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RESEARCH ARTICLE

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Subjective recursive expected utility

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Abstract We axiomatize a subjective version of the recursive expected utility model. This development extends the seminal results of Kreps and Porteus (Econometrica 46:185–200 (1978)) to a subjective framework and provides foundations that are easy to relate to axioms familiar from timeless models of decision making under uncertainty. Our analysis also clarifies what is needed in going from a represention that applies within a single filtration to an across filtration representation.

Keywords Recursive utility · Subjective expected utility · Uncertainty · Risk information

JEL Classification Numbers D80 · D81 · D90

1 Introduction

We describe and provide axiomatic foundations for a subjective version of the recursive expected utility model. In a seminal paper, Kreps and Porteus (1978) provide

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an axiomatic analysis of preferences in a dynamic framework that delivers recursive expected utility with exogenously specified probabilities. The Kreps–Porteus framework has been tremendously influential in the exploration of recursive preference models and it or its extensions have been successfully applied to finance, macroeconomics, game theory and behavioral economics.[1] Recursive expected utility allows for a number of features not present in the standard discounted expected utility model including discount factors that may vary with payoffs, the separation of intertemporal substitution from intratemporal risk aversion, and preferences for the timing of the resolution of uncertainty. It does all this while retaining tractability, especially the ability to use dynamic programming and optimization.

Extending recursive expected utility to subjective domains is important for a number of reasons. First, issues of learning, updating beliefs and information acquisition cannot be effectively addressed in a solely objective framework. Second, many applications and real-world problems do not come with probabilities pre-specified. Moreover, in atemporal models of decision-making under risk, the importance of providing foundations for models with subjective beliefs has been well-recognized. For example, subjective analogues to the objective probability expected utility foundations of von Neumann and Morgenstern were provided by (among others) Savage (1954) and Anscombe and Aumann (1963).

The main body of the paper is organized as follows. In Section 2.1 we develop an appropriate space of objects of choice: the space of temporal acts. In the Kreps–Porteus framework, preferences are defined over objects called temporal lotteries that are essentially probability trees. Risk is modeled through exogenously specified objective probabilities. Our temporal acts generalize temporal lotteries by introducing a state space and temporal resolution of uncertainty about the state in addition to the temporal risk structure of Kreps–Porteus. In Section 2.2, we formally define a subjective recursive expected utility (SREU) representation of preferences over temporal acts. In Section 3, we lay out the set of preference axioms that, in Section 4.1, we show characterizes SREU for subsets of temporal acts restricted to a given filtration specifying how information is revealed over time. In addition to standard weak order and continuity axioms, we introduce four axioms: a temporal sure-thing principle, a temporal substitution axiom, and two axioms which together yield state independence of preferences at each time. In Section 4.2, we give examples in which there is more than one filtration, and show that without further assumptions, direct information effects may interact with preference for the timing of the resolution of uncertainty. Such interaction is not allowed by overall (i.e., cross-filtration) SREU preferences. This motivates two additional axioms that, in Section 4.3, we show are necessary and sufficient to extend these within filtration representations to a single SREU representation across filtrations (and thus covering the whole domain of temporal acts). The first of these axioms says that how information is revealed does not matter if the prize received does not depend on the state of the world. This axiom added to the earlier ones characterizes an SREU representation across filtrations where only the prior distribution on the state space may vary with the filtration. The second axiom requires consistency across filtrations in the way that bets on one event conditional on another are evaluated relative to lotteries. With this final axiom added to the others, full

[1] To name just a few see Dumas et al. (2000), Duffie et al. (1997), Epstein and Zin (1989, 1991), Chew and Epstein (1990), Caplin and Leahy (2001), and Grant et al. (2001)

SREU obtains. The prior distribution no longer may vary with the filtration. Section 5 discusses related literature, including previous foundations for subjective recursive representations developed by Skiadas (1998), Wang (2003), and Hayashi (2005). Proofs and some mathematical descriptions are contained in an Appendix.

2 The model

2.1 The objects of choice

In this section we present a model where the objects of choice are temporal acts – acts that encode an explicit timing structure for the resolution of uncertainty. We do this in an Anscombe–Aumann (lottery acts) framework, thus acts pay off in terms of lotteries. Specifically, our temporal acts are a generalization of Anscombe–Aumann style acts in the same sense that Kreps–Porteus temporal lotteries are a generalization of standard lotteries. In the following, for an arbitrary set X we denote the set of all lotteries with finite support on X by $\Delta(X)$. For a lottery $l \in \Delta(X)$, we denote the probability that l assigns to outcome $x \in X$ by $l(x)$. A lottery that assigns probability p_i to outcome $x_i \in X$ with $\sum_{i=1}^{n} p_i = 1$ may be written $(x_1, p_1; ...; x_n, p_n)$.

The state space that represents all subjective uncertainty is denoted by a finite set Ω. Let $\mathcal{F}$ be an algebra on Ω. Events in this formulation are elements of $\mathcal{F}$. Suppose time is indexed by $t \in \{0, \ldots, T\}$. Let Z_t be the set of possible time t prizes. We assume that each Z_t is a compact Polish (i.e., complete separable metric) space.

Let $I = \{\mathcal{F}_{I,0}, \mathcal{F}_{I,1}, \ldots \mathcal{F}_{I,T}\}$ be a filtration, i.e., each $\mathcal{F}_{I,t}$ is an algebra on Ω and $\mathcal{F}_{I,t} \subseteq \mathcal{F}_{I,t+1} \subseteq \cdots \subseteq \mathcal{F}_{I,T} = \mathcal{F}$. Denote the set of all filtrations by $\mathcal{I}$. Note that there is a unique partition that generates the algebra $\mathcal{F}_{I,t}$. Denote this partition by $\mathcal{P}_{I,t}$ and let $\mathcal{P}_{I,t}(\omega)$ denote the element of this partition that contains $\omega \in \Omega$. The interpretation is that, just *after* time t (i.e., after any time t uncertainty/risk is resolved), the decision maker will know that the state lies in $\mathcal{P}_{I,t}(\omega)$.

Let $\mathcal{F}_{-1}$ be the trivial algebra, that is $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$, and let $\mathcal{F}_{I,-1} = \mathcal{F}_{-1}$ for all $I \in \mathcal{I}$. Therefore $\mathcal{P}_{I,-1} = \{\Omega\}$ and $\mathcal{P}_{I,-1}(\omega) = \Omega$ for all ω and all I.

Fixing $I \in \mathcal{I}$, we define the set of all temporal acts with respect to the filtration I recursively:

The set of time T-temporal acts, where the information thus far revealed is given by $A \in \mathcal{P}_{I,T-1}$, is denoted by $F_{I,T,A}$. An element, f, of the set $F_{I,T,A}$ is a function $f : A \to \Delta(Z_T)$ measurable with respect to $\mathcal{F}_{I,T}$. Thus the set of all time T-temporal acts given filtration I is simply $F_{I,T} \equiv \cup_{A \in \mathcal{P}_{I,T-1}} F_{I,T,A}$. For $f \in F_{I,T}$, we write $f(\omega, z)$ to denote the probability of receiving the prize z in state ω.

The set of time t-temporal acts, where the information thus far revealed is given by $A \in \mathcal{P}_{I,t-1}$, is denoted by $F_{I,t,A}$. An element, f, of the set $F_{I,t,A}$ is a function $f : A \to \Delta(Z_t, F_{I,t+1})$, measurable with respect to $\mathcal{F}_{I,t}$, with the property that if for any $\omega \in A$, $(z, g) \in \operatorname{supp} f(\omega)$ then $g \in F_{I,t+1,\mathcal{P}_{I,t}(\omega)}$. We write $f(\omega, z, g)$ to denote the probability of receiving the prize/$t+1$ -temporal act pair, (z, g), at state ω. We write $f(\omega, z)$ to denote the marginal probability of receiving the prize z at state ω. The set of all time t-temporal acts with respect to filtration I is $F_{I,t} \equiv \cup_{A \in \mathcal{P}_{I,t-1}} F_{I,t,A}$.

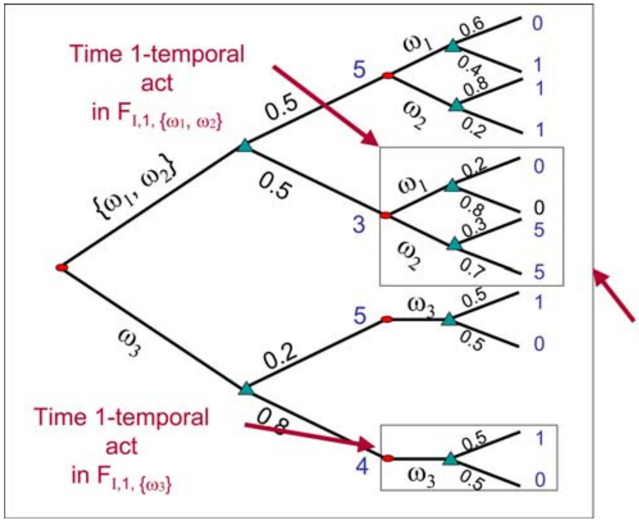

Fig. 1 A temporal act in a two period world with some time 1 continuations indicated

Since $\mathcal{P}_{I,-1} = \{\Omega\}$ notice that the set of all time 0-temporal acts with respect to filtration I, $F_{I,0} = F_{I,0,\Omega}$. We sometimes denote this set by simply F_I . [2]

Definition 1 (Temporal Acts) *The set of all* temporal acts *is* $\cup_{I \in \mathcal{I}} F_{I,0}$ *and is denoted* F.

Figure 1 illustrates a temporal act where there are three states, two time periods 0 and 1, and the partitions that generate the filtration are given by $\mathcal{P}_{I,0} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ and $\mathcal{P}_{I,1} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$. The oval nodes are "uncertainty" nodes, depicting the evolution of information about the state. The triangular nodes are "risk" nodes, depicting lotteries. The figure also illustrates time 1-temporal acts.

2.2 Subjective recursive expected utility

We now write down formally what is meant by a SREU representation of preferences over temporal acts:

Notation 1 *E_m denotes the expectation operator with respect to the measure m. (Similarly, $E_{m|A}$ denotes the expectation with respect to the measure m conditional on the event A.)*

[2] In general, $F_{I,T,A} \cap F_{I',T,A} \neq \emptyset$, so the same function may be a time T-temporal act with respect to several filtrations. The same will be true for times $t > 0$, but at time 0, we have $F_{I,0} \cap F_{I',0} = \emptyset$ for all $I' \neq I$. This last fact is true because since the filtrations differ, there must exist a time $t^* < T$ and a state ω^* such that $\mathcal{P}_{I,t^*}(\omega^*) \neq \mathcal{P}_{I',t^*}(\omega^*)$. So, at time $t^* + 1$, the continuation acts at state ω^* will be different because under filtration I they will have domain $\mathcal{P}_{I,t^*}(\omega^*)$ while under filtration I' they will have domain $\mathcal{P}_{I',t^*}(\omega^*)$.

Definition 2 (SREU Representation) *A preference relation, $\succeq$, over temporal acts has a* subjective recursive expected utility (SREU) representation *if there exists a probability measure μ on the state space, a continuous utility function $U : Z_T \to \mathbb{R}$ and continuous aggregator functions $u_t : Z_t \times \mathbb{R} \to \mathbb{R}$ for $t = 0, ..., T-1$ that combine current outcomes with continuation values such that (a) each u_t is strictly increasing in the continuation value, (b) if we define $U_T : Z_T \to \mathbb{R}$ by $U_T(z_T) = U(z_T)$ and recursively $U_t : Z_t \times \cup_{I \in \mathcal{I}} F_{I,t+1} \to \mathbb{R}$ by,*

$$U_t(z_t, f) = u_t\left(z_t, E_{\mu|A}\left[E_{f(\omega)} U_{t+1}(z_{t+1}, h)\right]\right) \quad (1)$$

where A is the domain of f, then the following holds:
For any temporal acts $f, g \in F$,

$$f \succeq g \iff E_\mu\left[E_{f(\omega)} U_0(z_0, h)\right] \geq E_\mu\left[E_{g(\omega)} U_0(z_0, k)\right]. \quad (2)$$

Observe that equation 1 is what makes the representation recursive, while equation 2 makes it recursive expected utility. It is subjective because the probability measure μ is subjective. Thus the name subjective recursive expected utility. This representation is related to a number of historically prominent recursive utility representations. Koopmans (1960) is the first, to our knowledge, to provide foundations for a recursive utility representation. His objects of choice are infinite-horizon consumption streams and his model does not consider risk or uncertainty. Epstein (1983) generalizes Koopmans' approach in order to incorporate (objective) risk and considers choice among lotteries over consumption streams. He provides foundations for expected utility representations over such lotteries where the utility function is recursive with aggregators of particular forms. Most directly related to the representation above, Kreps and Porteus (1978) model choice among temporal lotteries and provide foundations for the special case of SREU in which it is as if there is only a single state of the world, and so μ plays no role.[3] In Kreps and Porteus, as here, the timing of the resolution of lotteries (i.e., objective risk) may matter. SREU brings subjective uncertainty into the model and similarly allows the timing of the resolution of such uncertainty to matter.

2.3 Further notation and definitions

This subsection collects some definitions and notation used in the axiomatization and analysis that follows.

We define mixtures over elements of $F_{I,t,A}$ (t-temporal acts with common domain A and common filtration I).

Definition 3 (α-Mixture) *Let $f, g \in F_{I,t,A}$ where $A \in \mathcal{P}_{I,t-1}$. We denote the α-mixture of f and g by $\alpha f + (1-\alpha) g \in F_{I,t,A}$ where $\alpha \in [0, 1]$ and the mixture is taken statewise, over probability distributions (as in Anscombe and Aumann (1963)).*

[3] In its most general form, the Kreps and Porteus (1978) representation also allows utilities and aggregators to depend on the history of realized outcomes. We do not consider such history dependence here mainly because it simply adds to already heavy notation without adding much conceptual insight. If one wished (so as to capture habit formation for example), history dependence could be easily incorporated by adding the realized history as an additional argument of the functions above. Furthermore, the axiomatic foundations that we provide later in the paper could be similarly modified.

To conserve on notation and so as to treat time T together with times $t < T$, we will sometimes write as if an element of $F_{I,T}$ has range in $\Delta\left(Z_T \times F_{I,T+1}\right)$. Since $F_{I,T+1}$ is formally undefined, read $Z_T \times F_{I,T+1}$ as Z_T.

It will often be useful to be able to refer to time s-temporal acts that are, in a natural sense, continuations of time t-temporal acts where $s \geq t$. Loosely, a time s continuation should tell everything that may happen from time s onward. Since what may happen may depend on the state, we will want to talk about continuations at a given state. Furthermore, since even given the state, past randomizations may affect what may happen from s onward, it is important to note that there may be many such continuations. Formally:

Definition 4 (1-Step continuation) *For $f \in F_{I,t}$, say that g is a* 1-step continuation of f in state ω *if there exists a prize z_t such that $(z_t, g) \in \operatorname{supp} f(\omega)$.*

Definition 5 (Continuation) *For $f \in F_{I,t}$, say that g is a* continuation of f at time s in state ω *if either (a) $s = t$ and $g = f$ or (b) $s = t + 1$ and g is a 1-step continuation of f in state ω or (c) $s \geq t + 2$ and there exist $h_{t+1}, \ldots, h_{s-1}$ such that h_{t+1} is a 1-step continuation of f in state ω, h_{t+i} is a 1-step continuation of h_{t+i-1} in state ω for $i = 2, \ldots, s - t - 1$, and g is a 1-step continuation of h_{s-1} in state ω.*

We also refer to any continuation of f at time s as a *time s continuation of f*. Of special interest will be continuations that are " constant" in the sense that they do not depend on the state.

Definition 6 (Constant continuation) *For $f \in F_{I,t,A}$, say that f is* constant *if there exists $l = (l_t, l_{t+1}, \ldots, l_T) \in \Delta Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$ such that, for all times $s \geq t$ and for all states $\omega \in A$, for any g that is a continuation of f at time s in state ω, $g(\omega, \cdot) = l_s(\cdot)$. We say that f is* associated with l.

Denote the set of all constant elements of $F_{I,t,A}$ by $F^*_{I,t,A}$ and the set of all constant elements of $F_{I,t}$ by $F^*_{I,t}$. Notationally, we write $l \in \Delta Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$ to stand for a constant t-temporal act associated with l where the domain A will be clear from the context. For example, if $f \in F^*_{I,t,\mathcal{P}_{I,t-1}(\omega)}$ is associated with $m \in \Delta Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$, and $g \in F_{I,t-1,\mathcal{P}_{I,t-2}(\omega)}$, when we write $g(\omega, z_{t-1}, m)$ we mean $g(\omega, z_{t-1}, f)$. Note that this association makes sense because there is a bijection between $F^*_{I,t,\mathcal{P}_{I,t-1}(\omega)}$ and elements of $\Delta Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$. Similarly, if all elements in the support of $g(\omega)$ are in $\Delta\left(Z_{t-1} \times F^*_{I,t,\mathcal{P}_{I,t-1}(\omega)}\right)$ we may refer to $g(\omega)$ as an element of $\Delta(Z_{t-1} \times \Delta Z_t \times \cdots \times \Delta Z_T)$ without any confusion resulting. Thus $g(\omega, z_{t-1}, l)$ is the probability of receiving $(z_{t-1}, l) \in Z_{t-1} \times \Delta Z_t \times \cdots \times \Delta Z_T$ in state ω. Similarly, if $g(\omega)$ yields (z_i, l_i) with probability p_i for $i = 1, ..., n$, then we may write $g(\omega) = ((z_1, l_1), p_1; ...; (z_n, l_n), p_n)$.

Definition 7 (Constant act) *A temporal act f is a* constant act *if f is constant. We use l_f to denote the associated vector of lotteries.*

Denote the set of all constant acts by F^*.

Next we define some additional subsets of temporal acts, specifically those where all lotteries are degenerate up to (but not including) time t. For a fixed filtration I we denote the set of such temporal acts by F^t_I. We call this set F^t_I the

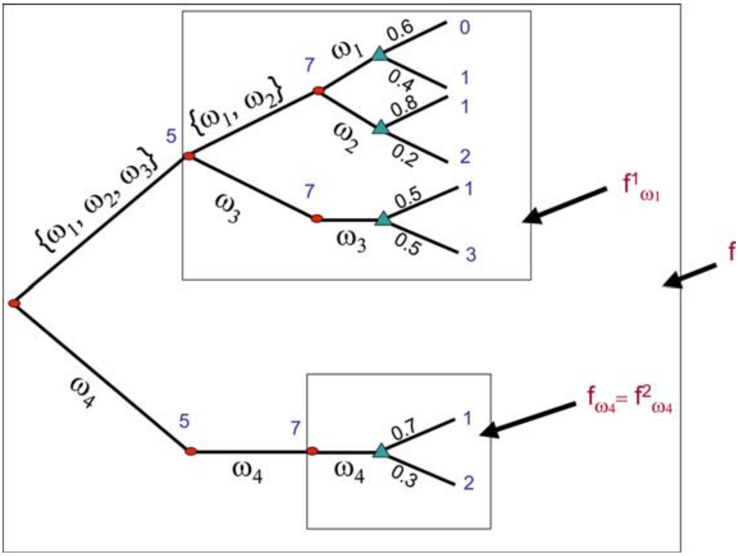

Fig. 2 $f \in F_I^2$

temporal acts degenerate up to time t (with respect to filtration I).[4] Any element of this set, given a time $s \leq t$ and a state ω, has a unique continuation at time s in state ω denoted by f_ω^s. To lighten the notation, whenever it is clear from the context, we refer to the time t continuation of an act $f \in F_I^t$ in state ω by f_ω. When two acts in F_I^t agree on the immediate prizes that they give at all states and times up to (but not including) time t, we say that they share the same prize history. Clearly, two acts that share the same prize history may have different continuations at time t. Also note that $F_I^0 = F_{I,0}$, i.e., the set of all temporal acts degenerate up to time 0 is nothing but the set of all temporal acts.

Figure 2 illustrates, $f \in F_I^2$, a temporal act degenerate up to time 2. There are four states, three time periods 0, 1 and 2, and the partitions that generate the filtration I are given by $\mathcal{P}_{I,0} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}$, $\mathcal{P}_{I,1} = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ and $\mathcal{P}_{I,2} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. At times 0 and 1, the act gives prizes 5 and 7 respectively at all states. The figure also illustrates, $f_{\omega_1}^1$, the unique time 1 continuation in state ω_1, and, $f_{\omega_4} = f_{\omega_4}^2$, the unique time 2 continuation in state ω_4.

3 Preference axioms

Our primitive is $\succeq$, a binary relation over the temporal acts, F. The following axioms will be imposed on $\succeq$:

Axiom 1 *[Weak Order] $\succeq$ is complete and transitive.*

[4] Formally, $f \in F_I^t$ if, for all ω and $s < t$, each time s continuation of f in state ω assigns probability 1 to some element of its range, $Z_s \times F_{I,s+1,\mathcal{P}_{I,s}(\omega)}$.

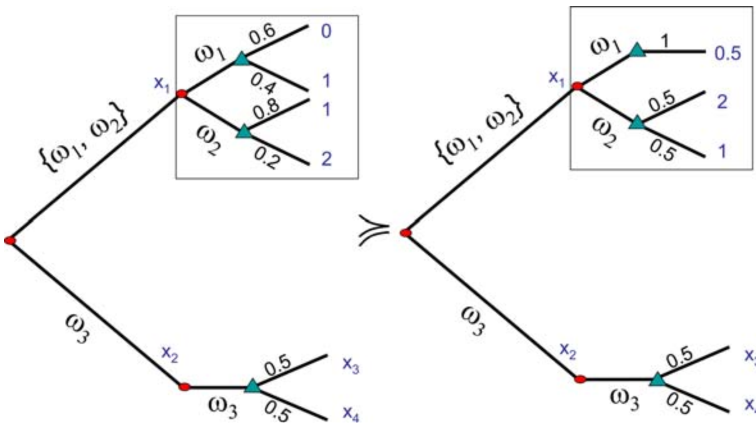

Fig. 3 $\hat{f} \succeq \hat{g}$

Axiom 2 *[Continuity]* $\succeq$ *is continuous. That is, for any* $f \in F$, *the sets,*

$$M(f) = \{g \in F \,|\, g \succeq f\}$$

and

$$W(f) = \{g \in F \,|\, f \succeq g\}$$

are closed.[5]

The weak order and continuity axioms are standard axioms in the literature and ensure the existence of a continuous real-valued representation of preferences. To understand our next axiom, consider two acts that are identical except on event $A \in \mathcal{P}_{I,t-1}$ and contain only degenerate lotteries before time t. The Temporal Sure-Thing Principle says that preference between such acts is preserved under any common change occurring in any part of the tree other than the continuation following event A. This axiom implies (a) separability from past prizes, and (b) separability from unrealized events. Figures 3 and 4 provide an illustration of the axiom.

Axiom 3 *[Temporal sure-thing principle] Fix a filtration* I *and time* t. *Let* $A \in \mathcal{P}_{I,t-1}$. *Suppose* $\hat{f}, \hat{g}, \tilde{f}, \tilde{g} \in F_I^t$ *are such that,* $\hat{f}$ *and* $\hat{g}$ *share the same prize history,* $\tilde{f}$ *and* $\tilde{g}$ *share the same prize history, and*

$$\hat{f}_\omega = \tilde{f}_\omega, \hat{g}_\omega = \tilde{g}_\omega \quad \textit{for all } \omega \in A,$$
$$\hat{f}_\omega = \hat{g}_\omega, \;\; \tilde{f}_\omega = \tilde{g}_\omega \quad \textit{otherwise,}$$

then $\hat{f} \succeq \hat{g}$ *if and only if* $\tilde{f} \succeq \tilde{g}$.

Using the temporal sure-thing principle, we may extend $\succeq$ from temporal acts to " continuation acts" (i.e., t-temporal acts) by filling in the rest of the tree in a common way as long as no risk resolves (i.e., only degenerate triangular nodes in the figures) before time t.

[5] The metric on F is defined in the appendix.

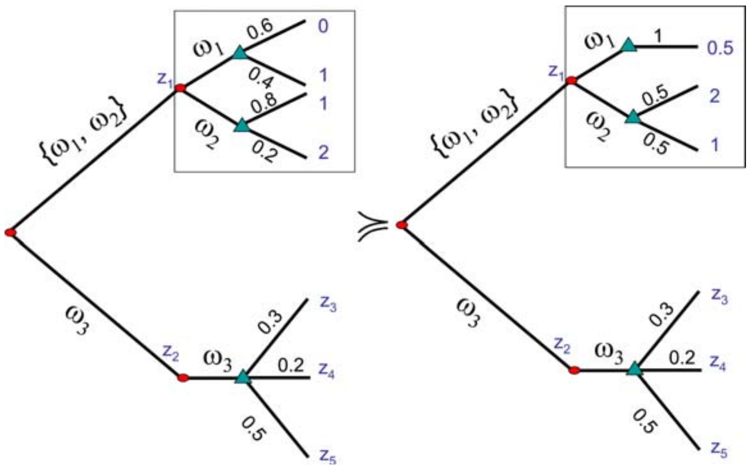

Fig. 4 $\tilde{f} \succeq \tilde{g}$

Definition 8 *[Conditional preference] For any* $f, g \in F_{I,t,A}$ *where* $A \in \mathcal{P}_{I,t-1}$ *we say that* $f \succeq g$ *if there exist temporal acts,* $\hat{f}, \hat{g} \in F_I^t$*, degenerate up to* t *and sharing the same prize history, such that* $\hat{f} \succeq \hat{g}$*,* $\hat{f}_\omega = f$*,* $\hat{g}_\omega = g$ *for all* $\omega \in A$*, and* $\hat{f}_\omega = \hat{g}_\omega$ *otherwise.*

Note that we use $\succeq$ to indicate both the overall preference relation on F_I and the induced preference relation on continuation acts in $F_{I,t,A}$. This should create no confusion. The next lemma proves that the preference relation induced on conditional acts by the definition of conditional preferences (Definition 8) is indeed a continuous weak order.

Lemma 1 *For any time* t*, filtration* I *and event* $A \in \mathcal{P}_{I,t-1}$*,* $\succeq$ *on* $F_{I,t,A}$ *is a continuous weak order.*

In what follows, we sometimes refer to the preference relation on $F_{I,t,A}$, rather than the overall preference relation on F_I. This is done just for notational convenience. Any statement involving the preference relation on $F_{I,t,A}$ may easily be restated in terms of the overall preference relation on F_I, by plugging the continuation acts into an overall reference act in F_I^t. The temporal sure-thing principle would then make sure that it does not matter which reference act is used in these comparisons.

Next we formulate a temporal substitution axiom for $\succeq$ on continuation acts. This axiom generalizes the temporal substitution axiom of Kreps and Porteus to our framework with subjective uncertainty. When there is only one state of the world, our axiom reduces to theirs. Just as in a static framework, substitution is crucial in characterizing an expected utility treatment of risk. The temporal aspect of the axiom is that the risk in question is limited to that occurring at a given time and event. This temporal aspect makes the axiom weaker than the well-known atemporal substitution/independence axiom to the extent that the decision maker is not indifferent to the timing of the resolution of uncertainty and risk (see e.g., the discussion on this point in Kreps and Porteus 1978).

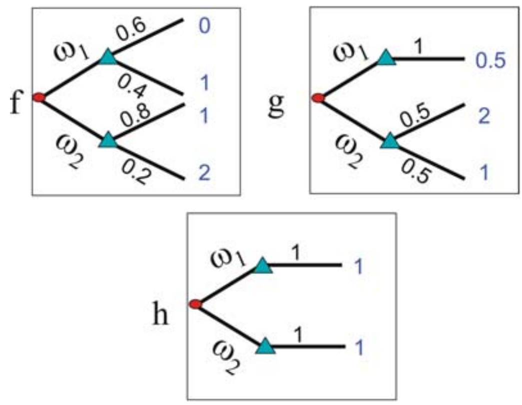

Fig. 5 f, g and h

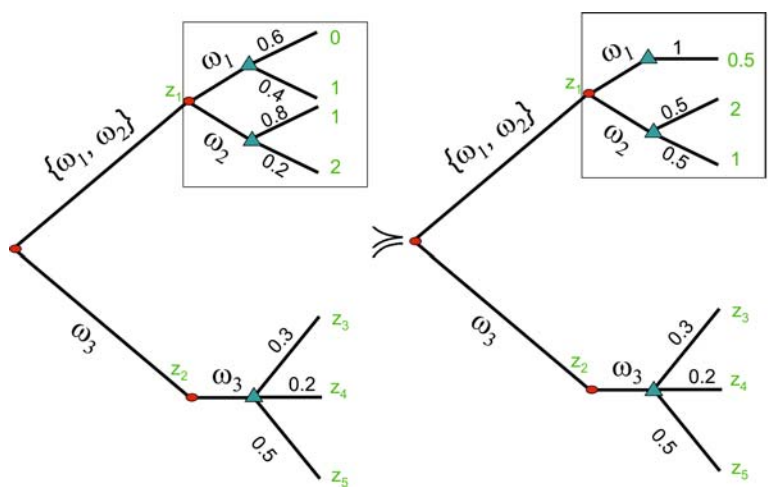

Fig. 6 $f \succeq g$

Axiom 4 *[Temporal substitution] Fix any filtration I and time t. Suppose $\alpha \in [0, 1]$ and $A \in \mathcal{P}_{I,t-1}$. For any $f, g, h \in F_{I,t,A}$.*

$$f \succeq g \text{ if and only if } \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h.$$

Figures 5-7 illustrate axiom 4.

In an atemporal Anscombe–Aumann style model, in addition to the basic weak order, continuity and substitution axioms, one needs an axiom implying state independence of preference over lotteries to deliver expected utility. The following monotonicity axiom is a temporal version of such state independence. Like temporal substitution, it is weaker than its atemporal counterpart. Specifically it requires state independence only within a given time, event and filtration. It does this by imposing monotonicity of preference with respect to dominance in constant continuations at a given time, event and filtration.

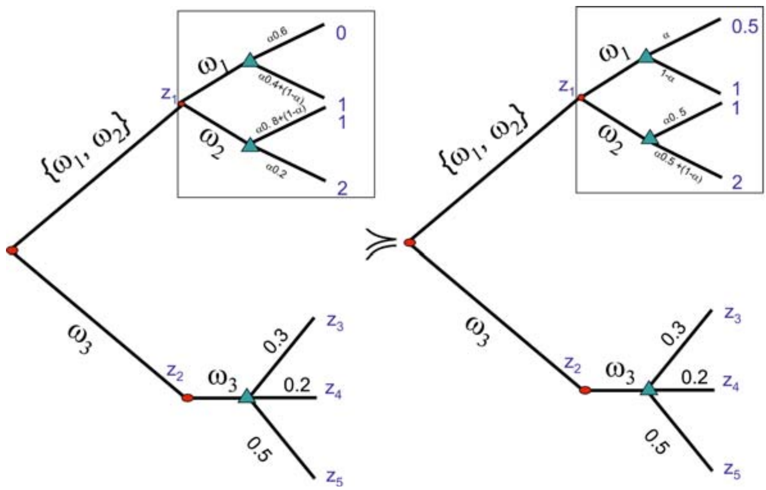

Fig. 7 $\alpha f + (1-\alpha)\, h \succeq \alpha g + (1-\alpha)\, h$

Axiom 5 *[Monotonicity] Fix any filtration I and time t. Given $A \in \mathcal{P}_{I,t-1}$, suppose that $f, g \in F_{I,t,A}$ and that all time $t+1$ continuations of f and of g are constant.*[6] *Define, for each $\omega \in A$, $f^{\omega}, g^{\omega} \in F_{I,t,A}$ as follows: For all $\omega' \in A$,*

$$f^{\omega}\left(\omega'\right) = f\left(\omega\right),$$

and,

$$g^{\omega}\left(\omega'\right) = g\left(\omega\right).$$

If $f^{\omega} \succeq g^{\omega}$ for all $\omega \in A$ then $f \succeq g$. Moreover if $f^{\omega} \succ g^{\omega}$ for some $\omega \in A$ then $f \succ g$.

Our next axiom, event independence, is also a form of state independence. It works across events, while still requiring a common time and filtration. One reason why we write monotonicity and event independence as separate axioms is that they play distinct roles in proving the main representation theorem. The first intermediate result in the proof is to show that the axioms up to monotonicity deliver a set of expected utility representations, one for each time, event and filtration. When event independence is additionally imposed, we show that the utilities in such representations may be taken to be event independent in the sense that they assign the same value to any given (prize, constant continuation) pair irrespective of the event on which it is realized.

Axiom 6 *[Event independence] Fix any filtration I and time t. Given $A, A' \in \mathcal{P}_{I,t-1}$, suppose that $f, g \in F_{I,t,A}$ and $f', g' \in F_{I,t,A'}$ have all time $t+1$ continuations constant. Further suppose that for all $\omega \in A$, $\omega' \in A'$,*

$$f\left(\omega\right) = f'\left(\omega'\right)$$

[6] Recall that this implies $f\left(\omega\right), g\left(\omega\right) \in \Delta\left(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T\right)$ for each $\omega \in A$.

and,

$$g(\omega) = g'(\omega').$$

Then $f \succeq g$ if and only if $f' \succeq g'$.

Finally, as is standard, to rule out the case where all acts are indifferent and deliver appropriate uniqueness of the representation a non-degeneracy axiom is needed. The version below is stronger than usual as it requires some strict preference at each event, time and filtration, thus implicitly ruling out events that are assigned zero weight by the preferences. This is done primarily for convenience, as dealing with null-events can be involved and is not the focus of our analysis.

Axiom 7 *[Non-degeneracy] For every filtration I, time t, and event $A \in \mathcal{P}_{I,t-1}$ there exist $f, g \in F_{I,t,A}$ for which all time $t+1$ continuations are constant and such that $f \succ g$.*

4 Representation results

4.1 Representation for a fixed filtration

Observe that except for continuity and weak order, all axioms so far concern only comparisons of temporal acts within the same filtration. We prove that the first six axioms yield a SREU representation within each filtration. Below we state the result and give a brief sketch of the main steps in the argument. The proof itself is contained in the Appendix.

Definition 9 (Within-filtration SREU representation) *A* SREU representation within a filtration *I is a SREU representation where the domain of the representation is restricted to temporal acts in F_I, the functions μ, U, and u_t in the representation are subscripted by I, and the domain of the derived $U_{I,t}$ is $Z_t \times F_{I,t+1}$ rather than $Z_t \times \cup_{I \in \mathcal{I}} F_{I,t+1}$.*

Proposition 1 *[Characterization of within-filtration SREU] Suppose preference $\succeq$ satisfies axioms weak order and continuity. Then $\succeq$ satisfies axioms temporal sure-thing principle, temporal substitution, monotonicity and event independence if and only if, for each filtration I, the restriction of $\succeq$ to F_I has a SREU representation within I.*
Furthermore, the following uniqueness properties hold. If $\left(\mu_I, U_I, \left\{u_{I,t}\right\}_{t=0}^{T-1}\right)$ and $\left(\mu'_I, U'_I, \left\{u'_{I,t}\right\}_{t=0}^{T-1}\right)$ both yield SREU representations of $\succeq$ restricted to F_I, then, for each t, the derived $U'_{I,t}$ must be a positive affine transformation of the derived $U_{I,t}$. If non-degeneracy holds, μ_I is strictly positive on its domain and μ'_I must equal μ_I.

Remark 1 The weak order and continuity axioms are stronger than necessary for the proposition above because the conclusion does not refer to any cross-filtration comparisons. If we had used versions of weak order and continuity that apply only within each F_I, the six axioms would together be necessary and sufficient for SREU within each I. We do not do so here because the stronger versions are needed for the overall SREU representation.

The proof works by first showing the SREU representation restricted to t -temporal acts for which all $t+1$ continuations are constant. Then this is extended to cover all temporal acts. To begin, we apply mixture space techniques (as in Anscombe–Aumann style theories) to show that, together with weak order and continuity, the next three axioms characterize continuous expected utility on the subset of $F_{I,t,A}$ having all $t+1$ continuations constant. This gives a set of subjective expected utility representations with the utilities and beliefs indexed by I, t, A.

Next, fixing I, we construct a measure μ_I over the whole state space such that for any event A known coming in to time t (i.e., $A \in \mathcal{P}_{I,t-1}$), the measure $\mu_{I,t,A}$ is the conditional $\mu_I \mid A$.

Then, we show that adding the axiom event independence is equivalent to being able to replace all the $U_{I,t,A}$ with a common $U_{I,t}$ that assigns the same value to any given pair of immediate prize and continuation stream of lotteries irrespective of the event on which the continuation is realized.

Next we show that there is a recursive relationship between $U_{I,t}$ and $U_{I,t+1}$ that holds when evaluating constant continuations. The proof works by exploiting the nested structure of temporal acts degenerate up to $t+1$ with constant time $t+1$ continuations (nested since they are also temporal acts degenerate up to t with constant time $t+1$ continuations). Consider two such temporal acts differing only in their time $t+1$ continuation on some event $B \in \mathcal{P}_{I,t}$ and let A be the event in $\mathcal{P}_{I,t-1}$ containing B. By the temporal sure-thing principle these may be compared either by comparing their time $t+1$ continuations on B or by comparing their time t continuations on A and, furthermore, these comparisons must come out the same. Applying the subjective expected utility representations derived above to these two pairs of continuations then yields the relation between the time t and $t+1$ utilities on event B. As described in the preceding step, event independence ensures that this holds across events as well. This relation across time is what determines the aggregator functions $u_{I,t}$.

Finally, we show that the representations that apply in the constant continuation case may be extended to cover all temporal acts for a fixed filtration. In broad strokes, the argument uses continuity together with temporal substitution to show that "constant act equivalents" exist and that replacing continuations by their constant equivalents preserves the representations derived in the earlier steps.

4.2 Filtration-dependence and timing attitudes

Kreps and Porteus (1978) show that the curvature of an aggregator like $u_{I,t}$ characterizes attitude towards timing of the resolution of risk. Specifically, if the aggregator is convex (resp. concave) in its second argument then the decision maker prefers early (resp. late) resolution of risk. In their model, risk (through lotteries) is the only source of uncertainty, whereas our model contains states of the world in addition to lotteries. This leads to at least two differences regarding attitude towards timing.

First, attitude towards timing of the resolution of (lottery) risk may vary with the timing of the resolution of uncertainty about the state of the world (i.e., with the filtration). In our model, this will occur when the aggregators depend on the filtration.

Second, even when the aggregators do not depend on the filtration, attitude towards timing of the resolution of (lottery) risk may be distinct from attitude towards the timing of the resolution of uncertainty about the state of the world. The latter is influenced not only by the aggregators, but also by the way beliefs, μ_I, may vary with the filtration.

The next example illustrates the first difference mentioned above. In it, the aggregator is convex for one filtration and concave for the other filtration. Using temporal acts for each filtration that do not depend on the state (analogues in our setting of Kreps–Porteus temporal lotteries) the example shows that the decision maker prefers early resolution of risk in the first filtration, but late resolution of risk in the other filtration.

Example 1 Suppose there are two states of the world, i.e., $\Omega = \{\omega_1, \omega_2\}$ and two time periods, i.e., $T = 1$. Let filtration I describe a situation where the decision maker learns the true state of the world coming out of time 0. Thus, I can be generated by the partitions $\mathcal{P}_{I,0} = \mathcal{P}_{I,1} = \{\{\omega_1\}, \{\omega_2\}\}$. In contrast, let filtration I' describe a situation where the decision maker does not learn the true state until the end. Thus, I' is generated by the partitions $\mathcal{P}_{I',0} = \{\{\omega_1, \omega_2\}\}$ and $\mathcal{P}_{I',1} = \{\{\omega_1\}, \{\omega_2\}\}$.

We define two pairs of temporal acts, one pair on filtration I and the other on I'. All four temporal acts ultimately result in the payoff stream $(0, 0)$ with probability $\frac{1}{2}$ and $(0, 9)$ with probability $\frac{1}{2}$. Within each pair, the first temporal act resolves this risk only at time 1 while the second resolves it at time 0. For all four acts, the resolution and timing of this risk does not depend on the state of the world. Yet, we will see that the timing of the resolution of information about the state of the world will affect preference.

Recall that $(z_1, p_1; \ldots; z_n, p_n)$ denotes a lottery and $f(\omega, z, (z_1, p_1; \ldots; z_n, p_n))$ denotes the probability that, in state ω, temporal act f yields time 0 prize z followed by the constant continuation that yields the lottery $(z_1, p_1; \ldots; z_n, p_n)$ at time 1. In reading the following definitions it may be helpful to look at Figures 8 and 9.

Let $h \in F_I$ be defined by

$$h\left(\omega_1, 0, \left(0, \frac{1}{2}; 9, \frac{1}{2}\right)\right) = 1 = h\left(\omega_2, 0, \left(0, \frac{1}{2}; 9, \frac{1}{2}\right)\right),$$

and $h' \in F_I$ be defined by

$$h'(\omega_1, 0, (0, 1)) = h'(\omega_1, 0, (9, 1)) = \frac{1}{2} = h'(\omega_2, 0, (0, 1)) = h'(\omega_2, 0, (9, 1)).$$

Similarly, let $k \in F_{I'}$ be defined by

$$k\left(\omega_1, 0, \left(0, \frac{1}{2}; 9, \frac{1}{2}\right)\right) = 1 = k\left(\omega_2, 0, \left(0, \frac{1}{2}; 9, \frac{1}{2}\right)\right),$$

and $k' \in F_{I'}$ be defined by

$$k'(\omega_1, 0, (0, 1)) = k'(\omega_1, 0, (9, 1)) = \frac{1}{2} = k'(\omega_2, 0, (0, 1)) = k'(\omega_2, 0, (9, 1)).$$

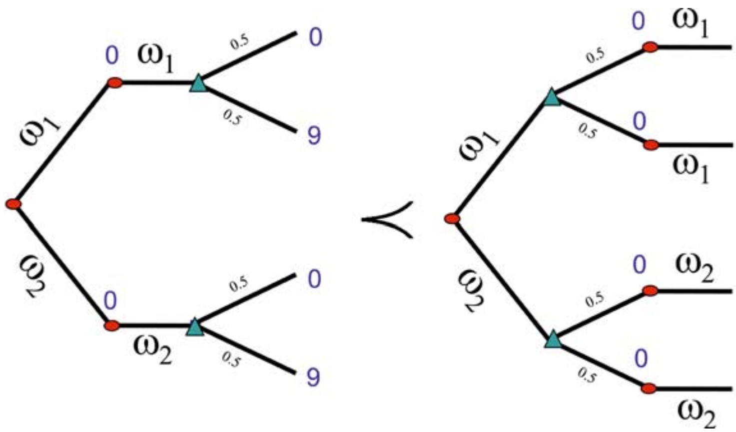

Fig. 8 $h \prec h'$

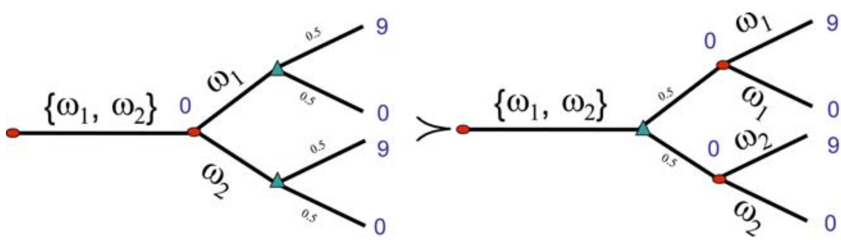

Fig. 9 $k \succ k'$

Suppose that $U_I(z) = U_{I'}(z) = z$, $u_{I,0}(z, \gamma) = z + \gamma^2$ and $u_{I',0}(z, \gamma) = z + \sqrt{\gamma}$. Applying Proposition 1, we see that $h' \succ h$ but $k \succ k'$since

$$40.5 = \frac{1}{2}(0 + 9^2) + \frac{1}{2}(0 + 0^2) > \left[0 + \left(\frac{1}{2}9 + \frac{1}{2}0\right)^2\right] = 20.25$$

and

$$1.5 = \frac{1}{2}(0 + \sqrt{9}) + \frac{1}{2}(0 + \sqrt{0}) < \left[0 + \sqrt{\left(\frac{1}{2}9 + \frac{1}{2}0\right)}\right] \approx 2.121.$$

So, with filtration I the decision maker prefers early resolution of risk, while with filtration I' late resolution of risk is preferred.

One of the things we do in the next section is provide an additional axiom that rules out dependence of attitude toward the timing of the resolution of (lottery) risk on the filtration. This is an important step in characterizing an SREU representation that applies across filtrations.

4.3 Representations across filtrations

Having obtained a representation for preference over temporal acts that share the same filtration, we now turn to comparisons of temporal acts across different filtrations. Such comparisons are crucial in many economically relevant choices. For example, any problems involving costly information acquisition – where an important decision is whether to bear a cost to learn information about the state of the world versus having the information revealed only later – are inherently comparisons between temporal acts defined on different filtrations. So far, the only axioms restricting cross-filtration preferences are weak order and continuity. They alone only guarantee an ordinal representation. To allow the recursive forms derived above to apply across filtrations as well, two invariance properties are required.

The first concerns acts that give a deterministic sequence of prizes up to time t and then give a lottery over time t prize/constant continuation pairs. For such acts, even though some information about the state of the world may be learned over time, this information has no consequences for the lotteries over prizes that the decision maker will receive. In theory, one could imagine that a decision maker might have some preference over the way information about the true state $\omega \in \Omega$ unfolds even when the state is utterly payoff irrelevant in this way. Or one might imagine that the way that information unfolds might somehow interact with the tastes (e.g., attitude toward timing of the resolution of risk (as in Example 1) or attitude toward risk) of the decision maker causing them to change with the filtration – a kind of dynamic state dependence. We wish to rule out such " pure information effect" behaviors, so that we may focus on the treatment of information in terms of *what* it conveys about outcomes and *when* it conveys it. This is done through the following axiom.

Axiom 8 *(Invariance to irrelevant information) For any I, I', t and $\ell \in \Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$, if $f \in F_I^t$ and $g \in F_{I'}^t$ give the same deterministic stream of prizes up to time t and $f_\omega(\omega) = g_\omega(\omega) = \ell$ for all ω then $f \sim g$.*

This axiom says that preference over acts giving a fixed stream of prizes followed by a lottery over prize/constant continuation pairs depends only on the identity of the stream of prizes and lottery, and in particular is independent of the filtration on which the acts are defined.

The next proposition shows that adding this axiom to the others is equivalent to SREU over all temporal acts with the modification that the beliefs may depend on the filtration.

Proposition 2 *(Characterization of SREU with filtration-dependent beliefs) Preferences $\succeq$ satisfy axioms weak order, continuity, temporal sure-thing principle, temporal substitution, monotonicity, event independence, and invariance to irrelevant information if and only if $\succeq$ has an SREU representation with the modification that instead of a single probability measure μ there is a collection of probability measures $\{\mu_I\}_{I\in\mathcal{I}}$ such that measure μ_I is used in evaluating temporal acts in F_I.*

Furthermore, the following uniqueness properties hold. If $\left(\{\mu_I\}_{I\in\mathcal{I}}, U, \{u_t\}_{t=0}^{T-1}\right)$ and $\left(\{\mu'_I\}_{I\in\mathcal{I}}, U', \{u'_t\}_{t=0}^{T-1}\right)$ both yield such an SREU representation of $\succeq$ then, for each t, the derived U'_t must be a positive affine transformation of the derived

U_t. If non-degeneracy holds, each μ_I is strictly positive and each μ'_I must equal the corresponding μ_I.

Consider the state space and filtrations in Example 1. We define two temporal acts, one on filtration I and the other on I'. Let $f \in F_I$ be defined by $f(\omega_1, 0, (9, 1)) = 1$ and $f(\omega_2, 0, (0, 1)) = 1$. Let $g \in F_{I'}$ be defined by $g(\omega_1, 0, g_1) = g(\omega_2, 0, g_1) = 1$ where $g_1(\omega_1, 9) = g_1(\omega_2, 0) = 1$.

Note that f and g both give a payoff of 0 at time 0 regardless of the state of the world, and at time 1 they both give 9 if the state is ω_1 and 0 if the state is ω_2. Yet, they differ in terms of when the payoff uncertainty is resolved. For f the resolution is immediate, whereas for g it is delayed.

Example 2 Suppose that $U(z) = z^{1/2}$, $u_0(z, \gamma) = z + \gamma^2$, and $\mu_I(\omega_1) = 0.5$ and $\mu_{I'}(\omega_1) = 0.8$. Applying Proposition 2, we see that $f \prec g$, since

$$0.5\left(0 + \left(9^{1/2}\right)^2\right) + 0.5\left(0 + \left(0^{1/2}\right)^2\right) < 0 + \left(0.8\left(9^{1/2}\right) + 0.2\left(0^{1/2}\right)\right)^2.$$

In example 2, the decision maker's beliefs are filtration dependent, and in particular, he assigns higher probability to ω_1 when the information is revealed later. Since these acts give a much better payoff in state ω_1, the decision maker prefers later resolution of uncertainty when comparing these two acts. This occurs even though these preferences reflect a preference for early resolution of (lottery) risk (since the aggregator, u_0, is convex in the continuation value).

We now turn to a second invariance property. This requires that uncertainty generated through the state space is calibrated with uncertainty generated through lotteries in the same way across filtrations.

Notation 2 *Let $I^e \in \mathcal{I}$ be the filtration where all information is learned at the* earliest *possible time, i.e., coming out of time* 0. *Formally, $\mathcal{F}_{I^e,0} = \mathcal{F}$.*

Axiom 9 *[Consistent beliefs] Fix any time t, filtration I, $\alpha \in [0, 1]$, event $A \in \mathcal{P}_{I,t}$, prizes $w, x \in Z_0$ and $y, z \in Z_t$ and streams of lotteries $l, m \in \Delta Z_1 \times \cdots \times \Delta Z_T$ and $l', m' \in \Delta Z_{t+1} \times \cdots \times \Delta Z_T$. Denote by B the unique event such that $B \in \mathcal{P}_{I,t-1}$ and $B \supseteq A$. Consider temporal acts $f, g, h, k \in F_{I^e}$ and $f', g', h', k' \in F_I^t$ where the latter share the same deterministic stream of prizes up to time t. Further suppose:*[7]

$$h(\omega) = ((w, l), 1) \text{ and } k(\omega) = ((x, m), 1) \text{ for all } \omega,$$

$$h'_\omega(\omega) = \left(\left(y, l'\right), 1\right) \text{ and } k'_\omega(\omega) = \left(\left(z, m'\right), 1\right) \text{ for all } \omega,$$

$$f(\omega) = \begin{cases} ((w, l), 1) & \text{if } \omega \in A \\ ((x, m), 1) & \text{if } \omega \notin A \end{cases},$$

$$g(\omega) = \begin{cases} ((w, l), \alpha; (x, m), (1 - \alpha)) & \text{if } \omega \in B \\ f(\omega) & \text{if } \omega \notin B \end{cases}$$

[7] Recall that $((w, l), 1)$ is a lottery yielding (w, l) with probability 1. Similarly, $((w, l), \alpha; (1 - \alpha))$ is a lottery yielding (w, l) with probability α and (x, m) with probability $(1 - \alpha)$.

$$f'_\omega(\omega) = \begin{cases} \left(\left(y, l'\right), 1\right) & \text{if } \omega \in A \\ \left(\left(z, m'\right), 1\right) & \text{if } \omega \notin A \end{cases}, \text{ and}$$

$$g'_\omega(\omega) = \begin{cases} \left(\left(y, l'\right), \alpha; (z, m'), (1-\alpha)\right) & \text{if } \omega \in B \\ f'_\omega(\omega) & \text{if } \omega \notin B \end{cases}.$$

Then

$$h \nsim k \text{ and } h' \nsim k'$$

$$\Longrightarrow$$

$$f \sim g \iff f' \sim g'.$$

As long as the decision maker cares which prize/continuation pair she gets (so as to rule out the trivial cases where, for example, $f \sim g$ no matter what α is), the axiom says that the lottery odds judged equivalent to betting on event A conditional on B are the same under any filtration. [8] To see this, note that given the representation from Proposition 2, $f \sim g \iff \mu_{I^e}(A\,|B) = \alpha$ and $f' \sim g' \iff \mu_I(A\,|B) = \alpha$. Therefore this axiom allows us to show that beliefs over the state space do not depend on the filtration, ruling out examples such as example 2.

With the addition of this axiom we can now state our main result, the SREU representation theorem:

Theorem 1 *[Characterization of SREU] Preferences $\succeq$ satisfy axioms weak order, continuity, temporal sure-thing principle, temporal substitution, monotonicity, event independence, invariance to irrelevant information and consistent beliefs if and only if $\succeq$ has an SREU representation.*
Furthermore, the following uniqueness properties hold. If $\left(\mu, U, \{u_t\}_{t=0}^{T-1}\right)$ and $\left(\mu', U', \left\{u'_t\right\}_{t=0}^{T-1}\right)$ both yield SREU representations of $\succeq$ then, for each t, the derived U'_t must be a positive affine transformation of the derived U_t. If non-degeneracy holds, μ is strictly positive and μ' must equal μ.

5 Discussion of related literature

To our knowledge, there are three previous papers that have provided foundations for subjective recursive classes of preferences that include recursive expected utility. These papers are Skiadas (1998) (see also the related Skiadas 1997; Wang 2003; Hayashi 2005).

We first discuss the Skiadas (1998) paper that, in a highly innovative framework, develops axioms describing a very general recursive form. Theorem 3 of Skiadas (1998) derives an SREU representation as a special case and, to our knowledge,

[8] This axiom is related to the "Horse/Roulette Replacement Axiom" of Machina and Schmeidler (1995) that they use as the main driver in characterizing probabilistically sophisticated beliefs in an Anscombe-Aumann setting and shares the same flavor of calibrating beliefs using lotteries to impose consistency. In their case consistency is across outcomes while in our case it is across filtrations.

is the first SREU result in the literature. There are substantial differences from our development in both the nature of the framework and the axioms. The whole flavor of the approach is quite different. For example, a crucial axiom for Skiadas's approach is Event Coherence. To state it, a little notation is required. In his framework, an act is a mapping from states and times into consumption together with a filtration that it is adapted to. Skiadas takes as primitive conditional preference relations over acts at any given time t and event E and denotes such preference by $\succeq_t^E$. Event Coherence says that for any disjoint events F and G and acts f and g where the associated filtrations have F and G as events in their respective time t algebras, if $f \succeq_t^F g$ and $f \succeq_t^G g$ then $f \succeq_t^{F \cup G} g$ (and a similar version with all preferences strict). In our framework, we do not refer to objects like $\succeq_t^{F \cup G}$as we condition only on elements of partitions rather than general events in the algebras generated by the partitions (in terms of our trees, we condition only on oval nodes, not sets of oval nodes at a given time). Thus, the parts of the preferences considered in the axioms are quite distinct in the two theories. One consequence is that the two approaches make connections with other theories more or less apparent. One thing that we feel is attractive about our approach is (as elaborated below) that it becomes quite easy to compare with the objective formulation of Kreps and Porteus (1978) and with standard timeless Anscombe-Aumann style models.

We next discuss the Wang and Hayashi papers. Each successfully integrates the treatment of ambiguity with that of timing of the resolution of uncertainty. To do so they axiomatize recursive forms including representations involving multiple priors. The relevant points of comparison with our work are those aspects of their results that do not involve ambiguity.

Both Wang and Hayashi work in infinite horizon environments and impose stationarity on preferences, while we work in a finite horizon setting. A key axiom for both Wang and Hayashi is a version of dynamic consistency. The essence of dynamic consistency is that if one continuation is preferred to another by tomorrow's preference no matter what is learned between today and tomorrow then a current outcome followed by the preferred continuation should be preferred according to today's preference to the same current outcome followed by the other continuation. Thus, dynamic consistency ties together conditional preference at different times. We don't assume dynamic consistency in our approach (though it is clearly satisfied by SREU). In fact, none of our axioms make explicit comparisons of conditional preferences (as derived in our Definition 8) across time. In contrast, the key axiom of our approach is temporal substitution at each time and event. Additionally, as we describe below, neither Wang nor Hayashi characterizes a full SREU representation.

The representation in Wang's Theorem 4.1 can be specialized to a representation like SREU by taking his "state-aggregator" μ to be conditional subjective expected utility. However, the conditions under which μ takes this form are not fully investigated by Wang. The most closely related result in his paper (Theorem 5.2) yields an expected utility form for μ with a conditional measure that varies with the filtration, but this is obtained through two assumptions – timing indifference and future independence – that are not generally satisfied by SREU. Conditions connecting beliefs across filtrations are also not investigated.

Hayashi's paper works with a fixed filtration. He describes how to specialize his main representation theorem to a SREU representation theorem within that fixed filtration. There are no developments comparable to the representations across filtrations described in our Section 4.3.

Compared to all of the above papers, the foundations we provide are more directly related to axioms familiar from timeless models of decision making under uncertainty. In particular, our approach emphasizes a temporal version of the usual substitution/independence axiom. We hope that this connection with timeless models allows more of the intuition and understanding built up there to be profitably exploited in the dynamic setting.

We think that the perspective offered by our development is especially useful in clarifying the distinction between an objective recursive expected utility model as in Kreps and Porteus (1978) and SREU. In particular, Theorem 2 in Kreps and Porteus (1978) shows that (with the addition of history independence) weak order, continuity and temporal substitution defined over their temporal lotteries are equivalent to (objective) recursive expected utility. Our framework and axioms are constructed in such a way that it is easy to see exactly how temporal substitution should be generalized and what other requirements are needed to obtain a version including subjective probabilities. Of special note is that our analysis makes clear what is needed in going from a representation that applies within a single filtration to an across filtration representation.

6 Appendix

6.1 Topology on the space of temporal acts

Let (S, d) be a metric space. For $\varepsilon > 0$ let

$$A^{\varepsilon} = \{y \in S \,|d\,(x, y) < \varepsilon, \qquad \text{for some } x \in A\}.$$

The *Prohorov metric* ρ on the set of Borel probability measures is defined as follows. For any two Borel probability measures μ_1 and μ_2 on S let

$$\rho\,(\mu_1, \mu_2) = \inf\,\{\varepsilon > 0\,|\mu_1\,(A) \leq \mu_2\,(A^{\varepsilon}) + \varepsilon \text{ for all Borel sets } A\}.$$

Suppose (Z_t, d_t) is a metric space for each $t \in \{0, \ldots, T\}$. As in Kreps and Porteus (1978), let $\Delta\,(Z_T)$ be endowed with the Prohorov metric. For any I and any $A \in \mathcal{P}_{I,T-1}$, let $\delta_{I,T,A}$ be a metric on $F_{I,T,A}$, defined by,

$$\delta_{I,T,A}\,(f, g) = \sup_{\omega \in A} \rho\,(f\,(\omega)\,, g\,(\omega))\,.$$

Iteratively, for any $A \in \mathcal{P}_{I,t}$, define the metric on $Z_t \times F_{I,t+1,A}$ to be the product metric $p_{I,t,A}$, specifically,

$$p_{I,t,A}\left((z_t, f), (z'_t, f')\right) = \frac{1}{2}\frac{d_t(z_t, z'_t)}{1 + d_t(z_t, z'_t)} + \frac{1}{4}\frac{\delta_{I,t+1,A}\left(f, f'\right)}{1 + \delta_{I,t+1,A}\,(f, f')}.$$

and take the metric on $\Delta\left(Z_t \times F_{I,t+1,A}\right)$ to be, $\rho_{I,t,A}$ the Prohorov metric with respect to $p_{I,t,A}$. Then, for any I and any $A \in \mathcal{P}_{I,t-1}$, let $\delta_{I,t,A}$ be a metric on $F_{I,t,A}$, defined by,

$$\delta_{I,t,A}\,(f, g) = \sup_{\omega \in A} \rho_{I,t,\mathcal{P}_{I,t}(\omega)}\,(f\,(\omega)\,, g\,(\omega))\,.$$

Iterating, we now have a metric on $F_{I,0,\Omega}$ for any given I. Finally, define a metric on F, by

$$\delta(f,g) = \begin{cases} \delta_{I,0,\Omega}(f,g) & \text{if } \exists I \in \mathcal{I} \text{ such that } f,g \in F_I, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that this is indeed a metric since the fact that $\delta_{I,0,\Omega}$ is bounded above by 1 for any I (because any Prohorov metric is) ensures that δ satisfies the triangle inequality.

6.2 Proof of Lemma 1

Weak order follows immediately from the weak order and temporal sure-thing principle axioms. To show that $\succeq$ on $F_{I,t,A}$ is continuous we need to prove that for any $f \in F_{I,t,A}$ the sets $\left\{g \in F_{I,t,A} \,|\, g \succeq f\right\}$ and $\left\{g \in F_{I,t,A} \,|\, f \succeq g\right\}$ are closed.

To show that the former set is closed, fix an $f \in F_{I,t,A}$ and a sequence $g^n \in F_{I,t,A}$ such that $g^n \succeq f$ for each n. Suppose $g^n \to g$ (i.e., $\delta_{I,t,A}(g, g^n) \to 0$). We now show $g \succeq f$. Construct $\hat{f} \in F_I^t$ and $\hat{g}^n \in F_I^t$ for each n sharing the same prize history $(z_0, \ldots z_{t-1})$ as follows. Let $\hat{f}_\omega = f$, $\hat{g}^n_\omega = g^n$ for all $\omega \in A$ and for all n. For each n, let $\hat{g}^n_\omega = \hat{f}_\omega$ for all $\omega \notin A$. Also construct $\hat{g} \in F_I^t$ having prize history $(z_0, \ldots z_{t-1})$ so that $\hat{g}_\omega = g$ for all $\omega \in A$,and $\hat{g}_\omega = \hat{f}_\omega$ otherwise.

Next, let $A \subseteq A_{t-2} \subseteq A_{t-3} \subseteq \cdots \subseteq A_0 \subseteq \Omega$ with $A_s \in \mathcal{P}_{I,s}$ denote the unique path to the event A in filtration I. Let $\bar{\omega} \in A$ and observe that[9]

$$\begin{aligned}
\delta\left(\hat{g}, \hat{g}^n\right) &= \delta_{I,0,\Omega}\left(\hat{g}, \hat{g}^n\right) \\
&= \sup_{\omega \in \Omega} \rho_{I,0,\mathcal{P}_{I,0}(\omega)}\left(\hat{g}(\omega), \hat{g}^n(\omega)\right) \\
&= \rho_{I,0,A_0}\left(\hat{g}(\bar{\omega}), \hat{g}^n(\bar{\omega})\right) \\
&= p_{I,0,A_0}\left((z_0, \hat{g}^1_{\bar{\omega}}), (z_0, \hat{g}^{n,1}_{\bar{\omega}})\right) \\
&= \frac{1}{4} \frac{\delta_{I,1,A_0}\left(\hat{g}^1_{\bar{\omega}}, \hat{g}^{n,1}_{\bar{\omega}}\right)}{1 + \delta_{I,1,A_0}\left(\hat{g}^1_{\bar{\omega}}, \hat{g}^{n,1}_{\bar{\omega}}\right)} \\
&\leq \delta_{I,1,A_0}\left(\hat{g}^1_{\bar{\omega}}, \hat{g}^{n,1}_{\bar{\omega}}\right) \\
&\quad \cdots \\
&\leq \delta_{I,t,A}\left(\hat{g}^t_{\bar{\omega}}, \hat{g}^{n,t}_{\bar{\omega}}\right) \\
&= \delta_{I,t,A}\left(g, g^n\right).
\end{aligned}$$

Since $\delta_{I,t,A}(g, g^n) \to 0$ by assumption, the above shows that $\delta\left(\hat{g}, \hat{g}^n\right) \to 0$ as well.

[9] Recall that, for $g \in F_I^t$, g^s_ω denotes the unique continuation of g in state ω at time $s < t$. So, $\hat{g}^{n,s}_{\bar{\omega}}$ is the unique time s continuation of $\hat{g}^n$ in state $\bar{\omega}$.

By the temporal sure thing principle axiom, $\hat{g}^n \succeq \hat{f}$ for each n. Since $\hat{g}^n \to \hat{g}$, the continuity axiom implies $\hat{g} \succeq \hat{f}$. By the construction of $\hat{g}$ and $\hat{f}$ and the definition of $\succeq$ on $F_{I,t,A}$, $g \succeq f$. This proves that $\left\{g \in F_{I,t,A} \,|g \succeq f\right\}$ is closed. The analogous arguments may be used to show closure for $\left\{g \in F_{I,t,A} \,|f \succeq g\right\}$.

6.3 Proof of Proposition 1

The proof works by first showing the SREU representation restricted to t -temporal acts for which all $t+1$ continuations are constant. Then this is extended to cover all temporal acts. To begin, we apply mixture space techniques (as in Anscombe–Aumann style theories) to show that, together with weak order and continuity, the next three axioms are sufficient for continuous expected utility on the subset of $F_{I,t,A}$ having all $t+1$ continuations constant. This gives a set of subjective expected utility representations with the utilities and beliefs indexed by I, t, A.

Proposition 3 *Suppose preference $\succeq$ satisfies axioms weak order, continuity, temporal sure-thing principle, temporal substitution, and monotonicity. Then there exists, for each $t \in \{0, ..., T\}$ and for each $A \in \mathcal{P}_{I,t-1}$ a function $U_{I,t,A} : Z_t \times \cup_{\omega\in A} F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)} \to \mathbb{R}$, continuous in both arguments, and a probability measure $\mu_{I,t,A}$ on the restriction of $\mathcal{F}_{I,t}$ to A such that if all time $t+1$ continuations of $f, g \in F_{I,t,A}$ are constant then,*

$$f \succeq g \Leftrightarrow \int_A \sum_{(z_t,h)\in \operatorname{supp} f(\omega)} f(\omega, z_t, h)\, U_{I,t,A}(z_t, h)\, d\mu_{I,t,A}$$

$$\geq \int_A \sum_{(z_t,k)\in \operatorname{supp} g(\omega)} g(\omega, z_t, k)\, U_{I,t,A}(z_t, k)\, d\mu_{I,t,A}$$

with $U_{I,t,A}(z_t, h) = U_{I,t,A}(z_t, k)$ if h and k are constant and associated with the same vector of lotteries. Moreover, each $U_{I,t,A}$ is unique up to positive affine transformations and if non-degeneracy holds each $\mu_{I,t,A}$ is unique and strictly positive on its domain.

Proof Fix $I \in \mathcal{I}$. Fix an event $A \in \mathcal{P}_{I,T-1}$. Elements of $F_{I,T,A}$ are functions from A to ΔZ_T. Observe that these are "Anscombe-Aumann"-style acts. By Lemma 1 $\succeq$ on $F_{I,T,A}$ induced from $\succeq$ on temporal acts via the temporal sure-thing principle satisfy weak order and continuity on that domain. Together with axioms Temporal Substitution and Monotonicity this allows us to apply a known Anscombe–Aumann-style expected utility representation theorem (see e.g., Schmeidler 1989) to deliver $U_{I,T,A}$ and $\mu_{I,T,A}$ satisfying the Proposition. Continuity ensures that $U_{I,T,A}$ is continuous. Given non-degeneracy, $U_{I,T,A}$ is unique up to positive affine transformations. The uniqueness of $\mu_{I,T,A}$ follows from nondegeneracy in the usual way. The strict positivity of $\mu_{I,T,A}$ follows from the strict part of monotonicity.

Next, fix a time $t = 0, ..., T-1$ and an event $A \in \mathcal{P}_{I,t-1}$. Elements of $F_{I,t,A}$ where all time $t+1$ continuations are constant are functions from A to $\Delta(Z_t \times \cup_{\omega\in A} F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)})$. Since each element of $\cup_{\omega\in A} F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)}$ has an associated element of $\Delta Z_{t+1} \times \cdots \times \Delta Z_T$, these functions may be taken to be maps from A to $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$. As above, taking the state space to be A and the outcome set to be $Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$ we are in an Anscombe–Aumann setting. As there is a bijection relating $F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)}$ and $\Delta Z_{t+1} \times \cdots \times \Delta Z_T$ for each $\omega \in A$, the Monotonicity axiom applied to the functions from A to $\Delta(Z_t \times \cup_{\omega \in A} F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)})$ yields the monotonicity in e.g., Schmeidler (1989) applied to the functions from A to $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$. Noting that the subset of $F_{I,t,A}$ where all time $t+1$ continuations are constant is closed under the mixture operations in the temporal substitution axiom, the other Anscombe–Aumann properties follow just as for the T case yielding an expected utility representation where the outcomes are elements of $Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T$. Denoting by $V_{I,t,A}(z_t, l)$ the utility function in this representation and setting $U_{I,t,A}(z_t, h) = V_{I,t,A}(z_t, l)$ if h is constant and associated with the the vector of lotteries l then the representation holds. Uniqueness follows as usual. □

Remark 2 Since the above Proposition shows that $U_{I,t,A}(z_t, h) = U_{I,t,A}(z_t, k)$ if h and k are constant and associated with the same vector of lotteries, we may write $U_{I,t,A}(z_t, l)$, for $l \in \Delta Z_{t+1} \times \cdots \times \Delta Z_T$, to mean $U_{I,t,A}(z_t, h)$ for any h associated with l.

Next we show that, fixing I, the $\mu_{I,t,A}$ are the conditionals of a single μ_I defined over the whole state space.

Proposition 4 *There exists a probability measure μ_I (unique given $\mu_{I,t,A}$'s) on $\mathcal{F}$ such that, for all $t \in \{0, \ldots, T\}$ and $A \in \mathcal{P}_{I,t-1}$,*

$$\mu_I | A \equiv \frac{\mu_I}{\mu_I(A)} = \mu_{I,t,A}, \tag{3}$$

on the domain of $\mu_{I,t,A}$.

Proof For $C \in \mathcal{F}$, set

$$\begin{aligned}&\mu_I(C)\\ &= \int_{\Omega} \int_{\mathcal{P}_{I,0}(\omega)} \cdots \int_{\mathcal{P}_{I,T-2}(\omega)} \left[\mu_{I,T,\mathcal{P}_{I,T-1}(\omega)}\left(C \cap \mathcal{P}_{I,T-1}(\omega)\right)\right]\\ &\qquad\qquad d\mu_{I,T-1,\mathcal{P}_{I,T-2}(\omega)} \cdots d\mu_{I,1,\mathcal{P}_{I,0}(\omega)} d\mu_{I,0,\Omega}.\end{aligned}$$

It is straightforward to check that this is a probability measure and that $\mu_I | A = \mu_{I,t,A}$ (on the domain of $\mu_{I,t,A}$) for any $A \in \mathcal{P}_{I,t-1}$. To show uniqueness given the $\mu_{I,t,A}$'s, suppose that ν_I is another such measure satisfying (3). Applying equation 3 to ν_I and plugging into the definition of $\mu_I(C)$ gives

$$
\begin{aligned}
&\mu_I(C) \\
&= \int\limits_{\Omega} \int\limits_{\mathcal{P}_{I,0}(\omega)} \cdots \int\limits_{\mathcal{P}_{I,T-2}(\omega)} \left[\frac{\nu_I\left(C \cap \mathcal{P}_{I,T-1}(\omega)\right)}{\nu_I\left(\mathcal{P}_{I,T-1}(\omega)\right)} \right] \\
&\qquad\qquad d\left(\frac{\nu_I}{\nu_I\left(\mathcal{P}_{I,T-2}(\omega)\right)}\right) \cdots d\left(\frac{\nu_I}{\nu_I\left(\mathcal{P}_{I,0}(\omega)\right)}\right) d\nu_I \\
&= \int\limits_{\Omega} \int\limits_{\mathcal{P}_{I,0}(\omega)} \cdots \int\limits_{\mathcal{P}_{I,T-2}(\omega)} \left[\nu_I\left(C \left| \mathcal{P}_{I,T-1}(\omega)\right.\right)\right] d\nu_I \left|\mathcal{P}_{I,T-2}(\omega) \ldots d\nu_I \right| \mathcal{P}_{I,0}(\omega)\, d\nu_I \\
&= \nu_I(C)
\end{aligned}
$$

for any $C \in \mathcal{F}$. □

The next result shows that adding the axiom event independence is equivalent to being able to replace all the $U_{I,t,A}$ with a common $U_{I,t}$ that assigns the same value to any given pair of immediate prize and continuation stream of lotteries irrespective of the event on which the continuation is realized.

Proposition 5 *Given the representation in Proposition 3, event independence holds if and only if, for $t \in \{0, ..., T\}$, there exist $U_{I,t} : Z_t \times F^*_{I,t+1} \to \mathbb{R}$ such that, for all $A \in \mathcal{P}_{I,t-1}$, $B \in \mathcal{P}_{I,t}$, and $B \subseteq A$ if $f \in F^*_{I,t+1,B}$ then $U_{I,t}(z_t, f) = U_{I,t,A}(z_t, f)$ and $U_{I,t}(z_t, f) = U_{I,t}(z_t, g)$ whenever f and g are associated with the same vector of lotteries. Such $U_{I,t}$'s are unique up to positive affine transformations.*

Proof ($\Longrightarrow$) Fix I, t and $\ell_1, \ell_2 \in \Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$. Given $A, A' \in \mathcal{P}_{I,t-1}$, suppose that $f, g \in F_{I,t,A}$ and $f', g' \in F_{I,t,A'}$ have all time $t+1$ continuations constant and that, for all $\omega \in A$, $\omega' \in A'$,

$$f(\omega) = f'\left(\omega'\right) = \ell_1$$

and,

$$g(\omega) = g'\left(\omega'\right) = \ell_2.$$

By event independence, $f \succeq g$ if and only if $f' \succeq g'$. By the representation in Proposition 3,

$$
\begin{aligned}
f \quad &\succeq \quad g \\
&\Longleftrightarrow \int_A \sum_{(z_t,h)\in \operatorname{supp} f(\omega)} f(\omega, z_t, h)\, U_{I,t,A}(z_t, h)\, d\mu_{I,t,A} \\
&\geq \int_A \sum_{(z_t,h)\in \operatorname{supp} g(\omega)} g(\omega, z_t, h)\, U_{I,t,A}(z_t, h)\, d\mu_{I,t,A} \\
&\Longleftrightarrow \sum_{(z_t,l)\in \operatorname{supp}\ell_1} \ell_1(z_t, l)\, U_{I,t,A}(z_t, l) \geq \sum_{(z_t,l)\in \operatorname{supp}\ell_2} \ell_2(z_t, l)\, U_{I,t,A}(z_t, l).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f' &\succeq g' \\
&\Longleftrightarrow \int_{A'} \sum_{(z_t,h)\in \operatorname{supp} f'(\omega)} f'(\omega, z_t, h)\, U_{I,t,A'}(z_t, h)\, d\mu_{I,t,A'} \\
&\geq \int_{A'} \sum_{(z_t,h)\in \operatorname{supp} g'(\omega)} g'(\omega, z_t, k)\, U_{I,t,A'}(z_t, k)\, d\mu_{I,t,A'} \\
&\Longleftrightarrow \sum_{(z_t,l)\in \operatorname{supp}\ell_1} \ell_1(z_t, l)\, U_{I,t,A'}(z_t, l) \geq \sum_{(z_t,l)\in \operatorname{supp}\ell_2} \ell_2(z_t, l)\, U_{I,t,A'}(z_t, l).
\end{aligned}
$$

Since the above holds for any ℓ_1 and ℓ_2, $U_{I,t,A}$ and $U_{I,t,A'}$ order $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$ identically. Therefore, any $U_{I,t,A}$ must be a positive affine transformation of any $U_{I,t,A'}$. Normalize all the $U_{I,t,A}$'s so that they are equal on $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$ and call this common normalization $U_{I,t}$. By Remark 2, specifying $U_{I,t,A}$ on $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$ determines it everywhere, and thus we have determined a common $U_{I,t}$ with the property that $U_{I,t}(z_t, f) = U_{I,t}(z_t, g)$ whenever f and g are associated with the same vector of lotteries. Since any choice of normalization works, the $U_{I,t}$ are unique only up to positive affine transformations.

($\Longleftarrow$) Follows immediately from substituting the $U_{I,t}$ in the representation of Proposition 3. □

Next we show that there is a recursive relationship between $U_{I,t}$ and $U_{I,t+1}$ that holds when evaluating constant continuations. The proof works by exploiting the nested structure of temporal acts degenerate up to $t+1$ with constant time $t+1$ continuations (nested since they are also temporal acts degenerate up to t with constant time $t+1$ continuations).

Proposition 6 *Suppose preference $\succeq$ satisfies axioms weak order, continuity, temporal sure-thing principle, temporal substitution, monotonicity and event independence. Then for each $I \in \mathcal{I}$, there exist continuous functions $U_I : Z_T \to \mathbb{R}$, and for $t = 0, ..., T-1$ functions $u_{I,t} : Z_t \times \mathbb{R} \to \mathbb{R}$ continuous in both arguments and a measure μ_I on $\mathcal{F}$ such that (a) each $u_{I,t}$ is strictly increasing in its second argument, (b) if we define $U_{I,T} : Z_T \to \mathbb{R}$ by $U_{I,T}(z_T) = U_I(z_T)$, and $U_{I,t} : Z_t \times F^*_{I,t+1} \to \mathbb{R}$ for $t = 0, ..., T-1$ recursively by*

$$
U_{I,t}(z_t, f) = u_{I,t}\left(z_t, \sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\right), \tag{4}
$$

where $f \in F^*_{I,t+1,B}$, $B \in \mathcal{P}_{I,t}$ and $\omega \in B$ then the representations in Proposition 3 hold using these $U_{I,t}$. Moreover, if another collection $(U'_I, u'_{I,t}, \mu'_I)$ satisfies the above, then the derived $U'_{I,t}$ must be positive affine transformations of the corresponding $U_{I,t}$. If non-degeneracy holds, μ_I is unique. [10,11]

[10] For the case $t = T-1$ the h arguments in equation 4 are superfluous and should be ignored.

[11] Since f is constant, by Propositions 3 and 5 the value of the second argument of $u_{I,t}$ is the same no matter which $\omega \in B$ is considered.

Proof From Propositions 3, 4, and 5, obtain $U_{I,T} : Z_T \to \mathbb{R}$, and for $t = 0, \ldots, T-1$, $U_{I,t} : Z_t \times F^*_{I,t+1} \to \mathbb{R}$, and a measure μ_I on $\mathcal{F}$. Fix these $U_{I,t}$'s and μ_I and use them to define the $u_{I,t}$ through equation 4. Observe that this will define the $u_{I,t}$ only for values of its second argument that correspond to continuation utilities that may be attained using constant acts. Denote the set of such continuation values by $\mathcal{R}^*_t$, i.e.,

$$\mathcal{R}^*_t = \left\{ x \in \mathbb{R} \,\middle|\, \sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h) = x \text{ for some } f \in F^*_{I,t+1} \right\}.$$

We now show that such $u_{I,t}$'s are indeed functions on $Z_t \times \mathcal{R}^*_t$ and, given the $U_{I,t}$'s and μ_I, are unique. Specifically, we show that the value of $u_{I,t}$ is completely determined by its two arguments. We then show the continuity of $u_{I,t}$ in its first argument. The proof that $u_{I,t}$ is continuous in its second argument will be delayed until the proof of Proposition 1. Given that continuity, $u_{I,t}$ may be continuously extended to $Z_t \times \mathbb{R}$ yielding the functions in the statement of the proposition.

Fix $t = 0, \ldots, T-1$ and $B \in \mathcal{P}_{I,t}$. Let $f, g \in F^*_{I,t+1,B}$. Suppose $\hat{f}, \hat{g} \in F^{t+1}_I$ are two temporal acts degenerate up to $t+1$ with the same prize history up to time $t+1$. Note that both $\hat{f}$ and $\hat{g}$ are also therefore temporal acts degenerate up to t with the same prize history up to time t. Suppose that for each $\omega \notin B$, $\hat{f}_\omega$ and $\hat{g}_\omega$ are constant and $\hat{f}_\omega = \hat{g}_\omega$. Further suppose that $\hat{f}_\omega = f$ and $\hat{g}_\omega = g$ for each $\omega \in B$. That is, $\hat{f}$ and $\hat{g}$ have constant time $t+1$ continuations that are identical on B^c and equal to f and g, respectively on B. By Propositions 3, 4, and 5, the temporal sure-thing principle and the definition of $\succeq$ applied to $F^*_{I,t+1,B}$,

$$\begin{aligned}
\hat{f} \quad &\succeq \quad \hat{g} \iff f \succeq g \\
&\iff \int_B \left(\sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h) \right) d\mu_I \,|B \\
&\geq \int_B \left(\sum_{(z_{t+1},h)\in \operatorname{supp} g(\omega)} g(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h) \right) d\mu_I \,|B \\
&\iff \sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h) \\
&\geq \sum_{(z_{t+1},h)\in \operatorname{supp} g(\omega)} g(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h),
\end{aligned} \tag{5}$$

where the last equivalence follows from the constancy of f and g.

Let A be the unique element in $\mathcal{P}_{I,t-1}$ such that $A \supseteq B$. Observe that $\hat{f}$ and $\hat{g}$ have unique time t continuations on the event A. Denote these by $h, j \in F_{I,t,A}$ respectively and note that all time $t+1$ continuations of h and j are constant and the lotteries given at time t by h and j are degenerate. Also recall that $\hat{f}$ and $\hat{g}$ agree outside of A by construction. Applying the definition of conditional preferences (Definition 8) and the temporal sure thing principle yields,

$$\hat{f} \succeq \hat{g} \iff h \succeq j.$$

Applying Propositions 3, 4, and 5,

$$\begin{aligned}
\hat{f} \ &\succeq \ \hat{g} \Longleftrightarrow h \succeq j \\
&\Longleftrightarrow \int_A U_{I,t}\left(z_t, \hat{f}_\omega\right) d\mu_I \,|A \geq \int_A U_{I,t}\left(z_t, \hat{g}_\omega\right) d\mu_I \,|A \\
&\Longleftrightarrow \int_B U_{I,t}\left(z_t, \hat{f}_\omega\right) d\mu_I \,|A \geq \int_B U_{I,t}\left(z_t, \hat{g}_\omega\right) d\mu_I \,|A \\
&\Longleftrightarrow U_{I,t}\left(z_t, f\right) \geq U_{I,t}\left(z_t, g\right) \qquad (6)
\end{aligned}$$

where the third equivalence follows because $\hat{f}$ and $\hat{g}$ agree on B^c, and the fourth because $\hat{f}_\omega = f$ and $\hat{g}_\omega = g$ when $\omega \in B \in \mathcal{P}_{I,t}$. Equations 5, 6 together imply that,

$$\begin{aligned}
U_{I,t}\left(z_t, f\right) \geq U_{I,t}\left(z_t, g\right) \iff &\sum_{(z_{t+1},h)\in \mathrm{supp} f(\omega)} f\left(\omega, z_{t+1}, h\right) U_{I,t+1}\left(z_{t+1}, h\right) \\
\geq &\sum_{(z_{t+1},h)\in \mathrm{supp} g(\omega)} g\left(\omega, z_{t+1}, h\right) U_{I,t+1}\left(z_{t+1}, h\right). \qquad (7)
\end{aligned}$$

The above shows equation 7 holds when both f and g are in $F^*_{I,t+1,B}$. Next, we show this continues to hold when $f \in F^*_{I,t+1,B}$ and $g' \in F^*_{I,t+1,B'}$ for any $B, B' \in \mathcal{P}_{I,t}$. Fix such f and $g'$and let $\ell \equiv g'\left(\omega'\right)$. Define $g \in F^*_{I,t+1,B}$ by $g\left(\omega\right) = \ell$. Since g and g' are associated with the same vector of lotteries it follows (by Proposition 5) that $U_{I,t}\left(z_t, g\right) = U_{I,t}\left(z_t, g'\right)$. For the same reason,

$$\begin{aligned}
&\sum_{(z_{t+1},h)\in \mathrm{supp} g(\omega)} g\left(\omega, z_{t+1}, h\right) U_{I,t+1}\left(z_{t+1}, h\right) \\
&= \sum_{(z_{t+1},l)\in \mathrm{supp} \ell} \ell\left(z_{t+1}, l\right) U_{I,t+1}\left(z_{t+1}, l\right) \\
&= \sum_{(z_{t+1},h)\in \mathrm{supp} g'(\omega')} g'\left(\omega', z_{t+1}, h\right) U_{I,t+1}\left(z_{t+1}, h\right).
\end{aligned}$$

Therefore equation 7 continues to hold when $f \in F^*_{I,t+1,B}$ and $g' \in F^*_{I,t+1,B'}$ for any $B, B' \in \mathcal{P}_{I,t}$.

This shows that the $u_{I,t}$'s are uniquely defined on $Z_t \times \mathcal{R}^*_t$ (given the $U_{I,t}$'s) through equation 4 and are strictly increasing in the second argument. Continuity of $u_{I,t}$ in its first argument follows directly from the continuity of $U_{I,t}$ in its first argument. The proof that $u_{I,t}$ is continuous in its second argument will be delayed until the proof of Proposition 1. Given that continuity, $u_{I,t}$ may be continuously extended to $Z_t \times \mathbb{R}$ yielding the functions in the statement of the proposition. The uniqueness result in the Proposition follows from uniqueness of the $U_{I,t}$ and μ_I shown in Propositions 3, 4, and 5. □

Finally, we show that the representations that apply in the constant continuation case may be extended to cover all temporal acts for a fixed filtration. In broad strokes, the argument uses continuity together with temporal substitution to show that " constant act equivalents" exist and that replacing continuations by their constant equivalents preserves the representations derived in the earlier steps. We make use of four intermediate lemmas, stated and proved below, before proving the main result of this section, Proposition 1.

Lemma 2 *Under the assumptions of Proposition 6, for any $I \in \mathcal{I}$, there exists $\hat{z}_t, \check{z}_t \in Z_t$ for $t \in \{0, ..., T\}$ such that $U_{I,t}\left(\hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \geq U_{I,t}(z_t, f) \geq U_{I,t}\left(\check{z}_t, \check{z}_{t+1}, \ldots, \check{z}_T\right)$ for any t, $z_t \in Z_t$ and $f \in F^*_{I,t+1}$.*

Proof We will prove the existence of the $\hat{z}_t$'s. Existence of the $\check{z}_t$'s follows from similar arguments. Fix I and $U_{I,t}$'s from Proposition 5 and define the corresponding $u_{I,t}$'s using equation 4. Since $U_{I,T}$ is continuous and Z_T is compact we can find $\hat{z}_T \in Z_T$ such that $U_{I,T}\left(\hat{z}_T\right) \geq U_{I,T}(z_T)$ for all $z_T \in Z_T$.

Now, suppose that for some $t \leq T$ we have $\hat{z}_s$ for $s \geq t$ such that $U_{I,t}\left(\hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \geq U_{I,t}(z_t, f)$ for any $z_t \in Z_t$ and $f \in F^*_{I,t+1}$. We will show that there exists $\hat{z}_{t-1}$ such that $U_{I,t-1}\left(\hat{z}_{t-1,}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \geq U_{I,t-1}(z_{t-1}, f)$ for any $z_{t-1} \in Z_{t-1}$ and $f \in F^*_{I,t}$. To this end, let $\hat{u}_t \equiv U_{I,t}\left(\hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right)$. Since $u_{I,t-1}$ is continuous in its first argument and Z_{t-1} is compact there exists $\hat{z}_{t-1} \in Z_{t-1}$ such that

$$u_{I,t-1}\left(\hat{z}_{t-1}, \hat{u}_t\right) \geq u_{I,t-1}\left(z_{t-1}, \hat{u}_t\right).$$

By Proposition 6,

$$U_{I,t-1}\left(z_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) = u_{I,t-1}\left(z_{t-1}, \hat{u}_t\right)$$

for all $z_{t-1} \in Z_{t-1}$. For $f \in F^*_{I,t}$ (with associated with vector of lotteries $l = (l_t, m)$ where $m \in \Delta Z_{t+1} \times \cdots \times \Delta Z_T$),

$$\begin{aligned} U_{I,t-1}\left(\hat{z}_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) &= u_{I,t-1}\left(\hat{z}_{t-1}, \hat{u}_t\right) \\ &\geq u_{I,t-1}\left(z_{t-1}, \hat{u}_t\right) \\ &\geq u_{I,t-1}\left(z_{t-1}, \sum_{z_t \in \operatorname{supp} l_t} l_t(z_t)\, U_{I,t}(z_t, m)\right) \\ &= U_{I,t-1}(z_{t-1}, f). \end{aligned}$$

The first equality is direct from the recursive representation, the first inequality follows from the definition of $\hat{z}_{t-1}$, the second inequality follows from the definition of $\hat{u}_t$ and the fact that $u_{I,t-1}$ is strictly increasing in its second argument and the final equality from the recursive representation and the definition of f. □

Lemma 3 *Fix I, t. Let $A \in \mathcal{P}_{I,t-1}$ and $A = \cup_{j=1}^K B_j$ where $B_j \in \mathcal{P}_{I,t}$. Let $f^i_j, g^i_j \in F_{I,t+1,B_j}$ for $i \in \left\{1, ..., N_j\right\}$, $j \in \{1, ..., K\}$ satisfy $f^i_j \succeq g^i_j$. If $f, g \in F_{I,t,A}$ are such that for $\omega \in B_j$,*

$$\begin{aligned} f(\omega) &= \left(\left(z^1_j, f^1_j\right), \alpha^1_j; ...; \left(z^{N_j}_j, f^{N_j}_j\right), \alpha^{N_j}_j\right) \\ g(\omega) &= \left(\left(z^1_j, g^1_j\right), \alpha^1_j; ...; \left(z^{N_j}_j, g^{N_j}_j\right), \alpha^{N_j}_j\right) \end{aligned}$$

then $f \succeq g$.

Proof First we prove the result when $N_j = 1$ for $j \in \{1, ..., K\}$. Fix $f_j^1, g_j^1 \in F_{I,t+1,B_j}$ for $j \in \{1, ..., K\}$ satisfying $f_j^1 \succeq g_j^1$. Define $f, g \in F_{I,t,A}$ as in the lemma. Let $\tilde{g} \in F_I^{t+1}$ be such that $\tilde{g}_\omega^t = g$ for $\omega \in A$. Since $f_1^1 \succeq g_1^1$, by the definition of $\succeq$ on $F_{I,t+1,B_1}$, $\tilde{g}_1 \succeq \tilde{g}$ where $\tilde{g}_1 \in F_I^{t+1}$ shares the same prize history with $\tilde{g}$ and is equal to f_1^1 on B_1 and to $\tilde{g}$ on B_1^c. This argument may be continued using $\tilde{g}_1$ in place of $\tilde{g}$ and creating $\tilde{g}_2$ by substituting f_2^1 on B_2. Since $f_2^1 \succeq g_2^1$, by the definition of $\succeq$ on $F_{I,t+1,B_2}$, $\tilde{g}_2 \succeq \tilde{g}_1$. Continuing in this way until all of A is covered, we find that $\tilde{g}_K \succeq \tilde{g}_{K-1} \succeq \cdots \succeq \tilde{g}_1 \succeq \tilde{g}$. Since $\tilde{g}_K$ is equal to f on A, $\tilde{g}$ is equal to g on A, and both are equal on A^c, by the definition of $\succeq$ on $F_{I,t,A}$, $f \succeq g$.

Next, we prove the result for the case where $N_1 \geq 1$ and $N_j = 1$ for $j \in \{2, ..., K\}$. Fix $f_j^i, g_j^i \in F_{I,t+1,B_j}$ for $i \in \left\{1, ..., N_j\right\}$, $j \in \{1, ..., K\}$ with $N_j = 1$ for $j \in \{2, ..., K\}$ satisfying $f_j^i \succeq g_j^i$. Define $f, g \in F_{I,t,A}$ as in the lemma. Let $h_i \in F_{I,t,A}$, $i \in \{1, ..., N_1\}$, be such that for $\omega \in B_1$,

$$h_i(\omega) = \left(\left(z_1^i, f_1^i\right), 1\right)$$

and $h_i(\omega) = f(\omega)$ otherwise. Similarly, Let $k_i \in F_{I,t,A}$, $i \in \{1, ..., N_1\}$, be such that for $\omega \in B_1$,

$$k_i(\omega) = \left(\left(z_1^i, g_1^i\right), 1\right)$$

and $k_i(\omega) = g(\omega)$ otherwise. Note that $f = \alpha_1^1 h_1 + \cdots + \alpha_1^{N_1} h_{N_1}$ and $g = \alpha_1^1 k_1 + \cdots + \alpha_1^{N_1} k_{N_1}$. Moreover, since $f_1^i \succeq g_1^i$ and $f_j^1 \succeq g_j^1$ for $j \in \{2, ..., K\}$, by the earlier case, $h_i \succeq k_i$. This holds for all $i \in \{1, ..., N_1\}$. Applying temporal independence then implies that $f \succeq g$.

The rest of the proof will be by induction. Fix $r \geq 2$. Suppose the lemma holds for the case where $N_j \geq 1$ for $j \in \{1, ..., r-1\}$ and $N_j = 1$ for $j \in \{r, ..., K\}$. We will show that then the lemma must hold for the case where $N_j \geq 1$ for $j \in \{1, ..., r\}$ and $N_j = 1$ for $j \in \{r+1, ..., K\}$. (Note that $r = K$ corresponds to the statement in the lemma.) Fix $f_j^i, g_j^i \in F_{I,t+1,B_j}$ for $i \in \left\{1, ..., N_j\right\}$, $j \in \{1, ..., K\}$ with $N_j = 1$ for $j \in \{r+1, ..., K\}$ satisfying $f_j^i \succeq g_j^i$. Define $f, g \in F_{I,t,A}$ as in the lemma. Let $h_i \in F_{I,t,A}$, $i \in \{1, ..., N_r\}$, be such that for $\omega \in B_r$,

$$h_i(\omega) = \left(\left(z_r^i, f_r^i\right), 1\right)$$

and $h_i(\omega) = f(\omega)$ otherwise. Similarly, Let $k_i \in F_{I,t,A}$, $i \in \{1, ..., N_r\}$, be such that for $\omega \in B_r$,

$$k_i(\omega) = \left(\left(z_r^i, g_r^i\right), 1\right)$$

and $k_i(\omega) = g(\omega)$ otherwise. Note that $f = \alpha_r^1 h_1 + \cdots + \alpha_r^{N_r} h_{N_r}$ and $g = \alpha_r^1 k_1 + \cdots + \alpha_r^{N_r} k_{N_r}$. Moreover, since $f_j^{i'} \succeq g_j^{i'}$ for $i' \in \left\{1, ..., N_j\right\}$, $j \in \{1, ..., r-1\}$, $f_r^i \succeq g_r^i$ and $f_j^1 \succeq g_j^1$ for $j \in \{r+1, ..., K\}$, by the induction hypothesis, $h_i \succeq k_i$. This holds for all $i \in \{1, ..., N_r\}$. Applying temporal independence then implies that $f \succeq g$. This completes the proof of the lemma. □

Lemma 4 *For any* $I, t, A \in \mathcal{P}_{I,t-1}$, $F_{I,t,A}^*$ *is connected.*

Proof Fix $f, g \in F^*_{I,t,A}$. Define $r(\alpha) = \alpha f + (1-\alpha) g$, $\alpha \in [0,1]$. Note that r is continuous in the topology generated by the metric $\delta_{I,t,A}$ and connects f and g within $F^*_{I,t,A}$. Thus $F^*_{I,t,A}$ is path-connected. Any path connected set is connected (e.g., Munkres 1975, p. 155). □

Lemma 5 *For any I, t and $f \in F_{I,t,A}$, where $A \in \mathcal{P}_{I,t-1}$ there exists an $f^* \in F^*_{I,t,A}$ such that $f^* \sim f$.*

Proof Let $\tilde{M}(f)$ be the set of all $g^* \in F^*_{I,t,A}$ such that $g^* \succeq f$. Similarly let $\tilde{W}(f)$ be the set of all $g^* \in F^*_{I,t,A}$ such that $f \succeq g^*$. We first show that $\tilde{M}(f)$ is non-empty.

Fix $f \in F_{I,T,A}$, where $A \in \mathcal{P}_{I,T-1}$. Note that by Lemma 2,

$$U_{I,T}\left(\hat{z}_T\right) \geq \int_A \sum_{z_T \in \operatorname{supp} f(\omega)} f(\omega, z_T)\, U_{I,T}(z_T)\, d\mu_I \,|A\ ,$$

which in turn implies by Proposition 3 that $\hat{z}_T \succeq f$.

Inductively, assume that $\left(\hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \succeq f$ for all $f \in F_{I,t,B}$ where $B \in \mathcal{P}_{I,t-1}$. We now show that $\left(\hat{z}_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \succeq f$ for all $f \in F_{I,t-1,A}$ where $A \in \mathcal{P}_{I,t-2}$. Fix some $f \in F_{I,t-1,A}$. Let $\tilde{f} \in F_{I,t-1,A}$ be such that

$$\tilde{f}\left(\omega, \cdot, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) = f(\omega, \cdot), \qquad \text{for all } \omega \in A.$$

By the induction hypothesis, $g \in F_{I,t,B}$, $\left(\hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \succeq g$. Therefore any time t continuation of $\tilde{f}$ is better than any time t continuation of f, and so, by Lemma 3, $\tilde{f} \succeq f$. By Lemma 2,

$$U_{I,t-1}\left(\hat{z}_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \geq \int_A U_{I,t-1}(z_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T) d\mu_I \,|A\ ,$$

which in turn implies by Proposition 3 that $\left(\hat{z}_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \succeq \tilde{f}$ and thus, by the previous sentence, $\left(\hat{z}_{t-1}, \hat{z}_t, \hat{z}_{t+1}, \ldots, \hat{z}_T\right) \succeq f$. This proves the induction argument and shows that $\tilde{M}(f)$ is non-empty.

A similar argument using $\left(\check{z}_{t-1}, \check{z}_t, \check{z}_{t+1}, \ldots, \check{z}_T\right)$ shows that $\tilde{W}(f)$ is non-empty. Axiom 2 (Continuity) and Lemma 1 imply that these sets are closed. $F^*_{I,t,A}$ is connected by Lemma 4. Since, by Lemma 1, $\tilde{M}(f) \cup \tilde{W}(f) = F^*_{I,t,A}$, there must exist $f^* \in \tilde{M}(f) \cap \tilde{W}(f)$,which completes the proof. □

Proposition 1 **(Characterization of within-filtration SREU)** *Suppose preference $\succeq$ satisfies axioms weak order and continuity. Then $\succeq$ satisfies axioms temporal sure-thing principle, temporal substitution, monotonicity and event independence if and only if, for each filtration I, the restriction of $\succeq$ to F_I has an SREU representation within I.*

Furthermore, the following uniqueness properties hold. If $\left(\mu_I, U_I, \left\{u_{I,t}\right\}_{t=0}^{T-1}\right)$ and $\left(\mu'_I, U'_I, \left\{u'_{I,t}\right\}_{t=0}^{T-1}\right)$ both yield SREU representations of $\succeq$ restricted to F_I, then, for each t, the derived $U'_{I,t}$ must be a positive affine transformation of the derived $U_{I,t}$. If non-degeneracy holds, μ_I is strictly positive on its domain and μ'_I must equal μ_I.

Proof of Proposition 1 We need to prove that, for each filtration I, there exists a probability measure μ_I on the state space, a continuous utility function $U_I : Z_T \to \mathbb{R}$ and continuous aggregator functions $u_{I,t} : Z_t \times \mathbb{R} \to \mathbb{R}$ for $t = 0, ..., T-1$ that combine current outcomes with continuation values such that (a) each $u_{I,t}$ is strictly increasing in the continuation value, (b) if we define $U_{I,T} : Z_T \to \mathbb{R}$ by $U_{I,T}(z_T) = U_I(z_T)$ and recursively $U_{I,t} : Z_t \times F_{I,t+1} \to \mathbb{R}$ by,

$$U_{I,t}(z_t, f) = u_{I,t}\left(z_t, E_{\mu_I|A}\left[E_{f(\omega)} U_{I,t+1}(z_{t+1}, h)\right]\right) \tag{8}$$

where A is the domain of f, then the following holds:
For any temporal acts $f, g \in F_I$,

$$f \succeq g \Longleftrightarrow$$

$$E_{\mu_I}\left[E_{f(\omega)} U_{I,0}(z_0, h)\right] \geq E_{\mu_I}\left[E_{g(\omega)} U_{I,0}(z_0, k)\right]. \tag{9}$$

To begin, for any $t \in \{0, ..., T\}$ and I obtain $U_{I,t}$'s and μ_I from Proposition 6. For any $z_t \in Z_t$, $A \in \mathcal{P}_{I,t}$ and $h \in F_{I,t+1,A}$ let,

$$U_{I,t}(z_t, h) = U_{I,t}\left(z_t, h^*\right)$$

where $h \sim h^*$ and $h^* \in F^*_{I,t+1,A}$. We know that such an h^* exists by Lemma 5. From Lemma 3 and Proposition 6, if $k^* \in F^*_{I,t+1,A}$ and $k^* \sim h^*$, then $U_{I,t}(z_t, k^*) = U_{I,t}(z_t, h^*)$ so $U_{I,t}(z_t, h)$ is well-defined.

For any $f, g \in F_{I,t,A}$, define $\hat{f}, \hat{g} \in F_{I,t,A}$ as follows. For each $(z_t, h) \in \operatorname{supp} f(\omega)$, choose some $h^* \sim h$, $h^* \in F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)}$ and let,

$$\hat{f}\left(\omega, z_t, h^*\right) = f(\omega, z_t, h).$$

Similarly, for all $(z_t, k) \in \operatorname{supp} g(\omega)$, choose some $k^* \sim k$, $k^* \in F^*_{I,t+1,\mathcal{P}_{I,t}(\omega)}$ and let,

$$\hat{g}\left(\omega, z_t, k^*\right) = g(\omega, z_t, k).$$

Now note that,

$$f \succeq g \Leftrightarrow \hat{f} \succeq \hat{g}$$

$$\Leftrightarrow \int_A \sum_{(z_t, h^*) \in \operatorname{supp} \hat{f}(\omega)} \hat{f}\left(\omega, z_t, h^*\right) U_{I,t}\left(z_t, h^*\right) d\mu_I |A$$

$$\geq \int_A \sum_{(z_t, k^*) \in \operatorname{supp} \hat{g}(\omega)} \hat{g}\left(\omega, z_t, k^*\right) U_{I,t}\left(z_t, k^*\right) d\mu_I |A$$

$$\Leftrightarrow \int_A \sum_{(z_t, h) \in \operatorname{supp} f(\omega)} f(\omega, z_t, h) U_{I,t}(z_t, h) d\mu_I |A$$

$$\geq \int_A \sum_{(z_t, k) \in \operatorname{supp} g(\omega)} g(\omega, z_t, k) U_{I,t}(z_t, k) d\mu_I |A,$$

where the first equivalence follows from Lemma 3, the second equivalence follows from the representation in Proposition 6, and the third equivalence follows from the construction of $\hat{f}$, $\hat{g}$ and $U_{I,t}$ as above.

Using these $U_{I,t}$'s and μ_I, define the $u_{I,t}$ through equation 8. We now show that such $u_{I,t}$'s are indeed functions and, given the $U_{I,t}$'s and μ_I, are unique. We need to show that for any $f \in F_{I,t+1,B}$ and $g \in F_{I,t+1,B'}$ where $B, B' \in \mathcal{P}_{I,t}$, $U_{I,t}(z_t, f) \geq U_{I,t}(z_t, g)$ if and only if

$$\int_B \sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B$$
$$\geq \int_{B'} \sum_{(z_{t+1},h)\in \operatorname{supp} g(\omega)} g(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,\Big|B'. \tag{11}$$

Find $g^* \in F^*_{I,t+1,B'}$ such that $g^* \sim g$. Define $\ell \equiv g^*(\omega)$. By (10),

$$\int_{B'} \sum_{(z_{t+1},h)\in \operatorname{supp} g(\omega)} g(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,\Big|B'$$
$$= \sum_{(z_{t+1},l)\in \operatorname{supp}\ell} \ell(z_{t+1}, l)\, U_{I,t+1}(z_{t+1}, l)\,. \tag{12}$$

For $A' \supseteq B'$, $A' \in \mathcal{P}_{I,t-1}$, let $\hat{g}, \tilde{g} \in F_{I,t,A'}$ be identical outside of B' and on B', $\hat{g}$ gives (z_t, g) and $\tilde{g}$ gives (z_t, g^*). Since $g^* \sim g$, by Lemma 3, $\hat{g} \sim \tilde{g}$. From (10), $\hat{g} \sim \tilde{g}$ if and only if $U_{I,t}(z_t, g) = U_{I,t}(z_t, g^*)$. Let $g^{**} \in F^*_{I,t+1,B}$ be associated with the same vector of lotteries as g^*. By Proposition 5, $U_{I,t}(z_t, g^{**}) = U_{I,t}(z_t, g^*)$. So, $U_{I,t}(z_t, g) = U_{I,t}(z_t, g^{**})$.

For $A \supseteq B$, $A \in \mathcal{P}_{I,t-1}$, let $\hat{f}, \check{g} \in F_{I,t,A}$ be identical outside of B and on B, $\hat{f}$ gives (z_t, f) and $\check{g}$ gives (z_t, g^{**}) . By (10), $\hat{f} \succeq \check{g}$ if and only if $U_{I,t}(z_t, f) \geq U_{I,t}(z_t, g^{**})$. By Lemma 3, $\hat{f} \succeq \check{g}$ if and only if $f \succeq g^{**}$.

So, $U_{I,t}(z_t, f) \geq U_{I,t}(z_t, g^{**}) = U_{I,t}(z_t, g)$ if and only if $f \succeq g^{**}$ if and only if

$$\int_B \sum_{(z_{t+1},h)\in \operatorname{supp} f(\omega)} f(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B$$
$$\geq \sum_{(z_{t+1},l)\in \operatorname{supp}\ell} \ell(z_{t+1}, l)\, U_{I,t+1}(z_{t+1}, l)$$
$$= \int_{B'} \sum_{(z_{t+1},h)\in \operatorname{supp} g(\omega)} g(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,\Big|B'.$$

This shows that the $u_{I,t}$'s are uniquely defined (given the $U_{I,t}$'s) through equation 8 and are strictly increasing in the second argument. Continuity of $u_{I,t}$ in its first argument follows directly from the continuity of $U_{I,t}$ in its first argument. By Lemma 5 , the set of attainable continuation utilities is exactly $\mathcal{R}^*_t$ (defined in the proof of Proposition 6). To show continuity in the second argument, fix $x \in \mathcal{R}^*_t$. By definition there exists $f^* \in F^*_{I,t+1}$ such that

$$\sum_{(z_{t+1},h)\in \operatorname{supp} f^*(\omega)} f^*(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h) = x.$$

From our earlier arguments, there exist $B \in \mathcal{P}_{I,t}$ and $\bar{f}, \underline{f} \in F_{I,t+1,B}$ such that $\bar{f} \succeq f^* \succeq \underline{f}$ (with at least one preference strict).

Suppose that we can find $\bar{f}$ and $\underline{f}$ with $\bar{f} \succ f^* \succ \underline{f}$. Then we can find $\alpha_x \in (0, 1)$ such that

$$x = \alpha_x \int_B \sum_{(z_{t+1},h)\in \operatorname{supp} \bar{f}(\omega)} \bar{f}(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B$$
$$+ (1-\alpha_x) \int_B \sum_{(z_{t+1},h)\in \operatorname{supp} \underline{f}(\omega)} \underline{f}(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B\ .$$

Now, consider a sequence $x^n \in \mathcal{R}_t^*$, such that $x^n \to x$. For each n large enough, there exists a corresponding $\alpha_{x^n} \in (0, 1)$ such that

$$x^n = \alpha_{x^n} \int_B \sum_{(z_{t+1},h)\in \operatorname{supp} \bar{f}(\omega)} \bar{f}(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B$$
$$+ (1-\alpha_{x^n}) \int_B \sum_{(z_{t+1},h)\in \operatorname{supp} \underline{f}(\omega)} \underline{f}(\omega, z_{t+1}, h)\, U_{I,t+1}(z_{t+1}, h)\, d\mu_I \,|B\ .$$

By equation 8 and the above,

$$U_{I,t}\left(z_t, \alpha_{x^n}\bar{f} + (1-\alpha_{x^n})\,\underline{f}\right) = u_{I,t}\left(z_t, x^n\right),$$

and similarly,

$$U_{I,t}\left(z_t, \alpha_x\bar{f} + (1-\alpha_x)\,\underline{f}\right) = u_{I,t}(z_t, x).$$

Since $\alpha_{x^n}\bar{f} + (1-\alpha_{x^n})\,\underline{f}$ converges to $\alpha_x\bar{f} + (1-\alpha_x)\,\underline{f}$ (in the Prohorov metric),

$$u_{I,t}\left(z_t, x^n\right) \to u_{I,t}(z_t, x).$$

Now suppose that there do not exist $\bar{f}$ and $\underline{f}$ such that $\bar{f} \succ f^* \succ \underline{f}$. The remaining two cases (i.e., either $\bar{f} \sim f^* \succ \underline{f}$ or $\bar{f} \succ f^* \sim \underline{f}$) can be proved analogously, taking into account that in these cases x can be approached only from one direction. This completes the argument for continuity of the $u_{I,t}$ in their second arguments on $\mathcal{R}_t^*$. Finally, continuously extend the $u_{I,t}$ to $Z_t \times \mathbb{R}$ preserving monotonicity in the second argument. Uniqueness follows from uniqueness in Proposition 6.

Necessity is all that remains to be shown.

Necessity of temporal sure-thing principle: Fix a filtration I and time t. Let $A \in \mathcal{P}_{I,t-1}$. Suppose $\hat{f}, \hat{g}, \tilde{f}, \tilde{g} \in F_I^t$ are such that, $\hat{f}$ and $\hat{g}$ share the same prize history $(z_0, z_1, ..., z_{t-1})$, and $\tilde{f}$ and $\tilde{g}$ share the same prize history $\left(z_0', z_1', ..., z_{t-1}'\right)$. Moreover suppose that,

$$\hat{f}_\omega = \tilde{f}_\omega, \hat{g}_\omega = \tilde{g}_\omega \quad \text{for all } \omega \in A,$$
$$\hat{f}_\omega = \hat{g}_\omega, \ \tilde{f}_\omega = \tilde{g}_\omega \quad \text{otherwise.}$$

Using the representation,

$$\begin{gathered}\hat{f} \succeq \hat{g} \\ \Leftrightarrow \\ \sum_{\omega\in\Omega} U_{I,0}\left(z_0, \hat{f}^0_\omega\right)\mu_I(\omega) \geq \sum_{\omega\in\Omega} U_{I,0}\left(z_0, \hat{g}^0_\omega\right)\mu_I(\omega).\end{gathered}$$

By construction $\hat{f}^0_\omega = \hat{g}^0_\omega$ for all $\omega \in A^c$, so

$$\begin{gathered}\hat{f} \succeq \hat{g} \\ \Leftrightarrow \\ U_{I,0}\left(z_0, \hat{f}^1_{\omega'}\right) \geq U_{I,0}\left(z_0, \hat{g}^1_{\omega'}\right)\end{gathered}$$

for some $\omega' \in A$. Iterating this argument we find that,

$$\begin{gathered}\hat{f} \succeq \hat{g} \\ \Leftrightarrow \\ U_{I,t-1}\left(z_{t-1}, \hat{f}_{\omega'}\right) \geq U_{I,t-1}\left(z_{t-1}, \hat{g}_{\omega'}\right)\end{gathered}$$

for some $\omega' \in A$. By the representation,

$$\begin{gathered}U_{I,t-1}\left(z_{t-1}, \hat{f}_{\omega'}\right) = u_{I,t-1}\left(z_{t-1}, E_{\mu_I|A}\left[E_{\hat{f}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\right) \\ \text{and} \\ U_{I,t-1}\left(z_{t-1}, \hat{g}_{\omega'}\right) = u_{I,t-1}\left(z_{t-1}, E_{\mu_I|A}\left[E_{\hat{g}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\right).\end{gathered}$$

Thus,

$$\begin{gathered}\hat{f} \succeq \hat{g} \\ \Leftrightarrow \\ u_{I,t-1}\left(z_{t-1}, E_{\mu_I|A}\left[E_{\hat{f}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\right) \geq u_{I,t-1}\left(z_{t-1}, E_{\mu_I|A}\left[E_{\hat{g}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\right) \\ \Leftrightarrow \\ E_{\mu_I|A}\left[E_{\hat{f}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right] \geq E_{\mu_I|A}\left[E_{\hat{g}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\end{gathered}$$

where the second equivalence follows since $u_{I,t-1}$ is increasing in its second argument. By the same argument we can show that,

$$\begin{gathered}\tilde{f} \succeq \tilde{g} \\ \Leftrightarrow \\ E_{\mu_I|A}\left[E_{\tilde{f}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right] \geq E_{\mu_I|A}\left[E_{\tilde{g}_{\omega'}(\omega)} U_{I,t}(z_t, h)\right]\end{gathered}$$

But note that $\hat{f}_{\omega'} = \tilde{f}_{\omega'}$ and $\hat{g}_{\omega'} = \tilde{g}_{\omega'}$ since $\omega' \in A$. Thus,

$$\hat{f} \succeq \hat{g} \Leftrightarrow \tilde{f} \succeq \tilde{g}.$$

This proves that the representation implies the temporal sure thing principle.

Necessity of temporal substitution Fix any filtration I and time t. Suppose $\alpha \in [0, 1]$ and $A \in \mathcal{P}_{I,t-1}$. For any $f, g, h \in F_{I,t,A}$,

$$\begin{aligned}
f &\succeq g \\
&\Leftrightarrow E_{\mu_I|A}\left[E_{f(\omega)} U_{I,t}(z_t, k)\right] \geq E_{\mu_I|A}\left[E_{g(\omega)} U_{I,t}(z_t, k)\right] \\
&\Leftrightarrow \alpha E_{\mu_I|A}\left[E_{f(\omega)} U_{I,t}(z_t, k)\right] + (1-\alpha) E_{\mu_I|A}\left[E_{h(\omega)} U_{I,t}(z_t, k)\right] \\
&\geq \alpha E_{\mu_I|A}\left[E_{g(\omega)} U_{I,t}(z_t, k)\right] + (1-\alpha) E_{\mu_I|A}\left[E_{h(\omega)} U_{I,t}(z_t, k)\right] \\
&\Leftrightarrow E_{\mu_I|A}\left[\alpha E_{f(\omega)} U_{I,t}(z_t, k) + (1-\alpha) E_{h(\omega)} U_{I,t}(z_t, k)\right] \\
&\geq E_{\mu_I|A}\left[\alpha E_{g(\omega)} U_{I,t}(z_t, k) + (1-\alpha) E_{h(\omega)} U_{I,t}(z_t, k)\right] \\
&\Leftrightarrow E_{\mu_I|A}\left[E_{\alpha f(\omega)+(1-\alpha)h(\omega)} U_{I,t}(z_t, k)\right] \geq E_{\mu_I|A}\left[E_{\alpha g(\omega)+(1-\alpha)h(\omega)} U_{I,t}(z_t, k)\right] \\
&\Leftrightarrow \alpha f + (1-\alpha) h \succeq \alpha g + (1-\alpha) h.
\end{aligned}$$

Necessity of monotonicity Fix any filtration I and time t. Given $A \in \mathcal{P}_{I,t-1}$, suppose that $f, g \in F_{I,t,A}$ and $f^\omega, g^\omega \in F_{I,t,A}$ for each $\omega \in A$. Further suppose that all time $t+1$ continuations of f and of g are constant. Define f^ω, g^ω as follows: For all $\omega' \in A$,

$$f^\omega\left(\omega'\right) = f(\omega),$$

and,

$$g^\omega\left(\omega'\right) = g(\omega).$$

Suppose $f^\omega \succeq g^\omega$ for all $\omega \in A$. This implies that,

$$\int_A \sum_{(z_t,l)\in \operatorname{supp} f^\omega(\omega')} U_{I,t}(z_t, l)\, d\mu_I\, |A\left(\omega'\right) \geq \int_A \sum_{(z_t,l)\in \operatorname{supp} g^\omega(\omega')} U_{I,t}(z_t, l)\, d\mu_I\, |A\left(\omega'\right)$$

where (z_t, l) denotes an immediate consumption/constant continuation pair. (Note that the representation implies that evaluation of the pair (z_t, l) does not depend on the state that it occurs, and it is for this reason that $U_{I,t}(z_t, l)$ is well-defined.) The previous inequality implies by the construction of f^ω and g^ω that

$$\sum_{(z_t,l)\in \operatorname{supp} f(\omega)} U_{I,t}(z_t, l) \geq \sum_{(z_t,l)\in \operatorname{supp} g(\omega)} U_{I,t}(z_t, l).$$

Finally the previous inequality holds for all $\omega \in A$ so,

$$\begin{aligned}
\int_A \sum_{(z_t,l)\in \operatorname{supp} f(\omega)} U_{I,t}(z_t, l)\, d\mu_I\, |A(\omega) &\geq \int_A \sum_{(z_t,l)\in \operatorname{supp} g(\omega)} U_{I,t}(z_t, l)\, d\mu_I\, |A(\omega) \\
&\Leftrightarrow f \succeq g.
\end{aligned}$$

The strict part of monotonicity follows from similar arguments and by noticing that $\mu_I\, |A$ is strictly positive for all $\omega \in A$.

Necessity of event independence: Shown in Proposition 5. This completes the proof. □

6.4 Proof of Proposition 2
(Characterization of SREU with filtration-dependent beliefs)

We need to prove that there exists a probability measure μ_I on the state space for each filtration I, a continuous utility function $U : Z_T \to \mathbb{R}$ and continuous aggregator functions $u_t : Z_t \times \mathbb{R} \to \mathbb{R}$ for $t = 0, ..., T-1$ that combine current outcomes with continuation values such that (a) each u_t is strictly increasing in the continuation value, (b) if we define $U_T : Z_T \to \mathbb{R}$ by $U_T(z_T) = U(z_T)$ and recursively $U_t : Z_t \times \cup_{I\in\mathcal{I}} F_{I,t+1} \to \mathbb{R}$ by,

$$U_t(z_t, f) = u_t\left(z_t, E_{\mu_I|A}\left[E_{f(\omega)}U_{t+1}(z_{t+1}, h)\right]\right) \tag{13}$$

where A is the domain of f, then the following holds:

For any temporal acts $f \in F_I$ and $g \in F_{I'}$,

$$f \succeq g \Longleftrightarrow$$

$$E_{\mu_I}\left[E_{f(\omega)}U_0(z_0, h)\right] \geq E_{\mu_{I'}}\left[E_{g(\omega)}U_0(z_0, k)\right]. \tag{14}$$

To begin, from the characterization of within-filtration SREU (Proposition 1) for each $I \in \mathcal{I}$ obtain U_I and for $t = 0, ..., T-1$ functions $u_{I,t}$ and a probability measure μ_I on $\mathcal{F}$.

Next fix some I, I', t and $\ell, \ell' \in \Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$. Suppose $f, g \in F_I^t$ and $f', g' \in F_{I'}^t$ give the same deterministic stream of prizes, $z_0, z_1, \ldots, z_{t-1}$, up to time t and $f_\omega(\omega) = f'_\omega(\omega) = \ell$ and $g_\omega(\omega) = g'_\omega(\omega) = \ell'$ for all ω. By the Invariance to Irrelevant Information axiom $f \sim f'$ and $g \sim g'$. Thus $f \succeq g$ if and only if $f' \succeq g'$. Applying Proposition 1 and recalling that the $u_{I,t}$'s are increasing in their second arguments,

$$f \succeq g$$
$$\Leftrightarrow$$
$$u_{I,0}\left(z_0, u_{I,1}\left(z_1 \cdots u_{I,t-1}\left(z_{t-1}, \sum_{(z_t,l)\in\text{supp}\ell} U_{I,t}(z_t, l)\right)\cdots\right)\right)$$
$$\geq u_{I,0}\left(z_0, u_{I,1}\left(z_1 \cdots u_{I,t-1}\left(z_{t-1}, \sum_{(z_t,l)\in\text{supp}\ell'} U_{I,t}(z_t, l)\right)\cdots\right)\right)$$
$$\Leftrightarrow$$
$$\sum_{(z_t,l)\in\text{supp}\ell} U_{I,t}(z_t, l) \geq \sum_{(z_t,l)\in\text{supp}\ell'} U_{I,t}(z_t, l).$$

Similarly,

$$f' \succeq g'$$
$$\Leftrightarrow$$

$$u_{I',0}\left(z_0, u_{I',1}\left(z_1 \cdots u_{I',t-1}\left(z_{t-1}, \sum_{(z_t,l)\in \mathrm{supp}\ell} U_{I',t}(z_t,l)\right)\cdots\right)\right)$$
$$\geq u_{I',0}\left(z_0, u_{I',1}\left(z_1 \cdots u_{I',t-1}\left(z_{t-1}, \sum_{(z_t,l)\in \mathrm{supp}\ell'} U_{I',t}(z_t,l)\right)\cdots\right)\right)$$

$$\Leftrightarrow$$

$$\sum_{(z_t,l)\in \mathrm{supp}\ell} U_{I',t}(z_t,l) \geq \sum_{(z_t,l)\in \mathrm{supp}\ell'} U_{I',t}(z_t,l).$$

Thus, elements of $\Delta(Z_t \times \Delta Z_{t+1} \times \cdots \times \Delta Z_T)$ are ranked identically by taking expectations over $U_{I,t}$ and $U_{I',t}$. By the standard uniqueness properties for expected utility, $U_{I,t}$ and $U_{I',t}$ must be related by a positive affine transformation. The above reasoning holds for any I' so without loss of generality we may normalize all the $U_{I',t}$'s to a common U_t. This may be done for each $t \in \{0, \ldots, T\}$.[12] Given these U_t's, the u_t's are uniquely defined through equation 13 as before. Proposition 1 and the fact that the U_t's are simply renormalizations guarantee that (14) holds whenever f and g share the same filtration. Next, we show that the same is true when f and g are temporal acts with different filtrations.

Fix $f \in F_I$ and $g \in F_{I'}$. Let $f^* \in F_I^*$ and $g^* \in F_{I'}^*$ be such that $f^* \sim f$ and $g^* \sim g$ (these exist by Lemma 5). Let $\hat{g}^* \in F_I^*$ be such that $l_{\hat{g}^*} = l_{g^*}$. By Invariance to Irrelevant Information, $\hat{g}^* \sim g^*$. Denoting l_{f^*} by (l_0, m) where $l_0 \in \Delta Z_0$ and $m \in \Delta Z_1 \times \cdots \times \Delta Z_T$ and $l_{\hat{g}^*}$ by $\left(\hat{l}_0, \hat{m}\right)$ where $\hat{l}_0 \in \Delta Z_0$ and $\hat{m} \in \Delta Z_1 \times \cdots \times \Delta Z_T$,

$$
\begin{aligned}
f &\succeq g \Leftrightarrow \\
f^* &\succeq \hat{g}^* \Leftrightarrow \\
\sum_{z_0\in \mathrm{supp} l_0} l_0(z_0)\, U_0(z_0, m)\, d\mu_I &\geq \sum_{z_0 \in \mathrm{supp}\hat{l}_0} \hat{l}_0(z_0)\, U_0\left(z_0, \hat{m}\right) d\mu_I \\
&\Leftrightarrow \\
\int_\Omega \sum_{(z_0,h)\in \mathrm{supp} f(\omega)} f(\omega, z_0, h)\, U_0(z_0, h)\, d\mu_I &\geq \int_\Omega \sum_{(z_0,k)\in \mathrm{supp} g(\omega)} g(\omega, z_0, k) \\
&\qquad U_0(z_0, k)\, d\mu_{I'}.
\end{aligned}
$$

where the second equivalence follows from the representation applied within filtration I and the third equivalence follows from the construction of f^*, g^* and $\hat{g}^*$ and the within filtration representations. This proves that (14) holds.

The uniqueness and strict positivity and continuity statements follow directly from the corresponding results in our earlier representations. Necessity follows from the characterization of within-filtration SREU (Proposition 1) and the obvious necessity of invariance to irrelevant information, weak order and continuity.

[12] For $t = 0$, the middle step in the above displayed inequalities is not necessary, as no $u_{I,t}$'s are involved.

6.5 Proof of Theorem 1 (Characterization of SREU)

We begin with a trivial case. Suppose that for all h and k as constructed in the consistent beliefs axiom, $h \sim k$. By (14), this implies U_0 is constant on $Z_0 \times \Delta Z_1 \times \cdots \times \Delta Z_T$, and, since any continuation is indifferent to some constant continuation, is thus constant overall. By (13), this implies all the U_t's are constant as well. In this case beliefs are irrelevant and the theorem follows straight from the characterization of SREU with filtration-dependent beliefs (Proposition 2). From here on we assume there does exist some $h \nsim k$. Define $\mu \equiv \mu_{I^e}$ from Proposition 2. The strict part of the Monotonicity axiom together with $h \nsim k$ imply that μ is everywhere strictly positive.

Fix some such $h \nsim k$. We will show that for any filtration I, μ_I may be set equal to μ. Fix any I and time t. If $h' \sim k'$ for all h' and k' as constructed in the consistent beliefs axiom then, by similar reasoning as in the previous paragraph, the only beliefs that may matter for filtration I are those over events in $\mathcal{F}_{I,t-1}$ and conditional probabilities over finer events in $\mathcal{F}$ may be freely set to those in μ. Otherwise fix an $h' \nsim k'$. Take any $A \in \mathcal{P}_{I,t}$ and $B \in \mathcal{P}_{I,t-1}$ with $B \supseteq A$ and construct temporal acts f, g, f' and g' as in axiom consistent beliefs. By (14) and some manipulation,

$$
\begin{aligned}
f &\sim g \\
&\Longleftrightarrow \mu(A)U_0(w,l) + (1-\mu(A))\,U_0(x,m) \\
&= \mu(B)\left(\alpha U_0(w,l) + (1-\alpha)U_0(x,m)\right) + (1-\mu(B))\,U_0(x,m) \\
&\Longleftrightarrow \mu(A\,|B)U_0(w,l) + (1-\mu(A\,|B))\,U_0(x,m) \\
&= \alpha U_0(w,l) + (1-\alpha)U_0(x,m) \\
&\Longleftrightarrow \mu(A\,|B) = \alpha.
\end{aligned}
$$

Next, let $B \subseteq B_{t-2} \subseteq B_{t-3} \subseteq \cdots \subseteq B_0 \subseteq \Omega$ with $B_s \in \mathcal{P}_{I,s}$ denote the unique path to the event B in filtration I. By (13) and (14),

$$
\begin{aligned}
f' &\sim g' \\
&\Longleftrightarrow u_0(z_0, \mu_I(B_0\,|\Omega)u_1(z_1, \cdots \mu_I(B\,|B_{t-2})u_{t-1}(z_{t-1}, \mu_I(A\,|B)U_t\left(y,l'\right) \\
&\quad + (1-\mu_I(A\,|B))\,U_t\left(z,m'\right)) \\
&\quad +(1-\mu_I(B\,|B_{t-2}))u_{t-1}\left(z_{t-1}, U_t\left(z,m'\right)\right)\cdots) \\
&\quad +(1-\mu_I(B_0\,|\Omega))u_1\left(z_1,\cdots u_{t-1}\left(z_{t-1}, U_t\left(z,m'\right)\right)\cdots\right)) \\
&= u_0(z_0, \mu_I(B_0\,|\Omega)u_1(z_1, \cdots \mu_I(B\,|B_{t-2})u_{t-1}(z_{t-1}, \alpha U_t\left(y,l'\right) \\
&\quad + (1-\alpha)\,U_t\left(z,m'\right)) + (1-\mu_I(B\,|B_{t-2}))u_{t-1}\left(z_{t-1}, U_t\left(z,m'\right)\right)\cdots) \\
&\quad +(1-\mu_I(B_0\,|\Omega))u_1\left(z_1,\cdots u_{t-1}\left(z_{t-1}, U_t\left(z,m'\right)\right)\cdots\right)) \\
&\Longleftrightarrow \mu_I(A\,|B)U_t\left(y,l'\right) + (1-\mu_I(A\,|B))\,U_t\left(z,m'\right) \\
&= \alpha U_t\left(y,l'\right) + (1-\alpha)\,U_t\left(z,m'\right) \\
&\Longleftrightarrow \mu_I(A\,|B) = \alpha.
\end{aligned}
$$

As the above argument may be made for any α, $f \sim g \iff f' \sim g'$ from Axiom Consistent Beliefs delivers

$$\mu_I(A\,|B) = \mu(A\,|B)$$

for all $A \in \mathcal{P}_{I,t}$ and $B \in \mathcal{P}_{I,t-1}$ with $B \supseteq A$. The argument may be repeated to show this equality for any t , and so μ_I may be replaced with μ in Proposition 2. The same holds for all I, and so the theorem is proved. Necessity follows as in the characterization of SREU with filtration-dependent beliefs (Proposition 2) with the equality of conditional beliefs implying the consistent beliefs axiom.

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