Math. Control Signals Systems (2005) 17: 101–127
DOI 10.1007/s00498-005-0151-x

ORIGINAL ARTICLE

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Geometric homogeneity with applications to finite-time stability

Received: 17 April 2003 / Revised: 28 March 2004 / Published online: 16 May 2005

Abstract This paper studies properties of homogeneous systems in a geometric, coordinate-free setting. A key contribution of this paper is a result relating regularity properties of a homogeneous function to its degree of homogeneity and the local behavior of the dilation near the origin. This result makes it possible to extend previous results on homogeneous systems to the geometric framework. As an application of our results, we consider finite-time stability of homogeneous systems. The main result that links homogeneity and finite-time stability is that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity. We also show that the assumption of homogeneity leads to stronger properties for finite-time stable systems.

Keywords Geometric homogeneity · Homogeneous systems · Stability · Finite-time stability · Lyapunov theory

1 Introduction

Homogeneity is the property whereby objects such as functions and vector fields scale in a consistent fashion with respect to a scaling operation on the underlying space. Geometrically, a function that is homogeneous with respect to a scaling

A preliminary version of this paper appears in the proceedings of the 1997 American Control Conference. This research was supported in part by the Air Force Office of Scientific Research under grant F49620-95-1-0019.

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operation has the property that every scaled level set of the function is also a level set, while a homogeneous vector field has the property that every scaled orbit of the vector field is also an orbit.

Homogeneity is defined in relation to a scaling operation or a *dilation*, which is essentially an action of the multiplicative group of positive real numbers on the state space. The familiar operation of scalar multiplication on $\mathbb{R}^n$ yields the *standard dilation* $\Delta_\lambda(x) = \lambda x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$. Homogeneity with respect to the standard dilation is one of the two axioms for linearity, the other being additivity. Many familiar properties of linear systems follow, in fact, from homogeneity alone. Early work on homogeneous systems was restricted to the standard dilation. For instance, the stability of systems that are homogeneous with respect to the standard dilation was considered in [C,H1]. More recently, [R5] contains results on the input–output properties as well as the universal stabilization of such systems. Vector fields whose components are all homogeneous polynomials of the same degree form an important subclass of systems that are homogeneous with respect to the standard dilation. References related to such polynomial systems can be found in [DM,IO].

Recent years have seen increasing interest in systems that are homogeneous with respect to dilations of the form

$$\Delta_\lambda(x) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n),\ \lambda > 0,\ x = (x_1, \dots, x_n) \in \mathbb{R}^n\,, \tag{1}$$

where r_i, $i = 1, \dots, n$, are positive real numbers [H4,H5,H7,H8,HHX,K4,K6, R1,SA1,SA3]. The standard dilation is a special case of (1) with $r_1 = \cdots = r_n = 1$. Many of the recent results on homogeneous systems are generalizations of familiar properties of linear systems. For instance, for a system that is homogeneous with respect to the dilation (1), asymptotic stability of the origin implies global asymptotic stability as well as the existence of a C^1 Lyapunov function that is also homogeneous with respect to the same dilation [R1]. This property of homogeneous systems is an extension of the familiar fact that an asymptotically stable linear system has a quadratic Lyapunov function, both of which are homogeneous with respect to the standard dilation. The stability of a homogeneous system is determined by that of its restriction to certain invariant sets [K6] just as the stability of a linear system is determined by its restriction to its eigenspaces.

An important application of homogeneity is in deducing the stability of a nonlinear system from the stability of a homogeneous approximation. A general result of this kind, which appears in [H4], states that if a vector field can be written as the sum of several vector fields, each of which is homogeneous with respect to a fixed dilation of the form (1), then asymptotic stability of the lowest degree vector field implies local asymptotic stability of the original vector field. Similar results can also be found in [H1, Sect. 57] for the special case of the standard dilation. A special case of these results is Lyapunov's well known first method of stability analysis, where the Taylor series expansion is used to write a given analytic vector field as a sum of vector fields homogeneous with respect to the standard dilation, and stability of the given vector field is deduced from the stability of the lowest degree vector field in the sum, which is the linearization of the given vector field.

Homogeneous stabilization of homogeneous systems is considered in [K4,K6, SA1], while connections between stabilizability and homogeneous feedback stabilization are explored in [H7,SA3]. Dilations of the form (1) play an important role in the theory of nilpotent approximations of control systems, which are useful in studying local controllability properties of nonlinear control systems. See, for instance, [H5]. Dilations of the form (1) were also used for finite-time stabilization using state feedback [BB3, H8, R4] and output feedback [HHX].

Since the description of dilations of the form (1) clearly involves the use of coordinates, homogeneity with respect to a given dilation of the form (1) is a coordinate-dependent property. Thus a system that is homogeneous in one set of coordinates may not be homogeneous in another. A coordinate-free generalization of the notion of homogeneity is proposed in [K7]. Specifically, it is observed in [K7] that a smooth dilation satisfying the axioms given in [K7] gives rise to a one-parameter subgroup of diffeomorphisms ϕ on $\mathbb{R}^n$ given by $\phi_t(x) = \Delta_{e^t}(x),\ t \in \mathbb{R},\ x \in \mathbb{R}^n$, such that the origin is an asymptotically stable equilibrium in reverse time under the infinitesimal generator ν of ϕ. This same idea also appears in [H6] for the case in which the dilation is of the form (1). Homogeneity has a particularly simple characterization in terms of the vector field ν. For instance, a smooth vector field f on $\mathbb{R}^n$ is homogeneous of degree m with respect to Δ if and only if the Lie derivative $L_\nu f$ of f with respect to ν satisfies $L_\nu f = mf$. Based on these ideas, [K8] develops a geometric notion of homogeneity in terms of the vector field ν, called the Euler vector field of the dilation Δ. According to an anonymous reviewer, the main ideas of [K7, K8] were also investigated independently in [R2]. A recent treatment of geometric homogeneity may be found in [BR, Ch. 5]. On a related note, [P] gives examples of vector fields that are homogeneous with respect to a general linear Euler vector field with degrees of homogeneity that are, loosely speaking, functions of the state.

While [K8] develops a geometric notion of homogeneity, it does not consider extensions to the geometric framework of previous results on systems that are homogeneous with respect to dilations of the form (1). Extension of results that involve non-topological properties of the dilation (1) presents a special challenge, because these results depend explicitly on the parameters $r_1, \ldots, r_n$, and the geometric significance of these parameters is not obvious. A case in point is the main result of [R1], which asserts that an asymptotically stable system that is homogeneous with respect to a dilation of the form (1) admits a C^1 homogeneous Lyapunov function with a degree of homogeneity that is greater than $\max\{r_1, \ldots, r_n\}$. A key contribution of the present paper is to identify the spectral abscissa $\overline{\sigma}$ of the linearization of the Euler vector field at the origin as the correct generalization of the parameter $\max\{r_1, \ldots, r_n\}$, thus making it possible to fully extend results such as that of [R1] to the geometric framework. We provide the first such extensions of several results.

The coordinate-free framework that we adopt highlights the distinction between topological and non-topological aspects of homogeneity. Topologically, homogeneity involves a continuous Euler vector field that has the origin as an asymptotically stable equilibrium in reverse time. Functions and vector fields that are homogeneous with respect to such an Euler vector field possess topological properties such as properness of sign-definite homogeneous functions, global asymptotic stability of attractive equilibria of homogeneous vector fields, and existence of continuous homogeneous Lyapunov functions for asymptotically stable equilibria of homogeneous vector fields. However, homogeneity as a purely topological property is not sufficiently strong to obtain results that relate regularity properties of homogeneous objects to their homogeneity properties. Such results require growth bounds on the trajectories of the Euler vector field, and are possible in the case where the Euler vector field is C^1. A principal contribution of this paper is to quantify the relationship between the regularity properties of a homogeneous function, its degree of homogeneity with respect to a C^1 Euler vector field, and the local behavior of the integral curves of the Euler vector field near the origin. A key parameter in this quantification is the spectral abscissa $\overline{\sigma}$ of the linearization of the Euler vector field at the origin. Proposition 3.1 in Sect. 3 relates the local behavior of the integral curves of the Euler vector field near the origin to the parameter $\overline{\sigma}$.

We consider homogeneous functions and their properties in Sect. 4. The main result of this section, Theorem 4.1, lays out the relationship between the degree of homogeneity of a homogeneous function, the regularity properties of the function and the parameter $\overline{\sigma}$. A special case of this result is the observation that the scalar function $V(x) = |x|^\alpha$ on $\mathbb{R}$, which is homogeneous of degree $\alpha \geq 0$ with respect to the standard dilation on $\mathbb{R}$ with $\overline{\sigma} = 1$, is Hölder continuous at $x = 0$ for $\alpha > 0$, Lipschitz continuous at $x = 0$ for $\alpha \geq 1$ and C^1 for $\alpha > 1$. Theorem 4.1 plays a crucial role in all subsequent results involving assertions of regularity.

We introduce homogeneous vector fields in Sect. 5, and consider the stability of homogeneous systems in Sect. 6. As an improvement over a previous result, we show that an attractive equilibrium of a homogeneous system is not merely globally attractive as asserted in [H1, Sect. 17] and [R1], but is, in fact, globally asymptotically stable. We prove a new topological stability result for homogeneous systems, which states that if all solutions of a homogeneous system that start in a compact set subsequently remain in the interior of that set, then the origin is a globally asymptotically stable equilibrium for the system. We give a stronger version of Theorem 5.12 of [BR] (which, according to an anonymous reviewer, appears as Proposition 2 on page 35 of [R2]) giving the existence of homogeneous Lyapunov functions for asymptotically stable homogeneous systems in a geometric setting. While Theorem 5.12 of [BR] does not address the regularity of the Lyapunov function at the origin, our result asserts the existence of a continuous (C^1) Lyapunov function with a continuous Lyapunov derivative for a system that is homogeneous with respect to a continuous (C^1) Euler vector field. The main result of [R1] asserts stronger regularity properties for the Lyapunov function in the case of dilations of the form (1). However, unlike the proof given in [R1], which depends on explicit coordinate-based computations, our proof makes use of Theorem 4.1 to relate the regularity of the Lyapunov function to the degree of homogeneity of the system and the parameter $\overline{\sigma}$.

As an application of our results, we consider finite-time stability of homogeneous systems in Sect. 7. Finite-time stability is the property, whereby the trajectories of a non-Lipschitzian system reach a Lyapunov stable equilibrium state in finite time. Classical optimal control theory provides several examples of systems that exhibit convergence to the equilibrium in finite time [R3]. A well-known example is the double integrator with time-optimal bang-bang feedback control. These examples typically involve closed-loop dynamics that are discontinuous. Finite-settling-time behavior of systems with continuous dynamics is considered in [BB3] and the references contained therein.

A detailed analysis of continuous, time invariant finite-time-stable systems including Lyapunov and converse Lyapunov results was given in [BB4]. Differential inequalities provide the main tool for analyzing finite-time stability of general systems [BB4]. However, differential inequalities can be difficult to verify in practice, especially for high-dimensional systems. This dependence on differential inequalities renders the analysis and design of finite-time-stable systems difficult. To overcome this difficulty, we consider finite-time stability of homogeneous systems and show that the assumption of homogeneity leads to simpler sufficient conditions for finite-time stability as well as stronger properties for finite-time-stable systems.

The main result that links homogeneity to finite-time stability is a topological result that asserts that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has negative degree of homogeneity. This connection is not surprising in view of the fact that finite-time stability is an inherently non-Lipschitzian phenomenon. The ideas involved in the proof of this result were used in constructing finite-time stabilizing controllers for second-order systems in [BB3, R4]. A proof of this result for dilations of the form (1) appears in [HHX] and [BR, Corr. 5.4]. This result was also applied to output-feedback finite-time stabilization of second-order systems in [HHX] and, subsequently, to state-feedback finite-time stabilization of a class of higher-order systems in [H8]. In all these applications, the dilations involved were of the form (1). In Sect. 7, we prove this result in our more general setting.

In Sect. 7, we also show that a finite-time-stable system that is homogeneous with respect to a C^1 Euler vector field admits a C^1 homogeneous Lyapunov function satisfying a differential inequality, and the settling time function of such a system is Hölder continuous at the equilibrium. These results are significant in view of the counterexamples provided in [BB4], which demonstrate that in the general (non-homogeneous) case, a finite-time-stable system may not necessarily admit a C^1 Lyapunov function satisfying a differential inequality, and the settling-time function of such a system may not necessarily be continuous. The strengthened converse Lyapunov result of Sect. 7 is used to prove a non-Lipschitzian analog of the result given in [H4], namely, if a vector field can be written as the sum of several vector fields, each of which is homogeneous with respect to a given C^1 Euler vector field, then finite-time stability of the lowest (most negative) degree vector field implies finite-time stability of the original vector field. Finally, we use these results in Sect. 8 to demonstrate the existence of a class of finite-time stabilizing controllers for a chain of integrators and show that every controllable linear system is finite-time stabilizable through continuous state feedback.

A novel feature of the results that we present is that the spectral abscissa $\overline{\sigma} \geq 0$ of the linearization of the Euler vector field at the origin need not be positive, that is, the origin need not be an exponentially stable equilibrium for the vector field $-\nu$ unlike in the case of dilations of the form (1). Example 5.2 illustrates homogeneity with respect to an Euler vector field having $\overline{\sigma} = 0$. We also give a scalar example to demonstrate that, while vector fields having bounded components cannot be homogeneous with respect to dilations of the form (1), such vector fields can be homogeneous in a geometric sense. The treatment of this paper thus extends previous work on homogeneous systems and increases the scope of applicability of techniques that depend on homogeneity by allowing for more general classes of systems and Euler vector fields.

2 Preliminaries

Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. The notions of openness, convergence, continuity and compactness that we use refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$. We use $\mathbb{R}_+$ to denote the nonnegative real numbers. Let $\mathcal{A}^c$, $\overline{\mathcal{A}}$, bd $\mathcal{A}$ and int $\mathcal{A}$ denote the complement, closure, boundary and interior of the set $\mathcal{A} \subseteq \mathbb{R}^n$, respectively. A set $\mathcal{A} \subset \mathbb{R}^n$ is *bounded* if $\overline{\mathcal{A}}$ is compact. We denote the composition of two functions $U : \mathcal{A} \to \mathcal{B}$ and $V : \mathcal{B} \to \mathcal{C}$ by $V \circ U : \mathcal{A} \to \mathcal{C}$. By an *open neighborhood* of a set $\mathcal{K} \subseteq \mathbb{R}^n$, we mean an open set in $\mathbb{R}^n$ containing $\mathcal{K}$. If $\{x_i\}$ is a sequence in $\mathbb{R}^n$ and $\mathcal{K} \subset \mathbb{R}^n$, we write $x_i \to \mathcal{K}$ if, for every open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there exists a positive integer k such that $x_i \in \mathcal{U}$ for all $i > k$. In a similar fashion, given a function $y : \mathbb{R}_+ \to \mathbb{R}^n$, we write $y(t) \to \mathcal{K}$ if, for every open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there exists $\tau \in \mathbb{R}_+$ such that $y(t) \in \mathcal{U}$ for all $t > \tau$.

Throughout this paper, we let f denote a continuous vector field on $\mathbb{R}^n$ with the property that, for every initial condition $y(0) \in \mathbb{R}^n$, the system of differential equations

$$\dot{y}(t) = f(y(t)) \tag{2}$$

has a unique right-maximally-defined solution, and this unique solution is defined on $[0, \infty)$. Under these assumptions on f, the solutions of (2) are jointly continuous functions of time and the initial condition [H3, Thm. V.2.1] and thus define a continuous *global semiflow* [BH] $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^n$. In particular, ψ satisfies

$$\psi(0, x) = x, \tag{3}$$

$$\psi(t, \psi(h, x)) = \psi(t + h, x) \tag{4}$$

for all $t, h \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. Given $t \in \mathbb{R}_+$, we denote the map $\psi(t, \cdot)$ by $\psi_t(\cdot)$. By the continuity of ψ, $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for every $t \in \mathbb{R}_+$.

Given $x \in \mathbb{R}^n$, the *integral curve* of f starting at x is the continuously differentiable map $\psi^x(\cdot) \stackrel{\triangle}{=} \psi(\cdot, x)$. The *$f$-orbit*, alternatively, the *ψ-orbit*, of x is the set $\mathcal{O}^x = \psi^x(\mathbb{R}_+)$.

A set $\mathcal{A} \subseteq \mathbb{R}^n$ is *positively invariant* under f if $\psi_t(\mathcal{A}) \subseteq \mathcal{A}$ for all $t > 0$. A set $\mathcal{A}$ that is positively invariant under f is *invariant* (called *weakly invariant* in [BH]) or *strictly positively invariant* under f if, respectively, $\psi_t(\mathcal{A}) = \mathcal{A}$ or $\psi_t(\mathcal{A}) \subset \text{int}\,\mathcal{A}$ for all $t > 0$.

A nonempty set $\mathcal{K} \subset \mathbb{R}^n$ is *attractive* under f if there exists an open neighborhood $\mathcal{V}$ of $\mathcal{K}$ such that $\psi^x(t) \to \mathcal{K}$ for all $x \in \mathcal{V}$. In this case, the set doa$(\mathcal{K})$ of all points x such that $\psi^x(t) \to \mathcal{K}$ is the *domain of attraction* of $\mathcal{K}$. If $\mathcal{K}$ is attractive with domain of attraction $\mathbb{R}^n$, then $\mathcal{K}$ is *globally attractive*. The domain of attraction of a nonempty attractive set $\mathcal{K}$ is an open, invariant set containing $\mathcal{K}$ [BH, Prop. 8.8], [BS, Sect. V.1].

A nonempty, compact set $\mathcal{K} \subset \mathbb{R}^n$ is *Lyapunov stable* under f if, for every open neighborhood $\mathcal{U}_\epsilon$ of $\mathcal{K}$, there exists an open neighborhood $\mathcal{U}_\delta$ of $\mathcal{K}$ such that $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\epsilon$ for all $t \in \mathbb{R}_+$. $\mathcal{K}$ is *(globally) asymptotically stable* under f if $\mathcal{K}$ is Lyapunov stable and (globally) attractive. Finally, the origin is said to be Lyapunov stable, (globally) attractive or (globally) asymptotically stable under f if the set $\{0\}$ is, respectively, Lyapunov stable, (globally) attractive or (globally) asymptotically stable under f. Note that if a nonempty compact set $\mathcal{K}$ is Lyapunov stable under

f, then $\mathcal{K}$ is necessarily positively invariant. In particular, if the origin is Lyapunov stable under f, then $\psi^0 \equiv 0$ and, consequently, $f(0) = 0$.

It will often be convenient to call a set invariant, attractive or stable under ψ whenever the set has the respective property under f.

The following result links the concepts of positive invariance and attractiveness.

Lemma 2.1 *Let $\mathcal{A} \subset \mathbb{R}^n$ be nonempty, compact and positively invariant under ψ. Then the largest subset $\mathcal{K}$ of $\mathcal{A}$ that is invariant under ψ is nonempty, compact and, for every $x \in \mathcal{A}$, $\psi^x(t) \to \mathcal{K}$. In addition, if $\mathcal{A}$ is strictly positively invariant under ψ, then $\mathcal{K} \subset \operatorname{int} \mathcal{A}$ and $\mathcal{K}$ is asymptotically stable under ψ.*

Proof See Appendix 9. □

Remark 2.1 Given $x \in \mathbb{R}^n$, (4) implies that $\mathcal{O}^x$ is positively invariant under ψ. Therefore, $\overline{\mathcal{O}^x}$ is positively invariant since, for every $t \in \mathbb{R}_+$, $\psi_t(\overline{\mathcal{O}^x}) \subseteq \overline{\psi_t(\mathcal{O}^x)} \subseteq \overline{\mathcal{O}^x}$, where the first inclusion follows from the continuity of ψ [M, Thm. 7.1, p.103] and the second from the positive invariance of $\mathcal{O}^x$. It can be shown that the largest invariant set contained in $\overline{\mathcal{O}^x}$ is the *positive limit set* $\mathcal{O}^x_+ \triangleq \bigcap_{t \geq 0} \overline{\psi_t(\mathcal{O}^x)}$ of x [BH, Ch. 5], [BS, pp. 19–24]. If $\mathcal{O}^x$ is bounded, then Lemma 2.1 with $\mathcal{A} = \overline{\mathcal{O}^x}$ yields the familiar result [BH, Thm. 5.5, 5.9], [BS, p. 24], [K9, p. 114] that the positive limit set of x is nonempty, compact and $\psi^x(t) \to \mathcal{O}^x_+$. Thus the first part of Lemma 2.1 is a generalization of well-known results on positive limit sets.

The following technical result will be needed later. The last part of this result also follows from Theorem V.1.16 in [BS] in the case where solutions of (2) are unique in reverse time as well.

Lemma 2.2 *Suppose $\mathcal{K} \subset \mathbb{R}^n$ is nonempty, compact and attractive under ψ, and let $\mathcal{M} \subset \operatorname{doa}(\mathcal{K})$ be nonempty and compact. Then $\psi(\mathbb{R}_+ \times \mathcal{M})$ is bounded. In addition, if $\mathcal{K}$ is asymptotically stable under ψ, then, for every open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there exists $\tau > 0$ such that $\psi_t(\mathcal{M}) \subset \mathcal{U}$ for all $t > \tau$.*

Proof See Appendix 9. □

A function $V : \mathbb{R}^n \to \mathbb{R}$ is *proper* if the inverse image $V^{-1}(M)$ of M is compact for every compact set $M \subset V(\mathbb{R}^n)$. V is *radially unbounded* if V is proper and $V(\mathbb{R}^n)$ is unbounded. V is *positive (negative) definite* if $V(0) = 0$ and V takes only positive (negative) values on $\mathbb{R}^n \backslash \{0\}$. Finally, V is *sign definite* if V is either positive or negative definite.

A continuous function $V : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is Fréchet differentiable at x with Fréchet derivative $dV_x : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ [F2, pp. 264–266], if

$$\lim_{z \to x} \frac{V(z) - V(x) - dV_x(z - x)}{\|z - x\|} = 0. \tag{5}$$

V is continuously differentiable, that is, C^1, on an open set $\mathcal{U} \subseteq \mathbb{R}^{n_1}$ if and only if V is Fréchet differentiable on $\mathcal{U}$ and the map $x \mapsto dV_x \in \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ is continuous on $\mathcal{U}$, where $\mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ denotes the set of linear maps from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_2}$ with the induced norm $\|\cdot\|_{\mathrm{i}}$. Equivalently, V is C^1 on $\mathcal{U}$ if and only if V is Fréchet differentiable on $\mathcal{U}$ and, for every $v \in \mathbb{R}^{n_1}$, the map $x \mapsto dV_x(v)$ is continuous.

Let $\varphi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ be a C^1 diffeomorphism. If $V : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is C^1 on an open set $\mathcal{U} \subseteq \mathbb{R}^{n_1}$, then the function $V \circ \varphi$ is C^1 on the open set $\varphi^{-1}(\mathcal{U})$ and the chain rule holds in the form

$$d(V \circ \varphi)_x(v) = dV_{\varphi(x)}(d\varphi_x(v)), \quad x \in \varphi^{-1}(\mathcal{U}), \ \ v \in \mathbb{R}^{n_1}. \tag{6}$$

By letting $n_1 = n_2$ and $V = \varphi^{-1}$ in (6), it is easy to see that $d\varphi_x$ is invertible and $(d\varphi_x)^{-1} = (d\varphi^{-1})_{\varphi(x)}$.

The *Lie-derivative* of a continuous function $V : \mathbb{R}^{n_1} \to \mathbb{R}$ with respect to f is given by

$$L_f V(x) = \lim_{t \to 0^+} \frac{1}{t}[V(\psi_t(x)) - V(x)], \tag{7}$$

whenever the limit exists. If V is C^1 on $\mathbb{R}^n$, then $L_f V$ is defined and continuous on $\mathbb{R}^n$, and given by $L_f V(x) = dV_x(f(x))$.

The origin is a *finite-time-stable* equilibrium under f (or ψ), if and only if 0 is Lyapunov stable under f and there exist an open neighborhood $\mathcal{N}$ of 0 that is positively invariant under f and a positive-definite function $T : \mathcal{N} \to \mathbb{R}$ called the *settling-time function* such that $\psi(T(x), x) = 0$ for all $x \in \mathcal{N}$ and $\psi(t, x) \neq 0$ for all $x \in \mathcal{N}\backslash\{0\}$, $t < T(x)$. The origin is a *globally finite-time-stable* equilibrium under f (or ψ) if 0 is finite-time stable with $\mathcal{N} = \mathbb{R}^n$. Note that by the uniqueness assumption, it necessarily follows that $\psi(T(x) + t, x) = 0$ for all $t \in \mathbb{R}_+$ and, therefore,

$$T(x) = \min\{t \in \mathbb{R}_+ : \psi(t, x) = 0\} \tag{8}$$

for all $x \in \mathcal{N}$. Also, finite-time stability of the origin implies asymptotic stability of the origin. Various properties of the settling-time function are given in [BB4]. Versions of the following sufficient condition for the origin to be a finite-time-stable equilibrium of f appears in [BB1,BB3,H2]. A proof as well as a converse is given in [BB4].

Theorem 2.1 *Suppose there exists a continuous, positive-definite function* $V : \mathcal{V} \to \mathbb{R}$ *defined on an open neighborhood* $\mathcal{V}$ *of the origin such that* $L_f V$ *is defined everywhere on* $\mathcal{V}$ *and satisfies* $L_f V(\cdot) \leq -c[V(\cdot)]^\alpha$ *on* $\mathcal{V}$ *for some* $c > 0$ *and* $\alpha \in (0, 1)$. *Then the origin is a finite-time-stable equilibrium under* f, *and the settling-time function is continuous on the domain of attraction of the origin. In addition, if* $\mathcal{V} = \mathbb{R}^n$ *and* V *is proper, then the origin is a globally finite-time-stable equilibrium under* f.

In the sequel, we will need to consider a complete vector field ν on $\mathbb{R}^n$ such that the solutions of the differential equation $\dot{y}(t) = \nu(y(t))$ define a continuous *global flow* $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$. For each $s \in \mathbb{R}$, the map $\phi_s(\cdot) = \phi(s, \cdot)$ is a homeomorphism and $\phi_s^{-1} = \phi_{-s}$. Notions of invariance, Lyapunov stability and attractivity are similarly defined for ϕ in terms of its restriction to $\mathbb{R}_+ \times \mathbb{R}^n$. However, since ϕ_s is a bijection for each $s \in \mathbb{R}$, it is easy to show that a set $\mathcal{K} \subseteq \mathbb{R}^n$ is invariant under ϕ if and only if there exists $\epsilon > 0$ such that $\phi_s(\mathcal{K}) \subseteq \mathcal{K}$ for all $s \in (-\epsilon, \epsilon)$.

In the case where ν is C^1, ϕ_s is a diffeomorphism and $(d\phi_s)_x$ a linear isomorphism for every $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Moreover, for every $x \in \mathbb{R}^n$, the function $s \mapsto (d\phi_{-s})_x$ satisfies the differential equation

$$\frac{d}{ds}(d\phi_{-s})_x = -d\nu_{\phi(-s,x)} \circ (d\phi_{-s})_x \tag{9}$$

on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ [H3, Thm. V.3.1]. Also, in this case, we define the Lie derivative of f with respect to ν to be the vector field $L_\nu f$ given by

$$(L_\nu f)(x) \stackrel{\triangle}{=} \lim_{s\to 0} \frac{1}{s}[(d\phi_{-s})_{\phi_s(x)}(f(\phi_s(x))) - f(x)] \tag{10}$$

wherever the limit exists. $L_\nu f$ is also the Lie bracket $[\nu, f]$ of the vector fields ν and f.

3 Euler vector fields and dilations

In the sequel, we will assume that the vector field ν is an *Euler vector field*, that is, the origin is a globally asymptotically stable equilibrium under $-\nu$. Thus, $\lim_{s\to\infty} \phi(-s, x) = 0$ for all $x \in \mathbb{R}^n$. Also, given a bounded open neighborhood $\mathcal{U}$ of 0, the continuity of ϕ implies that, for every $x \notin \mathcal{U}$, there exists $s > 0$ such that $\phi_{-s}(x) \in \operatorname{bd} \mathcal{U}$, while, for every $x \in \mathcal{U}\backslash\{0\}$, there exists $s > 0$ such that $\phi_s(x) \in \operatorname{bd} \mathcal{U}$. Moreover, $\{0\}$ is the only nonempty compact set that is invariant under ν. Since ν defines a global flow on $\mathbb{R}^n$, $\phi(s, x) = 0$, $(s, x) \in \mathbb{R} \times \mathbb{R}^n$, implies $x = 0$.

In the case that ν is C^1, we let $\overline{\sigma}$ denote the spectral abscissa, that is, the largest of the real parts of the eigenvalues, of the linearization $d\nu_0$ of ν at 0. Since the origin is asymptotically stable under $-\nu$, it follows that $\overline{\sigma} \geq 0$. The following technical result relates the local behavior near 0 of the integral curves of $-\nu$ and the solutions of (9) to the parameter $\overline{\sigma}$, and plays a key role in the proof of the main result of the next section.

Proposition 3.1 *Suppose ν is C^1. Let $\mathcal{M}$ be a nonempty compact set such that $0 \notin \mathcal{M}$ and let $\sigma > \overline{\sigma}$. Then the following hold.*

(i) There exists an open neighborhood $\mathcal{U}$ of 0 such that $e^{\sigma s}\phi_{-s}(x) \notin \mathcal{U}$ for all $s \in \mathbb{R}_+$ and $x \in \mathcal{M}$.
(ii) There exists an open neighborhood $\mathcal{U}$ of 0 such that $e^{\sigma s}(d\phi_{-s})_x(v) \notin \mathcal{U}$ for all $s \in \mathbb{R}_+$, $x \in \mathcal{M}$ and $v \in \mathbb{R}^n$ such that $\|v\| = 1$.

Proof We begin by noting that in this proof, we make explicit use of the identification between $\mathbb{R}^n$ and each of its tangent spaces. Choose $\sigma > \overline{\sigma}$ and denote the vector field $x \mapsto \nu(x) - \sigma x$ by g. All the eigenvalues of the linearization dg_0 at 0 of g have negative real parts. Hence there exists a C^1, positive-definite quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for the linear system $\dot{v}(t) = dg_0(v(t))$ such that the quadratic function $v \mapsto dV_v(dg_0(v))$ is negative definite.

(i) V is also a Lyapunov function locally for the vector field g, that is, there exists an open neighborhood $\mathcal{V}$ of 0 such that $L_g V$ takes nonpositive values on $\mathcal{V}$. By Lemma 2.2, there exists $T > 0$ such that $\phi_{-s}(\mathcal{M}) \in \mathcal{V}$ for all $s > T$. Let β

denote the minimum value attained by V on the compact set $\phi([-T,0]\times\mathcal{M})$. Since $\phi(s,x)=0$ implies $x=0$, $0\notin\phi(\mathbb{R}\times\mathcal{M})$ and hence $\beta>0$. Now, let $x\in\mathcal{M}$ and denote $y:\mathbb{R}_+\to\mathbb{R}^n$ by $y(s)=e^{\sigma s}\phi_{-s}(x)$ so that $\dot y(s)=-e^{\sigma s}g(\phi_{-s}(x))$. Note that since V is quadratic, $dV_{y(s)}=e^{\sigma s}dV_{\phi_{-s}(x)}$. Therefore, $\frac{d}{ds}V\circ y(s)=dV_{y(s)}(\dot y(s))=-e^{2\sigma s}L_gV(\phi_{-s}(x))\geq 0$ for every $s>T$. Thus $V(e^{\sigma s}\phi_{-s}(x))\geq\beta$ for all $(s,x)\in\mathbb{R}_+\times\mathcal{M}$. The result now follows by letting $\mathcal{U}=V^{-1}([0,\beta/2))$.

(ii) Since g is C^1, dg_x is continuous in x and, therefore, there exists an open neighborhood $\mathcal{V}$ of 0 such that, for every $x\in\mathcal{V}$, the quadratic function $v\mapsto dV_v(dg_x(v))$ is negative definite. By Lemma 2.2, there exists $T>0$ such that $\phi_{-s}(\mathcal{M})\in\mathcal{V}$ for all $s>T$. Let β denote the minimum value attained by V on the compact set $\mathcal{K}=\{e^{\sigma s}(d\phi_{-s})_x(v): x\in\mathcal{M}, v\in\mathbb{R}^n, \|v\|=1, s\in[0,T]\}$. Since ϕ_{-s} is a diffeomorphism for all $s\geq 0$, it follows that $0\notin\mathcal{K}$ and hence $\beta>0$. Now, let $x\in\mathcal{M}$ and $v\in\mathbb{R}^n$ be such that $\|v\|=1$. Denote $y:\mathbb{R}_+\to\mathbb{R}^n$ by $y(s)=e^{\sigma s}(d\phi_{-s})_x(v)$. It follows from (9) that $\dot y(s)=-e^{\sigma s}dg_{\phi(-s,x)}\circ(d\phi_{-s})_x(v)=-dg_{\phi(-s,x)}(y(s))$. We compute $\frac{d}{ds}V\circ y(s)=dV_{y(s)}(\dot y(s))=-dV_{y(s)}(dg_{\phi(-s,x)}(y(s)))\geq 0$ for all $s>T$. Thus $V(e^{\sigma s}(d\phi_{-s})_x(v))\geq\beta$ for all $s\in\mathbb{R}_+$, $x\in\mathcal{M}$ and $v\in\mathbb{R}^n$ such that $\|v\|=1$. The result now follows by letting $\mathcal{U}=V^{-1}([0,\beta/2))$. □

The flow ϕ induces an action of the multiplicative group of positive real numbers on $\mathbb{R}^n$ given by $\Delta_\lambda(\cdot)=\phi_{\ln(\lambda)}(\cdot)$, $\lambda>0$. Δ is called the *dilation* associated with the Euler vector field ν.

The dilations often considered in the literature [DM, H4, H5, H7, K3, K4, K6, R1, SA2] are of the form

$$\Delta_\lambda(x_1,\ldots,x_n)=(\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n), \tag{11}$$

where $x_1,\ldots,x_n$ are suitable coordinates on $\mathbb{R}^n$ and $r_1,\ldots,r_n$ are positive real numbers. The dilation corresponding to $r_1=\cdots=r_n=1$ is the standard dilation on $\mathbb{R}^n$.

The Euler vector field of the dilation (11) is linear and is given by [H6, K6, K8]

$$\nu=r_1x_1\frac{\partial}{\partial x_1}+\cdots+r_nx_n\frac{\partial}{\partial x_n}, \tag{12}$$

with $\overline{\sigma}=\max\{r_1,\ldots,r_n\}$. The global flow of ν is given by

$$\phi_s(x_1,\ldots,x_n)=(e^{r_1s}x_1,\ldots,e^{r_ns}x_n)=\Delta_{e^s}(x_1,\ldots,x_n). \tag{13}$$

4 Homogeneous functions

Following [K8], we define a function $V:\mathbb{R}^n\to\mathbb{R}$ to be *homogeneous* of degree $l\in\mathbb{R}$ with respect to ν if

$$V\circ\phi_s(x)=e^{ls}V(x),\quad s\in\mathbb{R},\ x\in\mathbb{R}^n. \tag{14}$$

It necessarily follows from (14) that if V is homogeneous of degree $l\neq 0$, then $V(0)=0$. Equation (14) is equivalent to

$$\phi_s(V^{-1}(M))=V^{-1}(e^{ls}M),\quad M\subseteq\mathbb{R},\ s\in\mathbb{R}. \tag{15}$$

In particular, the image under ϕ_s of a level set of a homogeneous function V is a level set of V.

If V is a continuous homogeneous function of degree $l > 0$, then $L_\nu V$ is defined everywhere and satisfies

$$L_\nu V = lV. \tag{16}$$

Equation (16) is easily verified by using (14) in (7). See also [K3, K8, SA2]. In the case that ν is the Euler vector field of the standard dilation on $\mathbb{R}^n$ and V is C^1, equation (16) yields the so called Euler identity

$$\frac{\partial V}{\partial x_1} + \cdots + \frac{\partial V}{\partial x_n} = lV.$$

It is often convenient to call functions that are homogeneous with respect to ν as homogeneous with respect to the corresponding dilation Δ. Using (11) and (13) in (14), it is easy to see that a function $V : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree l with respect to the dilation (11) if and only if

$$V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^l V(x_1, \ldots, x_n), \quad k > 0. \tag{17}$$

Homogeneous polynomial functions of n variables form a common example of homogeneous functions. The dilation in this case is the standard dilation on $\mathbb{R}^n$. Positive-definite functions homogeneous of degree 1 are usually referred to as homogeneous norms [K6, MM].

The following proposition is the main result of this section, and relates the regularity properties of a homogeneous function to its homogeneity properties. The result is a generalization of the simple observation that the scalar function $V(x) = |x|^\alpha$ on $\mathbb{R}$, which is homogeneous of degree $\alpha \geq 0$ with respect to the standard dilation on $\mathbb{R}$, is Hölder continuous at 0 for $\alpha > 0$, Lipschitz continuous at 0 for $\alpha \geq 1$ and C^1 for $\alpha > 1$. Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *Hölder continuous* with exponent $\alpha > 0$ at $x \in \mathbb{R}^n$ if there exist $k > 0$ and an open neighborhood $\mathcal{U}$ of x such that

$$|V(x) - V(z)| \leq k\|x - z\|^\alpha, \quad z \in \mathcal{U}. \tag{18}$$

V is simply said to be Hölder continuous at x if V is Hölder continuous at x with some exponent $\alpha > 0$. Note that Hölder continuity at x implies continuity at x and that Lipschitz continuity is the same as Hölder continuity with exponent $\alpha \geq 1$.

Theorem 4.1 *Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n \backslash \{0\}$ and homogeneous of degree l with respect to ν. Then statements (i), (ii) and (iii) below hold. If, in addition, ν is C^1, then the statements (iv), (v), and (vi) below hold.*

- *(i) If $l < 0$, then V is continuous on $\mathbb{R}^n$ if and only if $V \equiv 0$.*
- *(ii) If $l = 0$, then V is continuous on $\mathbb{R}^n$ if and only if $V \equiv V(0)$.*
- *(iii) If $l > 0$, then V is continuous on $\mathbb{R}^n$.*
- *(iv) If $l > 0$, then, for every $\sigma > \overline{\sigma}$, V is Hölder continuous at 0 with exponent l/σ.*
- *(v) If $l > \overline{\sigma}$, then V is Fréchet differentiable at 0 and $dV_0 \equiv 0$.*

(vi) If $l > \overline{\sigma}$ and V is C^1 *on* $\mathbb{R}^n \backslash \{0\}$*, then*

$$dV_x(v) = e^{-ls} dV_{\phi_s(x)}((d\phi_s)_x(v)), \ (s, x) \in \mathbb{R} \times \mathbb{R}^n, \ v \in \mathbb{R}^n, \tag{19}$$

and V is C^1 *on* $\mathbb{R}^n$.

Proof First note that continuity on $\mathbb{R}^n$ and homogeneity imply

$$V(0) = V\left(\lim_{s\to\infty} \phi(-s, x)\right) = \lim_{s\to\infty} V(\phi(-s, x)) = \lim_{s\to\infty} e^{-ls} V(x), \ x \in \mathbb{R}^n. \tag{20}$$

(i) Clearly, $V \equiv 0$ implies continuity on $\mathbb{R}^n$. Now, let $l < 0$ and suppose V is continuous on $\mathbb{R}^n$. If $V(z) \neq 0$ for some $z \in \mathbb{R}^n$, then the limit in (20) does not exist for $x = z$, which contradicts continuity. Therefore, we conclude that $V \equiv 0$.

(ii) Clearly, $V \equiv V(0)$ implies continuity on $\mathbb{R}^n$. On the other hand, if $l = 0$ and V is continuous on $\mathbb{R}^n$, then (20) yields $V(0) = V(x)$ for all $x \in \mathbb{R}^n$.

(iii) Let $l > 0$ and consider $\epsilon > 0$. Let $\mathcal{V}$ be a bounded open neighborhood of 0 and denote $L = \max_{x \in \text{bd } \mathcal{V}} |V(x)|$. Choose $T > 0$ such that $Le^{-ls} < \epsilon$ for all $s > T$. The set $\mathcal{M} = \phi([-T, 0] \times \text{bd } \mathcal{V})$ is compact and does not contain 0. Hence there exists an open neighborhood $\mathcal{U}_\delta \subset \mathcal{V}$ of 0 such that $\mathcal{U}_\delta \cap \mathcal{M} = \varnothing$. Consider $x \in \mathcal{U}_\delta \backslash \{0\}$. There exists $z \in \text{bd } \mathcal{V}$ and $s \in \mathbb{R}_+$ such that $x = \phi_{-s}(z)$. By construction, $s > T$. Therefore, $|V(x) - V(0)| = |V(x)| = e^{-ls}|V(z)| < Le^{-ls} < \epsilon$. Thus $|V(x) - V(0)| < \epsilon$ for all $x \in \mathcal{U}_\delta$ and hence V is continuous at 0.

(iv) Choose $\sigma > \overline{\sigma}$ and let $\mathcal{V}$ be a bounded open neighborhood of 0. Denote $\mathcal{K} = \{e^{\sigma s}\phi_{-s}(x) : s \in \mathbb{R}_+, x \in \text{bd } \mathcal{V}\}$. By Proposition 3.1 *(i)*, there exists a bounded open neighborhood $\mathcal{U} \subset \mathcal{V}$ such that $\mathcal{U} \cap \mathcal{K} = \varnothing$. Let $L_1 = \max_{x \in \text{bd } \mathcal{V}} |V(x)|$ and $L_2 = \inf_{z \notin \mathcal{U}} \|z\|$. Consider $x \in \mathcal{U}\backslash\{0\}$ and let $z \in \text{bd } \mathcal{V}$ and $s \in \mathbb{R}_+$ be such that $x = \phi_{-s}(z)$. By construction, $e^{\sigma s} x \notin \mathcal{U}$. Therefore,

$$\frac{|V(x)|}{\|x\|^{l/\sigma}} = \frac{|V(z)|}{\|e^{\sigma s}x\|^{l/\sigma}} < \frac{L_1}{L_2^{l/\sigma}}.$$

Thus $V(x)/\|x\|^{l/\sigma}$ is uniformly bounded for $x \in \mathcal{U}\backslash\{0\}$ and V is Hölder continuous at 0 with exponent l/σ.

(v) Let $l > \overline{\sigma}$ and choose $\sigma \in (\overline{\sigma}, l)$. By *(iv)*, there exist an open neighborhood $\mathcal{U}$ of 0 and $k > 0$ such that for all $x \in \mathcal{U}\backslash\{0\}$, $|V(x)|/\|x\|^{l/\sigma} \leq k$. Therefore, for every $x \in \mathcal{U}\backslash\{0\}$,

$$\frac{|V(x) - V(0)|}{\|x - 0\|} \leq k\|x\|^{\frac{l}{\sigma}-1}. \tag{21}$$

Since, $\frac{l}{\sigma} - 1 > 0$, $\|x\|^{\frac{l}{\sigma}-1} \to 0$ as $x \to 0$, so that (21) implies that V is Fréchet differentiable at 0 and $dV_0 \equiv 0$.

(vi) Let $l > \overline{\sigma}$. By *(v)*, V is Fréchet differentiable at 0 and $dV_0 \equiv 0$. Now, suppose V is C^1 on $\mathbb{R}^n\backslash\{0\}$. Equation (19) clearly holds for $x = 0$. For $x \neq 0$, (19) follows from (6) and (14) with $\varphi = \phi_s$ and $\mathcal{U} = \phi_{-s}(\mathcal{U}) = \mathbb{R}^n\backslash\{0\}$. To prove that V is C^1 on $\mathbb{R}^n$, it suffices to show that, for every $v \in \mathbb{R}^n$, $dV_x(v) \to dV_0(v) = 0$ as $x \to 0$.

Choose $\sigma \in (\overline{\sigma}, l)$ and let $\mathcal{V}$ be a bounded open neighborhood of 0. By Proposition 3.1, there exists an open neighborhood $\mathcal{U} \subset \mathcal{V}$ such that $e^{\sigma s}\phi_{-s}(z) \notin \mathcal{U}$ and $e^{\sigma s}(d\phi_{-s})_z(v) \notin \mathcal{U}$ for all $s \geq 0$, $z \in \text{bd}\,\mathcal{V}$ and $v \in \mathbb{R}^n$ such that $\|v\| = 1$. Let $L_1 = \max_{z\in \text{bd}\,\mathcal{V}} \|dV_z\|_{\text{i}}$ and $L_2 = \inf_{z\notin\mathcal{U}} \|z\| > 0$. Now consider $v \in \mathbb{R}^n$, $x \in \mathcal{U}\backslash\{0\}$ and let $z \in \text{bd}\,\mathcal{V}$ and $s \geq 0$ be such that $x = \phi_{-s}(z)$. By construction, $\|e^{\sigma s}x\| \geq L_2$. Also, $e^{\sigma s}\|v\| = e^{\sigma s}\|(d\phi_{-s})_z \circ (d\phi_s)_x(v)\| \geq L_2\|(d\phi_s)_x(v)\|$, that is, $\|(d\phi_s)_x(v)\| \leq e^{\sigma s}\|v\|/L_2$. Now, using (19) we compute

$$\begin{aligned}\frac{|dV_x(v)|}{\|x\|^{\frac{l}{\sigma}-1}} &= \frac{|dV_{\phi(-s,z)} \circ (d\phi_{-s})_z \circ (d\phi_s)_x(v)|}{\|x\|^{\frac{l}{\sigma}-1}} \\ &= \frac{e^{-ls}|dV_z((d\phi_s)_x(v))|}{\|x\|^{\frac{l}{\sigma}-1}} \\ &\leq \frac{L_1 e^{-ls}\|(d\phi_s)_x(v)\|}{\|x\|^{\frac{l}{\sigma}-1}} \leq \frac{L_1\|v\|}{L_2\|e^{\sigma s}x\|^{\frac{l}{\sigma}-1}} \leq \frac{L_1\|v\|}{L_2^{l/\sigma}}. \end{aligned} \tag{22}$$

It follows from the inequality (22) that, for every $v \in \mathbb{R}^n$, $dV_x(v) \to 0 = dV_0(v)$ as $x \to 0$. Thus V is C^1 on $\mathbb{R}^n$. □

The following lemma asserts that sign-definite, homogeneous functions are radially unbounded.

Lemma 4.1 *Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is continuous and homogeneous with respect to ν.*

(i) If V is sign definite, then V is radially unbounded.
(ii) If $n > 1$ and V is proper, then V is sign definite.

Proof See Appendix 9 □

It is interesting to note that the second part of Lemma 4.1 is false if $n = 1$. The function $V(x) = x$, $x \in \mathbb{R}$, provides a counterexample since V is proper and homogeneous of degree 1 with respect to the standard dilation on $\mathbb{R}$, but not sign definite.

The following lemma provides a useful comparison between homogeneous functions.

Lemma 4.2 *Suppose V_1 and V_2 are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degrees $l_1 > 0$ and $l_2 > 0$, respectively, and V_1 is positive definite. Then, for every $x \in \mathbb{R}^n$,*

$$\left[\min_{\{z:V_1(z)=1\}} V_2(z)\right][V_1(x)]^{\frac{l_2}{l_1}} \leq V_2(x) \leq \left[\max_{\{z:V_1(z)=1\}} V_2(z)\right][V_1(x)]^{\frac{l_2}{l_1}}. \tag{23}$$

Proof Since $l_1, l_2 > 0$, $V_1(0) = V_2(0) = 0$ and (23) holds for $x = 0$. Therefore, suppose $x \neq 0$ and let $s = -\frac{1}{l_1}\ln[V_1(x)]$. Then, by homogeneity, $V_1(\phi_s(x)) = 1$ so that

$$\min_{\{z:V_1(z)=1\}} V_2(z) \leq V_2(\phi_s(x)) \leq \max_{\{z:V_1(z)=1\}} V_2(z). \tag{24}$$

Note that by Lemma 4.1, V_1 is proper, so that $V_1^{-1}(\{1\})$ is compact and the minimum and the maximum in (24) are well defined. Equation (23) now follows from (24) by noting that since V_2 is homogeneous of degree l_2, $V_2(\phi_s(x)) = e^{l_2 s} V_2(x) = [V_1(x)]^{-\frac{l_2}{l_1}} [V_2(x)]$. □

It is interesting to note that if we let V_1 denote the square of the Euclidean norm on $\mathbb{R}^n$ and V_2 a quadratic form on $\mathbb{R}^n$, then (23) yields the well known Rayleigh–Ritz inequality for quadratic forms [HJ, p. 176].

5 Homogeneous vector fields

Following [K7, K8], we define the vector field f to be homogeneous of degree $m \in \mathbb{R}$ with respect to ν if, for every $t \in \mathbb{R}_+$ and $s \in \mathbb{R}$,

$$\psi_t \circ \phi_s = \phi_s \circ \psi_{e^{ms}t}. \tag{25}$$

Geometrically, homogeneity of f implies that the image under ϕ_s of the ψ-orbit of $x \in \mathbb{R}^n$ is the ψ-orbit of the image under ϕ_s of x. Equation (25) implies that if $\mathcal{K}$ is (positively) invariant under f, then so is $\phi_s(\mathcal{K})$ for every $s \in \mathbb{R}$ since $\psi_t(\phi_s(\mathcal{K})) = \phi_s(\psi_{e^{ms}t}(\mathcal{K}))\,(\subseteq) = \phi_s(\mathcal{K})$ for all $t \in \mathbb{R}_+$ and $s \in \mathbb{R}$. In addition, if $V : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree l such that $L_f V$ is defined everywhere, then $L_f V$ is a homogeneous function of degree $l + m$. This fact follows easily by using (14) for V and (25) for f in (7) to verify (14) for $L_f V$. See also, [H4, H6, K7, K8].

In the case that ν is a smooth vector field, equation (25) is equivalent to

$$f(\phi_s(x)) = e^{ms}(d\phi_s)_x(f(x)), \quad s \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{26}$$

In this case, if f is homogeneous of degree m with respect to ν, then (26) implies that the limit in (10) exists for all $x \in \mathbb{R}^n$. Consequently, $L_\nu f$ is defined everywhere and satisfies

$$L_\nu f = mf. \tag{27}$$

Using (13) in (26), it is easy to see that the vector field f is homogeneous of degree m with respect to the dilation (11) if and only if the ith component f_i is a homogeneous function of degree $m + r_i$ with respect to Δ, that is,

$$f_i(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^{m+r_i} f_i(x_1, \ldots, x_n), \quad \lambda > 0, \; i = 1, \ldots, n. \tag{28}$$

It is clear from (28) that, a continuous, non-constant vector field that is homogeneous with respect to a dilation of the form (11) cannot have bounded components. The following example demonstrates that it is possible for vector fields having bounded components to be homogeneous in the generalized sense that we consider, and thus opens up the possibility of applying homogeneity techniques to control systems involving, for instance, input saturation.

Example 5.1 Consider the continuous vector field $f(x) = -\text{sat}(x^{1/3})\partial/\partial x$ on $\mathbb{R}$, where $\text{sat}(x) = x$ if $|x| \leq 1$ and $\text{sat}(x) = \text{sign}(x)$ if $|x| > 1$. The vector field f is homogeneous of degree $-2/3$ with respect to the continuous Euler vector field $\nu(x) = g(x)\partial/\partial x$, where $g : \mathbb{R} \to \mathbb{R}$ is the continuous, piecewise linear function given by

$$g(x) = \begin{cases} x, & |x| \leq 1, \\ = \frac{1}{3}(\text{sign}(x) + 2x), & |x| > 1. \end{cases}$$

Note that ν is C^1 on $\mathbb{R}\backslash\{-1, 1\}$, while f is C^1 on $\mathbb{R}\backslash\{-1, 0, 1\}$. Equation (27) can easily be verified to hold with $m = -2/3$ at every point where f and ν are differentiable. Homogeneity can be rigorously established by using the flows of the vector fields f and ν to show that (25) holds with $m = -2/3$. Note that unlike vector fields that are homogeneous with respect to dilations of the form (11), the vector field f considered in this example is globally bounded.

Our next example involves an Euler vector field whose linearization at the origin is zero.

Example 5.2 Let $m > 0$, and consider the vector field $f(x) = h(x)\partial/\partial x$, where the function $h : \mathbb{R} \to \mathbb{R}$ is given by

$$h(x) = \begin{cases} 0, & x = 0, \\ -x^3 e^{-\frac{m}{2}(x^{-2}-1)}, & 0 < |x| \leq 1, \\ -\text{sign}(x)|3x - 2\text{sign}(x)|^{(1+\frac{m}{3})}, & |x| > 1. \end{cases}$$

It is easy to verify that f is C^1 on $\mathbb{R}\backslash\{-1, 1\}$. Furthermore, f is homogeneous of degree m with respect to the C^1 Euler vector field $\nu(x) = g(x)\partial/\partial x$, where the function $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} x^3, & |x| \leq 1, \\ 3x - 2\text{sign}(x), & |x| > 1. \end{cases}$$

Equation (27) can easily be verified to hold at every point where f is differentiable. Homogeneity can be rigorously established by using the flow of ν to verify (26). Note that unlike the Euler vector fields of dilations of the form (11), the spectral abscissa of the linearization at $x = 0$ of the Euler vector field in this example is zero, that is, $x = 0$ is a non-hyperbolic, non-exponentially stable equilibrium for the vector field $-\nu$.

Remark 5.1 Equation (25) appears in [K7, K8], while (26) appears in [H6]. According to an anonymous reviewer, (25)–(27) also appear in [R2]. Equation (27), which is also given in [K7, K8], is adopted as the definition of a homogeneous vector field in [H6]. However, the degree of homogeneity of a vector field f satisfying (27) is defined to be $m + 1$ in [H6]. A similar convention is adopted in [H1] for the case of the standard dilation. Thus a linear vector field, which is homogeneous of degree 0 with respect to the standard dilation by our definition, is homogeneous of degree 1 according to [H1, H6]. As argued in [K8], the definition given in [K8] and adopted here is more appropriate as it leads to a consistent notion of homogeneity for scalar functions, vector fields and other objects. This consistency becomes evident, for instance, on comparing (16) and (27).

6 Stability of homogeneous systems

In this section, we re-derive in a more general setting some stability results that appear in the literature for the special case of systems that are homogeneous with respect to dilations of the form (11). We also prove a new stability result involving strictly positively invariant sets.

The following result is a stronger version of a result given in [H1, Sect. 17] and [R1].

Proposition 6.1 *Let f be homogeneous with respect to ν and suppose* 0 *is an attractive equilibrium under f. Then,* 0 *is a globally asymptotically stable equilibrium under f.*

Proof Suppose 0 is attractive under f and let $\mathcal{A}$ denote the domain of attraction of 0 under f. Let $\mathcal{U}_\epsilon$ be an open neighborhood of 0 and consider a bounded open neighborhood $\mathcal{V}$ of 0 such that $\overline{\mathcal{V}} \subset \mathcal{A}$. The first part of Lemma 2.2 implies that the set $\mathcal{M} = \overline{\psi(\mathbb{R}_+ \times \overline{\mathcal{V}})}$ is compact. Hence, by the second part of Lemma 2.2, there exists $\tau_2 > 0$ such that $\phi_{-\tau_2}(\mathcal{M}) \subset \mathcal{U}_\epsilon$. Now, the set $\mathcal{U}_\delta = \phi_{-\tau_2}(\mathcal{V}) \subset \mathcal{U}_\epsilon$ is open and, for every $t > 0$, equation (25) implies that $\psi_t(\mathcal{U}_\delta) = \phi_{-\tau_2}(\psi_{e^{-m\tau_2}t}(\mathcal{V})) \subset \phi_{-\tau_2}(\mathcal{M}) \subset \mathcal{U}_\epsilon$, where m is the degree of homogeneity of f. Thus, 0 is Lyapunov stable under f.

Next, consider $x \in \mathbb{R}^n$. Since $\mathcal{A}$ is an open neighborhood of 0, there exists $s > 0$ such that $z = \phi_{-s}(x) \in \mathcal{A}$. Equation (25) implies that $\psi^x(t) \to \{0\}$ if and only if $\psi^{\phi_{-s}(x)}(t) = \psi^z(t) \to \{0\}$. It follows that $x \in \mathcal{A}$ and hence $\mathcal{A} = \mathbb{R}^n$. Global asymptotic stability now follows. □

Standard converse Lyapunov results for asymptotic stability imply that if the origin is an asymptotically stable equilibrium under f, then the origin is contained in a compact set that is strictly positively invariant with respect to f, since a sufficiently small sublevel set of a Lyapunov function is compact and strictly positively invariant. The next result shows that under the assumption of homogeneity, the reverse implication is also valid, that is, the existence of a nonempty compact set that is strictly positively invariant with respect to f is sufficient to conclude asymptotic stability of the origin. Interestingly, the proof does not involve the construction of a Lyapunov function.

Theorem 6.1 *Suppose the vector field f is homogeneous with respect to ν. If $\mathcal{A} \subset \mathbb{R}^n$ is a bounded open set that contains* 0 *and is positively invariant under f, then* 0 *is Lyapunov stable under f. If $\mathcal{A} \subset \mathbb{R}^n$ is compact and strictly positively invariant under f, then $0 \in \mathcal{A}$ and* 0 *is globally asymptotically stable under f.*

Proof Suppose $\mathcal{A}$ is bounded, open, positively invariant under f and contains 0, and let $\mathcal{U}_\epsilon$ be an open neighborhood of 0. Since the origin is asymptotically stable under $-\nu$, by Lemma 2.2, there exists $s > 0$ such that $\mathcal{U}_\delta = \phi_{-s}(\mathcal{A}) \subset \mathcal{U}_\epsilon$. $\mathcal{U}_\delta$ is open since ϕ_{-s} is a homeomorphism and positively invariant under f by homogeneity. Moreover, $0 = \phi_{-s}(0) \in \phi_{-s}(\mathcal{A}) = \mathcal{U}_\delta$. Therefore, for every $t \geq 0$, $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\delta \subset \mathcal{U}_\epsilon$, thus proving Lyapunov stability.

Now, suppose $\mathcal{A}$ is compact and strictly positively invariant under f and let $\mathcal{K}$ denote the largest subset of $\mathcal{A}$ that is invariant under f. By Lemma 2.1, $\mathcal{K}$ is nonempty, compact, contained in int $\mathcal{A}$ and asymptotically stable under f. Since $\mathcal{K}$

is compact, $\mathcal{K} \subset \operatorname{int} \mathcal{A}$ and ϕ is continuous, there exists $\epsilon > 0$ such that $\phi_s(\mathcal{K}) \subset \mathcal{A}$ for all $|s| \leq \epsilon$. By homogeneity, $\phi_s(\mathcal{K})$ is invariant under f for every $s \in (-\epsilon, \epsilon)$. However, $\mathcal{K}$ is the largest subset of $\mathcal{A}$ that is invariant under f, so that $\phi_s(\mathcal{K}) \subseteq \mathcal{K}$ for all $s \in (-\epsilon, \epsilon)$, that is, $\mathcal{K}$ is invariant under ϕ. Since the only compact, non-empty set that is invariant under ϕ is $\{0\}$, we conclude that $\mathcal{K} = \{0\}$. Thus $0 \in \mathcal{A}$ and 0 is asymptotically stable under f. Global asymptotic stability follows from Proposition 6.1. □

Our next result asserts that an asymptotically stable homogeneous system admits a homogeneous Lyapunov function. The well known result from [R1] is a stronger version of our result for the special case of dilations of the form (11). Theorem 5.12 of [BR] extends the result of [R1] to more general dilations. However, the extension in [BR] is only partial because, unlike [R1], the result from [BR] does not address the regularity of the Lyapunov function at the origin. Our result below makes use of Theorem 4.1 and strengthens the result from [BR] by giving sufficient conditions on the degree of homogeneity of the Lyapunov function for the Lyapunov function to be continuous or C^1 at the origin.

Theorem 6.2 *Suppose f is homogeneous of degree m with respect to ν and 0 is an asymptotically stable equilibrium under f. Then, for every $l > \max\{-m, 0\}$, there exists a continuous, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is homogeneous of degree l with respect to ν, C^1 on $\mathbb{R}^n \backslash \{0\}$, and such that $L_f V$ is continuous and negative definite. Furthermore, if ν is C^1, then, for every $l > \max\{-m, \overline{\sigma}\}$, there exists a positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is homogeneous of degree l with respect to ν, C^1 on $\mathbb{R}^n$ and such that $L_f V$ is continuous and negative definite.*

Proof Fix $l > \max\{-m, 0\}$. Theorem 5.12 of [BR] implies that there exists a continuous, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is homogeneous of degree l with respect to ν, C^1 on $\mathbb{R}^n \backslash \{0\}$, and such that $L_f V$ is negative definite. It follows that $L_f V$ is continuous on $\mathbb{R}^n \backslash \{0\}$ and homogeneous of degree $l + m > 0$ with respect to ν. Hence *iii)* of Theorem 4.1 implies that $L_f V$ is continuous. If, in addition, ν is C^1 and $l > \overline{\sigma}$, then *vi)* of Theorem 4.1 implies that V is C^1 on $\mathbb{R}^n$. □

Remark 6.1 For the case where ν is the Euler vector field of a dilation of the form (11), Theorem 6.2 was first proved in [R1]. In the case of a dilation of the form (11), two main simplifications occur in the proof of Theorem 6.2. First, the dilation (11) is specially adapted to the usual coordinates on $\mathbb{R}^n$. As a result, the partial derivatives of a homogeneous function along the coordinate directions are also homogeneous. This property makes it possible to prove regularity by simply computing the partial derivatives and using homogeneity to check continuity of the partial derivatives. In the more general setting that we consider, this simplification is not possible. Instead, our proof of Theorem 6.2 makes use of Theorem 4.1 to prove regularity of the candidate Lyapunov function. Indeed, equation (19) in Theorem 4.1 is a generalization of the fact that the partial derivatives of a function homogeneous with respect to the dilation (11) are also homogeneous. The second simplification is related to the fact that the relationship between the regularity properties of a homogeneous function and its degree of homogeneity depend on the Euler vector field. In the case of dilations of the form (11), the Euler vector field is characterized by the n parameters $r_1, \ldots, r_n$, and the dependence of regularity properties of a homogeneous function on these parameters is easy to see. In the general case that we consider, the dependence of the regularity properties of a homogeneous function on the Euler vector field is not straightforward. The latter part of Theorem 4.1, which depends very crucially on Proposition 3.1, makes this dependence precise with the help of the parameter $\overline{\sigma}$. While it is possible to assert the existence of a continuous (but not necessarily C^1) homogeneous Lyapunov function without using Theorem 4.1 (see, for instance, Theorem 5.12 of [BR]), a complete extension of the main result of [R1] to our general setting is possible only by using Theorem 4.1.

Remark 6.2 The vector field $-\nu$ is easily seen to be homogeneous of degree 0 with respect to ν. Furthermore, the origin is asymptotically stable under $-\nu$. Therefore, Theorem 6.2 implies that, for every $l > 0$, there exists a continuous, positive-definite function that is homogeneous of degree l with respect to ν and C^1 on $\mathbb{R}\backslash\{0\}$. Moreover, if ν is C^1, then for every $l > \overline{\sigma}$, there exists a positive-definite function that is homogeneous of degree l with respect to ν and C^1 on $\mathbb{R}^n$.

7 Finite-time stability of homogeneous systems

It is instructive to first study the finite-time stability of a scalar homogeneous system. For $\alpha > 0$, the scalar system

$$\dot{x} = -k\text{sign}(x)|x|^\alpha \tag{29}$$

represents a continuous vector field on $\mathbb{R}$ that is homogeneous of degree $\alpha - 1$ with respect to the standard dilation $\Delta_\lambda(x) = \lambda x$. Equation (29) can be readily integrated to obtain the semiflow of (29) as

$$\mu(t,x) = \begin{cases} \text{sign}(x)\left(\frac{1}{|x|^{\alpha-1}} + k(\alpha-1)t\right)^{-\frac{1}{\alpha-1}}, & & \alpha > 1, \\ e^{-kt}x, & & \alpha = 1, \\ \text{sign}(x)(|x|^{1-\alpha} - k(1-\alpha)t)^{\frac{1}{1-\alpha}}, & 0 \le t < \frac{|x|^{1-\alpha}}{k(1-\alpha)}, & \alpha < 1, \\ 0, & t \ge \frac{|x|^{1-\alpha}}{k(1-\alpha)}, & \alpha < 1. \end{cases} \tag{30}$$

It is clear from (30) that the origin is asymptotically stable under (29) if and only if $k > 0$ and finite-time stable if and only if $k > 0$ and $\alpha < 1$. In other words, the origin is finite-time stable under (29) if and only if the origin is asymptotically stable under (29) and the degree of homogeneity of (29) is negative. Moreover, in the case that $k > 0$ and $\alpha < 1$, the settling-time function is given by $T(x) = \frac{1}{k(1-\alpha)}|x|^{1-\alpha}$, which is easily seen to be Hölder continuous at the origin and homogeneous of degree $1 - \alpha$. This section contains extensions of these simple observations to multi-dimensional homogeneous systems. The following result represents the main application of homogeneity to finite-time stability and finite-time stabilization.

Theorem 7.1 *Suppose f is homogeneous of degree m with respect to ν. Then the origin is a finite-time-stable equilibrium under f if and only if the origin is an asymptotically stable equilibrium under f and $m < 0$.*

Proof As noted in Sect. 2, finite-time stability of the origin implies asymptotic stability. Therefore, it suffices to prove that if the origin is an asymptotically stable equilibrium under f, then the origin is a finite-time-stable equilibrium under f if and only if $m < 0$.

Suppose the origin is an asymptotically stable equilibrium of f and let $l > \max\{-m, 0\}$. By Theorem 6.2, there exists a continuous, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is homogeneous of degree l and is such that $L_f V$ is continuous, negative definite, and homogeneous of degree $l + m$. Applying Lemma 4.2 with $V_1 = V$ and $V_2 = L_f V$, we get

$$-c_1[V(x)]^{\frac{l+m}{l}} \leq L_f V(x) \leq -c_2[V(x)]^{\frac{l+m}{l}}, \quad x \in \mathbb{R}^n, \tag{31}$$

where $c_1 = -\min_{\{z:V(z)=1\}} L_f V(z)$ and $c_2 = -\max_{\{z:V(z)=1\}} L_f V(z)$. Note that both c_1 and c_2 are positive since $L_f V$ is negative definite.

Now, if $m \geq 0$ and $0 \neq x \in \mathbb{R}^n$, then applying the comparison principle [K1, Sect. 5.2], [K9, Sect. 2.5], [RHL, ch. IX], [Y, Sect. 4] to the first inequality in (31) yields $V(\psi(t, x)) \geq \mu(t, V(x))$ where μ is given by (30) with $k = c_1 > 0$ and $\alpha = l + m/l \geq 1$. Since, in this case, $\mu(t, V(x)) > 0$ for all $t \geq 0$, we conclude that $\psi(t, x) \neq 0$ for every $t \geq 0$, that is, the origin is not a finite-time-stable equilibrium under f. Thus finite-time stability of the origin implies $m < 0$.

Conversely, if $m < 0$, then the second inequality in (31) implies that the hypotheses of Theorem 2.1 hold with $c = c_2 > 0$ and $0 < \alpha = l + m/l < 1$. Thus, by Theorem 2.1, the origin is a finite-time-stable equilibrium under f. □

Remark 7.1 The proof of Theorem 7.1 involves constructing a homogeneous Lyapunov function, applying Lemma 4.2 to the Lyapunov function and its derivative to obtain a differential inequality for the Lyapunov function, and then applying Theorem 2.1 to conclude finite-time stability. Any application of Theorem 7.1 to finite-time stabilization involves the additional step of rendering the closed-loop system asymptotically stable and homogeneous with negative degree. References [BB3, R4] achieved finite-time stabilization of second-order systems by explicitly carrying out the steps listed above including the construction of a homogeneous Lyapunov function. Theorem 7.1 first appears as a result in [BB2] and was proven in the case of dilations of the form (11) in [HHX] and as Corollary 5.4 in [BR]. The result was applied to output-feedback finite-time stabilization of second-order systems in [HHX] and, subsequently, to state-feedback finite-time stabilization of a class of higher-order systems in [H8]. In all these applications, the dilations involved were of the form (11). A crucial step in the extension of these ideas to our general setting is the construction of a homogeneous Lyapunov function using Theorem 6.2. As explained in Remark 6.1, this construction depends very strongly on Propositions 3.1 and Theorem 4.1, both of which are relatively straightforward in the case of dilations of the form (11). Thus, Theorem 7.1 represent a nontrivial extension of ideas used in [BB3, H8, HHX, R4] to our more general setting.

Reference [BB4] contains a converse Lyapunov result for finite-time stability. A stronger version of the same result is provided by the following theorem under the assumption of homogeneity.

Theorem 7.2 *Suppose f is homogeneous of degree m with respect to ν and let $\alpha \in (0, 1)$. If the origin is a finite-time-stable equilibrium under f, then there* *exist $c > 0$ and a continuous, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is C^1 on $\mathbb{R}^n\backslash\{0\}$, homogeneous of degree $l = -m/(1-\alpha)$ and is such that $L_f V$ is continuous on $\mathbb{R}^n$ and satisfies*

$$L_f V(x) \leq -c[V(x)]^\alpha,\ x \in \mathbb{R}^n. \tag{32}$$

In addition, if ν is C^1, then, for every $\alpha \in (0, 1)$ such that $\overline{\sigma} + m < \overline{\sigma}\alpha$, the above assertion holds with V a C^1 function on $\mathbb{R}^n$.

Proof Fix $\alpha \in (0, 1)$. By Theorem 7.1, $m < 0$ and 0 is an asymptotically stable equilibrium for f. By Theorem 6.2, there exists a continuous, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ that is C^1 on $\mathbb{R}^n\backslash\{0\}$ and homogeneous of degree $l = -m/(1-\alpha) > -m > 0$ with respect to ν and is such that $L_f V$ is continuous and negative definite on $\mathbb{R}^n$. Moreover, $L_f V$ is homogeneous of degree $l+m > 0$ with respect to ν. Therefore, Lemma 4.2 applies with $V_1 = V$, $V_2 = L_f V$ and $l_2/l_1 = l + m/l = \alpha$ and (32) follows from (23) with $c = -\max_{\{z: V(z)=1\}} L_f V(z) > 0$.

If, in addition, ν is C^1 and $\overline{\sigma} + m < \overline{\sigma}\alpha$, then $l = -m/(1-\alpha) > \overline{\sigma}$ and *vi)* of Theorem 4.1 implies that V is C^1 on $\mathbb{R}^n$. □

Examples given in [BB4] show that finite-time stability implies neither Hölder continuity nor continuity of the settling-time function. The following result shows that these regularity properties of the settling-time function follow under the additional assumption of homogeneity.

Theorem 7.3 *Let f be homogeneous of degree m with respect to ν. Suppose the origin is a finite-time-stable equilibrium under f and let T denote the settling-time function. Then the origin is a globally finite-time-stable equilibrium under f, T is homogeneous of degree $-m$ with respect to ν and T is continuous on $\mathbb{R}^n$. If, in addition, ν is C^1, then, for every $\sigma > \overline{\sigma}$, T is Hölder continuous at the origin with exponent $-m/\sigma$.*

Proof Let $\mathcal{N}$ denote the domain of definition of T as given by (8). By finite-time stability, $\mathcal{N}$ contains an open neighborhood of the origin. Let $x \in \mathbb{R}^n$. Since the origin is a globally asymptotically stable equilibrium under $-\nu$, there exists $s > 0$ such that $z = \phi_{-s}(x) \in \mathcal{N}$. Equation (25) implies that $\psi(t, x) = \psi(t, \phi_s(z)) = \phi_s(\psi(e^{ms}t, z))$ so that $\psi(t, x) = 0$ if and only if $\psi(e^{ms}t, z) = 0$. It now follows from (8) that $T(x)$ is defined, that is, $x \in \mathcal{N}$, and

$$T(\phi_{-s}(x)) = T(z) = e^{ms} T(x). \tag{33}$$

Thus $\mathcal{N} = \mathbb{R}^n$ and the origin is a globally finite-time-stable equilibrium under f. On comparing (33) and (14), it follows that T is homogeneous with respect to ν with degree $-m$. By Theorem 7.2, there exists a Lyapunov function satisfying the hypotheses of Theorem 2.1. Therefore, by Theorem 2.1, T is continuous on $\mathbb{R}^n$. By Theorem 7.1, $-m > 0$. Hence, in the case that ν is C^1, the assertion about Hölder continuity follows from *(vi)* of Theorem 4.1. □

It was shown in [BB4] that finite-time stability does not imply the existence of a C^1 function satisfying equation (32), while the settling-time function of a system with a finite-time-stable equilibrium may not be Hölder continuous or even continuous at the origin. Theorems 7.2 and 7.3 show that stronger results hold under the assumption of homogeneity and are thus significant.

It was shown in [H4, H6, R1] that if a vector field can be written as a sum of several vector fields, each homogeneous with respect to a certain fixed dilation, then the given vector field is asymptotically stable if the homogeneous vector field having the lowest degree of homogeneity is. The following theorem provides an analogous result for finite-time stability.

Theorem 7.4 *Let ν be C^1 and suppose $f = g_1 + \cdots + g_k$, where, for each $i = 1, \ldots, k$, the vector field g_i is continuous, homogeneous of degree m_i with respect to ν and $m_1 < m_2 < \cdots < m_k$. If the origin is a finite-time-stable equilibrium under g_1, then the origin is a finite-time-stable equilibrium under f.*

Proof Suppose ν is C^1 and let the origin be a finite-time-stable equilibrium under g_1. Choose $l > \max\{-m_1, \overline{\sigma}\}$. By Theorem 6.2, there exists a positive-definite, C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ that is homogeneous of degree l and is such that $L_{g_1}V$ is negative definite. For each $i = 1, \ldots, k$, $L_{g_i}V$ is continuous and homogeneous of degree $l + m_i > 0$ with respect to ν. By Lemma 4.2, there exist $c_1 > 0$ and $c_2, \ldots, c_k \in \mathbb{R}$ such that

$$L_{g_i}V(x) \leq -c_i[V(x)]^{\frac{l+m_i}{l}}, \quad x \in \mathbb{R}^n, \ i = 1, \ldots, k. \tag{34}$$

Therefore, for every $x \in \mathbb{R}^n$,

$$L_f V(x) \leq -c_1[V(x)]^{\frac{l+m_1}{l}} - \cdots - c_k[V(x)]^{\frac{l+m_k}{l}} = [V(x)]^{\frac{l+m_1}{l}}[-c_1 + U(x)], \tag{35}$$

where $U(x) = -c_2(V(x))^{\frac{m_2-m_1}{l}} - \cdots - c_k(V(x))^{\frac{m_k-m_1}{l}}$. Since $m_i - m_1 > 0$ for every $i > 1$, it follows that the function U, which takes the value 0 at the origin, is continuous. Therefore, there exists an open neighborhood $\mathcal{V}$ of the origin such that $U(x) < c_1/2$ for all $x \in \mathcal{V}$. Equation (35) now yields

$$L_f V(x) \leq -\frac{c_1}{2}[V(x)]^{\frac{l+m_1}{l}}, \quad x \in \mathcal{V}. \tag{36}$$

By Theorem 7.1, $m_1 < 0$, so that the hypotheses of Theorem 2.1 are satisfied with $c = c_1/2$ and $\alpha = \frac{l+m_1}{l} \in (0, 1)$. Hence, by Theorem 2.1, the origin is a finite-time-stable equilibrium under f. □

8 Finite-time stabilization of linear control systems

The following proposition proves the existence of a continuous finite-time-stabilizing feedback controller for a chain of integrators by giving an explicit construction involving a small parameter. The controller renders the closed-loop system asymptotically stable and homogeneous of negative degree with respect to a suitable dilation so that finite-time stability follows by Theorem 7.1. Theorem 6.1 plays a key role in the proof of asymptotic stability along with a continuity argument.

Proposition 8.1 *Let $k_1, \ldots, k_n > 0$ be such that the polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$ is Hurwitz, and consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= u. \end{aligned} \tag{37}$$

There exists $\epsilon \in (0, 1)$ such that, for every $\alpha \in (1 - \epsilon, 1)$, the origin is a globally finite-time-stable equilibrium for the system (37) under the feedback

$$u = \chi_\alpha(x_1, \ldots, x_n) = -k_1 \mathrm{sign} x_1 |x_1|^{\alpha_1} - \cdots - k_n \mathrm{sign} x_n |x_n|^{\alpha_n}, \tag{38}$$

where $\alpha_1, \ldots, \alpha_n$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \quad i = 2, \ldots, n, \tag{39}$$

with $\alpha_{n+1} = 1$ and $\alpha_n = \alpha$.

Proof Let $k_1, \ldots, k_n > 0$ be chosen as in the proposition and, for each $\alpha > 0$, let f_α denote the closed-loop vector field obtained by using the feedback (38) in (37). For each $\alpha > 0$, the vector field f_α is continuous. It is also easy to verify that, for each $\alpha > 0$, the vector field f_α is homogeneous of degree $(\alpha - 1)/\alpha$ with respect to the Euler vector field

$$\nu_\alpha = \frac{1}{\alpha_1} x_1 \frac{\partial}{\partial x_1} + \cdots + \frac{1}{\alpha_n} x_n \frac{\partial}{\partial x_n}, \tag{40}$$

where $\alpha_n = \alpha$ and $\alpha_1, \ldots, \alpha_{n-1}$ satisfy (39). Moreover, the vector field f_1 is linear with the Hurwitz characteristic polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$. Therefore, by Theorem 6.2, there exists a positive-definite, radially unbounded, Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $L_{f_1} V$ is continuous and negative definite. Let $\mathcal{A} = V^{-1}([0, 1])$ and $\mathcal{S} = \mathrm{bd}\, \mathcal{A} = V^{-1}(\{1\})$. Then $\mathcal{A}$ and $\mathcal{S}$ are compact since V is proper and $0 \notin \mathcal{S}$ since V is positive definite. Define $\varphi : (0, 1] \times \mathcal{S} \to \mathbb{R}$ by $\varphi(\alpha, z) = L_{f_\alpha} V(z)$. Then φ is continuous and satisfies $\varphi(1, z) < 0$ for all $z \in \mathcal{S}$, that is, $\varphi(\{1\} \times \mathcal{S}) \subset (-\infty, 0)$. Since $\mathcal{S}$ is compact, it follows from Lemma 5.8 in [M, p. 169] that there exists $\epsilon > 0$ such that $\varphi((1 - \epsilon, 1] \times \mathcal{S}) \subset (-\infty, 0)$. It follows that for $\alpha \in (1 - \epsilon, 1]$, $L_{f_\alpha} V$ takes negative values on $\mathcal{S}$. Thus, $\mathcal{A}$ is strictly positively invariant under f_α for every $\alpha \in (1 - \epsilon, 1]$. By Theorem 6.1, the origin is a globally asymptotically stable equilibrium under f_α for every $\alpha \in (1 - \epsilon, 1]$. The result now follows from Theorems 7.1 and 7.3 by noting that, for every $\alpha \in (1 - \epsilon, 1)$, the degree of homogeneity of f_α with respect to ν_α is negative. □

Remark 8.1 Since the results of this paper were derived under the assumption of forward uniqueness, a final remark on the uniqueness of solutions for the various systems considered in the proof of Proposition 8.1 is in order. Each of the vector fields considered in Proposition 8.1 is locally Lipschitz everywhere except on a finite collection of submanifolds. Moreover, in each case, the vector field is transverse to each such submanifold everywhere except at the origin. Hence forward uniqueness for all initial conditions except the origin follows from [F1, Lem. 2, p. 107], [K2, Prop. 4.1] or [K5, Prop. 2.2], while forward uniqueness at the origin follows from Lyapunov stability.

Figure 1 shows an initial condition response along with the corresponding control input for the triple integrator plant $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = u$ under the feedback $u = -\text{sign}(x_1)|x_1|^{1/2} - 1.5\text{sign}(x_2)|x_2|^{3/5} - 1.5\text{sign}(x_3)|x_3|^{3/4}$, which is obtained from (38) with $n = 3$, $k_1 = 1$, $k_2 = k_3 = 1.5$ and $\alpha = 3/4$. Note that for this example, Proposition 8.1 does not guarantee finite-time stability specifically for $\alpha = 3/4$. Instead, stability has to be inferred from Fig. 1.

The following result uses the controller described in Proposition 8.1 to show that every controllable linear system is finite-time stabilizable through continuous state feedback. It should be pointed out that Theorem 8.1 is not a new result and is included here only for completeness. For instance, it was shown in [GKS, K10] that every controllable linear system can be finite-time stabilized using bounded, continuous feedback control while [H8] proves the following result using an alternative construction of a finite-time stabilizing controller for a chain of integrators.

Theorem 8.1 *Every controllable linear control system on $\mathbb{R}^n$ is globally finite-time stabilizable through continuous state feedback.*

Proof Every controllable linear system is feedback equivalent to a linear system in Brunovsky canonical form which is simply a collection of decoupled, independently controlled chains of integrators [S, Sect. 4.2]. The result now follows by noting that Proposition 8.1 can be used to finite-time stabilize each chain of integrators in the Brunovsky canonical form. □

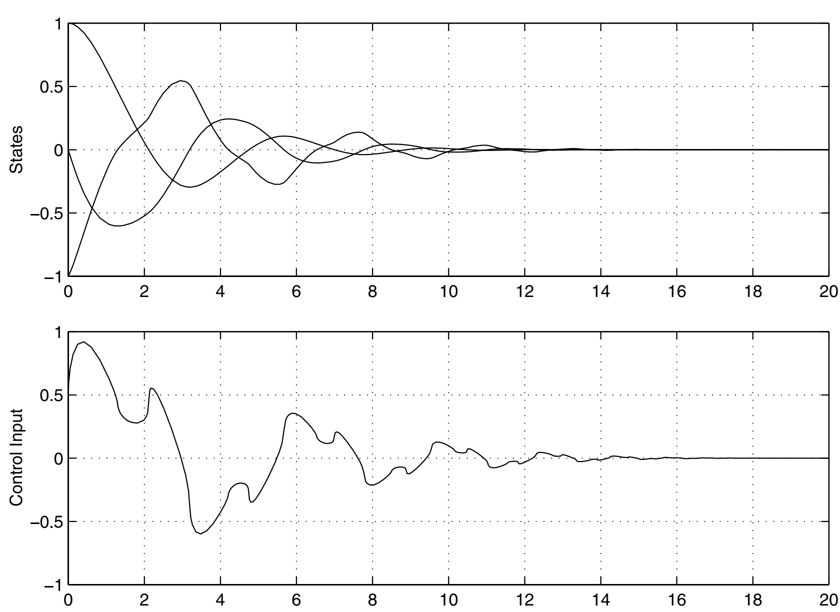

Fig. 1 Initial condition response of a finite-time-stabilized triple integrator

9 Appendix

First we recall a consequence of compactness. If $\{\mathcal{M}_t\}_{t\geq 0}$ is a collection of nonempty, compact sets that are nested in the sense that $\mathcal{M}_{t_2} \subseteq \mathcal{M}_{t_1}$ for every $t_2 \geq t_1$, then $\bigcap_{t\geq 0} \mathcal{M}_t$ is nonempty [M, Thm. 5.9, p. 170].

Proof of Lemma 2.1 Since $\mathcal{A}$ is positively invariant, $\psi_h(\psi_t(\mathcal{A})) = \psi_t(\psi_h(\mathcal{A})) \subseteq \psi_t(\mathcal{A})$ for every $t, h \in \mathbb{R}_+$ so that $\psi_t(\mathcal{A})$ is positively invariant for every $t \in \mathbb{R}_+$. By the compactness of $\mathcal{A}$ and the continuity of ψ, $\psi_t(\mathcal{A})$ is compact for every $t \in \mathbb{R}_+$. Thus $\{\psi_h(\mathcal{A})\}_{h\geq 0}$ is a collection of nested nonempty compact sets and hence $\mathcal{K} \stackrel{\triangle}{=} \bigcap_{h\geq 0} \psi_h(\mathcal{A})$ is nonempty and compact. $\mathcal{K}$ is also the intersection of positively invariant sets and hence positively invariant [BH, Lem. 3.3], [BS, Thm. II.1.2]. Therefore, to show that $\mathcal{K}$ is invariant, it suffices to show that $\mathcal{K} \subseteq \psi_h(\mathcal{K})$ for all $h \in \mathbb{R}_+$. Let $h > 0$ and consider $x \in \mathcal{K}$. Then $x \in \psi_{h+t}(\mathcal{A}) = \psi_h(\psi_t(\mathcal{A}))$ for every $t \in \mathbb{R}_+$, so that $\psi_h^{-1}(\{x\}) \bigcap \psi_t(\mathcal{A})$ is nonempty for every $t \in \mathbb{R}_+$. Moreover, $\psi_h^{-1}(\{x\})$ is closed by continuity, so that $\psi_h^{-1}(\{x\}) \bigcap \psi_t(\mathcal{A})$ is compact for every $t \in \mathbb{R}_+$. Thus $\{\psi_h^{-1}(\{x\}) \cap \psi_t(\mathcal{A})\}_{t\geq 0}$ is a collection of nested nonempty compact sets and hence $\bigcap_{t\geq 0}[\psi_h^{-1}(\{x\}) \cap \psi_t(\mathcal{A})] = \psi_h^{-1}(\{x\}) \bigcap \mathcal{K}$ is nonempty. It follows that $x \in \psi_h(\mathcal{K})$ and $\mathcal{K}$ is invariant.

If $\mathcal{C} \subset \mathcal{A}$ is invariant under ψ, then $\mathcal{C} = \psi_t(\mathcal{C}) \subseteq \psi_t(\mathcal{A})$ for every $t \geq 0$, so that $\mathcal{C} \subseteq \mathcal{K}$. Thus $\mathcal{K}$ is the largest subset of $\mathcal{A}$ that is invariant under ψ.

Let $\mathcal{U}$ be an open neighborhood of $\mathcal{K}$. To show that $\psi^x(t) \to \mathcal{K}$ for all $x \in \mathcal{A}$, it suffices to show that there exists $\tau > 0$ such that $\psi_t(\mathcal{A}) \subset \mathcal{U}$ for all $t > \tau$. The sets $\{\psi_t(\mathcal{A}) \bigcap \mathcal{U}^{\mathrm{c}}\}_{t\geq 0}$ form a collection of nested compact sets. If $\psi_t(\mathcal{A}) \bigcap \mathcal{U}^{\mathrm{c}}$ is nonempty for every $t \in \mathbb{R}_+$, then by compactness, $\varnothing \neq \bigcap_{t\geq 0} [\psi_t(\mathcal{A}) \cap \mathcal{U}^{\mathrm{c}}] = \left[\cap_{t\geq 0}\psi_t(\mathcal{A})\right] \bigcap \mathcal{U}^{\mathrm{c}} = \mathcal{K} \bigcap \mathcal{U}^{\mathrm{c}} = \varnothing$ which is a contradiction. Therefore, there exists $\tau \in \mathbb{R}_+$ such that $\psi_\tau(\mathcal{A}) \bigcap \mathcal{U}^{\mathrm{c}}$ is empty, that is, $\psi_\tau(\mathcal{A}) \subset \mathcal{U}$. By the positive invariance of $\psi_\tau(\mathcal{A})$, $\psi_t(\mathcal{A}) = \psi_{t-\tau}(\psi_\tau(\mathcal{A})) \subseteq \psi_\tau(\mathcal{A}) \subset \mathcal{U}$ for all $t > \tau$. Thus $\psi^x(t) \to \mathcal{K}$ for all $x \in \mathcal{A}$.

Now, suppose $\mathcal{A}$ is strictly positively invariant. Then, for a given $t > 0$, $\mathcal{K} \subseteq \psi_t(\mathcal{A}) \subset \operatorname{int} \mathcal{A}$. Also, $\mathcal{K}$ is attractive since $\mathcal{K} \subset \operatorname{int} \mathcal{A}$ and $\psi^x(t) \to \mathcal{K}$ for all $x \in \operatorname{int} \mathcal{A}$.

If $\mathcal{K}$ is not Lyapunov stable, then there exists an open neighborhood $\mathcal{U}$ of $\mathcal{K}$ and a sequence $\{(t_i, x_i)\}$ in $\mathbb{R}_+ \times \mathbb{R}^n$ such that $x_i \to \mathcal{K}$ and $\psi(t_i, x_i) \notin \mathcal{U}$ for $i = 1, 2, \ldots$. However, as shown above, there exists a $\tau > 0$ such that $\psi_t(\mathcal{A}) \subset \mathcal{U}$ for all $t > \tau$. This implies that $t_i \leq \tau$ for all $i = 1, 2, \ldots$. Therefore, without any loss of generality, we may assume that $t_i \to t \in \mathbb{R}_+$. Also, since $\mathcal{A}$ is compact, we may assume that $x_i \to x \in \mathcal{K}$. Then, by continuity, $\psi(t_i, x_i) \to \psi(t, x)$. However, $\psi(t, x) \in \mathcal{K} \subset \mathcal{U}$ by invariance while $\psi(t_i, x_i) \notin \mathcal{U}$ by construction. This contradiction proves that $\mathcal{K}$ is Lyapunov stable. Attractivity and Lyapunov stability imply asymptotic stability. □

Proof of Lemma 2.2 Let $\mathcal{V}$ be a bounded open neighborhood of $\mathcal{K}$, and let $\mathcal{S} = \mathcal{M} \cup \operatorname{bd} \mathcal{V}$. Then $\mathcal{S}$ is compact. Define $T : \operatorname{doa}(\mathcal{K}) \to \mathbb{R}_+$ by $T(x) = \inf\{t \in \mathbb{R}_+ : \psi_t(x) \in \mathcal{V}\}$.

We claim that T is upper semicontinuous. To see this, consider $x \in \operatorname{doa}(\mathcal{K})$ and let $\{x_i\}$ be a sequence in $\operatorname{doa}(\mathcal{K})$ converging to x. Choose $\epsilon > 0$. There exists

$t \leq T(x) + \epsilon$ such that $\psi_t(x) \in \mathcal{V}$. By continuity of ψ, there exists M such that $\psi_t(x_i) \in \mathcal{V}$ for every $i > M$. It follows that, for every $i > M$, $T(x_i) \leq T(x) + \epsilon$. Since ϵ was chosen arbitrarily, it follows that $\limsup_{i\to\infty} T(x_i) \leq T(x)$. Thus T is upper semicontinuous.

The upper semicontinuous function T is bounded above on the compact set $\mathcal{S}$. Let $\tau = \sup_{x\in\mathcal{S}} T(x)$. Consider $x \in \mathcal{M}$. For every $t \leq \tau$, $\psi(t, x) \in \psi([0, \tau]\times\mathcal{M})$. Let $t > \tau$. We claim that $\psi(t, x) \in \psi([0, \tau] \times \overline{\mathcal{V}})$. Indeed, this is trivially true if $\psi(t, x) \in \overline{\mathcal{V}}$. Hence consider the case where $\psi(t, x) \notin \overline{\mathcal{V}}$. Since $t > \tau \geq T(x)$, there exists $s \leq t$ such that $\psi(s, x) \in \mathcal{V}$. By the continuity of ψ, it follows that there exists $h \geq 0$ such that $\psi(h, x) \in \text{bd}\ \mathcal{V}$ and $\psi(s, z) \notin \overline{\mathcal{V}}$ for every $s \in (h, t]$. It follows by our definition of T that $t - h \leq T(\psi(h, x)) \leq \tau$. Hence $\psi(t, x) = \psi_{t-h}(\psi(h, x)) \subseteq \psi([0, \tau] \times \overline{\mathcal{V}})$. Thus, for every $x \in \mathcal{M}$ and $t \geq 0$, $\psi(t, x) \in \psi([0, \tau] \times (\mathcal{M} \cup \overline{\mathcal{V}}))$, that is, $\psi(\mathbb{R}_+ \times \mathcal{M}) \subseteq \psi([0, \tau] \times (\mathcal{M} \cup \overline{\mathcal{V}}))$. Since the set $\psi([0, \tau] \times (\mathcal{M} \cup \overline{\mathcal{V}}))$ is clearly compact, it follows that $\psi(\mathbb{R}_+ \times \mathcal{M})$ is bounded.

Next, assume that $\mathcal{K}$ is asymptotically stable under ψ, and let $\mathcal{U}$ be an open neighborhood of $\mathcal{K}$. By Lyapunov stability, there exists an open neighborhood $\mathcal{V}$ of $\mathcal{K}$ such that $\psi_t(\mathcal{V}) \subseteq \mathcal{U}$ for all $t \in \mathbb{R}_+$. Consider the collection of nested sets $\{\mathcal{M}_t\}_{t>0}$ where $\mathcal{M}_t = \{x \in \mathcal{M} : \psi_h(x) \notin \mathcal{V},\ h \in [0, t]\} = \mathcal{M} \bigcap \left(\bigcup_{h\in[0,t]} \psi_h^{-1}(\mathcal{V})\right)^{\mathrm{c}}$, $t > 0$. For each $t > 0$, $\mathcal{M}_t$ is a compact set. Therefore, if $\mathcal{M}_t$ is nonempty for each $t > 0$, then there exists $x \in \bigcap_{t>0} \mathcal{M}_t$, that is, there exists $x \in \mathcal{M}$ such that $\psi_t(x) \notin \mathcal{V}$ for all $t > 0$, which contradicts the assumption that $\mathcal{M} \subset \text{doa}(\mathcal{K})$. Thus there exists $\tau > 0$ such that $\mathcal{M}_\tau = \varnothing$, that is, $\mathcal{M} \subset \bigcup_{h\in[0,\tau]} \psi_h^{-1}(\mathcal{V})$. Therefore, for every $t > \tau$, $\psi_t(\mathcal{M}) \subset \bigcup_{h\in[0,\tau]} \psi_t\left(\psi_h^{-1}(\mathcal{V})\right) = \bigcup_{h\in[0,\tau]} \psi_{t-h}(\mathcal{V}) \subseteq \mathcal{U}$, where the last inclusion follows from Lyapunov stability. □

Proof of Lemma 4.1 *(i)* Suppose V is sign definite. Then $V(0) = 0$, while *(i)* and *(ii)* of Theorem 4.1 imply that the degree of homogeneity l of V is positive. Without any loss of generality, we may assume that V is positive definite. Let $\mathcal{K} \subset \mathbb{R}^n$ be a bounded open neighborhood of 0 and let $\beta = \min_{z\in\text{bd}\ \mathcal{K}} V(z) > 0$. Now, suppose $x \notin \overline{\mathcal{K}}$, and let $s > 0$ be such that $z = \phi_{-s}(x) \in \text{bd}\ \mathcal{K}$. Then, homogeneity implies that $V(x) = e^{ls} V(z) > V(z) \geq \beta$. Thus, $V^{-1}([0, \beta])$, which is closed by continuity, is contained in the compact set $\overline{\mathcal{K}}$ and hence compact. Now, given $\gamma \in V(\mathbb{R}^n)$, equation (15) implies that $V^{-1}([0, \gamma]) = V^{-1}([0, e^{ls}\beta]) = \phi_s(V^{-1}([0, \beta]))$ for $s = \frac{1}{l}(\ln\gamma - \ln\beta)$, so that by the continuity of ϕ_s, $V^{-1}([0, \gamma])$ is compact. Since every compact set M in $V(\mathbb{R}^n)$ is contained in an interval of the form $[0, \gamma]$, it follows that V is proper. Radial unboundedness follows from equation (14) by letting $s \to \infty$.

(ii) Let $n > 1$ and suppose V is proper so that $\mathcal{S} = V^{-1}(\{0\})$ is compact. If $l \leq 0$, then it follows from *i), ii)* of Theorem 4.1 that $V^{-1}(\{V(0)\}) = \mathbb{R}^n$, which contradicts properness. Therefore, $l > 0$ and by (14), $V(0) = 0$ so that $\mathcal{S}$ is nonempty. Applying equation (15) with $M = \{0\} \subset \mathbb{R}$ yields $\phi_s(\mathcal{S}) = \mathcal{S}$ for all $s \in \mathbb{R}$, that is, $\mathcal{S}$ is invariant under ν. Since the only compact, nonempty set that is invariant under ν is $\{0\}$, it follows that $\mathcal{S} = \{0\}$. Also, $\mathbb{R}^n\backslash\{0\}$ is connected. Since V is continuous and $\mathcal{S} = \{0\}$, it follows that $V(\mathbb{R}^n\backslash\{0\}) = V(\mathbb{R}^n\backslash\mathcal{S})$ is a connected subset of $\mathbb{R}$ that does not contain 0, and hence V is sign definite. □

References

[BR] Bacciotti A, Rosier L (2001) Liapunov functions and stability in control theory. Springer-Verlag, London

[BB1] Bhat SP, Bernstein DS (1995) Lyapunov analysis of finite-time differential equations. In: Proceedings American control conference, Seattle, pp 1831–1832

[BB2] Bhat SP, Bernstein DS (1997) Finite-time stability of homogeneous systems. In: Proceedings of American control conference, Albuquerque, pp 2513–2514

[BB3] Bhat SP, Bernstein DS (1998) Continuous, finite-time stabilization of the translational and rotational double integrators. IEEE Trans Automatic Control 43:678–682

[BB4] Bhat SP, Bernstein DS (2000) Finite-time stability of continuous autonomous systems. SIAM J Control Optim 38:751–766

[BH] Bhatia NP, Hajek O (1969) Local semi-dynamical systems. Lecture Notes in Mathematics, vol 90. Springer, Berlin Heidelberg New York

[BS] Bhatia NP, Szegö GP (1970) Stability theory of dynamical systems. Springer, Berlin Heidelberg New York

[C] Coleman C (1960) Asymptotic stability in 3-space. In: Cesari L, LaSalle J, Lefschetz S (eds) Contributions to the theory of nonlinear oscillations, vol V. Princeton University Press, pp 257–268

[DM] Dayawansa WP, Martin CF (1989) Some sufficient conditions for the asymptotic stabilizability of three dimensional homogeneous polynomial systems. In: Proceedings of conference on decision and control. Tampa, pp 1366–1369

[F1] Filippov AF (1988) Differential equations with discontinuous righthand sides. Kluwer, Dordrecht

[F2] Flett TM (1980) Differential analysis. Cambridge University Press, Cambridge

[GKS] Gavrilyako VM, Korobov VI, Skylar GM (1986) Designing a bounded control of dynamic systems in entire space with the aid of controllability function. Automation Remote Control 47:1484–1490

[H1] Hahn W (1967) Stability of motion. Springer, Berlin Heidelberg New York

[H2] Haimo VT (1986) Finite time controllers. SIAM J Control Optim. 24:760–770

[H3] Hartman P (1982) Ordinary differential equations. 2nd edn. Birkhäuser, Boston

[H4] Hermes H (1991) Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. In: Elaydi S (ed) Differential equations, stability and control. Lecture Notes in Pure and Applied Mathematics, vol 127. Marcel Dekker, New York, pp 249–260

[H5] Hermes H (1991) Nilpotent and high-order approximations of vector field systems. SIAM Rev 33:238–264

[H6] Hermes H (1992) Vector field approximations; flow homogeneity. In: Wiener J, Hale JK (eds) Ordinary and delay differential equations. Longman Sci Tech, pp 80–89

[H7] Hermes H (1995) Homogeneous feedback controls for homogeneous systems. Syst Control Lett 24:7–11

[H8] Hong Y (2002) Finite-time stabilization and stabilizability of a class of controllable systems. Systems Control Lett 46:231–236

[HHX] Hong Y, Huang J, Xu Y (2001) On an output feedback finite-time stabilization problem. IEEE Trans Automatic Control 46:305–309

[HJ] Horn RA, Johnson CR (1985) Matrix analysis. Cambridge University Press, Cambridge

[IO] Iggidr A, Outbib R (1995) Feedback stabilization of homogeneous polynomial systems. In: Preprint, IFAC Symposium on nonlinear control systems design, Lake Tahoe

[K1] Kartsatos AG (1980) Advanced ordinary differential equations. Mariner Tampa

[K2] Kawski M (1988) Controllability, approximations and stabilization. In: Bowers K, Lund J (eds) Computation and control, vol 1. Birkhäuser, Boston, pp 155–167

[K3] Kawski M (1988) Stabilizability and nilpotent approximations. In: Proceedings of conference on decision and control, Austin, pp 1244–1248

[K4] Kawski M (1989) Homogeneous feedback laws in dimension three. In: Proceedings of conference on decision and control, Tampa, pp 1370–1375

[K5] Kawski M (1989) Stabilization of nonlinear systems in the plane. Syst Control Lett 12:169–175

[K6] Kawski M (1990) Homogeneous stabilizing feedback laws. Control-Theory Adv Technol 6:497–516

[K7] Kawski M (1991) Families of dilations and asymptotic stability. In: Bonnard B, Bride B, Gauthier JP, Kupka I (eds) Analysis of controlled dynamical systems, progress in systems and control theory, vol 8. Birkhäuser, Boston, pp 285–294

[K8] Kawski M (1999) Geometric homogeneity and stabilization. In: Krener A, Mayne D (eds) IFAC postprint volumes series. Elsevier. Amsterdam

[K9] Khalil HK (1996) Nonlinear systems. 2nd edn. Prentice-Hall, Upper Saddle River

[K10] Komarov VA (1984) Design of constrained controls for linear systems. Automation Remote Control 45:1280–1286

[MM] M'Closkey RT, Murray RM (1997) Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. IEEE Trans Automatic Control 42:614–628

[M] Munkres JR (1975) Topology a first course. Prentice-Hall, Englewood Cliffs

[P] Praly L (1997) Generalized weighted homogeneity and state dependent time scale for controllable systems. In: Proceedings of conference on decision and control, San Diego pp 4342–4347

[R1] Rosier L (1992) Homogeneous Lyapunov function for homogeneous continuous vector field. Syst Control Lett 19:467–473

[R2] Rosier L (1993) Etude de quelques problèmes de stabilisation. PhD Thesis, Ecole Normale Supérieure de Cachan, France

[RHL] Rouche N, Habets P, Laloy M (1977) Stability theory by Liapunov's direct method. Springer, Berlin Heidelberg New York

[R3] Ryan EP (1982) Optimal relay and saturating control system synthesis. IEE control engineering series, vol 14. Peter Peregrinus Ltd

[R4] Ryan EP (1991) Finite-time stabilization of uncertain nonlinear planar systems. Dyn Control 1:83–94

[R5] Ryan EP (1995) Universal stabilization of a class of nonlinear systems with homogeneous vector fields. Syst Control Lett 26:177–184

[SA1] Sepulchre R, Aeyels D (1995) Adding an integrator to a non-stabilizable homogeneous planar system. In: Preprint, IFAC symposium on nonlinear control systems design, Lake Tahoe

[SA2] Sepulchre R, Aeyels D (1996) Homogeneous Lyapunov functions and necessary conditions for stabilization. Math Control Signals Syst 9:34–58

[SA3] Sepulchre R, Aeyels D (1996) Stabilizability does not imply homogeneous stabilizability for controllable homogeneous systems. SIAM J Control Optimization 34:1798–1813

[S] Sontag ED (1990) Mathematical control theory, vol 6 of texts in applied mathematics. Springer, Berlin Heidelberg New York

[Y] Yoshizawa T (1966) Stability theory by Liapunov's second method. The Mathematical Society of Japan