The Moment Problem in a Certain Function Space of G. G. LORENTZ

By

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In this paper I obtain necessary and sufficient conditions for the existence of a function $f(x)$, belonging to a suitable function space, so that a given sequence of (real) constants $\{\mu_n\}$ may be the sequence of moment constants of the function $f(x)$; i.e. in order that μ_n may have the representation

$$\mu_n = \int_0^1 x^n f(x)\, dx \tag{1}$$

with $f(x)$ belonging to the specified space of functions. In particular the above problem is solved here for the space $X(C)$ of LORENTZ [2], the definition of which follows in the sequel. For this space, LORENTZ has given a solution and we obtain here a different set of conditions in order that (1) may hold, with $f(x) \in X(C)$.

Let C denote a class of positive integrable functions on $[0,1]$ and let C have the following properties:

(i) $1 \in C$;

(ii) C is *normal* in the sense that if $c_1(x) \in C$ and $c_2(x)$ is a measurable function such that $0 \leqq c_2(x) \leqq c_1(x)$ p.p. on $[0,1]$, then $c_2(x) \in C$;

(iii) the integrals $\int_0^1 c(x)\, dx$, $c \in C$, are bounded.

Evidently, all bounded measurable functions on [0, 1] belong to C. Now, in relation to C the class $X(C)$ is defined to be the class of all measurable functions $f(x)$ for which

$$\|f\| = \sup_{c \in C} \int_0^1 |f(x)|\, c(x)\, dx < \infty .$$

It may easily be verified that $X(C)$ is a Banach space (with the above indicated norm) and that $X(C)$ is normal in the sense stated earlier. A suitable choice of C will yield the spaces $L^p (p > 1)$, $\Lambda(\varphi, p)$ and $M(\varphi, p)$, defined by LORENTZ [2].

The following definitions are required in the sequel.

Two measurable functions $f(x)$ and $g(x)$, on $[0,1]$, are called *rearrangements* of each other if, for each real a, the sets $[f(x) \geqq a]$ and $[g(x) \geqq a]$ have equal measures. We shall say that the space $X(C)$ has the property of *rearrangement invariant norm* if $\|\hat{f}\| = \|f\|$ for each rearrangement $\hat{f}(x)$ of $f(x)$. (The spaces L^p, $\Lambda(\varphi, p)$ etc. cited earlier have this property.)

We shall, for our investigation, focus our attention on the spaces $X(C)$ which are endowed with the property that the integrals

$$F(e) = \int_e f(x)\,dx \tag{2}$$

are such that for each $\varepsilon > 0$, there exists a positive δ in such a manner that (measure of e) $\leqq \delta$ implies $|F(e)| \leqq \varepsilon$, for all $f(x) \in X(C)$ with $\|f\| \leqq 1$. This fact will be briefly denoted by the statement that "the integrals in (2) have the *property of uniform absolute continuity*".

For the space $X(C)$, which has the property of rearrangement invariant norm and for which the integrals in (2) are uniformly absolutely continuous, LORENTZ [2] has shown that μ_n will have the representation in (1) with $f \in X(C)$ and $\|f\| \leqq M$ if and only if the norms of the functions

$$f_n(x) = (n+1)\binom{n}{\nu}\Delta^{n-\nu}\mu_\nu, \quad \frac{\nu}{n+1} \leqq x < \frac{\nu+1}{n+1}, \; \nu = 0, 1, 2, \ldots, n \tag{3}$$

satisfy the condition $\|f_n\| \leqq M$, $n = 0, 1, 2, \ldots$.

LORENTZ's proof of the above solution rests, among other lemmas, on the uniform approximation of the function $f(x)$, which is continuous in $[0, 1]$, by the sequence of Bernstein polynomials $\{B_n^f(x)\}$,

$$B_n^f(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu}(1-x)^{n-\nu}x^\nu.$$

We shall also employ the same technique as LORENTZ's; however, our starting point is the following lemma, due in substance to MEYER-KÖNIG and ZELLER [3].

Lemma 1[1]). *Let $f(x)$ defined in $[0,1]$ be continuous there. Then the sequence $\{P_n^f(x)\}$ of Bernstein power series defined by*

$$P_n^f(x) = \sum_{k=n}^{\infty}\binom{k}{n}(1-x)^{k-n}x^{n+1}f\left(\frac{n+1}{k+1}\right), \qquad P_n^f(0) = f(0)$$

uniformly approximates $f(x)$ in $[\delta, 1]$, any $\delta > 0$.

We need also the following three lemmas.

Lemma 2. *Let the space $X(C)$ have the rearrangement invariant norm property. Then each transformation $F(x) = \int_0^1 K(x,t)\,f(t)\,dt$, with the properties*

$$\int_0^1 |K(x,t)|\,dt \leqq A \quad \text{and} \quad \int_0^1 |K(x,t)|\,dx \leqq A$$

is a continuous linear operator of norm not exceeding A, which maps $X(C)$ into itself.

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1) MEYER-KÖNIG and ZELLER [3] do not state lemma 1 in exactly this form. But the content of the lemma follows from a stronger result they prove in Satz 1 of their paper.

Lemma 3. *If the functions $F_n(x)$, $0 \leqq x \leqq 1$, have uniformly absolutely continuous integrals, then there exists a subsequence $F_{n_k}(x)$ and an integrable function $F(x)$ such that for each bounded integrable g, and $g_k(x) \to g(x)$ uniformly*

$$\int_0^1 F_{n_k}(x)\, g_k(x)\, dx \to \int_0^1 F(x)\, g(x)\, dx\,.$$

Also, if the $F_n(x)$ all belong to $X(C)$ and $\| F_n \| \leqq 1$, then $F(x) \in X(C)$ and $\| F \| \leqq 1$.

Lemmas 2 and 3 are due to Lorentz ([2], pp. 79—80).

Lemma 4. *Let μ_n be a sequence of real constants. Then*

$$\sum_{k=n}^{\infty} \binom{k}{n} \left| \Delta^{k-n} \mu_{n+1} \right| \leqq M \qquad (n = 0, 1, 2, \ldots)$$

if and only if

$$\sum_{k=0}^{n} \binom{n}{k} \left| \Delta^{n-k} \mu_k \right| \leqq N \qquad (n = 0, 1, 2, \ldots)\,.$$

Lemma 4 is a consequence of two known theorems, one due to the present author ([4], Theorem 1) and another due to Hausdorff ([2], Theorem 3.3.1); a direct proof of lemma 4 is due to Kuttner [1].

Let $\{\mu_n\}$, as before, be a sequence of real constants. We now define a sequence of functions as follows:

$$f_n^*(x) = \frac{(k+1)(k+2)}{(n+1)} \binom{k}{n} \Delta^{k-n} \mu_{n+1}\,, \quad \frac{n+1}{k+2} < x \leqq \frac{n+1}{k+1}\,, \quad \begin{array}{l} k = n, n+1, \ldots, \\ n = 0, 1, 2, \ldots\,. \end{array}$$

With this definition, we are now in a position to prove our main result.

Theorem. *Let the space $X(C)$ have the property of rearrangement invariant norm and let the integrals in (2) be uniformly absolutely continuous. Then in order that the sequence μ_n $(n = 1, 2, \ldots)$ may have the representation*

$$\mu_n = \int_0^1 x^n f(x)\, dx \tag{1}$$

with $f \in X(C)$ and $\| f \| \leqq M$ it is necessary and sufficient that for each n, $\| f_n^* \| \leqq M$.

Proof of the necessity. Let μ_n have the representation $\int_0^1 x^n f(x)\, dx$, $n = 1, 2, 3, \ldots$. Then

$$f_n^*(x) = \int_0^1 K_n^*(x, t)\, f(t)\, dt$$

where

$$K_n^*(x, t) = \frac{(k+1)(k+2)}{(n+1)} \binom{k}{n} (1-t)^{k-n}\, t^{n+1}\,, \quad \frac{n+1}{k+2} < x \leqq \frac{n+1}{k+1}\,, \quad k = n, n+1, \ldots\,.$$

It may easily be verified that

$$\int_0^1 \left| K_n^*(x, t) \right| dt = 1 \quad \text{and} \quad \int_0^1 \left| K_n^*(x, t) \right| dx = 1\,.$$

The proof of the necessity is now complete after Lemma 2.

Proof of the sufficiency. Let $f(x)$ be any continuous function in $[0,1]$ and let $P_n^f(x)$ have the meaning defined in Lemma 1. We shall define $P_{nm}^f(x)$ for $n, m = 0, 1, 2, \ldots$ by

$$P_{nm}^f(x) = \sum_{k=n}^{n+m} \binom{k}{n} (1-x)^{k-n} x^{n+1} f\left(\frac{n+1}{k+1}\right)$$

with the understanding that $P_{nm}^f(0) = f(0)$, for all n, m. We make the following preliminary comments.

(i) For each fixed n and m, $P_{nm}^f(x)$ is a polynomial in x;
(ii) $P_{nm}^f(x)$ converges uniformly to $P_n^f(x)$, as $m \to \infty$, for $x \in [\delta, 1]$,
(iii) $P_n^f(x)$ converges uniformly to $f(x)$ in $[\delta, 1]$.

Let us now assume that $\| f_n^* \| \leqq M$, for each n. Then since $1 \in C$ and

$$\| f_n^* \| = \sup_{c \in C} \int_0^1 | f_n^*(x) | \, c(x) \, dx$$

it follows that $\| f_n^* \| \leqq M$ implies

$$\sum_{k=n}^{\infty} \binom{k}{n} | \Delta^{k-n} \mu_{n+1} | \leqq M$$

and therefore by Lemma 4, also, that

$$\sum_{k=0}^{n} \binom{n}{k} | \Delta^{n-k} \mu_k | \leqq N,$$

for each n.

Now the polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with $\{a_n\}$ real and $x \in [0,1]$ form a linear subspace $P[0,1]$ of the space $C[0,1]$ of continuous functions in $[0,1]$. Then when the $\{\mu_n\}$ satisfies the above condition, we have, as shown by LORENTZ ([2], pp. 58—59), that

$$L(p) = a_0 \mu_0 + \cdots + a_m \mu_m$$

is a linear form over $P[0,1]$, which can be extended to $C[0,1]$ by setting, for

$$f \in C[0,1], \quad L(f) = \lim_n L(f_n),$$

where $f_n(x)$ is any sequence of polynomials uniformly approximating $f(x)$ in $[0,1]$; also such an extended linear form is continuous over C. The same result holds for $[\delta, 1]$ instead of $[0,1]$, for any $\delta > 0$.

Thus it follows from the observations made above and from the comments (i)—(iii) that for $x \in [\delta, 1]$, whatever be $\delta > 0$, that $L(P_{nm}^f) \to L(P_n^f)$ as $m \to \infty$ and that $L(P_n^f) \to L(f)$ as $n \to \infty$. Taking $f(x) = x^p$, $p = 0, 1, 2, \ldots$ we obtain, after a brief calculation, that

$$\sum_{k=n}^{\infty}\left(\frac{n+1}{k+1}\right)^p \binom{k}{n} \Delta^{k-n}\mu_{n+1} \to \mu_p \quad \text{as} \quad n \to \infty .$$

But the expression on the left hand side above is $\int_0^1 f_n^*(x)\, g_n(x)\, dx$ where

$$g_n(x) = \left(\frac{n+1}{k+1}\right)^p \quad \text{for} \quad \frac{n+1}{k+2} < x \leqq \frac{n+1}{k+1}, \qquad k = n, n+1, \ldots .$$

By an application of Lemma 3 it follows that $\mu_p = \int_0^1 x^p f(x)\, dx$ with $f \in X(C)$, $\|f\| \leqq M$.

This completes the proof of the theorem.

Application. In the special case of the space L^p $(p > 1)$, for example, the condition that $\|f_n^*\| \leqq M$ can be expressed in the form

$$\sum_{k=n}^{\infty}\left[\frac{(k+1)(k+2)}{(n+1)}\right]^{p-1}\left|\binom{k}{n}\Delta^{k-n}\mu_{n+1}\right|^p \leqq M, \qquad n = 0, 1, \ldots$$

while LORENTZ's condition for the same space is

$$\sum_{k=0}^{n}(n+1)^{p-1}\left|\binom{n}{k}\Delta^{n-k}\mu_k\right|^p \leqq M, \qquad n = 0, 1, \ldots .$$

Similar conditions for various other special cases of the space $X(C)$ can be derived from the expression that $\|f_n^*\| \leqq M$.

It will be interesting to know whether a direct equivalence of the two conditions $\|f_n\| \leqq M$ and $\|f_n^*\| \leqq M$, without any appeal to the theory of moment sequences, can be given.

References

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