole bandwidth forward curve $Y_1(t,\cdot):[t,\tau]\to\mathbf{R}_+$ at time t. Using the pricing formula for exchange options and the martingale measure for X_1 we get the following proposition.

Proposition 3.1 *The forward price is given by*

$$Y_1(t,T)=S_1(t,T)-H(t,T,S_1,X_1)\quad\text{for all}\quad t\in[0,T],\quad T\in[0,\tau],\tag{3.3}$$

where the exchange option

$$H(t,T,S_1,X_1)=S_1(t,T)N(z_t)-X_1(t,T)N\left(z_t-\sigma\sqrt{T-t}\right),\tag{3.4}$$

and

$$z_t=\frac{\ln\left(\frac{S_1(t,T)}{X_1(t,T)}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$\sigma=\sqrt{\sigma_1^2+\omega_{1,2}^2\sigma_2^2+\omega_{1,3}^2\sigma_3^2+2\omega_{1,2}\omega_{1,3}\rho_{2,3}\sigma_2\sigma_3-2\sigma_1\left(\omega_{1,2}\rho_{1,2}\sigma_2+\omega_{1,3}\rho_{1,3}\sigma_3\right)}$$

and $N(\cdot)$ *is the cumulative standard normal distribution.*

Proof. From equation (3.2) we get

$$\begin{aligned}H(t,T,S_1,X_1)&=E\left(X_1(T)\max\left[\frac{S_1(T)}{X_1(T)}-1,0\right]|F_t\right)\\&=X_1(t,T)E\left(\frac{X_1(T)}{X_1(t,T)}\max\left[\frac{S_1(T)}{X_1(T)}-1,0\right]|F_t\right)\\&=X_1(t,T)E^{Q_1}\left(\max\left[\frac{S_1(T,T)}{X_1(T,T)}-1,0\right]|F_t\right),\end{aligned}\tag{3.5}$$

where $S_1(t,T)=E[S_1(T)|F_t]$, $X_1(t,T)=E[X_1(T)|F_t]$, $E^{Q_1}(\cdot)$ in the last line of (3.5) is the expectation with respect to the martingale measure Q_1 for the numeraire process $X_1(\cdot,T)$ [see e.g. Geman (1989) and Björk (1998, chapter 19)], and the Radon-Nikodym derivative is given by

$$\frac{dQ_1}{dP}=\frac{X_1(T,T)}{X_1(t,T)}\quad\text{on } F_T.$$

Equation (3.5) implies that $H(t,T,S_1,X_1)$ is equal to $X_1(t,T)$ numbers of Black-Scholes type T-maturity call options on $\frac{S_1(t,T)}{X_1(t,T)}$ with zero discount rate and unit strike price. The process of $\frac{S_1(t,T)}{X_1(t,T)}$ is a martingale under the martingale measure Q_1. Therefore, it has zero expected change and the process is given by [see e.g. Björk (1998, chapter 19)]

$$d\left[\frac{S_1(t,T)}{X_1(t,T)}\right]=\frac{S_1(t,T)}{X_1(t,T)}\left[\sigma_1d\hat{B}_1(t)-\omega_{1,2}\sigma_2d\hat{B}_2(t)-\omega_{1,3}\sigma_3d\hat{B}_3(t)\right],\tag{3.6}$$

where $\hat{B}_1, \hat{B}_2$, and $\hat{B}_3$ are Brownian motions under Q_1. Note that the correlation structure between the Q_1-Brownian motions is the same as the correlation structure between the P-Brownian motions of Assumption 2.1. Using equations (3.5) and (3.6) and Black-Scholes formula we get (3.3) and (3.4). □

As, e.g., in Davis (1998) we end up Black-Scholes type pricing equation even though we started with the general pricing function of Assumption 2.3. Note that the change of measure in the proof of Proposition 3.1 is just an efficient technique to calculate the expectation of Assumption 2.3.

By using equations (3.3) and (3.4) the forward price can also be written as follows

$$\begin{aligned} Y_1(t,T) &= S_1(t,T)(1-N(z_t)) + X_1(t,T)N\left(z_t - \sigma\sqrt{T-t}\right) \\ &= S_1(t,T)N(-z_t) + X_1(t,T)N\left(z_t - \sigma\sqrt{T-t}\right), \end{aligned} \tag{3.7}$$

where $-z_t = \frac{\ln\left(\frac{X_1(t,T)}{S_1(t,T)}\right) - \frac{1}{2}\sigma^2(-t)}{\sigma\sqrt{T-t}}$ and $z_t - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_1(t,T)}{X_1(t,T)}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$.

Thus, equation (3.7) is just the expected marker price because if $S_1(T) < X_1(T)$ then the market selects the S_1-price and direct routing and, otherwise, the price X_1 and the longer routing are used.

Because $Y_1(t,T)$ is a nonlinear function of S-prices, the instantaneous correlations between the forward prices (Y-prices) are not constant and the correlations depend on time and the S-prices. That is, from equation (3.7) and Itô's lemma we get

$$\begin{aligned} dY_1(t,T) = &\frac{\partial Y(t,S_1,X_1)}{\partial t}dt + \frac{\partial Y(t,S_1,X_1)}{\partial S_1(t,T)}dS_1(t,T) + \frac{\partial Y(t,S_1,X_1)}{\partial X_1(t,T)}dX_1(t,T) \\ &+ \frac{1}{2}\frac{\partial^2 Y(t,S_1,X_1)}{\partial S_1(t,T)^2}(dS_1(t,T))^2 + \frac{1}{2}\frac{\partial^2 Y(t,S_1,X_1)}{\partial X_1(t,T)^2}(dX_1(t,T))^2 \\ &+ \frac{\partial^2 Y(t,S_1,X_1)}{\partial S_1(t,T)\partial X_1(t,T)}dX_1(t,T)dS_1(t,T), \end{aligned} \tag{3.8}$$

where $Y(t,S,X)$ is the forward price given by the right-hand side of equation (3.7) and it is a function of time and direct and alternative routing prices.

Because the partial derivatives in (3.8) are not constant, the parameters of the forward price dynamics are neither constant even though the S-process parameters are constant. That is, the drift and diffusion terms of Y_1 are changing all the time. According to equation (3.1) the forward prices (Y-prices) are martingale under P. Therefore, the drift term in (3.8) is zero and we get by using (2.2), (2.3), and (3.7)

$$dY_1(t,T) = v_{1,1}(t,T)dB_1(t) + v_{1,2}(t,T)dB_2(t) + v_{1,3}(t,T)dB_3(t), \tag{3.9}$$

where

$$v_{1,1}(t,T) = N(-z_t)S_1(t,T)\sigma_1$$
$$v_{1,2}(t,T) = N\left(z_t - \sigma\sqrt{T-t}\right)S_2(t,T)\sigma_2$$
$$v_{1,3}(t,T) = N\left(z_t - \sigma\sqrt{T-t}\right)S_3(t,T)\sigma_3.$$

According to equation (3.9) the forward price uncertainty is equal to the probability of direct routing multiplied by the direct routing's uncertainty plus the probability of alternative routing multiplied by the alternative routing's uncertainty. Thus, we have $0 \leq v_{1,i}(t,T) \leq S_i(t,T)\sigma_i$ for all $i \in \{1,2,3\}$. Further, equation (3.9) implies the facts that are already mentioned earlier: the market forward prices are martingales and the correlation structure of the forward prices is not constant even though the correlation between S-prices is constant (see Assumption 2.1). Note that even if S_1, S_2, and S_3 were independent, i.e., even if the Brownian motions were independent the forward prices would depend on each other because of the routing options. In order to see this, let us use equation (3.9) and assume momentarily that the S-processes are independent. Then the instantaneous covariance between $Y_1(t,T)$ and $Y_2(t,T)$ is given by

$$\mathrm{cov}[Y_1, Y_2] = v_{1,1}(t,T)v_{2,1}(t,T) + v_{1,2}(t,T)v_{2,2}(t,T) + v_{1,3}(t,T)v_{2,3}(t,T) > 0, \tag{3.10}$$

where the process of $Y_2(t,T)$ follows

$$dY_2(t,T) = v_{2,1}(t,T)dB_1(t) + v_{2,2}(t,T)dB_2(t) + v_{2,3}(t,T)dB_3(t)$$

and according to (3.9) all the diffusion terms of $Y_2(t,T)$ are positive. Because these terms are positive and because the correlations between the Brownian motions were assumed to be zero, we get (3.10). That is, the instantaneous covariance between $Y_1(t,T)$ and $Y_2(t,T)$ is positive and if there are no routing options then the partial derivatives in equations (3.8) and (3.9) are zero and we get $\mathrm{cov}[Y_1, Y_2] = 0$. Thus, the network arbitrage creates positive correlation between capacity prices. This is natural since network routing smoothens the point-to-point demands over the whole network and therefore prices move together. From now on we again assume the general correlation structure between the S-prices.

4. Option pricing

In addition to the routing options described in section 2 and 3 some bandwidth service contracts include also other option type characteristics. For instance, the seller can have a right to disconnect the service for a predefined penalty payment. These rights can be modeled as bandwidth options and, therefore, the understanding of option pricing is important in the bandwidth markets even though there do not exist traded option contracts.

Bandwidth options are modeled as options on forwards. Therefore, a European bandwidth call option's payoff on the T-maturity forward at the option's expiration date $T_C \in [0,T]$ is $\max[Y_1(T_c,T) - K, 0]$, where K is the strike price. Because Y_1 can be viewed as an exchange option, the call is a kind of compound option on S_1 and, in the pricing of bandwidth options we utilize the pricing theory of compound options [see Geske (1979) and Rubinstein (1992)]. In the same way, bandwidth put options are compound put options and the terminal payoff is $\max[K - Y_1(T_c, T), 0]$. We concentrate on the European bandwidth call option pricing because the corresponding put option price can be solved by using the following put-call parity

$$C(t, T_c, S_1, X_1, K) - P(t, T_c, S_1, X_1, K) = [Y_1(t, T) - K]\exp(-r(T_c - t)), \quad (4.1)$$

where $C(t, T_C, S_1, X_1, K)$ and $P(t, T_c, S_1, X_1, K)$ are the bandwidth call and put prices at time $t \in [0, T_C]$ and $0 \le t \le T_c \le T \le \tau$. At the options' expiration date T_c the left hand side of (4.1) is $Y_1(T_c,T) - K$. Using Assumption 2.3 we get

$$E[Y_1(T_c, T) - K|F_t]\exp(-r(T_c - t)) = [Y_1(t, T) - K]\exp(-r(T_c - t)).$$

This is the same as the right hand side of (4.1) and, therefore, equation (4.1) holds. Thus, this put-call parity is got from Assumption 2.3.

In the same way, according to Assumption 2.3 before expiration date T_c the bandwidth call option price has to be given by the discounted expected payoff:

$$\begin{aligned} C(t, T_c, S_1, X_1, K) = \exp(-r(T_c - t))E\Bigg[\max\Bigg(&S_1(T_c, T)N(-z_{T_c}) \\ &+ X_1(T_c, T)N\left(z_{T_c} - \sigma\sqrt{T - T_c}\right) - K, 0\Bigg)\Bigg|F_t\Bigg]. \end{aligned} \quad (4.2)$$

Equation (4.2) is solved in two steps. Firstly, we calculate the conditional option price on the value of $X_1(T_c,\mathrm{T})$ and, secondly, we integrate the conditional option price over the X_1's probability distribution. The first step, i.e., the calculation of the conditional call option price means integrating the option's payoff over the S_1's probability distribution and because the underlying forward price formula includes cumulative normal distributions, this conditional option price is described in terms of bivariate normal distributions. This gives the fact that the final option price formula includes trivariate normal distributions because we have to consider also the X_1's distribution.

The derivation of European bandwidth call option formula is presented in Appendix 2. The result of the first step, i.e., the conditional call price is given in the following lemma.

Lemma 4.1 *The conditional call option price is given by*

$$\begin{aligned} C(t, T_c, S_1, X_1, K)\ |_{X_1(T_c,T)=x} = \exp(-r(T_c - t))[&S_1(t, T)(N(q_t(x)) \\ &- M\left(q_t(x), z_t, \rho_{q,z}\right)) \\ &+ xM\left(q_t(x) - \sigma_1\sqrt{T_c - t}, z_t - \sigma\sqrt{T - t}, \rho_{q,z}\right) \\ &-KN\left(q_t(x) - \sigma_1\sqrt{T_c - t}\right)\Big], \end{aligned} \quad (4.3)$$

where $q_t(x) = \frac{\ln\left(\frac{S_1(t,T)}{\theta_S(x)}\right) + \frac{1}{2}\sigma_1^2(T_c - t)}{\sigma_1\sqrt{T_c - t}}$, $\theta_S(x)$ solves

$$\begin{aligned} &\theta_S(x)N\left(-\frac{1}{\sigma\sqrt{T - T_c}}\left[\ln\left(\frac{\theta_S(x)}{x}\right) + \tfrac{1}{2}\sigma^2(T - T_c)\right]\right) \\ &\qquad + xN\left(\frac{1}{\sigma\sqrt{T - T_c}}\left[\ln\left(\frac{\theta_S(x)}{x}\right) - \tfrac{1}{2}\sigma^2(T - T_c)\right]\right) = K, \end{aligned}$$

$z_t = \frac{\ln\left(\frac{S_1(t,T)}{x}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $M(q_t, z_t, \rho_{q,z})$ *is the area under a standard bivariate normal distribution function covering the region from* $-\infty$ *to* q_t and $-\infty$ *to* z_t, and the two random variables have correlation $\rho_{q,z} = \sqrt{\frac{T_c-t}{T-t}}\rho_S$, and $\rho_S = \frac{1}{\sigma}\left(\sigma_1 - \omega_{1,2}\rho_{1,2}\sigma_2 - \omega_{1,3}\ \rho_{1,3}\sigma_3\right)$ *is the instantaneous correlation between* S_1 *and* $\frac{S_1}{X_1}$.

Proof. See Appendix 2. *Q.E.D.*

The first term after the discount factor

$$S_1(t,T)\big(N(q_t(x)) - M\big(q_t(x), z_t, \rho_{q,z}\big)\big) = E[S_1(t,T)I\{Y_1(T_c,T) \geq K\} \\ I\{X_1(T) \geq S_1(T)\}|X_1(T_c,T) = x],$$

where I is the indicator function, i.e.

$$I\{x \geqslant S_1(T)\} = \begin{cases} 1 & \text{if } x \geq S_1(T) \\ 0 & \text{if } x < S_1(T) \end{cases}.$$

Thus, $I\{Y_1(T_c,T) \geq K\}I\{X_1(T) \geq S_1(T)\} = 1$ if the bandwidth option expires in the money and if at the expiration of the forward contract $X_1(T) \geq S_1(T)$. In this case the bandwidth market price at time T is $S_1(T)$. The second term $xM\big(q_t(x) - \sigma_1\sqrt{T_c - t}, z_t - \sigma\sqrt{T-t}, \rho_{q,z}\big)$ is x multiplied by the probability that the bandwidth option expires in the money and that at the expiration of the forward contract $X_1(T) < S_1(T)$. In this case at time T the bandwidth market price is $X_1(T)$. The last term $KN\big(q_t(x) - \sigma_1\sqrt{T_c - t}\big)$ is K multiplied by the probability that the bandwidth option expires in the money. Thus, equation (4.3) is the discounted expected value of the bandwidth call option given that $X_1(T_c,\mathrm{T}) = x$. Note that if $x \leq K$ then always $Y_1(T_c,\mathrm{T}) \leq K$ and, therefore, the call option expires worthless.

The conditional option price, equation (4.3), is similar to the compound option pricing formula in Geske (1979) and Rubinstein (1992). However, we have to consider also the uncertainties in the alternative routing price X_1 and, therefore, we integrate the conditional option price over the X_1's probability distribution. In order to get analytical option pricing formula in this integration we assume that the upper boundaries of the cumulative normal distributions in (4.3) are independent of $X_1(T_c,T)$'s outcome. The derivation of the option pricing formula is presented in Appendix 2 and the result is given by the following proposition.

Proposition 4.1 *Bandwidth call option price is given by*

$$\begin{aligned} C(t,T_c,S_1,X_1,K) = {} & \exp(-r(T_c-t))\Big[S_1(t,T)\Big(M\Big(q_t(x_q), w_t - \sigma_X\sqrt{T_c-t}, \rho_{q,w}\Big) \\ & - G\Big(q_t(x_q), z_t, w_t - \sigma_X\sqrt{T_c-t}, \rho_{q,z}, \rho_{q,w}, \rho_{z,w}\Big)\Big) \\ & + X_1(t,T)G\Big(q_t(x_q) - \sigma_1\sqrt{T_c-t}, z_t - \sigma\sqrt{T-t}, w_t, \rho_{q,z}, \rho_{q,w}, \rho_{z,w}\Big) \\ & - KM\Big(q_t(x_q) - \sigma_1\sqrt{T_c-t}, w_t - \sigma_X\sqrt{T_c-t}, \rho_{q,w}\Big)\Big], \end{aligned} \tag{4.4}$$

where $x_q = \frac{X_1(t,T)N(w_t)}{N\left(w_t - \sigma_X\sqrt{T_c - t}\right)}$, $w_t = \frac{\ln\left(\frac{X_1(t,T)}{K}\right) + \frac{1}{2}\sigma_X^2(T_c - t)}{\sigma_X\sqrt{T_c - t}}$, $G\left(q_t, z_t, w_t, \rho_{q,z}, \rho_{q,w}, \rho_{z,w}\right)$ *is the area under a standard trivariate normal distribution function covering the region from* $-\infty$ *to* q_t, $-\infty$ *to* z_t, and $-\infty$ *to* w_t and the three random variables have correlations $\rho_{q,z} = \sqrt{\frac{T_c - t}{T - t}}\rho_S$, $\rho_{q,w} = \rho_{S,X}$, and $\rho_{z,w} = \sqrt{\frac{T_c - t}{T - t}}\rho_X$,

$$\rho_{S,X} = \frac{1}{\sigma_X \sigma_1}\left[\omega_{1,2}\rho_{1,2}\sigma_1\sigma_2 + \omega_{1,3}\rho_{1,3}\sigma_1\sigma_3\right]$$

is the correlation between S_1 and X_1,

$$\rho_X = \frac{1}{\sigma_X \sigma}\left[\sigma_1\left(\omega_{1,2}\rho_{1,2}\sigma_2 + \omega_{1,3}\rho_{1,3}\sigma_3\right) - \sigma_X^2\right]$$

is the correlation between X_1 and $\frac{S_1}{X_1}$,

$$\sigma_X = \sqrt{\omega_{1,2}^2\sigma_2^2 + \omega_{1,3}^2\sigma_3^2 + 2\omega_{1,2}\omega_{1,3}\rho_{2,3}\sigma_2\sigma_3}$$

is the volatility of X_1, and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_X^2 - 2\sigma_1\left(\omega_{1,2}\rho_{1,2}\sigma_2 + \omega_{1,3}\rho_{1,3}\sigma_3\right)}$$

is the volatility of $\frac{S_1}{X_1}$.

Proof. See Appendix 2. □

Equation (4.4) is the discounted expected value of the bandwidth call option. The first term after the discount factor

$$\begin{aligned} &S_1(t,T)\Big(M\Big(q_t(x_q), w_t - \sigma_X\sqrt{T_c - t}, \rho_{q,w}\Big) \\ &\quad - G\Big(q_t(x_q), z_t, w_t - \sigma_X\sqrt{T_c - t}, \rho_{q,z}, \rho_{q,w}, \rho_{z,w}\Big)\Big) \\ &\quad = E[S_1(t,T)I\{Y_1(T_c,T) \geq K\}I\{X_1(T) \geq S_1(T)\}|F_t] \end{aligned}$$

is now the expected value of $S_1(t,T)I\{Y_1(T_c,T) \geq K\}I\{X_1(T) \geq S_1(T)\}$. Thus, $I\{Y_1(T_c,T) \geq K\}I\{X_1(T) \geq S_1(T)\} = 1$ if the bandwidth option expires in the money and if at the expiration of the forward contract $X_1(T) \geq S_1(T)$. The second term

$$\begin{aligned} &X_1(t,T)G\Big(q_t(x_q) - \sigma_1\sqrt{T_c - t}, z_t - \sigma\sqrt{T - t}, w_t, \rho_{q,z}, \rho_{q,w}, \rho_{z,w}\Big) \\ &\quad = E[S_1(t,T)I\{Y_1(T_c,T) \geqslant K\}I\{X_1(T) \leq S_1(T)\}|F_t] \end{aligned}$$

and $I\{Y_1(T_c,T) \geq K\}I\{X_1(T) \leq S_1(T)\} = 1$ if the option expires in the money and if at the expiration of the forward contract $X_1(T) \leq S_1(T)$. The last term $KM\left(q_t(x_q) - \sigma_1\sqrt{T_c - t}, w_t - \sigma_X\sqrt{T_c - t}, \rho_{q,z}\right)$ is again K multiplied by the probability that the bandwidth option expires in the money.

The selection of q_t's variable according to $x_q = \frac{X_1(t,T)N(w_t)}{N\left(w_t - \sigma_X\sqrt{T_c - t}\right)}$ is an approximation and this way the upper boundaries of the integrals in equation (4.4) are independent of $X_1(T_c,T)$'s outcome. This selection implies that

$$x_q E[I\{X_1(T_c,T) \geq K\}|F_t] = E[X_1(T_c,T)I\{X_1(T_c,T) \geq K\}|F_t],$$

i.e., x_q is the constant corresponding to $X_1(T_c,T)$ in the sense of the above equation. Appendix 3 illustrates the error from x_q by comparing our analytical option pricing formula with Monte Carlo simulation with 200,000 outcomes. According to the results, the higher the volatility of X_1 and the smaller the $X_1(t,T)/S_1(t,T)$ ratio the higher the approximation error, and the error is zero if X_1 is deterministic.

In the deriving of the option pricing formula (4.4) we have made two approximations: the process of $X_1(t,T) = S_2(t,T) + S_3(t,T)$ was assumed to follow a geometric Brownian motion [equation (2.3)] and the normal integrals' upper boundary q_t was assumed to be deterministic [equation (4.4)]. Both these approximations are due to the stochastic alternative routing price X_1 and in the case of deterministic X_1 equation (4.3) gives the correct call option price.

5. Example

In this section we illustrate our pricing models with numerical examples. Firstly, we analyze bandwidth forward price as a function of the S_1's volatility σ_1. Secondly, we study how the volatility affects bandwidth call option price and compare our option pricing model with Black-76 formula, which is a traditional commodity option pricing method.

Let us assume the following parameter values. Current time $t = 0$, forward maturity $T = 2$ years, bandwidth call option maturity $T_c = 1$ year, option strike price $K = 2.8$, and, for simplicity, discount rate $r = 0$. Price $S_1(0,1) = S_1(0,2) = 2.8$, $S_2(0,1) = S_2(0,2) = 1$, $S_3(0,1) = S_3(0,2) = 2$, $\sigma_1 = 0.2$, and σ_2, $\sigma_3 = 0$. That is, the expected direct routing prices are 2.8, 1, and 2, and they satisfy the network arbitrage condition. For simplicity, we assume that S_2 and S_3 are constant and, therefore, $X_1(t) = 3$ for all t. The annual volatility of S_1 is 20%. Because X_1 is constant we do not have to use the approximation of equations (2.3) and (4.4). Therefore, Proposition 3.1 and Proposition 4.1 give the correct forward and call option prices. Using the above parameter values and the bandwidth forward and call pricing functions we get 2.564 for the forward price and 0.0275 for the call price.

Figure 2 illustrates the bandwidth forward price as a function of volatility σ_1. Figure 2 indicates that the volatility affects the forward price and the function is decreasing. The volatility widens the probability distribution of future direct routing price S_1. However, due to the existence of the alternative routing the future market prices Y_1 are bounded above and, therefore, the volatility mainly lengthens the lower tail of Y_1's probability distribution. This implies that the expected future spot price and the forward price decrease when the volatility increases. Thus, the bandwidth forward pricing is different than in other commodity markets. For instance, if the underlying asset is a storable commodity then the forward prices are given by the cost-of-carry model and the volatility does not affect the commodity forward prices [see e.g. Hull (1997)].

Next we analyze how the volatility affects the bandwidth call option prices. Figure 3 illustrates the situation. The solid line is the call option price based on our model and the broken line is the corresponding option price given by the Black-76 formula, which does not consider the upper boundary

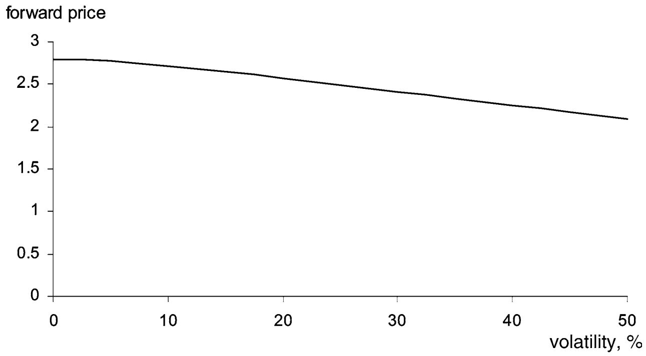

Fig. 2. Bandwidth forward price $S_1(t = 0, T = 2)$ as a function of annual volatility σ_1

of the underlying forward price. The increased volatility has two effects on bandwidth call price. Firstly, it widens the probability distribution of future spot prices. This increases the option price. Secondly, according to Figure 2 the increased volatility lowers the forward price and, therefore, the volatility also lowers the call option price. This is a natural option character as the underlying asset decreases also the call option value decreases. Thus, the total effect of volatility can be positive or negative. As can be seen from Figure 3, with the given parameters our call option price is first increasing and then decreasing function of the volatility. With low volatility values the first effect is greater, but with higher than 5% volatility the second effect starts to lower the option price. This is because the S_1-probability distribution is bounded above and, therefore, increasing volatility does not any more lengthen the upper probability tail. With the Black-76 model the first effect is greater all the time, because the model does not consider the alternative routing, i.e., the probability distribution is not bounded above. According to figures 2 and 3, the error term of Black-76 is significant in the situations where the second effect is strong, i.e., in the situations where there is a high probability that the alternative routing is used.

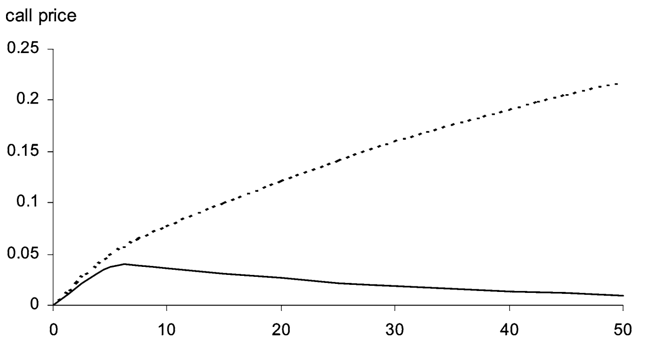

Fig. 3. Bandwidth call option price $C(t = 0, T_c = 1, S_1 = 2.8, X_1 = 3, K = 2.8)$ as a function of annual volatility σ_1 (solid line) and the corresponding call option price by using Black-76 model (broken line)

6. Conclusions

In this paper we have modeled point-to-point bandwidth contracts under the network arbitrage condition. In order to understand the effects from this condition we used the fixed routing bandwidth prices as the underlying assets for these contracts. Due to the network arbitrage condition bandwidth forwards include routing exchange options and, therefore, they are nonlinear instruments of the fixed routing prices. According to our bandwidth forward model, the network arbitrage creates positive correlation between forward prices and shortens the upper tails of bandwidth market price distributions. The more the upper tails are truncated the greater the correlation between the market prices. In the numerical examples, we have illustrated that the underlying bandwidth volatility affects the forward prices and the bandwidth forward prices are decreasing functions of the underlying volatility.

Because a bandwidth forward contract can be viewed as an exchange option, a bandwidth option is a kind of compound option. The analytical option price approximation is described in terms of bivariate and trivariate normal distributions and the approximation error is zero if the alternative routing price is deterministic. Because the forward price is a decreasing function of the underlying volatility and because the network arbitrage condition shortens the upper tail of the bandwidth price distribution, a bandwidth call option can be a decreasing function of the volatility. In the numerical examples, we have illustrated the difference between our option pricing model and a traditional commodity option pricing formula. The difference of these approaches is significant in the situations when there is a high probability that the underlying point-to-point routing is changed.

Appendix 1: Geometric Brownian motion approximation for alternative routing

In this appendix we analyze the geometric Brownian motion approximation for alternative routing price X_1. First, the approximation error's two first moments are calculated. Second, we will compare the geometric Brownian motion assumption with Monte Carlo Simulation with 20,000 outcomes.

The geometric Brownian motion assumption for X_1 can be written as

$$X_1(T) = [S_2(t,T) + S_3(t,T)] \exp\left(-\frac{1}{2}\sigma_X^2(T-t) + \sigma_X\sqrt{T-t}Z_X\right), \tag{A1.1}$$

where $S_2(t,T) = E[S_2(T)|F_t]$, $X_1(t,T) = E[S_2(T) + S_3(T)|F_t]$, Z_X is a standard normal variable,

$$\sigma_X = \sqrt{\omega_{1,2}^2\sigma_2^2 + \omega_{1,3}^2\sigma_3^2 + 2\omega_{1,2}\omega_{1,3}\rho_{2,3}\sigma_2\sigma_3},$$

$$\omega_{1,2} = \frac{S_2(t,T)}{S_2(t,T) + S_3(t,T)}, \text{ and } \omega_{1,3} = \frac{S_3(t,T)}{S_2(t,T) + S_3(t,T)}.$$

The distribution of (A1.1) is a lognormal with mean $S_2(t,T) + S_3(t,T)$ and variance

$$[S_2(t,T) + S_3(t,T)]^2\left[\exp\left(\sigma_X^2(T-t)\right) - 1\right]. \tag{A1.2}$$

The true distribution of $S_2(T) + S_3(T)$ is

$$S_2(T) + S_3(T) = S_2(t,T)\exp\left(-\frac{1}{2}\sigma_2^2(T-t) + \sigma_2\sqrt{T-t}Z_2\right) + S_3(t,T)\exp\left(-\frac{1}{2}\sigma_3^2(T-t) + \sigma_3\sqrt{T-t}Z_3\right), \quad \text{(A1.3)}$$

where Z_2 and Z_3 are standard normal variables correspondingly to S_2 and S_3, and equation (A1.3) is not a lognormal distribution. The mean of (A1.3) is again $S_2(t,T) + S_3(t,T)$ and the variance

$$S_2(t,T)^2\left[\exp\left(\sigma_2^2(T-t)\right) - 1\right] + S_3(t,T)^2\left[\exp\left(\sigma_3^2(T-t)\right) - 1\right] + 2S_2(t,T)S_3(t,T)\left[\exp\left(\rho_{2,3}\sigma_2\sigma_3(T-t)\right) - 1\right]. \quad \text{(A1.4)}$$

Comparing the variance equations (A1.2) and (A1.4) we see that the variances are different almost every time and can be equal only momentarily.

Next we compare our X_1-process approximation with the results from Monte Carlo Simulation. Let us assume the following parameter values: $S_2(0,1) = 1$, $S_3(0,1) = 2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$, and $\rho_{2,3} = 0$. That is, the expected bandwidth prices are 1 and 2, the annual volatilities are 20%, and the S-processes are independent of each other. Using equation (A1.1) our volatility estimate for $X_1(t,1)$ is 14.9071% and, therefore, based on our geometric Brownian motion assumption the variance of $X_1(1)$ is 0.2022. Using equation (A1.4) the true variance of $X_1(1)$ is 0.2041, so the error is 0.9%. Monte Carlo Simulation with 20,000 outcomes yields 0.2142 for the variance of $X_1(1)$ and the difference with the true variance is 4.9675%. Thus, with these parameter values and simulation outcomes our analytical approximation is closer to the true variance. Therefore, we should expect that our analytical model is also more accurate in the pricing of bandwidth contracts than the Monte Carlo Simulation with 20,000 outcomes.

Appendix 2: Bandwidth call option price

In this appendix we derive analytical pricing formula for European bandwidth call options in terms of integrals of the bivariate and trivariate normal distributions. This is possible because we have assumed that

$$S_1(T_c,T) = S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c-t) + \sigma_1\sqrt{T_c-t}Z_S\right), \quad \text{(A2.1)}$$

$$X_1(T_c,T) = X_1(t,T)\exp\left(-\frac{1}{2}\sigma_X^2(T_c-t) + \sigma_X\sqrt{T_c-t}Z_X\right), \quad \text{(A2.2)}$$

and under the martingale measure of X_1

$$\frac{S_1(T_c,T)}{X_1(T_c,T)} = \frac{S_1(t,T)}{X_1(t,T)}\exp\left(-\frac{1}{2}\sigma^2(T_c-t) + \sigma\sqrt{T_c-t}Z\right), \quad \text{(A2.3)}$$

where

$$\sigma_X = \sqrt{\omega_{1,2}^2\sigma_2^2 + \omega_{1,3}^2\sigma_3^2 + 2\omega_{1,2}\omega_{1,3}\rho_{2,3}\sigma_2\sigma_3},$$

$$\sigma = \sqrt{\sigma_1^2 + \omega_{1,2}^2\sigma_2^2 + \omega_{1,3}^2\sigma_3^2 - 2\omega_{1,2}\rho_{1,2}\sigma_1\sigma_2 - 2\omega_{1,3}\rho_{1,3}\sigma_1\sigma_3 + 2\omega_{1,2}\omega_{1,3}\rho_{2,3}\sigma_2\sigma_3},$$

Z, Z_S, and Z_X are standard normal variables and the correlation between S_1 and $\frac{S_1}{X_1}$ is ρ_S, the correlation between X_1 and $\frac{S_1}{X_1}$ is ρ_X, and the correlation between S_1 and X_1 is $\rho_{S,X}$.

According to Assumption 2.3 the European bandwidth call option price is

$$\begin{aligned} C(t, T_c, S_1, X_1, K) &= \exp(-r(T_c - t)) \cdot \\ &E\Big[C\Big(T_c, T_c, S_1(t,T)\exp\Big(-\frac{1}{2}\sigma_1^2(T_c - t) + \sqrt{T_c - t}\sigma_1 Z_S\Big), \\ &X_1(t,T)\exp\Big(-\frac{1}{2}\sigma_X^2(T_c - t) + \sqrt{T_c \cdot\cdot t}\sigma_X Z_X\Big), K\Big)|F_t\Big] \end{aligned} \tag{A2.4}$$

where $C(t,T_c,S_1,X_1,K)$ is T_c-maturity bandwidth call option price at time t and K is the strike price.

First, we calculate the conditional option price on the value of $X_1(T_c,\mathrm{T})$. From equation (3.10) we get that the price of the underlying forward price

$$\begin{aligned} Y_1(t,T) &= S_1(t,T) - \int_{-\infty}^{z_t - \sigma\sqrt{T-t}} \Big[S_1(t,T)\exp\Big(-\frac{1}{2}\sigma(T_c - t) - \sigma\sqrt{T_c - t}s\Big) - X_1(t,T)\Big]f(s)ds \\ &= S_1(t,T) - S_1(t,T)N(z_t) + X_1(t,T)N(z_t - \sigma\sqrt{T-t}), \end{aligned} \tag{A2.5}$$

where f is the density of the standard normal distribution, i.e., $f(u) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)$, the alternative routing is used if $-Z_S \le \frac{\ln\left(\frac{S_1(t,T)}{X_1(t,T)}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = z_t - \sigma\sqrt{T-t}$, and $z_t = \frac{\ln\left(\frac{S_1(t,T)}{x}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$.

The conditional option price is

$$\begin{aligned} C(t, T_c, S_1, x, K)|_{X_1(T_c,T)=x} &= \exp(-r(T_c - t)) \\ &\cdot E\Big(\max\Big[S_1(T_c,T) - S_1(T_c,T)N(z_{T_c}) + xN\Big(z_{T_c} - \sigma\sqrt{T - T_c}\Big) - K, 0\Big]|F_t\Big). \end{aligned} \tag{A2.6}$$

If $K \ge x$ then $S_1(T_c,T)N(-z_{T_c}) + xN\left(z_{T_c} - \sigma\sqrt{T - T_c}\right) \le K$ for all $S_1(T_c,T) \in \mathbf{R}_+$ and the call option expires worthless. Now we assume that $K \le x$ and then at time T_c the call option $C(T_c, T_c, S_1, x, K) = 0$ if $\theta_S \ge S_1(T_c,T)$ where θ_S solves

$$\begin{aligned} &\theta_S(x)N\left(-\frac{1}{\sigma\sqrt{T - T_c}}\left[\ln\left(\frac{\theta_S(x)}{x}\right) + \frac{1}{2}\sigma^2(T - T_c)\right]\right) \\ &+ xN\left(\frac{1}{\sigma\sqrt{T - T_c}}\left[\ln\left(\frac{\theta_S(x)}{x}\right) - \frac{1}{2}\sigma^2(T - T_c)\right]\right) = K. \end{aligned} \tag{A2.7}$$

That is, when $S_1(T_c,\mathrm{T}) < \theta_S(x)$ then $S_1(T_c,T) - S_1(T_c,T)N(z_{T_c}) + xN\left(z_{T_c} - \sigma\sqrt{T - T_c}\right) < K$ and, therefore, $C(T_c, T_c, S_1, X_1, K) = 0$.

According to (A2.1) and (A2.7) the option is in the money if

$$-Z_S \leq \frac{\ln\left(\frac{S_1(t,T)}{\theta_S(x)}\right) - \frac{1}{2}\sigma_1^2(T_c - t)}{\sigma_1\sqrt{T_c - t}} = q_t - \sigma_1\sqrt{T_c - t}, \tag{A2.8}$$

where $q_t(x) = \dfrac{\ln\left(\dfrac{S_1(t,T)}{\theta_S(x)}\right) + \dfrac{1}{2}\sigma_1^2(T_c - t)}{\sigma_1\sqrt{T_c - t}}$.

From equations (A2.6) the conditional option price can be written as follows

$$\begin{aligned} C(t, T_c, S_1, X_1, K)|_{X_1(T_c,T)=x} &= \exp(-r(T_c - t)) \\ \cdot \int_{-\infty}^{\infty} &\left[S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t) + \sigma\sqrt{T_c - t}u\right)\right. \\ &- \left(S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t) + \sigma_1\sqrt{T_c - t}u\right)N(z_{T_c})\right. \\ &\left.\left. - xN\left(z_{T_c} - \sigma\sqrt{T - T_c}\right)\right) - K\right] f(u)du \end{aligned} \tag{A2.9}$$

and using (A2.7) and (A2.8) we get

$$\begin{aligned} C(t, T_c, S_1, X_1, K)|_{X_1(T_c,T)=x} &= \exp(-r(T_c - t)) \\ \cdot \int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} &\left[S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t) - \sigma\sqrt{T_c - t}u\right)\right. \\ &- \left(S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t) - \sigma_1\sqrt{T_c - t}u\right)N(z_{T_c})\right. \\ &\left.\left. - xN\left(z_{T_c} - \sigma\sqrt{T - T_c}\right)\right) - K\right] f(u)du. \end{aligned} \tag{A2.10}$$

Integrating the first component in (A2.10) gives

$$\begin{aligned} &\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t) - \sigma_1\sqrt{T_c - t}u\right) f(u)du \\ &= \frac{S_1(t,T)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sigma_1^2(T_c - t)\right) \int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} \exp\left(-\sigma_1\sqrt{T_c - t}u - \frac{1}{2}u^2\right) du \end{aligned}$$

$$= \frac{S_1(t,T)}{\sqrt{2\pi}} \int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} \exp\left(-\frac{1}{2}\left(u+\sigma_1\sqrt{T_c-t}\right)^2\right)du = S_1(t,T)N(q_t(x)). \tag{A2.11}$$

The second component of (A2.10),

$$\begin{aligned}
&\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} S_1(t,T)\exp\left(-\frac{1}{2}\sigma_1^2(T_c-t)-\sigma_1\sqrt{T_c-t}u\right)N(z_{T_c})f(u)du \\
&=\frac{S_1(t,T)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sigma_1^2(T_c-t)\right)\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} \exp\left(-\sigma_1\sqrt{T_c-t}u-\frac{1}{2}u^2\right)N(z_{T_c})du \\
&=\frac{S_1(t,T)}{\sqrt{2\pi}}\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} \exp\left(-\frac{1}{2}\left(u+\sigma_1\sqrt{T_c-t}\right)^2\right)N(z_{T_c})du \\
&=S_1(t,T)M\left(q_t(x),z_t,\rho_{q,z}\right),
\end{aligned} \tag{A2.12}$$

where $q_t(x) = \frac{\ln\left(\frac{S_1(t,T)}{\theta_S(x)}\right)+\frac{1}{2}\sigma_1^2(T_c-t)}{\sigma_1\sqrt{T_c-t}}$, $z_t = \frac{\ln\left(\frac{S_1(t,T)}{x}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$, $M(q_t,z_t,\rho_{q,z})$ is the area under a standard bivariate normal distribution function covering the region from $-\infty$ to q_t and $-\infty$ to z_t, and the two random variables have correlation $\rho_{q,z} = \sqrt{\frac{T_c-t}{T-t}}\rho_S$.

The third component in (A2.10) gives

$$\begin{aligned}
&x\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} N\left(z_{T_c}-\sigma\sqrt{T-T_c}\right)f(u)du \\
&= xM\left(q_t(x)-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\right)
\end{aligned} \tag{A2.13}$$

and the fourth integral

$$\int_{-\infty}^{q_t(x)-\sigma_1\sqrt{T_c-t}} Kf(u)du = KN\left(q_t(x)-\sigma_1\sqrt{T_c-t}\right). \tag{A2.14}$$

Combining (A2.10)–(A2.14) we get that the conditional bandwidth call option price is given by

$$\begin{aligned}
&C(t,T_c,S_1,X_1,K)|_{X_1(T_c,T)=x} \\
&= \exp(-r(T_c-t))\left[S_1(t,T)\cdot\left(N(q_t(x))-M\left(q_t(x),z_t,\rho_{q,z}\right)\right)\right. \\
&\quad \left.+xM\left(q_t(x)-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\right)-KN\left(q_t(x)-\sigma_1\sqrt{T_c-t}\right)\right].
\end{aligned} \tag{A2.15}$$

Next we calculate the affect of the X_1-distribution. Using (A2.2) and (A2.15) we get

$$C(t,T_c,S_1,X_1,K)=\exp(-r(T_c-t))\int_{-\infty}^{\infty}\Big[S_1(t,T)\big(N(q_t(x))-M\big(q_t(x),z_t,\rho_{q,z}\big)\big)$$
$$+X_1(t,T)\exp\left(-\frac{1}{2}\sigma_X^2(T_c-t)+\sqrt{T_c-t}\sigma_X y\right)$$
$$M\left(q_t(x)-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\right)-KN\left(q_t(x)-\sigma_1\sqrt{T_c-t}\right)\Big]f(y)dy. \tag{A2.16}$$

Because if $K \geq X_1(T_c,T)$ the call option expires worthless we get from (A2.2) that the option is in the money if

$$-Z_X \leqslant \frac{\ln\left(\frac{X_1(t,T)}{K}\right)-\frac{1}{2}\sigma_X^2(T_c-t)}{\sigma_X\sqrt{T_c-t}} = w_t-\sigma_X\sqrt{T_c-t}, \tag{A2.17}$$

where $w_t = \dfrac{\ln\left(\dfrac{X_1(t,T)}{K}\right)+\dfrac{1}{2}\sigma_X^2(T_c-t)}{\sigma_X\sqrt{T_c-t}}$.

The x variable in $q(x)$ is approximated as follows

$$x_q = \frac{\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} X_1(t,T)\exp\left(-\frac{1}{2}\sigma_X^2(T_c-t)-\sqrt{T_c-t}\sigma_X y\right)f(y)dy}{\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} f(y)dy}. \tag{A2.18}$$

That is, x_q is the expected X_1-price given that the option expires in the money. Note that if X_1 is deterministic then we have $x_q = X_1(t,T)$. Using (A2.11) we get

$$x_q = \frac{X_1(t,T)N(w_t)}{N\left(w_t-\sigma_X\sqrt{T_c-t}\right)}. \tag{A2.19}$$

Now we have

$$C(t,T_c,S_1,X_1,K)=\exp(-r(T_c-t))\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}}\Big[S_1(t,T)\big(N(q_t)-M\big(q_t,z_t,\rho_{q,z}\big)\big)$$
$$+X_1(t,T)\exp\left(-\frac{1}{2}\sigma_X^2(T_c-t)-\sqrt{T_c-t}\sigma_X y\right)$$
$$\cdot M\left(q_t-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\right)-KN\left(q_t-\sigma_1\sqrt{T_c-t}\right)\Big]f(y)dy, \tag{A2.20}$$

where $q_t = q_t(x_q)$.

Integrating the first component of (A2.20) gives

$$\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} S_1(t,T)\big(N(q_t)-M\big(q_t,z_t,\rho_{q,z}\big)\big)f(y)dy$$

$$=S_1(t,T)\Big(M\Big(q_t,w_t-\sigma_X\sqrt{T_c-t},\rho_{q,w}\Big)-G\Big(q_t,z_t,w_t-\sigma_X\sqrt{T_c-t},\rho_{q,z},\rho_{q,w},\rho_{z,w}\Big)\Big), \tag{A2.21}$$

where $G\big(q_t,z_t,w_t-\sigma_X\sqrt{T_c-t},\rho_{q,z},\rho_{q,w},\rho_{z,w}\big)$ is the area under a standard trivariate normal distribution function covering the region from $-\infty$ to q_t, $-\infty$ to z_t, and $-\infty$ to $w_t-\sigma_X\sqrt{T_c-t}$ and the three random variables have correlations $\rho_{q,z}=\sqrt{\frac{T_c-t}{T-t}}\rho_S$, $\rho_{q,w}=\rho_{S,X}$, and $\rho_{z,w}=\sqrt{\frac{T_c-t}{T-t}}\rho_X$.

The second component of (A2.20) gives

$$\begin{aligned}
&\frac{X_1(t,T)}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\sigma_X^2(T_c-t)\right)\cdot\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} M\Big(q_t-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\Big)\\
&\quad\cdot\exp\left(-\sqrt{T_c-t}\,\sigma_X y-\frac{1}{2}y^2\right)dy\\
&=\frac{X_1(t,T)}{\sqrt{2\pi}}\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} M\Big(q_t-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},\rho_{q,z}\Big)\\
&\quad\cdot\exp\left(-\frac{1}{2}\Big(y+\sqrt{T_c-t}\,\sigma_X\Big)^2\right)dy\\
&=X_1(t,T)G\Big(q_t-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},w_t,\rho_{q,z},\rho_{q,w},\rho_{z,w}\Big).
\end{aligned} \tag{A2.22}$$

The third component in (A2.20),

$$\begin{aligned}
&\int_{-\infty}^{w_t-\sigma_X\sqrt{T_c-t}} KN\big(q_t-\sigma_1\sqrt{T_c-t}\big)f(y)dy\\
&=KM\big(q_t-\sigma_1\sqrt{T_c-t},w_t-\sigma_X\sqrt{T_c-t},\rho_{q,w}\big).
\end{aligned} \tag{A2.23}$$

Combining (A2.20)–(A2.23) we get that the bandwidth call option price is given by

$$\begin{aligned}
C(t,T_c,S_1,X_1,K)&=\exp(-r(T_c-t))\Big[S_1(t,T)\Big(M\Big(q_t,w_t-\sigma_X\sqrt{T_c-t},\rho_{q,w}\Big)\\
&\quad-G\Big(q_t,z_t,w_t-\sigma_X\sqrt{T_c-t},\rho_{q,z},\rho_{q,w},\rho_{z,w}\Big)\Big)\\
&\quad+X_1(t,T)G\Big(q_t-\sigma_1\sqrt{T_c-t},z_t-\sigma\sqrt{T-t},w_t,\rho_{q,z},\rho_{q,w},\rho_{z,w}\Big)\\
&\quad-KM\Big(q_t-\sigma_1\sqrt{T_c-t},w_t-\sigma_X\sqrt{T_c-t},\rho_{q,w}\Big)\Big].
\end{aligned} \tag{A2.24}$$

Appendix 3: Comparing analytical option pricing formula with Monte Carlo simulation

The aim of this appendix is to compare analytical bandwidth call option prices with Monte Carlo simulation and this way to demonstrate the performance of the analytical approximation. The Monte Carlo simulation uses 200,000 random samples. The generous amount of random samples is

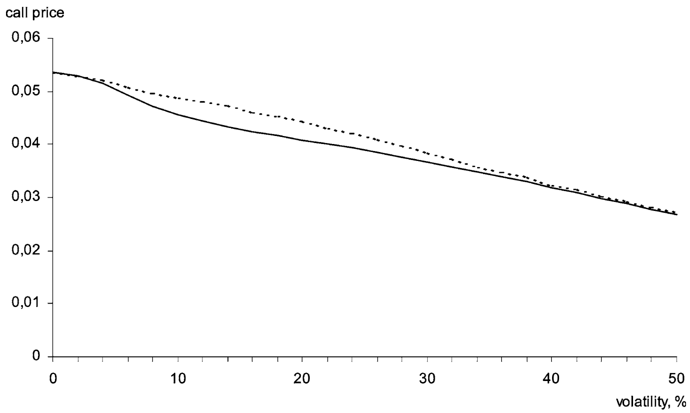

Figure A1. Analytical bandwidth call option price $C(t=0, T_c=1, T=2, S_1=2.8, \sigma_1=0.25, X_1=3.5, K=3)$ as a function of annual volatility σ_X (solid line) and the corresponding call option price by using Monte Carlo simulation (broken line)

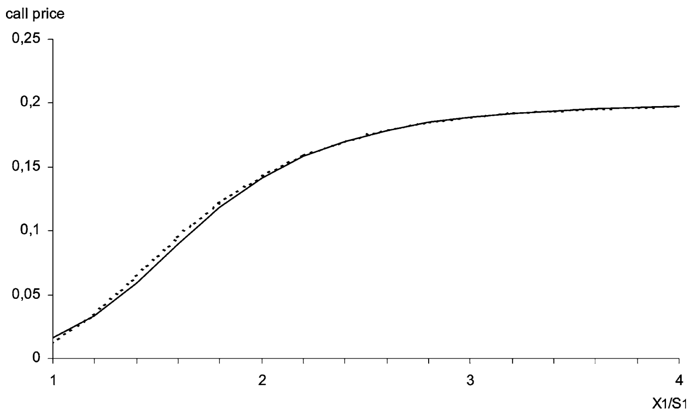

Figure A2. Analytical bandwidth call option price $C(t=0, T_c=1, T=2, S_1=2.8, \sigma_1=0.25, \sigma_X=0.25, K=3)$ as a function of the ratio $X_1(t,T)/S_1(t,T)$ (solid line) and the corresponding call option price by using Monte Carlo simulation (broken line)

crucial to ensure correct evaluation of rare events and we can assume that the simulation is close to the correct option price.

The call options are priced in different situations, which are selected in such a way that the possible shortcomings can be pointed out. The following variables affect the error term of the analytical approximation

- volatilities σ_1 and σ_X
- ratios between direct routing price $S_1(t,T)$, alternative routing price $X_1(t,T)$, and strike price K
- option maturity T_c and the underlying forward maturity T.

There are a few remarks to be made at this point. First, it is indeed the ratios of $S_1(t,T)$, $X_1(t,T)$, and K that are of importance. Second, the processes being simulated are $S_1(t,T)$ and $X_1(t,T)$ [as opposed to $S_1(t,T)$, $S_2(t,T)$, and $S_3(t,T)$]

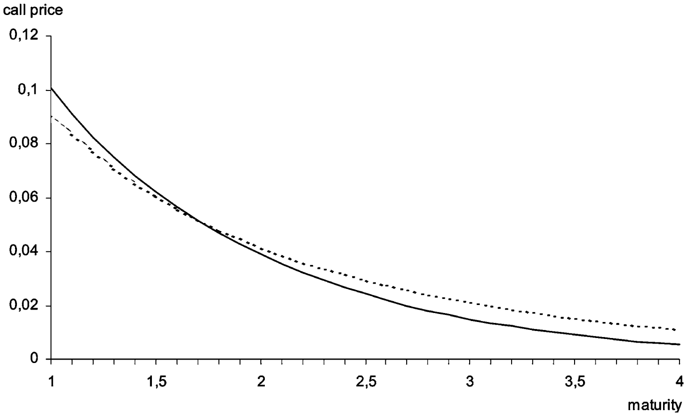

Figure A3. Analytical bandwidth call option price $C(t = 0, T_c = 1, S_1 = 2.8, \sigma_1 = 0.25, X_1 = 3.5, \sigma_X = 0.25, K = 3)$ as a function of the underlying forward contract's maturity T (solid line) and the corresponding call option price by using Monte Carlo simulation (broken line)

and that error hereof is not considered [for the error from X_1 process assumption see Appendix 1].

Some introductory understanding of the bandwidth call option formula must also be given. The inaccuracy in the pricing formula is partly due to the replacement of the call price's conditional expectation on $X_1(T_c,T)$ by the constant x_q. The error should cancel out when $X_1(T)$ is deterministic. Furthermore, the error should be smaller the greater the $X_1(t,T)/S_1(t,T)$ ratio, because then the probability distribution of $Y_1(t,T)$ is close to $S_1(t,T)$'s distribution. Thus, if $X_1(t,T)$ is high the bandwidth options are close to regular Black-Scholes options on $S_1(t,T)$.

In Figure A1 the difference between analytical pricing formula and the Monte Carlo simulation is illustrated as a function of X_1's volatility. Note that the difference is zero if the volatility is equal to zero and in this case the uncertainty in the option price is from S_1.

Figure A2 shows the impact from the ratio of $S_1(t,T)$ and $X_1(t,T)$. The difference between the analytical formula and the simulation decreases as a function of $X_1(t,T)/S_1(t,T)$. Combining figures A1 and A2 we see that the ratio impact is stronger the higher the X_1's volatility and with zero volatility the impact is zero.

Figure A3 illustrates how the option maturity affects the difference between the analytical and simulation option prices. According to the figure the error increases with the difference in maturities. This implies that the approximation x_q works better for T_c close to T. Using Figure A1 we realize that the maturity effect is stronger the higher the X_1's volatility and with zero volatility the effect is zero.

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