Arch. math. Logik 20 (1980), 85–94

CONSERVATIVE EXTENSIONS OF MODELS OF ARITHMETIC*

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Abstract

We give two characterizations of conservative extensions of models of arithmetic, in terms of the existence and uniqueness of certain amalgàmations with other models. We also establish a connection between conservativity and some combinatorial properties of ultrafilter mappings.

Arithmetic is, in this paper, the complete theory of the structure N whose universe is the set of natural numbers and whose relations and functions are all the relations and functions on this set. All models of arithmetic are elementary extensions of N, and, because of the presence of Skolem functions, all submodels of models of arithmetic are elementary submodels. If $A \subseteq B$ are models of arithmetic and if, for every subset $X \subseteq B$ that is definable in B with parameters from B, the intersection $X \cap A$ is definable in A with parameters from A, then B is called a *conservative* extension of A [9]. For example, because of the definition of N, all its extensions are conservative. In general, all conservative extensions are end extensions [9], but the converse fails, at least if the continuum hypothesis is true [3].

We shall prove two theorems characterizing conservativity in terms of the existence or uniqueness of certain special amalgamations. We shall also relate conservativity to some combinatorial properties of projections of ultrafilters. Before turning to these results, however, we need to know that models of arithmetic have proper conservative extensions. This can be shown by an iterated ultrapower (or limit ultrapower) argument [7, 3], but the following proof, though perhaps less efficient, seems more conceptual.

Let A be any model of arithmetic. A theorem of Keisler [8] says that A is a direct limit of ultrapowers of N with respect to ultrafilters on countable sets. Let $^*V$ be the corresponding limit of ultrapowers of the whole set-theoretic universe V. (Readers squeamish about proper classes may truncate V at some reasonably high rank; ω_1 is high enough.) For any $x \in V$, we write $^*x$ for the corresponding element of $^*V$. Thus, $^*N$ is a structure with the same universe as A and with all internal (in

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* Eingegangen am 14.4.1978.
[1] Partially supported by NSF grant MCS 76-06533.

$^*V$) relations and functions; A is the reduct of $^*N$ to the standard part of the language. For any $x \in {}^*V$, there is a countable set $C \in V$ such that $x \in {}^*C$. This is obvious if $^*V$ is an ultrapower of V with respect to an ultrafilter on a countable set I, for then x is the equivalence class of some function on I, and the range of this function serves as C; the general case, a limit of ultrapowers, follows immediately. Thus, any $x \in {}^*V$ has the form $({}^*f)(n)$ for some standard function f on N and some $n \in {}^*N$. In particular, if x is an internal subset of $^*N$, then there is a binary relation $R(p, q)$ on N, namely $p \in f(q)$ with f as above, and there is an $n \in {}^*N$ such that $(\forall p \in {}^*N)(p \in x \leftrightarrow ({}^*R)(p, n))$. This means that every internal subset of $^*N$ is parametrically definable in A. Now let $\bar{B} \in {}^*V$ be any proper elementary extension of $^*N$ in $^*V$, and let B be its reduct to the standard part of the language. Then B is a proper elementary extension of A. Furthermore, if X is parametrically definable in B (hence a fortiori in $\bar{B}$), then $X \cap A$ is internal in $^*V$ and therefore parametrically definable in A. Thus, B is a proper conservative extension of A, as required.

To formulate our main results, we need the notion of an amalgamation of two models. Let A be a model of arithmetic, and let B and C be elementary extensions of A. Replacing B or C with an isomorphic copy such that the isomorphism is the identity on A, we arrange that $B \cap C = A$. An amalgamation of B and C over A is a model D that contains B and C as submodels and is generated by $B \cup C$. Such a D is a model of the theory Complete diagram of B + Complete diagram of $C + \{b \neq c \mid b \in B - A,\ c \in C - A\}$ in the language $L(B \cup C)$ obtained from the language L of arithmetic by adding all the elements of $B \cup C$ as names for themselves. Conversely, in any model of this theory, the denotations of the closed terms form an amalgamation of B and C over A. It is easy to see that this theory is consistent, so amalgamations always exist [4]. Two amalgamations, D and D', are considered equivalent if there is an isomorphism between them that leaves $B \cup C$ pointwise fixed. Since D and D' are generated by $B \cup C$, such an isomorphism can only be the map defined by sending $f_D(b, c) \in D$ to $f_{D'}(b, c) \in D'$, where $b \in B$, $c \in C$, and f is a binary function on natural numbers with canonical extensions f_D and $f_{D'}$ in the models D and D'. It is routine to verify that this map is well-defined and is an isomorphism (i.e., that D and D' are equivalent) if and only if, for all L-formulas $\phi(x, y)$, all $b \in B$, and all $c \in C$, the truth value of $\phi(b, c)$ is the same in D as in D'.

Characterizations of Conservativity

Suppose B is an end extension of A, while C is an arbitrary extension of A. Will there always be an amalgamation that is an end extension of C? The answer is "no" in general, even if C is also an end extension of A, but it becomes "yes" if B is a conservative extension of A.

Theorem 1. *Let $A \subsetneq B$ be models of arithmetic. The following are equivalent.*

(1) *B is a conservative extension of A.*

(2) *For every extension C of A, there is an amalgamation of B and C over A that is an end extension of C.*

(3) *For some proper conservative extension C of A, there is an amalgamation of B and C over A that is an end extension of C.*

Proof. (2)⇒(3) is obvious because A has a proper conservative extension.
(3)⇒(1). Let C be a proper conservative extension of A, and let D be an amalgamation as in (3). As $A \subseteq B \subseteq D$, the conservativity of B will follow if we prove that D is a conservative extension of A. Consider any parametrically definable $X \subseteq D$, say

$$X = \{x \in D \mid D \models \phi(x, d)\}$$

for some formula ϕ and some parameter $d \in D$. Fix some $c \in C - A$, and use the fact that D is a model of arithmetic to find an element $q \in D$ such that

$$D \models \forall x\ (2^x \text{ occurs in the binary representation of } q \leftrightarrow x < c \text{ and } \phi(x, d)).$$

Then $q < 2^c \in C$, so, as D is an end extension of C, $q \in C$. Since C is a conservative extension, and therefore an end extension, of A, the set

$$\begin{aligned} X \cap A &= \{x \in A \mid D \models x < c \text{ and } \phi(x, d)\} \\ &= \{x \in A \mid D \models 2^x \text{ occurs in the binary representation of } q\} \\ &= \{x \in C \mid C \models 2^x \text{ occurs in the binary representation of } q\} \cap A \end{aligned}$$

is parametrically definable in A.
(1)⇒(2). We write sentences of $L(B \cup C)$ as $\phi(\boldsymbol{b}, \boldsymbol{c})$, where $\phi(\boldsymbol{x}, \boldsymbol{y})$ is an L-formula, $\boldsymbol{b} \in B$, and $\boldsymbol{c} \in C$. (Boldface letters denote finite sequences.) Given such a sentence, we can find a parametric definition in A of $\{\boldsymbol{a} \in A \mid B \models \phi(\boldsymbol{b}, \boldsymbol{a})\}$, because B is a conservative extension of A. Thus, we have a formula $\psi(z, \boldsymbol{y})$ of L and a parameter $p \in A$ such that

$$(\forall \boldsymbol{a} \in A) B \models \phi(\boldsymbol{b}, \boldsymbol{a}) \leftrightarrow \psi(p, \boldsymbol{a}).$$

The sentence $\psi(p, \boldsymbol{c})$, which involves constants from C only, will be called a C-transform of $\phi(\boldsymbol{b}, \boldsymbol{c})$. If ψ' and p' are another such formula and parameter, for the same $\phi(\boldsymbol{b}, \boldsymbol{c})$, then the sentence $\forall \boldsymbol{y}(\psi(p, \boldsymbol{y}) \leftrightarrow \psi'(p', \boldsymbol{y}))$ is true in A, hence also in B and C. Therefore, we can unambiguously define a theory T as the set of those sentences $\phi(\boldsymbol{b}, \boldsymbol{c})$ whose C-transforms are true in C. Since C-transforms can be chosen to commute with propositional connectives and to preserve logical validity, it is easy to see that T is a complete consistent theory.
If $\phi(\boldsymbol{b})$ is in the complete diagram of B (so it contains no $\boldsymbol{c}$), then, in the construction of its C-transform, $\{a \in A \mid B \models \phi(\boldsymbol{b})\} = A$, so we can take a tautology as ψ, so the C-transform of $\phi(\boldsymbol{b})$ is true in C, so $\phi(\boldsymbol{b}) \in T$. Thus, T includes the complete diagram of B.
If $\phi(\boldsymbol{c})$ is in the complete diagram of C, then, in the construction of its C-transform, we can take $\psi(\boldsymbol{y})$ to be $\phi(\boldsymbol{y})$ (no parameter p is needed), so the C-transform of $\phi(\boldsymbol{c})$

is $\varphi(c)$ itself, which is true in C, so $\phi(c) \in T$. Thus, T also includes the complete diagram of C.

If $b \in B-A$ and $c \in C-A$, then, in the construction of the C-transform of $b \neq c$, $\{a \in A \mid B \models b \neq a\} = A$, so we can take ψ to be a tautology, and therefore $(b \neq c) \in T$.

Therefore, we obtain an amalgamation D of B and C over A by taking any model of T and passing to the elementary submodel consisting of denotations of closed terms. It remains to show that D is an end extension of C. Suppose $d \in D$, $c \in C$, and $d < c$. As $B \cup C$ generates D, we have $d = f_D(b, c)$ for some $b \in B$, $c \in C$. We may assume, thanks to a pairing function, that the two c's in the last two sentences are the same. Note that, as D is a model of the complete theory T, the sentence $f(b, c) < c$ is in T.

Let $\theta(z, y_1, y_2)$ be an L-formula and let $p \in A$ be a parameter such that

$$(\forall a_1, a_2 \in A) B \models f(b, a_1) = a_2 \leftrightarrow \theta(p, a_1, a_2)\,;$$

they exist because B is a conservative extension of A. It follows, since B is an end extension of A, that

$$(\forall a \in A) B \models f(b, a) < a \leftrightarrow (\exists v < a) \theta(p, a, v)\,,$$

so $(\exists v < c)\theta(p, c, v)$ is a C-transform of the sentence $f(b, c) < c$ which we know to be in T. So the C-transform holds in C, and we have an $e \in C$ satisfying $\theta(p, c, e)$. But this is the C-transform of $f(b, c) = e$, so this equation is in T, hence is true in D. Therefore, $d = f_D(b, c) = e \in C$. $\square$

If we weaken clause (2) of the theorem by demanding only that $C < B - A$, i.e., that all elements of C precede all elements of $B - A$ in D, then the condition is satisfied by any end extension; see [4], Theorem 3. However, with this weakening, uniqueness of the amalgamation becomes a characterization of conservativity.

Theorem 2. *Let B be an end extension of A. It is a conservative extension if and only if, for every extension C of A, there is only one (up to equivalence) amalgamation of B and C with $C < B - A$.*

Proof. "Only if". The result is trivial if $A = B$, so we assume that B is a proper conservative extension of A and that D is an amalgamation of B and C over A with $C < B - A$. According to the remarks following the definition of equivalence, it suffices to show that, for any L-formula, $\phi(x, y)$, any $b \in B$ and any $c \in C$, the truth value of $\phi(b, c)$ in D can be determined from data in B and C without referring to D.

Let us therefore consider some particular ϕ, b, c as above. Using conservativity, we find a formula $\psi(z, y)$ and a parameter $p \in A$ such that

$$B \models \phi(b, a) \leftrightarrow \psi(p, a) \qquad (*)$$

for all $a \in A$ and therefore for all $a < q$, where $q \in B - A$ is the first point where $(*)$ fails (or an arbitrary point if $(*)$ never fails). Thus, the sentence

$$(\forall y < q)\,[\phi(b, y) \leftrightarrow \psi(p, y)]$$

is true in B, hence also in D. But, in D, $c<q$ because $C<B-A$. So $\phi(b,c)$ is equivalent in D to $\psi(p,c)$. This means that the truth value of $\phi(b,c)$ can be determined by first finding ψ and p (which can be done in B) and then evaluating $\psi(p,c)$ (which can be done in C, as C is an elementary submodel of D).

"If". Suppose that B is a non-conservative end extension of A, so there are a formula $\phi(x,y)$ and a parameter $b\in B$ such that no formula $\psi(x,y)$ and parameter $p\in A$ satisfy

$$(\forall a\in A)B\models\phi(b,a)\leftrightarrow\psi(p,a).$$

Fix such φ and b. Consider sets $\Gamma(x)$ of $L(A)$-formulas with only x free (i.e., not-necessarily-complete 1-types over A) such that

$$\begin{gathered}\text{For all finite conjunctions } \theta(x) \text{ of formulas in } \Gamma(x),\\ \text{for all standard functions } f, \text{ for all formulas } \psi(x) \text{ in } L(A),\\ \text{for all parameters } \mathbf{p}\in A, \text{ and for all } q\in B-A,\\ B\models\exists x[\theta(x)\wedge f(\mathbf{p},x)<q\cap\neg(\phi(b,x)\leftrightarrow\psi(x))].\end{gathered}\qquad (*)$$

By our choice of ϕ and b, $(*)$ is true when $\Gamma(x)$ is the empty set, since an appropriate x can be always be found in A. [The clause $f(\mathbf{p},x)<q$ will hold because B is an end extension of A.] Also, the union of a chain of such sets $\Gamma(x)$ is again such a set. By Zorn's lemma, let $\Gamma(x)$ be a maximal set satisfying $(*)$.

We shall show that $\Gamma(x)$ is a complete type over A; note that its consistency is obvious by $(*)$. If $\alpha(x)$ were an $L(A)$-formula, with only x free, such that neither $\alpha(x)$ nor its negation were in $\Gamma(x)$, then the maximality of $\Gamma(x)$ would provide us with two finite conjunctions $\theta^+(x)$ and $\theta^-(x)$ of formulas in $\Gamma(x)$, two standard functions f^+ and f^- with parameters $\mathbf{p}^+$ and $\mathbf{p}^-$ in A, two $L(A)$-formulas ψ^+ and ψ^-, and two elements q^+ and q^- of $B-A$ such that

$$B\models\forall x[\theta^\pm(x)\wedge\pm\alpha(x)\wedge f^\pm(\mathbf{p}^\pm,x)<q^\pm\rightarrow(\phi(b,x)\leftrightarrow\psi^\pm(x))],$$

where $+\alpha$ means α and $-\alpha$ means $\neg\alpha$. Let θ be the conjunction of θ^+ and θ^-, let $f(\mathbf{p},x)$ be the maximum of the $f^\pm(\mathbf{p}^\pm,x)$, let q be the smaller of $q^\pm$, and let ψ be $(\psi^+\wedge\alpha)\vee(\psi^-\wedge\neg\alpha)$. Then

$$B\models\forall x[\theta(x)\wedge f(\mathbf{p},x)<q\rightarrow(\phi(b,x)\leftrightarrow\psi(x))],$$

contrary to $(*)$ for $\Gamma(x)$. Therefore, $\Gamma(x)$ is complete.

Let C be an extension of A generated over A by a single element c that realizes the type $\Gamma(x)$ over A. We shall obtain two amalgamations of B and C over A, with $C<B-A$, that will be inequivalent because one will satisfy $\phi(b,c)$ while the other will not. To find these amalgamations, it suffices to prove the consistency of the two theories $T^\pm=$

$$\begin{gathered}\text{Complete diagram of } B\cup\\ \Gamma(c)\cup\{f(\mathbf{p},c)<q\mid\mathbf{p}\in A,\ q\in B-A,\ f \text{ standard}\}\cup\{\pm\phi(b,c)\}.\end{gathered}$$

To prove the consistency of any finite subtheory of $T^\pm$, we may assume (by taking the maximum of the f's and the minimum of the q's) that the subtheory contains only one inequality $f(p,c)<q$. Its consistency is then given by (∗), with $\psi=\mp$tautology. Therefore, by compactness, $T^{\pm}$ is consistent, and the proof is complete. □

Corollary. *Let B be an end extension of A. It is conservative if and only if, for every $C\supseteqq A$, if D is an amalgamation of B and C over A with $C<B-A$, then D is an end extension of C.*

Proof. If B is a conservative extension of A and C is any extension of A, then Theorem 1 provides an amalgamation D that is an end extension of C. It clearly satisfies $C<B-A$, and, by Theorem 2, it is the only amalgamation satisfying $C<B-A$, up to equivalence.
Conversely, recall (Theorem 3 of [4]) that we can always amalgate B and C over A with $C<B-A$, since B is assumed to be an end extension of A. If this amalgamation is necessarily an end extension, then B is a conservative extension of A, by Theorem 1. □
For applications of Theorem 2 to combinatorial properties of ultrafilters, see [5].

Minimal and Ramsey Extensions

In this section, we consider some combinatorial properties of ultrafilter mappings introduced by Baumgartner [1]. Let U and V be ultrafilters over N, and let f map U to V; this means that $f:N\to N$ and, for each $X\subseteqq N$, $X\in V$ if and only if $f^{-1}(X)\in U$. Such a map induces an elementary embedding of ultrapowers

$$f^*:V\text{-prod}\,N\to U\text{-prod}\,N:[g]_V\to[g\circ f]_U,$$

where the brackets mean equivalence class modulo the indicated ultrafilter.
Baumgartner calls a map of f of U to V *selective* if, for any function g on N, there is a set $X\in U$ such that either g is constant on each of the fibers $X\cap f^{-1}\{n\}$ or g is one-to-one on each of these fibers. (If g is constant on some of the fibers and one-to-one on others, a smaller $X\in U$ will satisfy the definition.) When f is constant and V is therefore principal, selectivity of f reduces to the usual definition of selectivity of U: every function on N is constant or one-to-one on some set in U. In general, f is selective if and only if there is no model A of arithmetic such that $f^*(V-\text{prod}\,N)\subsetneqq A\subsetneqq U-\text{prod}\,N$. To see this, it suffices to note that a-function g on N is fiberwise constant on a set in U if and only if $[g]_U\in f^*(V-\text{prod}\,N)$ and that g is fiberwise one-to-one on a set in U if and only if the canonical generator $[\text{id}]_U$ of $U-\text{prod}\,N$ is obtainable, by a standard binary function, from $[g]_U$ and $[f]_U=f^*[\text{id}]_V$, which means that $[g]_U$ and $f^*(V-\text{prod}\,N)$ together generate $U-\text{prod}\,N$. Thus, selective maps of ultrafilters correspond to minimal extensions of ultrapowers. For detailed information about such minimal extensions, see the thesis [11] of Eck.
We call f a *Ramsey* map from U to V if, whenever the set $[N]^2$ of two-element subsets of N is partitioned into two parts, there is a set $X\in U$ such that all the two-

element subsets $\{x, y\} \subseteq X$ with $f(x) = f(y)$ lie in the same part. In Baumgartner's terminology [1], this would be called a $[\ ]^2_2$-projection. When f is constant, the definition reduces to U being a Ramsey ultrafilter. In general, the definition can be reformulated as follows. Let

$$E_f = \{(x, y) \in N^2 \mid x < y \text{ and } f(x) = f(y)\};$$

then the filter generated by E_f and the sets X^2 with $X \in U$ is an ultrafilter. This condition can also be expressed model-theoretically. Consider two copies of $U - \operatorname{prod} N$, say A_1 and A_2 with isomorphisms $\alpha_i : U - \operatorname{prod} N \to A_i$ such that their intersection is at least $f^*(V - \operatorname{prod} N)$, in the sense that the restrictions of α_1 and α_2 to $f^*(V - \operatorname{prod} N)$ are equal. We do not exclude the possibility that $\alpha_1(a) = \alpha_2(b)$ for some a and b outside the range of f^*. An amalgamation of the two models A_i (over their intersection) is determined (up to equivalence) by the 2-type realized by the two generators $a_i = \alpha_i([\mathrm{id}]_U)$ of the A_i's. If we assume that $a_1 < a_2$ in the amalgamation, then this 2-type is an ultrafilter on N^2 containing E_f and X^2 for all $X \in U$, and any such ultrafilter arises from some amalgamation. Thus, to say that the sets E_f and X^2 for $X \in U$ generate an ultrafilter is to say that all such amalgamations are equivalent. Therefore, f is a Ramsey map if and only if there is at most one amalgamation of two copies of $U - \operatorname{prod} N$, with intersection at least $f^*(V - \operatorname{prod} N)$, and with the two generators a_i properly ordered ($a_1 < a_2$). (There is always at least one such amalgamation, unless f is an isomorphism.)

It follows easily from either the combinatorial definitions or the model-theoretic characterizations that all Ramsey maps are selective. For constant maps, the converse holds, by Kunen's result [6] that all selective ultrafilters are Ramsey. Baumgartner [1] has observed that the converse fails in general. We shall show, in the next theorem, that the converse holds if $U - \operatorname{prod} N$ is a conservative extension of $f^*(V - \operatorname{prod} N)$. Since all extensions of N are conservative, this result gives a new proof of Kunen's theorem. We shall also show that the conservativity assumption is necessary unless f is finite-to-one [i.e., unless $f^*(V - \operatorname{prod} N)$ is cofinal in $U - \operatorname{prod} N$].

Theorem 3. *Let $A \subsetneq B$ be models of arithmetic. Then the following are equivalent.*
(1) *B is a conservative minimal extension of A.*
(2) *A is not cofinal in B, and there is only one amalgamation of two copies of B, with intersection at least A but not all of B, up to equivalence and up to interchange of the two copies of B.*

Proof. (1)⇒(2). Assume (1). Then B is a proper end extension of A, so A is not cofinal in B. Consider any amalgamation D of two copies B_1 and B_2 of B, as in (2). If b is an arbitrary element of $B - A$, its images $b_1 \in B_1$ and $b_2 \in B_2$ are distinct in D, because otherwise the intersection of the two copies of B would include A as well as b and would therefore be all of B, by minimality. Interchanging the two copies of B if necessary, we may assume that $b_1 < b_2$ for a certain, henceforth fixed, $b \in B - A$.

Minimality of B implies that $B-A$ is a single sky, for if it contained two distinct skies then the lower of them, together with all skies below it, would constitute a model intermediate between A and B. (See [10] or [4] for the definition of sky; the latter reference contains all the facts about skies and amalgamations that will be needed here.) The two copies of this sky cannot lie in the same sky of D, for, if they did, they would have to intersect by [2], Theorem 1, and then the intersection of the two copies of B would be a model intermediate between A and B. Therefore, $B_1 - A < B_2 - A$ in D. Applying Theorem 2, with B_1 as C and B_2 as B, we find that the amalgamation D is unique.

(2)⇒(1). Assume (2). We show first that B is a minimal extension of A. Otherwise, if $A \subsetneq C \subsetneq B$, then Theorem 1 of [4] provides an amalgamation as in (2) with intersection precisely A and another with intersection C, contrary to (2).

The initial segment of B that contains A as a cofinal subset is a submodel of B. It is not B, by the first clause of (2), so, by the minimality just proved, it must be A. So B is an end extension of A.

Now let $\phi(x, y)$ be any L-formula and let $b \in B$. We shall find an $L(A)$-formula $\psi(x)$ such that

$$(\forall a \in A) B \models \phi(a, b) \leftrightarrow \psi(a).$$

This will establish the conservativity of B and thus complete the proof. If $b \in A$, then $\phi(x, b)$ is the required $\psi(x)$, so we assume $b \in B - A$.

Let D be the unique amalgamation of B_1 and B_2 as in (2), and let b_i be the image of b in B_i; we may assume $b_1 < b_2$. If an L-formula $\eta(u, v)$ is satisfied in D by the pair (b_1, b_2), then there is an $L(A)$-formula $\theta(u)$ satisfied in B by b such that

$$A \models \forall u, v [\theta(u) \wedge \theta(v) \wedge u < v \rightarrow \eta(u, v)],$$

for otherwise a compactness argument would produce an amalgamation as in (2) in which $\eta(b_1, b_2)$ is false.

If

$$D \models \forall x [\phi(x, b_1) \leftrightarrow \phi(x, b_2)],$$

then, by the preceding comment, we can find an $L(A)$-formula $\theta(u)$, satisfied in B by b, such that

$$A \models \forall u, v [\theta(u) \wedge \theta(v) \wedge u < v \rightarrow \forall x (\phi(x, u) \leftrightarrow \phi(x, v))].$$

Then the same sentence holds in the elementary extension B of A. If we take b as v and any element $a \in A$ satisfying θ as u, the clauses $\theta(u)$, $\theta(v)$ and $u < v$ are satisfied (the last because B is an end extension of A), so $B \models \forall x [\phi(x, a) \leftrightarrow \phi(x, b)]$. Of course such an a exists as $\exists x \theta(x)$ holds in B, hence in A. So we can take $\phi(x, a)$ as our $\psi(x)$.

Suppose, therefore, that the assumption of the preceding paragraph fails, so there is a smallest element $d \in D$ for which $\phi(d, b_1)$ and $\phi(d, b_2)$ are inequivalent; assume for definiteness that $\phi(d, b_1)$ is false but $\phi(d, b_2)$ is true. So $D \models \eta(b_1, b_2)$, where $\eta(u, v)$ is

$$\exists x[\neg \phi(x,u) \wedge \phi(x,v) \wedge (\forall y<x)(\phi(y,u) \leftrightarrow \phi(y,v))].$$

Therefore, there is an $L(A)$-formula $\theta(u)$, satisfied in B by b, such that

$$A \models \forall u, v[\theta(u) \wedge \theta(v) \wedge u<v \rightarrow \eta(u,v)]. \qquad (*)$$

As u ranges over the elements satisfying θ in A, in increasing order, the truth value of $\phi(a,u)$, for some fixed $a \in A$, can oscillate, but if the truth values of $\phi(c,u)$ are no longer varying for any $c<a$, then $(*)$ says that the truth value of $\phi(a,u)$ can change only from false to true, not vice versa. Hence, $\phi(a,u)$ can change truth value at most once after the truth values of $\phi(c,u)$ have stabilized for all $c<a$. Formalizing this argument, we obtain a proof, by internal induction in A, that the truth value of $\phi(a,u)$ must eventually stabilize. That is, for each $a \in A$ there is $a' \in A$ such that the sentence

$$\forall u, v[\theta(u) \wedge \theta(v) \wedge a'<u \wedge a'<v \rightarrow (\phi(a,u) \leftrightarrow \phi(a,v))]$$

holds in A. It must therefore also hold in B, and we can satisfy the clauses $\theta(v)$ and $a'<v$ by taking b as v. Thus, an $a \in A$ satisfies $\phi(a,b)$ if and only if it satisfies $\varphi(a,u)$ for all sufficiently large u satisfying $\theta(u)$. But this criterion for $\varphi(a,b)$ does not involve any parameters from $A-B$, so it can be checked in A. This means that we can take $\exists z \forall u[\theta(u) \wedge z<u \rightarrow \phi(x,z)]$ as $\psi(x)$ and the proof is complete. □

Minor changes in the first half of the proof would yield a stronger form of (2), where n copies of B are amalgamated, as a consequence of (1). This stronger form is, in the case of ultrapowers, equivalent to Baumgartner's definition of an $[\ ]^n_2$-projection, so we have, as a corollary, Baumgartner's result that a $[\ ]^2_2$-projection that isn't finite-to-one is an $[\ ]^n_2$-projection for all finite n.

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