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RESEARCH ARTICLE

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Type interaction models and the rule of six

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Abstract In this paper, I describe and analyze a class of type interaction models. In these models, an infinite population of agents with discrete types interact in groups of fixed size and possibly change their types as a function of those interactions. I then derive conditions for these models to produce multiple equilibria. These conditions demonstrate a trade off between the number of types and the size of the interacting groups. For deterministic interaction rules, I derive *the rule of six*: the number of agent types plus the group size must be at least six in order to support multiple equilibria given a spanning assumption.

Keywords Interactions · Multiple equilibria · Dynamics

JEL Classification Numbers C00

1 Introduction

In this paper, I construct a framework to model worlds populated by agents identified by discrete types that change as a result of interactions with other agents. I call these type interaction models. Type interaction models capture features inherent in a variety of social, biological, and physical phenomena including peer effects (Bikhchandani, Hirshleifer, and Welch 1992, 1998), sexual reproduction (Holland 1975), technological choice in the presence of strategic complementarities (Axelrod et al.

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1995; David 1985), information transmission, chemical reactions (Alberts et al. 1994), particle collisions and culture (Axelrod 1997). In each of these settings, agents or entities have discrete types. These types can denote behaviors, beliefs, strategies, or even velocities. As a result of interacting with other agents, an agent's type can change. These changes capture the influence of other agents. The nature of that influence depends upon the types of the other agents. Someone who does not use drugs (an abstaining type) may decide to experiment with drugs (and become a user type) after interacting with a drug user. Similarly, someone who votes for the republican party may switch allegiances to the democratic party if enough of her friends are democrats.

The main contributions of this paper are to define type interaction models, to show how they can generate multiple equilibria, and to demonstrate a trade off between the number of types and the size of interacting groups in the generation of multiple equilibria. This last result hinted at in the title of the paper is that if the number of types in the population plus the size of the interacting group equals or exceeds six, then multiple equilibria are possible. If the sum is less than six, multiple equilibria are not possible. In these models, an equilibrium is defined as a stable type distribution.

In the models described here, I consider type interactions models with random mixing of agents. Alternatively, interactions between agents can be determined by spatial or social location (Glaeser et al. 1996; Watts 1999; Arthur 1994). This is true of cellular automata models in which agents interact only with neighbors. (Wolfram 1994, 2002). I also assume an infinite number of agents each of whom has a type chosen from a finite set. These agents are then bundled into groups of fixed size simultaneously and their types are updated synchronously. It would also be possible to construct a model with finite populations, with noise, or with alternative grouping procedures.[1]

Given these assumptions, the initial population can be described as a probability distribution over the type space. Time is discrete. In each period, agents randomly form groups of some fixed size, possibly as small as two. As a result of a group interaction, agents choose /are assigned a type according to an *interaction rule*. As mentioned above, in the interesting cases, this new type depends on the types of the other interacting agents. The resulting transition probabilities depend upon the type distribution creating a non time homogeneous Markov process. This captures processes in which the distribution of types effects the transference of a trait or a behavior. For example, the more smokers in a population the more likely that other types are likely to transition into smokers.

These changes in types can be *deterministic:* a person's type could represent the information that she possesses. If so, when two people meet and share information, they may both change their type according to a deterministic rule. These type changes can also be *probabilistic:* in a model of sexual reproduction with exactly two offspring, such as a genetic algorithm model where type represents DNA, an

[1] Troy Tassier has performed computational experiments with type interaction models and found only small differences between his results in the finite case and the infinite case results that I describe. In the finite population case with noise and asynchronous updating, the population spends long periods of time at each equilibrium but eventually bounces between them. This produces a unique equilibrium distribution with meta stable states at the equilibria of the infinite population model.

offspring's DNA is a random draw that depends upon the parents' types. Deterministic type interaction models are a subclass of probabilistic type interaction models.

The conditions for multiple equilibria in type interaction models reveal a trade off between the number of types and the size of the interacting group. For deterministic type interaction models, binary interaction rules require four types of agents, interacting groups of size three require three types, and interacting groups of size four, need only two types to generate multiple equilibria. Combined, these results imply the rule of six for deterministic type interaction models. If the existence of multiple equilibria serves as a proxy for the amount of complexity, the rule of six formalizes the intuition that complexity increases with the number of types and the size of the groups interacting.

The fact that there is a rule of six is serendipitous. There is no underlying logic as to why six is sufficient in each case. To understand what drives the result requires close examination of the Markov transition matrix. The powers of the polynomials in that transition matrix depend on the size of the interacting group. The number of variables in those polynomials depends on the number of types. A higher degree polynomial is more likely to have multiple roots as is a polynomial of fixed degree with more variables. This explains why increasing group size or the number of types increases the number of roots.

The potential for multiple equilibria is significant for reasons that hardly need mention. Initial conditions play a role in determining the outcome as do the current state. As a result, accurate empirical predictions of outcomes are more difficult to make (Manski 1993). Further, interventions that shift the state of the process may exist and be beneficial. Such interventions cannot exist for irreducible Markov processes which generate unique equilibria. A policy that redistributes income to shift the income distribution has no long run effect if the process governing income dynamics is an irreducible Markov process. At most, a redistribution produces only temporary changes in the state of the system. Long run change requires a change in the underlying dynamic process, in the transition probabilities. That is not true of type distribution models. Alterations in the distribution over types can change the equilibrium of the system.

At first, the rule of six might appear to contradict the evolutionary game theory literature on coordination games (Kandori et al. 1992; Foster and Young 1990; Blume 1993). Those models also produce multiple meta stable states, yet they rely on only two types. However, in those models, the agents do not meet in small groups of fixed size. They meet the entire population. Therefore, the results are consistent with the rule of six. In models that do assume the agents play only a few other agents, the rule for updating the agents' types (their strategies) often does not depend solely on the types of the agents with whom they interact (Glaeser et al. 1996). Instead, the updating rule also depends on the types of the agents who interact with the agents with whom they interact. Thus, the results here are consistent with the extant literature.

Even though type interaction models could be applied far more generally in what follows. I frame type interaction models in the context of social processes. I bias the presentation of the results in this way for several reasons. First, the initial minimal model with multiple equilibria has a clean social science interpretation as peer effects. Second, the general class of interaction models has many such social science applications. Durlauf (1997), Picker (1997), Banerjee (1992), and others have advanced the argument that the micro level processes that drive many of our pressing social problems – crime, drug use, and educational performance include strong interaction and peer effects. Third, even though the main results can be interpreted the context of biological, physical and ecological processes, those environments often contain a enormous number of types. The results here focus on minimal conditions for multiple equilibria, so they may be less relevant in those contexts, even if the general class of models is germane to those environments.

The remainder of this paper is organized as follows. I begin with a toy example of a deterministic rule with four types of agents who interact in pairs that generates three equilibria, two of which are stable and have equal sized basins of attraction. In Sections 3 and 4, I describe a binary interaction models and interaction models with more types. In Section 5, I extend the binary type interaction model to allow for larger groups. In Section 6, I return to the example from Section 2 and discuss the implications for policy intervention, and in Section 7, I offer tentative conclusions and directions for future work.

2 An example of a type interaction model

I begin with a toy example of a binary interaction rule that generates two stable equilibria and one unstable equilibrium. My goal is to provide a clean minimal example of how type interaction models can produce multiple equilibria rather than to construct a realistic model of some real world phenomenon. That said, this example has a natural interpretation in the context of students' school performance or as a model with peer effects. The example belongs to the class of what I call *binary type interaction models.* In a binary type interaction model, agents interact in pairs and their types may change as a result of these interactions. In the general formulation, the groups within which agents interact may be of any fixed size.

The example is highly stylized, but because it allows four types of agents it will appear much richer than the standard two type models. These four types are indexed by the letters A, B, C, and D. They can be thought of as motivation levels of students.

A: Eager
B: Moderately interested
C: Occasionally interested
D: Disruptive

Let upper case letters denote the types and lower case letters denote the probabilities of those types. In this way, a distribution over the four types can be written as (a, b, c, d) where $a + b + c + d = 1$. At the beginning of each period, students are randomly assigned to pairs. As a result of these pairwise interactions, students change their types according to the deterministic type interaction rule shown in Table 1 below. Table 1 also gives the probability of each pair arising.

This interaction rule captures three phenomena that can be found in human interactions: first if two people of different types interact, then they tend to become more alike: when an A meets a D, the A student tends to become more studious and the other less so. Second, if two people are identical, they tend to differentiate. If two B's meet, then one becomes a C and the other becomes an A, and if two eager students (type A's) interact, then one of them becomes a B and the other remains an A. Third, more extreme people are more persuasive. If an eager student or a

Table 1 A binary interaction model with four types

Pair	Prob	Outcome
AA	a^2	AB
AB	$2ab$	AA
AC	$2ac$	AA
AD	$2ad$	BC
BB	b^2	AC
BC	$2bc$	BC
BD	$2bd$	DD
CC	c^2	BD
CD	$2cd$	DD
DD	d^2	CD

disruptive student meets with a moderately interested or occasionally interested student, the latter student switches his type to match the former student's type. For example, if a type D meets with a type B, both students become D's.

Notice that if the population begins with all A's, then in the next period there will be both A's and B's. In two periods, there will be A's, B's, and C's, and in three periods, all four types will be present in the population. The same phenomenon occurs if we begin with all agents being of any other single type as well. This can be thought of as the interaction rule *spanning* the set of types. Later, I place a constraint on interaction rules that requires this spanning. Without the spanning assumption, generating multiple equilibria would be trivial.

2.1 Equilibria

To solve for the equilibria of the system and to understand the dynamics, I construct the set of difference equations that result from this interaction rule. Given a distribution of agent types (a, b, c, d), let $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ denote the distribution across types in the next period. The table above shows that there are four ways that a type A agent can be created. The first occurs if two type A's meet (this happens with probability a^2), then one A is created. Similarly, if two type B's meet (probability b^2), then one A is also created. And finally, if an A and a B (probability $2ab$) meet or if an A and a C meet (probability $2ac$), then two type A's are created. Since the probabilities of all of the pairwise matchings must sum to one and matching produces two outputs, converting these raw numbers into the expected number of type A's requires dividing the number of type A's produced by two. This gives the following difference equation for the type A agents.

$$\hat{a} = \frac{a^2}{2} + \frac{b^2}{2} + 2ab + 2ac$$

Similar equations can be derived for each of the other types.

$$\hat{b} = \frac{a^2}{2} + \frac{c^2}{2} + ad + bc$$

$$\hat{c} = \frac{b^2}{2} + \frac{d^2}{2} + ad + bc$$

$$\hat{d} = \frac{c^2}{2} + \frac{d^2}{2} + 2cd + 2bd$$

This system of equations can then be solved analytically. In the symmetric equilibrium, $a = d$ and $b = c$. In this case, the difference equations reduce to the following:

$$a = \frac{a^2}{2} + \frac{b^2}{2} + 4ab$$

$$b = \frac{3a^2}{2} + \frac{3b^2}{2}$$

By symmetry $a + b = c + d$. Substituting the equality $a + b = \frac{1}{2}$ into the second equation yields.

$$b^* = \frac{5 - \sqrt{7}}{12}, a^* = \frac{1 + \sqrt{7}}{12}$$

This equilibrium turns out not to be stable. In numerical simulations of this process it never occurred. The system also has two stable asymmetric equilibria, which for ease of interpretation, I present in numerical form.

a	b	c	d
0.694	0.256	0.047	0.003
0.003	0.047	0.256	0.695

In the first of these equilibria, over 95% of the agents are either type A or type B. In the subsequent analysis, I refer to this as the *good* equilibrium and to the other stable equilibrium as the *bad* equilibrium. Small perturbations from the symmetric unstable equilibrium can lead to either of the two stable equilibria. Thus, this example shows how a small change in initial attitudes created by a new teacher, a new school, or a new principal could lead to drastic changes in average performance even though no structural changes occurred in the interaction process. Since both equilibria are stable, it suggests why once a school has settled into the bad equilibria, getting out could be difficult.

As we shall see, the reason that this model supports more than one equilibrium is that the transition probabilities in the induced Markov process written over types change as a function of the population proportion. Therefore, in the Markov transition matrix written over the four types, the transition probabilities at the good equilibrium are different from the transition probabilities at the bad equilibrium.

3 Type interaction models

I now describe a general class of type interaction models. In these models, there exists an infinite number of agents each assigned a type. The set of possible types is finite.

The set of agent types $N = \{1, 2, 3, \ldots, n\}$, *where* $n > 1$

This construction allows the population of agents to be represented as a probability distribution over the types.

A distribution over types $P(N) = \{p = (p_1, p_2, \ldots, p_n) : \sum_{i=1}^{n} p_i = 1\}$

The agents then interact in groups of some fixed size k that are randomly chosen.

A group of size k $x^k = (x_1, x_2, \ldots, x_k)$ *where* $x_i \in N$,

It is also useful then to define the set of all groups possible consisting of k agents.

The set of groups of size S^k $S^k = \{(x_1, x_2, \ldots, x_k) : x_i \in N\}$

When the k agents interact, they change their type according to an interaction rule. A deterministic type interaction rule maps S^k into itself.

A deterministic type interaction rule, $T : S^k \rightarrow S^k$.

Note that the interaction rule operates on subsets, so the order of the agents within the group does not matter. In the more general case, a type interaction rule maps the group of size k into a probability distribution over the groups of size k. A *type interaction rule,* $T : S^k \rightarrow \Delta(S^k)$, where $\Delta(S^k)$ equals the set of all probability distribution over the members of S^k.

With these definitions in place, I can formally define a type interaction model.

A type interaction model $\Omega = \{N, k, T\}$ *where N is the number of agent types, k is the size of groups that interact, and T is an interaction rule*

A type interaction equilibrium is a probability distribution that remains unchanged under an application of the type mapping generated by T, M_T

A type interaction equilibrium is a probability distribution $p^* \in P(N)$ *such that* $p^* = M_T(p^*)$

Generating multiple equilibria with a type interaction rule is trivial if some subset of types remains contained within itself through the interaction mapping. For example, if $k = 2$, and if $T(i, i) = (i, i)$ for all i, then the degenerate distribution e_i ($e_i^i = 1$ and $e_j^i = 0$ for all $j \neq i$) is an equilibrium for all i. To preclude this possibility, I assume that the interaction rule *spans*, i.e. beginning from any population every possible type must appear in the population with positive probability in finite time. A similar assumption is required for the unique equilibrium result in Markov theory. In both the interaction environment and the Markov environment, the spanning/irreducibility assumptions are made because they apply to real world situations – *an agent can get to any type from any other type with positive probability.*

To formalize the spanning assumption, it is helpful to introduce two new mappings. First, let ρ denote the mapping from S^k into $P(N)$ the probability distribution over the types. Suppose, for example, that there are three types, A,B, and C. Given the set (A, A, B) in S^k, $\rho(A, A, B) = (\frac{2}{3}, \frac{1}{3}, 0)$. Given ρ, it is possible to consider an interaction rule as a mapping from the distribution over types $P(N)$ into itself.

I denote that mapping by M_T and refer to it as the *type mapping generated by the interaction rule.* Given $P(N)$, this induces a probability distribution over S^k, call this distribution q. I can then write

$$M_T(p) = \sum_{x^k \in S^k} q(x^k)\rho(T(x^k))$$

Given an initial distribution over types, p^0, $M_T(p^0)$ gives the distribution over types after the first interaction. Hereafter, I adopt the convention that $M_T^s(p)$ equals the distribution over types in the population after s applications of the map M_T beginning with the distribution p.

An interaction rule $T : S^k \to S^k$ spans if the type mapping generated by the interaction rule, M_T is such that for any $p^0 \in P(N)$ and $j \in N$ $\exists\, t(p^0, j) > 0$ s.t. $M_T^{t(p^o,j)}(p^0)$ places strictly positive probability on type j.

The spanning assumption implies that beginning from any initial distribution, that for any type i, there exists some number t such that after t applications, t, of the interaction rule, there will be a positive probability of type i in the distribution.

As the example in the previous section shows, type interaction models can generate multiple equilibria even though the interaction rule spans. It is possible to recast a type interaction model as a Markov process. If the population distribution equals the state of the Markov process, then the interaction rule creates transition probabilities over the states. However, it requires infinitely many states. The resulting process is reducible. The system cannot get from some states to others even in infinite time. In particular, if a state is in the basin of attraction of one equilibrium, it is not possible to leap from that state to another equilibrium state.

4 Binary interaction models

I first consider binary interaction models. In these models, the agents meet in groups of size two. The first claim states that if there are only two types, then there does not exist a binary interaction rule with multiple equilibria. The mathematical intuition for why this is the case is not difficult. With only two types that meet in pairs, the transition probabilities belong to a restricted class of linear functions. These functions do not permit enough curvature to support multiple equilibria.

Claim 1 *There does not exists a two type binary interaction rule that supports multiple equilibria.*

Proof Note that the claim states that this is true for both probabilistic and deterministic binary interaction models. It suffices to show that it holds for probabilistic rules. Let $T_{IJ:K}$ equal the expected number of K's that result when an I meets a J. Let a equal the proportions of A's in the population and $(1 - a)$ equal the proportion of B's. I can then write the transition probabilities as a function the $T_{IJ:K}$'s. For example, the probability that an A remains an A equals the probability that an A meets an B (which is $(1 - a)$) times one half the expected number of A's produced by such pairings (which is $\frac{T_{AB:A}}{2}$) plus the probability that an A meets an

A (which is a) times one half the expected number that such meetings produce an A (which is $\frac{T_{AA:A}}{2}$). Thus, the probability that a type A becomes a type A equals

$$\frac{aT_{AA:A} + (1-a)T_{AB:A}}{2}$$

Using similar logic, the probability that a type B becomes a type A equals

$$\frac{aT_{AB:A} + (1-a)T_{BB:A}}{2}$$

Solving for a fixed point requires setting the sum of these probabilities equal to a

$$\frac{a(aT_{AA:A} + (1-a)T_{AB:A}) + (1-a)((1-a)T_{BB:A} + aT_{AB:A})}{2} = a$$

which reduces to

$$(T_{AA:A} - 2T_{AB:A} + T_{BB:A})a^2 + (2T_{AB:A} - 2T_{BB:A} - 2)a + T_{BB:A} = 0$$

This is a polynomial of degree two in a. It suffices to show that this polynomial has a single root in the interval (0, 1). At $a = 0$, the polynomial has a value of $T_{BB:A}/2$ which is strictly greater than zero by the spanning assumption. And, at $a = 1$, the polynomial has a value of $T_{AA:A}/2 - 1$ which is strictly less than one, again by the spanning assumption. Since the polynomial is of degree two in a single variable, there can be only one root in the interval [0, 1]. [2]

This first claim does not imply that there do not exist two type transition matrices with state dependent probabilities that can generate multiple equilibria. The following transition matrix has three equilibria $a = 0.5$, $a = \frac{11-\sqrt{77}}{11}$, and $a = \frac{11+\sqrt{77}}{11}$.

	Prob $\to$ A	Prob $\to$ B
Type A	$\frac{9a^2}{10} + 2ab$	$\frac{a^2}{10} + b^2$
Type B	$\frac{b^2}{10} + a^2$	$\frac{9b^2}{10} + 2ab$

The claim, however, does imply that the transition probabilities shown in the matrix above cannot lie within the class of allowable transition probabilities for binary interaction models. The assumption that the agents meet in pairs and that the new types depend only on the composition of the pair constrains the set of possible transition matrices to be a subset of the set of linear functions of p. If agents meet in larger groups, then the transition probabilities can be higher order polynomials. With higher order polynomials, multiple equilibria become possible. In fact, as I show later, these transition probabilities are consistent with a two type probabilistic ternary interaction model that produces multiple equilibria.

I now turn to binary interaction models with three types. I show that three types are insufficient to support multiple equilibria for either deterministic or probabilistic type interactions.

[2] I would like to thank Carl Simon for suggesting a simplification in the last step of this proof.

Claim 2 *Every three type binary interaction rule has a unique equilibrium.*

Proof Consider a type interaction with three types A, B, and C where a, b, and $(1-a-b)$ denote the probabilities of A, B, and C respectively. The proof relies on the two-simplex for the probabilities a and b. Let the curve $\dot{a}=0$ denote the set of probabilities $(a, b, 1-a-b)$ such that the probability of an agent being of type A is unchanged. Let the curve $\dot{b}=0$ denote the set of probabilities $(a, b, 1-a-b)$ such that the probability of an agent being of type B is unchanged. It suffices to show that these two curves can intersect at most once.

Recall that $T_{IJ:K}$ equals the expected number of Ks produced when an I meets a J. If when an A and a B meet they become a B and a C with probability two thirds and a C and an A with probability one third, then $T_{AB:B}=\frac{2}{3}$ $T_{AB:C}=1$ $T_{AB:A}=\frac{1}{3}$. It is straightforward to show that the curves $\dot{a}=0$ and $\dot{b}=0$ are quadratic functions of a and b of the following form:

$$\beta_0+\beta_a a+\beta_b b+\beta_{ab}ab+\beta_{a^2}a^2+\beta_{b^2}b^2=0$$

For the $\dot{a}=0$ curve the following are true

$\beta_0=T_{CC:A}/2\geq 0$
$\beta_a=-1+T_{AC:A}-T_{CC:A}$
$\beta_b=T_{BC:A}-T_{CC:A}$
$\beta_{a^2}=T_{AA:A}/2-T_{AC:A}+T_{CC:A}/2$
$\beta_{b^2}=T_{BB:A}/2-T_{BC:A}+T_{CC:A}/2$
$\beta_{ab}=T_{CC:A}+T_{AB:A}-T_{AC:A}-T_{BC:A}$

The proof relies on a two dimensional representation of the probabilities a, and b with a represented on the horizontal axis and b on the vertical axis. The 45° degree line corresponds to the set of points where $c=0$ and the origin corresponds to the point $c=1$. At the corner $a=1$ the value of $\dot{a}$ equals $T_{AA:A}/2-1$ which must be negative since all of the $T's$ lie in the interval $(0, 2)$. At the corner $b=1$ the value of $\dot{a}$ equals $T_{BB:A}/2$ which is weakly positive. At the corner $c=1$ the value of $\dot{a}$ equals $T_{CC:A}/2$ which is also weakly positive. Further, any point along the line $a=0$ the value of $\dot{a}$ equals $T_{CC:A}(1-b)^2/2+T_{BC:A}b(1-b)+b^2T_{BB:A}/2$, which must be *strictly positive* by the spanning assumption. Similarly, the value of $\dot{b}$ must be strictly negative at the corner $b=1$, weakly positive at the corners $a=1$ and $c=1$ and strictly positive on the line $b=0$.

First, it can be shown that these curves cannot intersect at a corner. The proof is by contradiction. Suppose that they could intersect at a corner. Given the restrictions on the curves $\dot{a}$ and $\dot{b}$, the only corner at which the curves can intersect is at $c=1$. This is impossible by the spanning assumption. Either $T_{CC:A}$ or $T_{CC:B}$ must be strictly positive.

Therefore, without loss of generality, assume that the curve $\dot{a}=0$ does not intersect at the origin where $c=1$. There exists a point $(a^*, 0, 1-a^*)$ on the line $b=0$ where the curve $\dot{a}=0$. Recall that the curve $\dot{a}=0$ is quadratic, that below the curve $\dot{a}$ is negative, and that at the corner $b=1$ it is weakly positive. Therefore, the curve can be written as an arc with either increasing or decreasing slope that intersects the line $c=0$ somewhere to the right of the point $a=1$. A similar argument shows that the curve $\dot{b}=0$ can be written as an arc with either

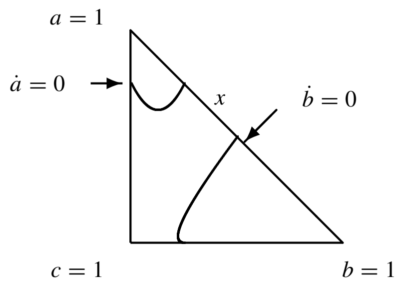

Fig. 1 Non intersecting curves

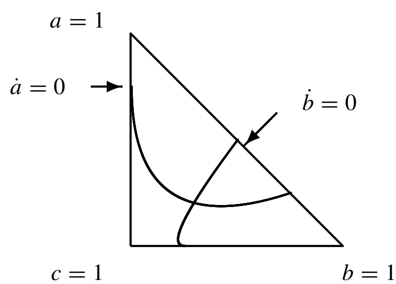

Fig. 2 Curves that intersect only once

increasing or decreasing slope that begins anywhere on the b axis except for the point $b = 1$ and intersects the line $c = 0$ somewhere to the left of the point $b = 1$.

Given the restriction on slope there are only three possibilities. The curves could fail to intersect, the curves could intersect once, or they could intersect twice. The first and third possibilities cannot occur. Suppose first that the the curves fail to intersect as in Figure 1. Consider the point x that lies between their respective intersections of the line $c = 0$.

By construction at x both $\dot{a} > 0$ and $\dot{b} > 0$. However, since $c = 0$, $\dot{c} \geq 0$, there exists a contradiction. Similarly, if they intersect twice, there must also exist a point on the line $c = 1$ such that $\dot{a} > 0$ and $\dot{b} > 0$, so here too there exists a contradiction. Therefore, the only possibility is a single interior equilibrium as shown in Figure 2 above. [3]

5 Type interactions models with larger groups

I now consider type interaction rules models with larger groups. The next claim states that if there are only two types then there does not exist a deterministic interaction rule in which agents meet in groups of size three that generates multiple equilibria.

Claim 3 *There does not exist a deterministic two type ternary interaction rule that supports multiple equilibria.*

[3] I would like to thank Jimmy John's Submarines for supplying the napkin used in this proof.

Proof See Appendix.

The next claim states that it is possible to generate multiple equilibria if the interaction rule is probabilistic.

Claim 4 *There exist probabilistic two type ternary interaction rules that support multiple equilibria.*

Proof The proof is by the construction of an example.

Triple	Outcome
AAA	AAA with $p = 0.9$ and AAB with $p = 0.1$
AAB	AAA
ABB	BBB
BBB	BBB with $p = 0.9$ and ABB with $p = 0.1$

This creates the transition probabilities that were described earlier as an example of a two type model that generates multiple equilibria.

	Prob $\rightarrow A$	Prob $\rightarrow B$
Type A	$\frac{9a^2}{10} + 2ab$	$\frac{a^2}{10} + b^2$
Type B	$\frac{b^2}{10} + a^2$	$\frac{9b^2}{10} + 2ab$

The three equilibria of this model are $a = 0.5$, $a = \frac{11-\sqrt{77}}{11}$, and $a = \frac{11+\sqrt{77}}{11}$. The symmetric equilibrium is unstable.

5.1 Three type interactions

The next two claims complete the proof of the rule of six. The first claim states that there exist interaction rules with three types of agents meeting in groups of three that support multiple equilibria.

Claim 5 *There exists a three type deterministic ternary interaction rule that supports multiple equilibria.*

Proof The proof is by the construction of an example

Pair	Outcome
AAA	AAB
AAB	AAA
ABB	AAA
AAC	AAA
ABC	ABC
ACC	CCC
BBC	CCC
BCC	CCC
CCC	BCC
BBB	ABC

Proof The following difference equations define the deterministic dynamical system:

$$\hat{a} = \frac{2a^3}{3} + 3a^2b + 3ab^2 + 3a^2c + 2abc + \frac{b^3}{3}$$

$$\hat{b} = \frac{a^3 + b^3c^3}{3} + 2abc$$

$$\hat{c} = \frac{2c^3}{3} + 3c^2b + +3cb^2 + +3ac^2 + +2abc + \frac{b^3}{3}$$

The two stable equilibria of this set of equations are $a = 0.81$, $b = 0.18$ and $c = 0.01$, and $a = 0.01$, $b = 0.18$ and $c = 0.81$. and the unstable symmetric equilibria is $a = c = 0.45$, and $b = 0.1$.

The next claim states that if agents interact in groups of four (*quaternary interactions*) then a deterministic rule will suffice to generate multiple equilibria with only two types of agents.

Claim 6 *There exists a two type deterministic quaternary interaction rules that supports multiple equilibria.*

Proof The proof is also by example. Consider the following interaction rule

Group	Outcome
$AAAA$	$AAAB$
$AAAB$	$AAAA$
$AABB$	$AABB$
$ABBB$	$BBBB$
$BBBB$	$ABBB$

This creates the following transition probabilities:

	Prob $\rightarrow$ A	Prob $\rightarrow$ B
Type A	$\frac{3a^3}{4} + 3a^2b + 3ab^2$	$\frac{a^3}{4} + b^3$
Type B	$\frac{b^3}{4} + a^3$	$\frac{3b^3}{4} + 3ab^2 + 3a^2b$

This system has three fixed points $a = \frac{3-\sqrt{3}}{6}$, $a = 0.5$, and $a = \frac{3+\sqrt{3}}{6}$. The first and the last are stable.

The rule of six can now be formally stated.

Claim 7 *(The rule of six) A deterministic type interaction rule can generate multiple equilibrium if and only if the number of types plus the group size is greater than or equal to six.*

Proof follows from the previous claims.

6 Policy interventions

A system that produces multiple equilibria is more easily manipulated by interventions. In the context of a type interaction model, a policy interventions could be either an alteration of the interaction rule, a change in the choice of equilibrium state or, a change in the the initial state of the system. The first type of intervention would require changing of how many types interact or in the transition mapping. The second could involve having a large percentage of the agents change their type. Both of these interventions may not be possible. The third type of intervention might only involve a relatively small change in the initial distribution of types. However, the fact that such an intervention could be successful does not imply that the intervention is easy to find. If the system exhibits extreme sensitivity to initial conditions, then it may be difficult to locate the basin of an equilibrium.

In this section, I show that it is possible to characterize large subsets of the basins of attraction of the two stable equilibria. This exercise proves that type interaction models do not necessarily exhibit extreme sensitivity to initial conditions for all initial points. In the two claims that follow, I describe the dynamics of two processes and show how to construct a successful intervention. Even though a full characterization of the basins of attraction may be difficult to accomplish, characterizing a large subset of the basin of attraction for a particular equilibrium need not. To guarantee that equilibrium it is sufficient to move the initial distribution into a subset of its basin of attraction. First, recall the *good* equilibrium from the four type binary interaction example presented in Section 2.

a	b	c	d
0.694	0.256	0.0467	0.003

Construct the following sets: S_a which is the set of probability distributions where there are more type A's than type D's and S_b which is the set of probability distributions where there are more type B's than type C's.

$$S_a = \{(a, b, c, d) : a > d\}$$

$$S_b = \{(a, b, c, d) : b > c\}$$

The next claim states that once the distribution enters the intersection of S_a and S_b it never fully escapes it.

Claim 8 *If the initial distribution over types lies in $S_a \cap S_b$, then in every subsequent period the type distribution lies in S_a. If the distribution ever leaves S_b then it reenters S_b in the subsequent period.*

Proof See Appendix.

The previous claim defines a subset of the basin of attraction for the good equilibrium. In other words, to avoid the bad equilibrium choose the initial distribution to lie in the intersection of the sets S_a and S_b. The claim implies that the distribution cannot go to the bad equilibrium, but it does not rule out cycling. Numerical simulations show that if the initial state of the system is placed in the intersection of the sets S_a and S_b, then it goes to the "good" equilibrium.

A similar result exists for the three type ternary interaction model that generates multiple equilibria.

Claim 9 *In the three type ternary interaction rule that generates multiple equilibrium described in Claim 5, if the distribution over types satisfies* $a \geq 2b$ *and* $a \geq 2c$*, then so do subsequent type distributions.*

Proof See Appendix.

These two claims demonstrate the possibility of choosing initial points so as to guarantee the good equilibrium. That, of course, would also be true for a system that exhibits extreme sensitivity to initial conditions, but if we assume that the population is finite, or that there is some noise, then we would not be able to manipulate a chaotic systems, whereas we could manipulate the outcomes in systems such as those described in this section. In fact, the dynamics are relatively well behaved. If the first system begins with many more type A agents, the good equilibrium is realized.

7 Discussion

This paper contributes to a growing literature on models with interaction effects. It shows how interactions among agents of diverse types produce multiple equilibria by creating state dependent transition probabilities. The main technical contribution of this paper is its characterization of minimal conditions that must be satisfied for a discrete type deterministic interaction model to support multiple equilibria: group size plus the number of types must sum to at least six. This result builds from a relatively simple insight: the power of the polynomials in the transition matrix depends on the group size and the number of variables in the polynomials depends on the number of types. Therefore, the number of equilibria should vary positively with each. The rule of six results summarizes this trade off between group size and the number of types. This result has clear and important implications. If many types interact in small groups or if a small number of types interacting in large groups, then the resulting system may not be predictable. That insight is important when trying to understand social, physical, and biological processes in which interactions can change type distributions.

Admittedly, the type interaction model presented in this paper is stark and unrealistic. Agents do not have sophisticated mental models. They change their types according to a fixed rule. The groups that form are random. They do not depend on the geographic location of the agents. Nevertheless, minimal benchmark conditions such as those derived in this paper are a valuable part of the larger enterprise to understand how systems of interacting agents generate complex phenomena. Thus, the results described in this paper should be seen as just a beginning. They should encourage further exploration of the linkages between the fineness of type categorizations, the size of group interactions and the multiplicity of equilibria. These linkages should be important to theorists and empiricists alike.

The preliminary analysis contained in this paper suggests several extensions. First, there is the important question of how the number of equilibria grows in the number of types. It seems likely that a type interaction model with $N = 2K + 1$ types could support at least $2K$ stable equilibria. But that lower bound is probably not binding for large K. It should be possible to create interaction rules that generate an even larger number of stable equilibria. That said, results on the number of solutions of systems of polynomials of a given degree from the mathematics literature should be seen as extremely crude upper bounds because the polynomials of degree k generated by interactions rules are strict subsets of the set of all polynomials of degree k. Second, if we restrict the interaction rules to those that may be plausible in an economic, political, chemical, biological, or physical context, we should further restrict the possibility of multiple equilibria. Thus, It might be possible to prove that there are analogs of the rule of six for subclasses of interaction rules. For example, assumptions that the types can be represented by binary strings and that new types can differ by at most one bit value, or assumptions that types have an implicit ranking and can rise of fall by at most one rank would restrict the set of transition rules. Perhaps, a rule of eight or a rule of ten exists for such models.

Appendix

Claim 5 *There does not exist a deterministic two type ternary interaction rule that supports multiple equilibria.*

Proof The set of all possible outcomes for each group of size three with a deterministic interaction rule can be written as follows:

Group	Possible outcomes
AAA	AAB, ABB, BBB
AAB	AAA, AAB, ABB, BBB
ABB	AAA, AAB, ABB, BBB
BBB	AAA, AAB, ABB

To simplify notation, let T_k equal the expected number of A's after an interaction with a group of exactly k agents of type A. Using the previous notation this was written as $T_{AAB:A}$ In the new notation, it is written as $T_2 = 1$ implies that a group with two agents of type A and one agent of type B creates exactly one agent of type A. Given that the rule is deterministic and that the rule must span, the following restrictions can be placed on the T_k's.

$T_3 \in \{0, 1, 2\}$
$T_2 \in \{0, 1, 2, 3\}$
$T_1 \in \{0, 1, 2, 3\}$
$T_0 \in \{1, 2, 3\}$

This creates 144 possible deterministic interaction rules. Each generates a polynomial of degree three or less which can be written as

$$f(a) = \alpha_3 a^3 + \alpha_2 a^2 + \alpha_1 a + \alpha_0$$

A fixed point of this equation satisfies $f(a) = a$. For convenience, define the function $g(a) = f(a) - a$. By construction it must be the case that, $g(0) > 0$, and $g(1) < 0$. Therefore, in order for there to be more than one equilibria, two conditions must be satisfied:

(1) the roots of $g'(a) = 0$ must be real valued at lie in the open interval (0, 1)
(2) the value of $g(a)$ at the smaller root must be negative and the value of $g(a)$ at the larger root must be positive

It is straightforward exercise to show that none of the polynomials belonging to the set can satisfy these two conditions simultaneously.

Claim 8 *If the initial distribution over types lies in $S_a \cap S_b$, then in every subsequent period the type distribution lies in S_a. If the distribution ever leaves S_b then it re-enters S_b in the subsequent period.*

Proof The first step is to show that in every period the distribution, lies in S_a so long as that in every other period it lies in S_b. Suppose that $(a_0, b_0, c_0, d_0) \in S_a \cap S_b$.

First, show that $(a_1, b_1, c_1, d_1) \in S_a$.

$$a_1 = \frac{a_0^2}{2} + \frac{b_0^2}{2} + 2a_0b_0 + 2a_0c_0$$
$$d_1 = \frac{c_0^2}{2} + \frac{d_0^2}{2} + 2c_0d_0 + 2b_0d_0$$

The result follows from $(a_0, b_0, c_0, d_0) \in S_a \cap S_b$.

Now there are two cases. If $b_1 > c_1$, then $a_2 > d_2$ by repeating the argument for $a_1 > d_1$. Suppose instead that $b_1 \leq c_1$.

$$(a_2 - d_2) = \frac{a_1^2}{2} + \frac{b_1^2}{2} + 2a_1b_1 + 2a_1c_1 - \frac{c_1^2}{2} - \frac{d_1^2}{2} - 2c_1d_1 - 2b_1d_0$$

This can be rewritten as

$$\frac{a_1^2}{2} - \frac{d_1^2}{2} + (a_1 - d_1)(b_1 + c_1) + \frac{b_1^2}{2} - \frac{c_1^2}{2} + (a_1 - d_1)(b_1 + c_1)$$

which reduces to

$$\frac{a_1 - d_1}{2}(a_1 + d_1 + 2b_1 + 2c_1) + \frac{b_1 + c_1}{2}(b_1 - c_1 + 2a_1 - 2d_1)$$

which can be reduced to the following

$$\frac{a_1 - d_1}{2}(1 + b_1 + c_1) + \frac{b_1 + c_1}{2}(b_1 - c_1 + 2a_1 - 2d_1)$$

It suffices to show that $(b_1 - c_1 + 2a_1 - 2d_1) > 0$ substituting in the values for a_1, b_1, c_1 and d_1 obtains

$$b_1 - c_1 + 2a_1 - 2d_1 = \frac{a_0^2}{2} + \frac{c_0^2}{2} - \frac{b_0^2}{2} - \frac{d_0^2}{2}a_0^2 + b_0^2 - c_0^2$$
$$-d_0^2 + 4(a_0 - d_0)(b_0 + c_0)$$

which reduces to

$$\frac{3a_0^2}{2} + \frac{b_0^2}{2} + \frac{c_0^2}{2} - \frac{3d_0^2}{2} + 4(a_0 - d_0)(b_0 + c_0)$$

which, by assumption is strictly greater than 0. This completes the proof that the distribution never leaves S_a. It remains to show that if the distribution leaves S_b, i.e. if $b_1 < c_1$ that in the next period it returns to S_b. The difference equations imply that

$$b_2 = \frac{a_1^2}{2} + \frac{c_1^2}{2} + a_1 d_1 + b_1 c_1$$

$$c_2 = \frac{b_1^2}{2} + \frac{d_1^2}{2} + a_1 d_1 + b_1 c_1$$

Subtracting the second equation from the first gives

$$b_2 - c_2 = \frac{a_1^2 - d_1^2}{2} + \frac{c_1^2 - b_1^2}{2}$$

which follows by assumption and completes the proof.

Claim 9 *In the three type ternary interaction rule that generates multiple equilibrium described in Claim 5, if the distribution over types satisfies $a \geq 2b$ and $a \geq 2c$, then so do subsequent type distributions.*

Proof It suffices to show $\hat{a} \geq 2\hat{b}$ and $a \geq 2\hat{c}$.

$\hat{a} \geq 2\hat{b}$: suffices to show

$$\frac{a^3}{3} + 3a^2b + 3ab^2 + 3a^2c + 2abc + \frac{b^3}{3} \geq \frac{2(a^3 + b^3)}{3} + 4abc + \frac{2c^3}{3}$$

which reduces to

$$3a^2b + 3ab^2 + 3a^2c \geq \frac{b^3}{3} + 2abc + \frac{2c^3}{3}$$

$a > 2b$ and $a > 2c$ implies that $3a^2c \geq 2abc + 8c^3$. Therefore, it suffices to show

$$3a^2b + 3ab^2 \geq \frac{b^3}{3}$$

Which follows from $a > 2b$.

$a \geq 2\hat{c}$:

$$\frac{a^3}{3} + 3a^2b + 3ab^2 + 3a^2c + 2abc + \frac{b^3}{3}$$
$$= \frac{4c^3}{3} + \frac{a^3}{2} + 6c^2b + \frac{3a^2b}{2} + 6cb^2 + 6ac^2 + 2abc + \frac{b^3}{3}$$

Suffices to show

$$\frac{4c^3}{3} + \frac{a^3}{2} + 6c^2b + \frac{3a^2b}{2} + 6cb^2 + 6ac^2 + 2abc + \frac{b^3}{3}$$
$$\geq \frac{4c^3}{3} + 6c^2b + +6cb^2 + +6ac^2 + +4abc + \frac{2b^3}{3}$$

which reduces to

$$\frac{a^3}{2} + \frac{3a^2b}{2} \geq +2abc\frac{b^3}{3}$$

which follows from the assumptions.

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