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THE ELASTIC CHARACTERISTICS OF THREE-PLY ORTHOTROPIC STRUCTURES

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NOMENCLATURE

English Letters:

a_{ij} Constants associated with generalized Hooke's law, using properties based on cord tension.

$T = t/t_R$ Ratio of ply thicknesses

E, F, G Elastic constants for orthotropic laminates with cords in tension.

E^T, F^T, G^T Elastic constants for three ply orthotropic structure

R Used as subscript and superscript to denote radial or third ply

$N = G_{xy}/E_x$

$N^* = G_{xy}/E_x^R$

$N^{**} = G_{xy}^R/G_{xy}$

x, y, z Orthogonal co-ordinates aligned along and normal to the cord direction.

t ply thickness

Greek Letters:

α One-half the included angle between cords in adjoining plies in a two-ply laminate.

ϵ Strain

μ Poisson's ratio

ξ, η, ζ Orthogonal co-ordinates aligned along and normal to the principal axes of elasticity, or orthotropic axes, in an orthotropic laminate.

σ Stress

σ' Interply stress

I. FOREWORD

Preliminary investigations carried out by the Tire and Suspension Systems Research Group from 1960 through 1963 centered around the determination of plane elastic characteristics of orthotropic structures, and this information eventually has become useful in the analysis of pneumatic tires. Most of the theoretical and experimental work on these properties was restricted to two and four-ply bias angle structures. Recent commerical developments have prompted an interest in the behavior of three-ply orthotropic structures, in which two plies are arranged such that their cords form angles of $\pm \alpha$ with a vertical axis, while the third ply is located such that its cords are perpendicular to this axis.

The general solution for plane elastic characteristics of orthotropic structures presented in Ref. 1 is completely applicable to the three-ply orthotropic structure. Thus, interply stresses as well as elastic properties can be investigated in the same light as previous discussions. In addition, a special case of the three-ply structure can be analyzed where the third ply is composed of a homogeneous, isotropic material. This can be used as a special model for investigating the internal reaction of a two-ply structure by treating the layer of material between the layers of cords as the third ply. This results in a somewhat different view of interply stresses.

II. SUMMARY

This report presents the expressions for the plane elastic characteristics of three-ply orthotropic laminates. The laminated structures considered are those consisting of two identical plies whose cords are separated by an included angle of 2α and a single ply whose cords are perpendicular to the bisector of the angle separating the plies.

In all cases the expression for an elastic stiffness of the total three-ply structure is a linear combination of the same property of a two-ply structure, equivalent to the two angular plies, and an elastic property of the third ply. Thus, the elastic properties can be computed having only a knowledge of the material and geometric properties of the individual plies.

Unlike the two-ply structure, whose elastic properties are described by two independent parameters, the properties of the three-ply structure are described by five independent parameters. This indicates there are many possible combinations of material and constructional properties for varying the elastic behavior of a three-ply orthotropic structure. Graphical representation of some of the more common combinations of the five parameters is included, along with several examples of their use. In addition, a graphical comparison is included showing the effect on the elastic properties of the inclusion of the third ply.

The behavior of interply shear stresses in the three-ply structure are discussed in the same manner as previous discussions by this group Ref. 3.

The form of the expressions for these stresses is identical to that of two-ply structures although the actual magnitudes can be different.

A special case of the three-ply structure, where the third ply is homogeneous and isotropic, is discussed in some detail; it provides a new means for examining a two-ply structure since the isotropic ply can represent the layer of material between the layers of cord. As a result of this investigation, another view-point is provided from which to discuss the origin and nature of interply stresses.

III. THEORETICAL DEVELOPMENT

In this section the plane elastic properties of a three-ply orthotropic structure will be developed in a manner similar to that presented in Ref. 1. Some of the material presented here is repeated merely for completeness and continuity.

An orthotropic structure is one exhibiting three mutually orthogonal planes of elastic symmetry. As has been pointed out in previous discussions, for laminated cord-rubber structures to exhibit othotropy there must be no elastic coupling between normal and shearing effects. This structure normally arises when the cords in all plies of a symmetrical structure are either in tension or in compression. Thus, in this development it will be assumed that the cords in all plies are in tension.

The three-ply structure considered here is that in which two identical plies are arranged so that their cords are separated angularly by an included angle 2α, while the third ply, not necessarily of the same construction, is located such that its cords are perpendicular to the bisector of the angle between the other two plies, as shown in Fig. 1. In that illustration, the heavy diagonal lines representing typical cords of the two identical plies are shown being bisected by coordinates ξ and η, two arms of the orthogonal ξ, η, ζ system.

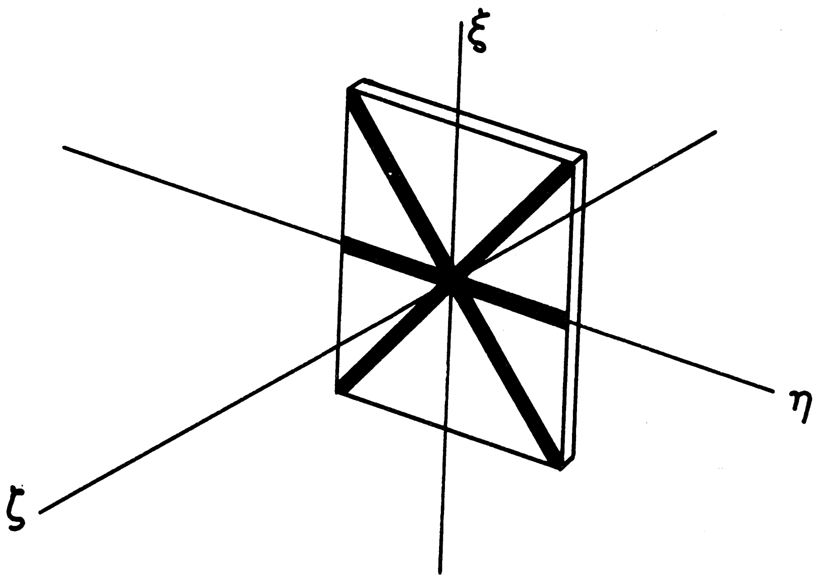


Fig. 1. Schematic view of a small section of a three-ply laminate.

This report will be confined to those structures in which effects in the ζ direction are negligibly small compared eith those in the ξ and η directions. This assumption will generally be true since this report is limited to structures which are large in the ξ and η directions compared to their thickness in the ζ direction.

If the three-ply structure is imagined to carry an applied load as shown in Fig. 2, then a strength of materials approach can be used to find the plane

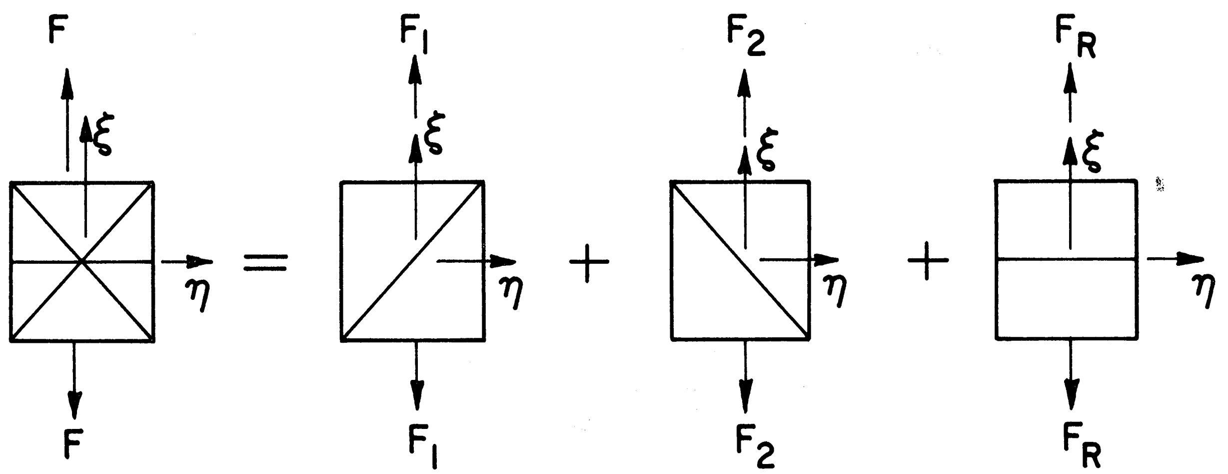


Fig. 2. Schematic representation of load distribution resulting from an applied load in the ξ direction.

elastic properties of the combined structure.

$$F = F_1 + F_2 + F_R$$

However, plies 1 and 2 are identical and their cords are assumed to be in tension:

$$F = 2F_1 + F_R \doteq 2\sigma_1 t\ell + \sigma_R t_R \ell$$

$$\sigma_\xi^T = \frac{F}{A} = \frac{(2\sigma_1 t + \sigma_R t_R)}{(2t + t_R)}$$

where:

$$\sigma_1 = E_1 (\epsilon_1)_\xi$$

$$\sigma_R = E_R (\epsilon_R)_\xi$$

Referring to Ref. 1, it is noted that E_1 is simply the extension modulus in the ξ-direction of a single ply of cord-rubber material while E_R is the extension modulus perpendicular to the cords in the third ply, denoted as E_y in previous discussions. Therefore:

$$E_1 = E_\xi$$

$$E_R = E_y^R$$

Also, from compatibility of strain:

$$(\epsilon_1)_\xi = (\epsilon_R)_\xi = \epsilon_\xi^T$$

Thus,

$$E_\xi^T = \frac{\sigma_\xi^T}{\epsilon_\xi^T} = \frac{2E_\xi t + E_y^R t_R}{(2t + t_R)} \qquad (1)$$

Carrying out similar arguments for the relationship between stress in the η-direction and strain in the η-direction, stress in the ξ-direction and strain in the η-direction or stress in the η-direction and strain in the ξ-direction, and shear stress and shear strain, the following expressions are obtained for E^T_η, $F^T_{\xi n}$, and $G^T_{\xi n}$, respectively:

$$E^T_\eta = \frac{2E_\eta t + E^R_x t_R}{(2t + t_R)} \tag{2}$$

$$F^T_{\xi\eta} = \frac{2F^T_{\xi\eta} t + F^R_{xy} t_R}{(2t + t_R)} \tag{3}$$

$$G^T_{\xi\eta} = \frac{2G^T_{\xi\eta} t + G^R_{xy} t_R}{(2t + t_R)} \tag{4}$$

In Eqs. (1) - (4), E^T_ξ, E^T_η, $F^T_{\xi\eta}$ and $G^T_{\xi\eta}$ refer to the elastic properties of the total three-ply structure. It is emphasized that E_ξ, E_η, $F_{\xi\eta}$ and $G_{\xi\eta}$ are exactly the same elastic constants obtained in Ref. 1 and, thus, are calculated as before. On the other hand, E^R_x, E^R_y, F^R_{xy} and G^R_{xy} are the elastic constants associated with a single ply in the directions along and perpendicular to the cords and thus are identical to those discussed in Ref. 5. As a result of these relations, it is clear that in all cases the elastic property of the three-ply structure is a linear combination of the same elastic property of the two identical plies and an elastic property of the third ply.

For computational convenience Eqs. (1) - (4) are made dimensionless by dividing by G_{xy}, the shear modulus in the x-y plane of a single sheet of the material from which the two identical plies are constructed.

$$\frac{E^T_\xi}{G_{xy}} = \frac{2\left(\frac{E_\xi}{G_{xy}}\right)t + \left(\frac{E^R_y}{G_{xy}}\right)t_R}{(2t + t_R)}$$

$$\frac{E^T_\eta}{G_{xy}} = \frac{2\left(\frac{E_\eta}{G_{xy}}\right)t + \left(\frac{E^R_x}{G_{xy}}\right)t_R}{(2t + t_R)}$$

$$\frac{F^T_{\xi\eta}}{G_{xy}} = \frac{2\left(\frac{F_{\xi\eta}}{G_{xy}}\right)t + \left(\frac{F^R_{xy}}{G_{xy}}\right)t_R}{(2t + t_R)} \qquad (5)$$

$$\frac{G^T_{\xi\eta}}{G_{xy}} = \frac{2\left(\frac{G_{\xi\eta}}{G_{xy}}\right)t + \left(\frac{G^R_{xy}}{G_{xy}}\right)t_R}{(2t + t_R)}$$

References 2 and 4 discuss the origin of the following ratios which will be used in the remainder of this report:

$$\frac{E_x}{F_{xy}} = 0.5; \quad \frac{G_{xy}}{E_y} = 0.25; \quad \frac{G_{xy}}{E_x} = N; \quad N = 10^{-2}, 10^{-3}, 10^{-4}$$

The numerical values of these ratios of elastic constants are based on some experimentally substantiated assumptions concerning the elastic behavior of individual sheets of ordinary ply stock. For instance, it is assumed that the extension in the direction perpendicular to the cords and the shearing deformation in the x-y plane take place primarily in the rubber between the cords. In effect this is implying that the cords are indefinitely rigid in comparison with the rubber. It is also assumed that the rubber used is nearly incompressible and thus exhibits a Poisson's ratio of approximately 0.5.

From these values the following expressions can be deduced:

$$\frac{F_{xy}}{G_{xy}} = \frac{2}{N}; \quad \frac{E^R_x}{F^R_{xy}} = 0.5; \quad \frac{G^R_{xy}}{E^R_y} = 0.25$$

Also let:

$$N^{*} = \frac{G_{xy}}{E_x^R} \; ; \quad N^{**} = \frac{G_{xy}^R}{G_{xy}} \, , \quad T = {}^{t}/_{t_R}$$

Then:

$$\frac{G_{xy}}{F_{xy}^R} = \frac{N^*}{2} \; ; \quad \frac{G_{xy}}{E_y^R} = \frac{1}{4N^{**}}$$

Using these values and expressions, Eqs. (5) become:

$$\frac{E_\xi^T}{G_{xy}} = \frac{2\left(\frac{E_\xi}{G_{xy}}\right)T + 4N^{**}}{(2T + 1)}$$

$$\frac{E_\eta^T}{G_{xy}} = \frac{2\left(\frac{E_\eta}{G_{xy}}\right)T + \left(\frac{1}{N^*}\right)}{(2T + 1)} \qquad (6)$$

$$\frac{F_{\xi\eta}^T}{G_{xy}} = \frac{2\left(\frac{F_{\xi\eta}}{G_{xy}}\right)T + \left(\frac{2}{N^*}\right)}{(2T + 1)}$$

$$\frac{G_{\xi\eta}^T}{G_{xy}} = \frac{2\left(\frac{G_{\xi\eta}}{G_{xy}}\right)T + N^{**}}{(2T + 1)}$$

Again referring to Ref. 2, it is seen that the first bracketed expression in the numerator of each property in Eqs. (6) depends on two independent parameters, the cord half angle α and the ratio N. Thus the plane elastic properties of the three-ply orthotropic structure depend on five independent parameters, α, N, N*, N** and T.

Many different combinations of the parameters can be used in computing the properties given by Eqs. (6). It is noted that the ratios (E_ξ/G_{xy}), (E_η/G_{xy}), $(F_{\xi\eta}/G_{xy})$ and $(G_{\xi\eta}/G_{xy})$ required in these computations can be obtained directly from the tables of Ref. 2. As has been mentioned before, this is completely acceptable since these expressions, as used in Eqs. (6), are identical to those described in Ref. 2.

The results of specific combinations of the independent parameters are illustrated graphically in Figs. 4, 6, and 7. Figure 4 shows the behavior of the overall elastic properties of the three-ply structure for the special case of all plies being identical construction. Even in this simplified structure it is interesting to note the variety of changes that can be obtained in the elastic properties by varying different parameters. Consider for example, a three-ply, thin-walled cylindrical tube such as shown in Fig. 3. Assume that

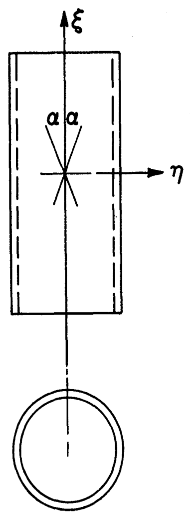


Fig. 3. Three-ply cylindrical tube.

all of its plies are of the same construction, and the angular lay up is $+\alpha$, $-\alpha$, r. Although this is not a pneumatic tire, the general effects experienced by this tube will be similar to those experienced by a tire. Thus, to make this example somewhat more realistic a cord half-angle of 10° is chosen to simulate the approximate angle used for the breaker plies in many radial ply tires. In addition, it will be assumed that the tube has a textile-rubber combination such that $N = 10^{-3}$ (a reasonable combination for certain carcass combinations). The numerical values of the elastic properties can be obtained from Fig. 4. Suppose, however, that the real interest is not in the actual values but in what will happen to these properties as certain parameters are changed. For example, what will happen if the cord half-angle is increased from 10° to 20°? From Fig. 4 it is seen that (E^T_ξ/G_{xy}), a measure of resistance to deformation in the ξ-direction (circumferential direction in the tire), decreases by a factor of approximately 5; $(G^T_{\xi\eta}/G_{xy})$, a measure of shear resistance in the ξ-η plane, increases by a factor of approximately 3.5; while (E^T_η/G_{xy}), resistance to deformation in η-direction (radial direction in the tire) remains constant and $(F^T_{\xi\eta}/G_{xy})$, a measure of the Poisson ratio effect, decreases only about 10%.

In another case, suppose the angle is to remain a constant but a stiffer cord is contemplated, a cord such that $N = 10^{-4}$ (a reasonable value for certain wire cord-rubber combinations). What effect will this change have on the elastic properties? Again from Fig. 4, it is seen that (E^T_ξ/G_{xy}) increases by a factor of approximately 3.5; $(G^T_{\xi\eta}/G_{xy})$ increases by a factor of approximately 10; (E^T_η/G_{xy}) increases by a factor of approximately 10; and $(F^T_{\xi\eta}/G_{xy})$ increases by a factor of approximately 10.

Other combinations and variations could be included here if space and time were available. However, it is clear from this simple illustration that large variations in elastic properties of three-ply structures of the kind described above can be accomplished with relatively simple changes in the parameters.

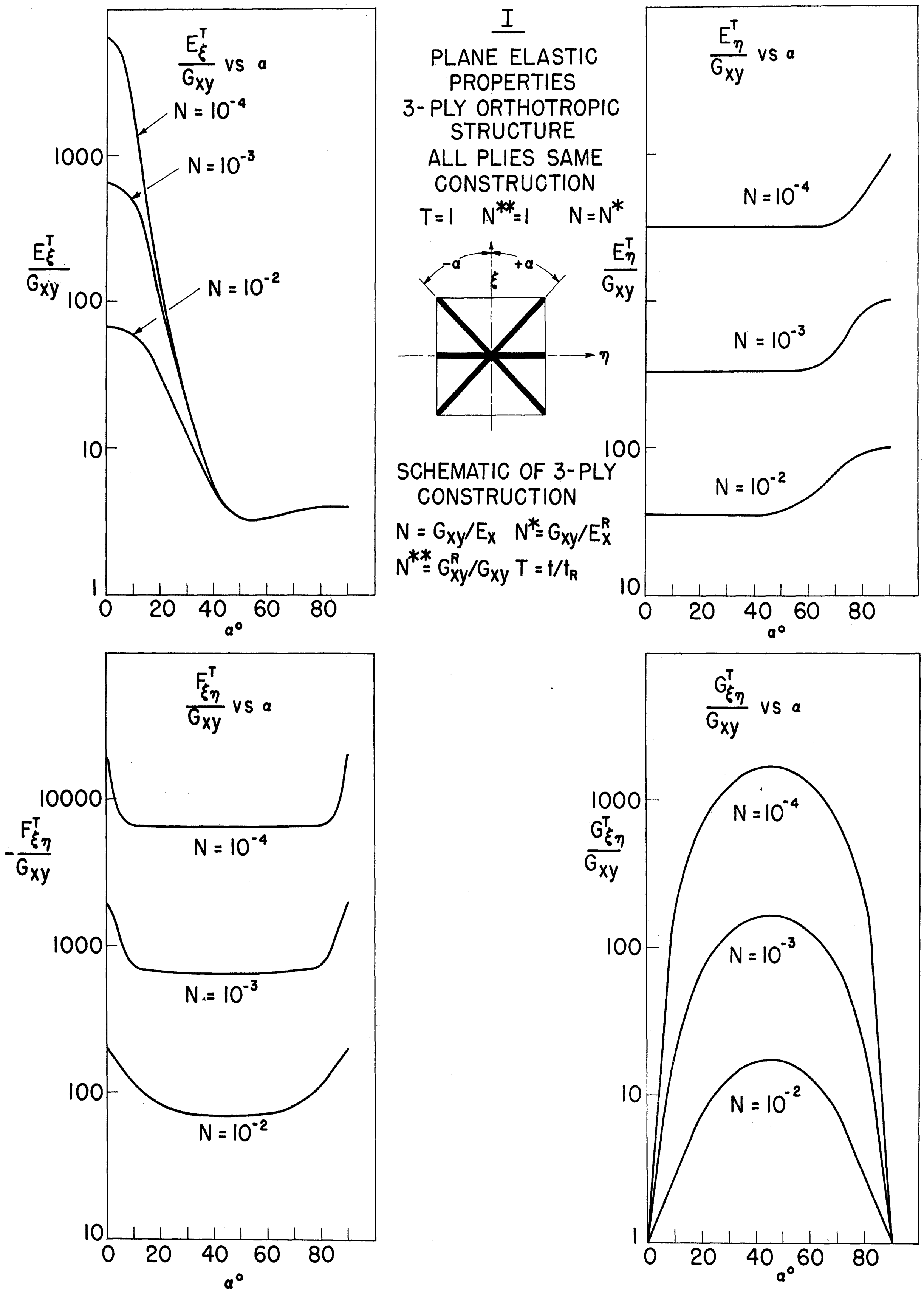


Fig. 4. Plane elastic properties of a three-ply orthotropic structure with all plies of the same material. N = N*, N** = 1, T = 1.

For the case under consideration, namely all plies being of the same construction, it is interesting to note the effect which the inclusion of the third ply has had on the overall properties of the structure. Figure 5 is a graphical comparison of the plane elastic properties of a two-ply structure and a three-ply structure in which all plies have been obtained from the same ply stock. As can be seen (E^T_ξ/G_{xy}) and (E_ξ/G_{xy}) are very nearly the same except at very small cord angles, (E^T_η/G_{xy}) and (E_η/G_{xy}) are very different except at very large cord angles, $(F^T_{\xi\eta}/G_{xy})$ and $(F_{\xi\eta}/G_{xy})$ are very different except at very small cord angles and very large cord angles, $(G^T_{\xi\eta}/G_{xy})$ and $(G_{\xi\eta}/G_{xy})$ are very nearly the same except when the cord half-angle is approximately 45°. The result of this seems to be that, in general, the inclusion of the third ply (when all plies are of the same construction) slightly decreases the resistance to deformation in the ξ-direction, slightly decreases the resistance to shear in the ξ-η plane, greatly increases the resistance to deformation in the η-direction and greatly increases the resistance due to lateral contraction effects. These comparisons illustrate some of the advantages and disadvantages that can be encountered in using two and three-ply orthotropic structures.

Another construction worth observing is that illustrated in Fig. 6 in which the third ply, or radial ply in a tire, is of weaker construction than the two symmetrical plies. This simulates the usual construction encountered in a radial ply tire in which the radial ply is constructed with textile cords and the breaker plies are reinforced with steel wire cords.

Again consider the tube illustrated in Fig. 3, choosing a cord half-angle of 10°. It will be assumed that the third ply is reinforced with a relatively

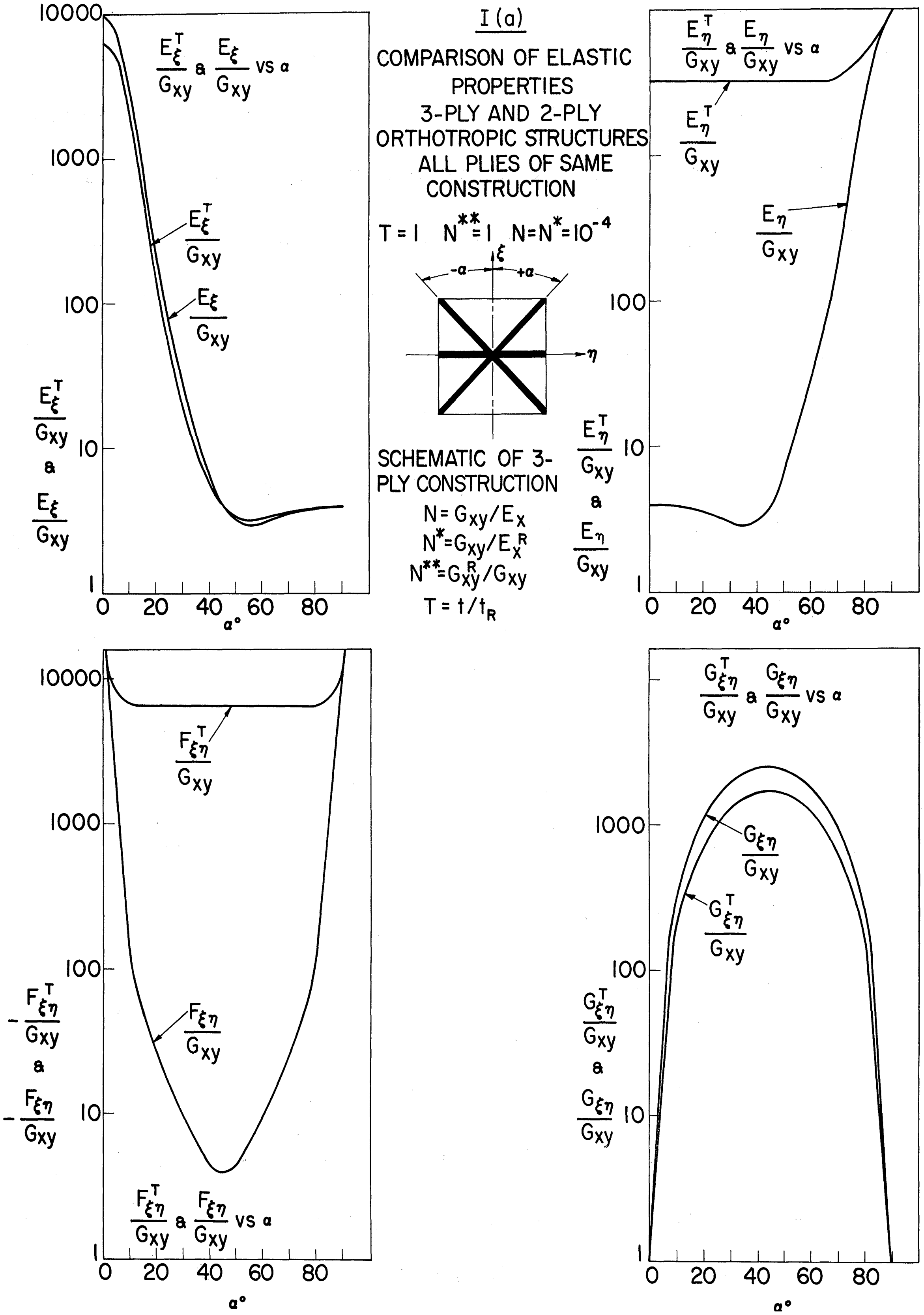


Fig. 5. Comparison of plane elastic properties of a two-ply and three-ply orthotropic structure - all plies of the same construction. $T = 1$, $N^{**} = 1$, $N = N^{*} = 10^{-4}$.

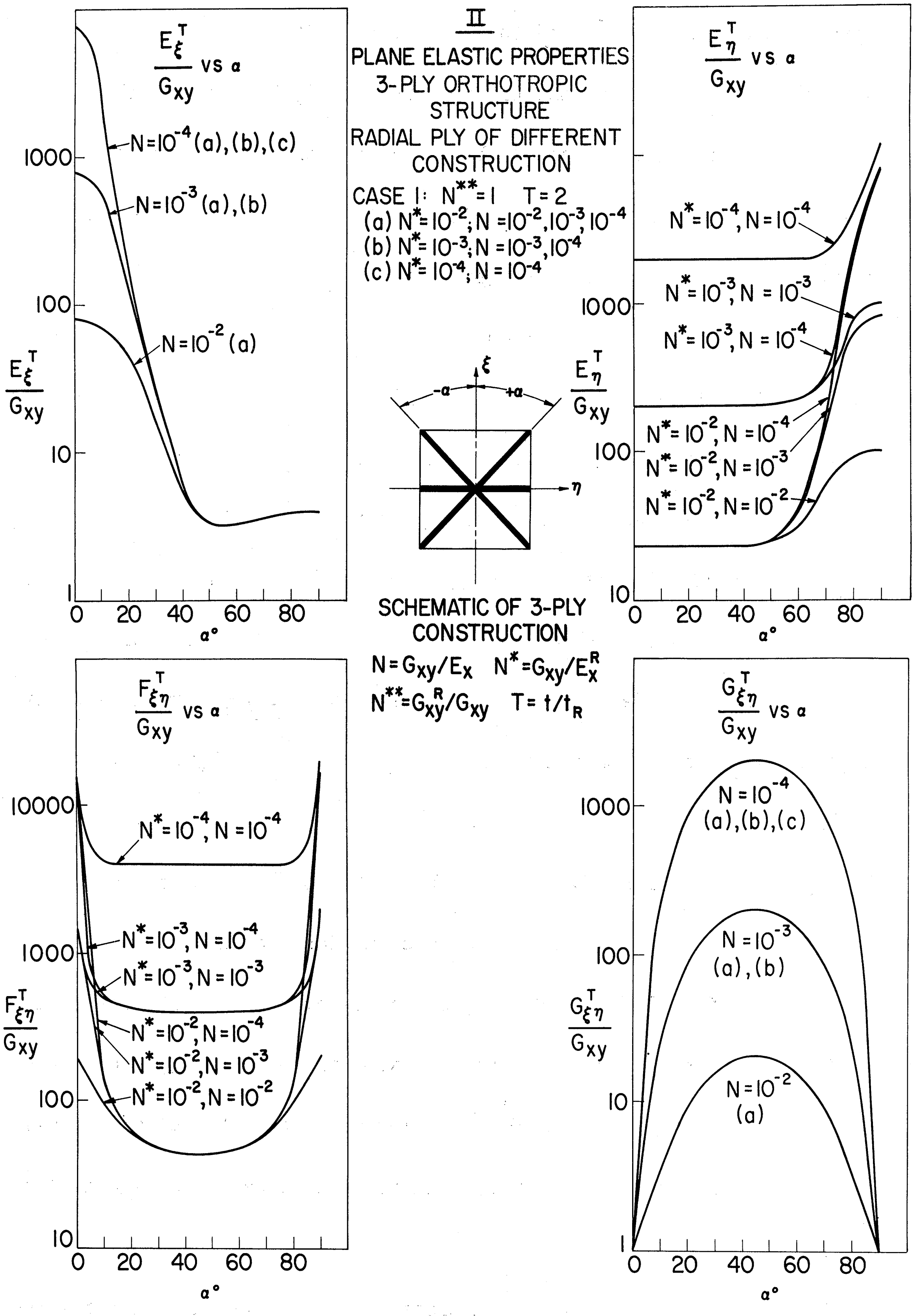


Fig. 6. Plane elastic properties of a three-ply orthotropic structure - third ply of different construction. Case 1: N** = 1, T = 2.

weak cord such that $N^* = 10^{-2}$ and the symmetrical plies are such that $N = 10^{-3}$. Also assume that the ratio of the ply thicknesses, T, becomes 2. The numerical values of the elastic properties can be obtained from Fig. 6. However, again it is assumed that the real interest is in what will happen to the properties of this tube as one varies different parameters.

First, what effects are observed if only the cord half-angle is changed from 10° to 20°? It is seen from Fig. 6 that (E^T_ξ/G_{xy}) decreases by a factor of approximately 5; (E^T_η/G_{xy}) remains constant; $(F^T_{\xi\eta}/G_{xy})$ decreases approximately 20%; and $(G^T_{\xi\eta}/G_{xy})$ decreases by a factor of approximately 2.

Second, what effects are observable if only the cords in the symmetrical plies are replaced by stiffer cords such that $N = 10^{-4}$? Again from Fig. 6, (E^T_ξ/G_{xy}) increases by a factor of approximately 3.5; (E^T_η/G_{xy}) remains constant; $(F^T_{\xi\eta}/G_{xy})$ remains nearly constant; and $(G^T_{\xi\eta}/G_{xy})$ increases by a factor of approximately 10.

Third, what effects are observed if both the cords in the symmetrical plies and the cords in the third or radial ply are stiffened such that $N = 10^{-4}$ and $N^* = 10^{-3}$? Again from Fig. 6, (E^T_ξ/G_{xy}) increases by a factor of approximately 3.5; (E^T_η/G_{xy}) increases by a factor of approximately 9; $(F^T_{\xi\eta}/G_{xy})$ increases by a factor of approximately 4; and $(G^T_{\xi\eta}/G_{xy})$ increases by a factor of approximately 10. As before, many other combinations and variations can be examined with similar ease.

In Fig. 7 another group of combinations is illustrated in which the radial ply is thicker than the symmetrical plies. These examples by no means cover all the possible three-ply structures. However, they give an indication of

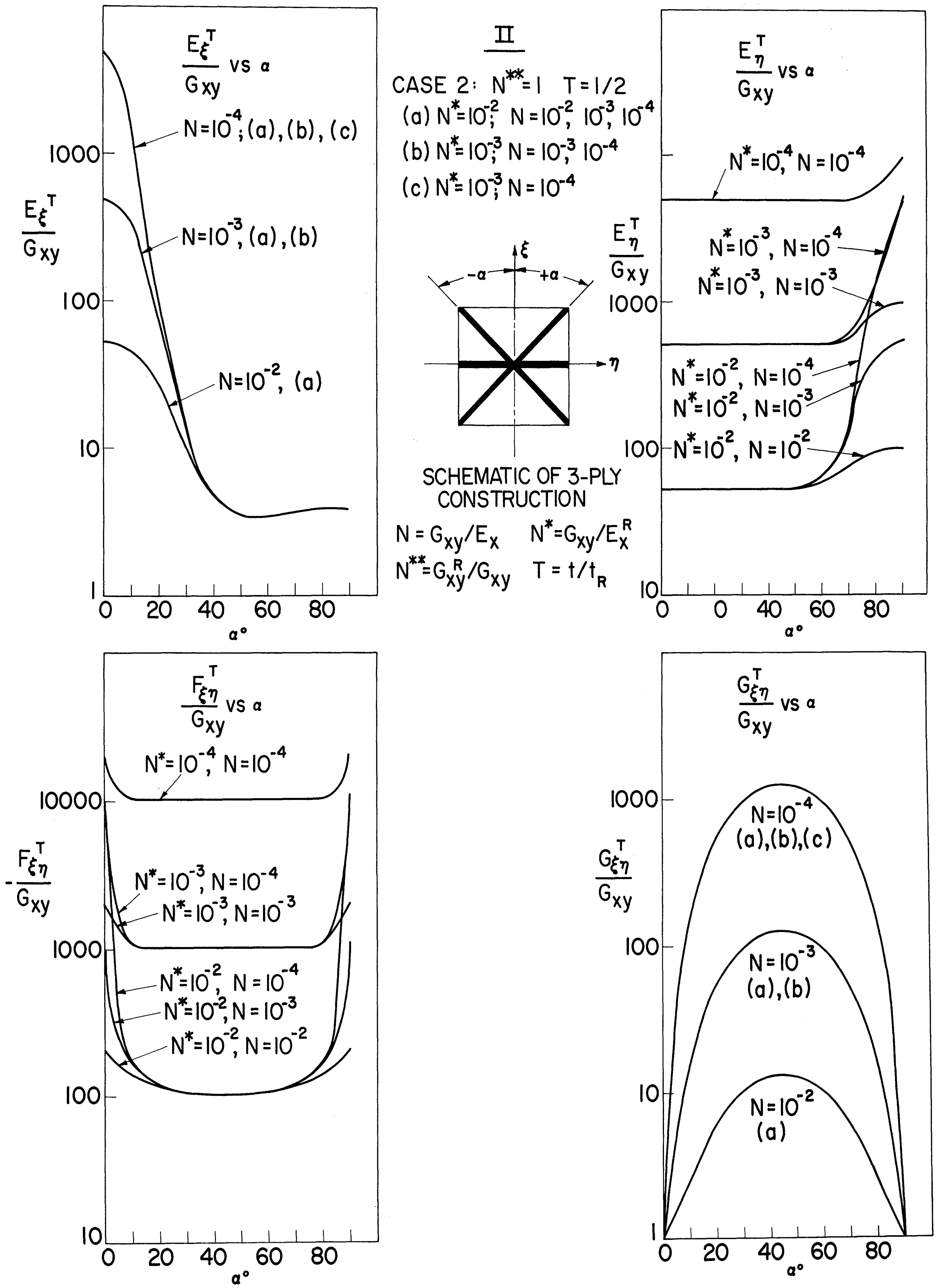


Fig. 7. Plane elastic properties of a three-ply orthotropic structure - third ply of different construction. Case 2: $N^{**} = 1$, $T = 1/2$.

the numerous possibilities for elastic properties changes due to the inclusion of a third ply.

IV. INTERPLY STRESSES

In this section an investigation will be made of the interply stresses existing in three-ply orthotropic structures of the kind described in the previous section. Also, the ideas developed for this structure will be used to analyze a two-ply structure in which the isotropic layer of material between the cords will be considered as the third ply.

In Ref. 1, interply stresses were introduced as those stress components arising from the elastic interply connection. They can have shearing (distortion-causing) or normal (extension-causing) effects on the plies. It was discovered that for structures whose cords are all in tension, applied normal stresses generate only the distortion-causing interply stresses, which can be calculated from knowledge of the structural properties of the individual plies and the magnitude of the applied stress. Also, it was noted that the application of shear stresses produce only the extension-causing interply stresses, which also can be calculated by knowing only the structural properties of the individual plies and the magnitude of the applied load.

Since Ref. 1 and other related discussions have dealt only with ortotropic structures composed of an even number of plies, it will be the purpose of this section to investigate the behavior of interply stresses for a three-ply orthotropic structure.

Returning to the construction of Fig. 1, and imagining that the load distribution is as shown in Fig. 8, it is possible to write the generalized Hooke's law about each of the plies. All symmetry properties of the two angular

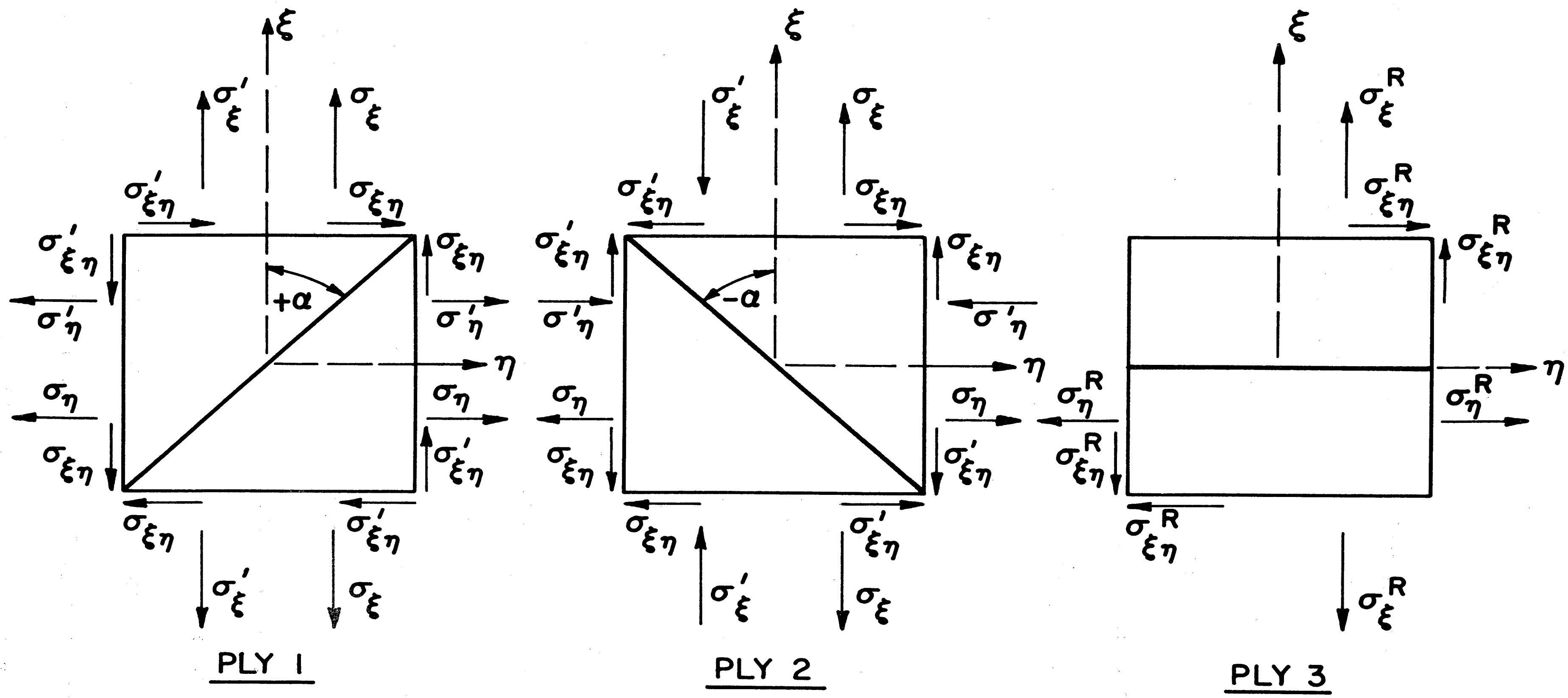


Fig. 8. General plane state of stress on each ply of a three-ply structure - all cords in tension.

plies discussed in Ref. 1 are utilized here.

Ply 1 :

$$\begin{aligned}
\epsilon_\xi &= a_{11}\,(\sigma_\xi+\sigma'_\xi) + a_{12}\,(\sigma_\eta+\sigma'_\eta) + a_{13}\,(\sigma_{\xi\eta}+\sigma'_{\xi\eta}) \\
\epsilon_\eta &= a_{21}\,(\sigma_\xi+\sigma'_\xi) + a_{22}\,(\sigma_\eta+\sigma'_\eta) + a_{23}\,(\sigma_{\xi\eta}+\sigma'_{\xi\eta}) \\
\epsilon_{\xi\eta} &= a_{31}\,(\sigma_\xi+\sigma'_\xi) + a_{32}\,(\sigma_\eta+\sigma'_\eta) + a_{33}\,(\sigma_{\xi\eta}+\sigma'_{\xi\eta})
\end{aligned} \tag{7}$$

Ply 2 :

$$\begin{aligned}
\epsilon_\xi &= a_{11}\,(\sigma_\xi-\sigma'_\xi) + a_{12}\,(\sigma_\eta-\sigma'_\eta) - a_{13}\,(\sigma_{\xi\eta}-\sigma'_{\xi\eta}) \\
\epsilon_\eta &= a_{21}\,(\sigma_\xi-\sigma'_\xi) + a_{22}\,(\sigma_\eta-\sigma'_\eta) - a_{23}\,(\sigma_{\xi\eta}-\sigma'_{\xi\eta}) \\
\epsilon_{\xi\eta} &= -a_{31}\,(\sigma_\xi-\sigma_\xi) - a_{32}\,(\sigma_\eta-\sigma'_\eta) + a_{33}\,(\sigma_{\xi\eta}-\sigma'_{\xi\eta})
\end{aligned} \tag{8}$$

Ply 3 :

$$\begin{aligned}
\epsilon_\xi &= a^R_{11}\,\sigma^R_\xi + a^R_{12}\,\sigma^R_\eta + a^R_{13}\,\sigma^R_{\xi\eta} \\
\epsilon_\eta &= a^R_{21}\,\sigma^R_\xi + a^R_{22}\,\sigma^R_\eta + a^R_{23}\,\sigma^R_{\xi\eta} \\
\epsilon_{\xi\eta} &= a^R_{31}\,\sigma^R_\xi + a^R_{32}\,\sigma^R_\eta + a^R_{33}\,\sigma^R_{\xi\eta}
\end{aligned} \tag{9}$$

Equations linking the three plies :

$$\begin{aligned}
\sigma^T_\xi &= (2t\sigma_\xi + t_R\sigma^R_\xi)/(2t + t_R) \\
\sigma^T_\eta &= (2t\sigma_\eta + t_R\sigma^R_\eta)/(2t + t_R) \\
\sigma^T_{\xi\eta} &= (2t\sigma_{\xi\eta} + t_R\sigma^R_{\xi\eta})/(2t + t_R)
\end{aligned} \tag{10}$$

It is noted there that these equations have this form only if it is assumed that the interply stresses exist as shown in Fig. 8. This implies the existence of interply stresses only between plies 1 and 2. This is not at all in disagreement with the concept of interply stresses as discussed in Ref. 1 and other reports, for interply stresses are only necessary to oppose the anisotropic nature of an individual ply when it is loaded in a direction other than perpendicular to the cords or parallel to the cords. The third ply of othotropic structure discussed here is always loaded so that it does not require the "extra" stresses for distortionless extensions and extensionless distortions.

In Eqs. (7)-(10) the unknowns may be considered to be ϵ_ξ, ϵ_η, $\epsilon_{\xi\eta}$, σ_ξ, σ_η, $\sigma_{\xi\eta}$, σ^R_ξ, σ^R_η, $\sigma^R_{\xi\eta}$, σ'_ξ, σ'_η, and $\sigma'_{\xi\eta}$, assuming that the averaged external stresses σ^T_ξ, σ^T_η, and $\sigma^T_{\xi\eta}$ are given. All the a_{ij}, t, and t_R are known. Thus, Eqs. (7)-(10) are determinate as they stand since twelve unknowns are present in twelve equations.

The strains of the three plies must be identical since the plies are assumed to deform as a unit. Hence, they can be equated, provided it is understood that the strains in question here are averaged over several times the cord spacing. Setting the first of Eqs. (7) equal to the first of Eqs. (8), the second equal to the second and the third equal to the third, one obtains three equations for the interply stress components σ'_ξ, σ'_η and $\sigma'_{\xi\eta}$:

$$\sigma'_\eta = \frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)} \sigma_{\xi\eta}$$

$$\sigma'_\xi = - \left[\left(\frac{a_{12}}{a_{11}}\right) \frac{(a_{12}a_{13} - a_{11}a_{23})}{(a_{11}a_{22} - a_{12}^2)} + \frac{a_{13}}{a_{11}} \right] \sigma_{\xi\eta} \qquad (11)$$

$$\sigma_{\xi\eta}' = -\left(\frac{a_{13}}{a_{33}}\right)\sigma_\xi - \left(\frac{a_{23}}{a_{33}}\right)\sigma_\eta$$

As would be expected, Eqs. (11) are exactly the same in form as those of Eqs. (20) in Ref. 1. However, there is a major difference in that the expressions in Eqs. (11) are not self-contained as they are in Ref. 1. This is brought about by the fact that in Eqs. (11) σ_ξ, σ_η, and $\sigma_{\xi\eta}$ are not the applied stresses but those absorbed by the two angular plies. Therefore, in order to numerically compute the actual values of the interply stress, one must first determine σ_ξ, σ_η, and $\sigma_{\xi\eta}$ in terms of the known applied loads σ_ξ^T, σ_η^T, and $\sigma_{\xi\eta}^T$. This illustrates the fact that a two-ply structure will generate an interply stress greater or less than a three-ply depending on the relative magnitudes of σ_ξ, σ_η, $\sigma_{\xi\eta}$ and σ_ξ^T, σ_η^T, $\sigma_{\xi\eta}^T$.

The external stress taken by each ply, σ_ξ, σ_η, $\sigma_{\xi\eta}$, σ_ξ^R, σ_η^R and $\sigma_{\xi\eta}^R$, are determined by making use of Eqs. (9) and (10). Equation (9) can be simplified by recalling the nature of the third ply:

$$a_{11}^R = \frac{1}{E_y^R}, \quad a_{12} = -\frac{1}{F_{xy}^R}, \quad a_{13} = 0$$

$$a_{21}^R = -\frac{1}{F_{xy}^R}, \quad a_{22}^R = \frac{1}{E_x^R}, \quad a_{23}^R = 0$$

$$a_{31}^R = 0, \quad a_{32}^R = 0, \quad a_{33}^R = \frac{1}{G_{xy}^R}$$

Thus, Eqs. (9) become:

$$\epsilon_\xi = \frac{\sigma_\xi^R}{E_y^R} - \frac{\sigma_\eta^R}{F_{xy}^R}$$

$$\epsilon_\eta = -\frac{\sigma_\xi^R}{E_{xy}^R} + \frac{\sigma_\eta^R}{E_x^R} \qquad (9a)$$

$$\epsilon_{\xi\eta} = \frac{\sigma^R_{\xi\eta}}{G_{xy}}$$

The stresses in the third ply, σ^R_ξ, σ^R_η, and $\sigma^R_{\xi\eta}$ are found in terms of σ_ξ, σ_η and $\sigma_{\xi\eta}$ from Eqs. (10) and substituted into Eqs. (9a). The resulting equations are then combined with Eqs. (7) and (11) to obtain σ_ξ, σ_η and $\sigma_{\xi\eta}$ in terms of the applied stresses σ^T_ξ , σ^T_η and $\sigma^T_{\xi\eta}$. Omitting the intermediate algebra, the resulting expressions are:

$$\sigma_\xi = \frac{(2T+1)}{(2T + \frac{E^R_y}{E})} \sigma^T_\xi$$

$$\sigma_\eta = \frac{(2T + 1)}{(2T + \frac{E^R_x}{E_\eta})} \sigma^T_\eta \qquad (12)$$

$$\sigma_{\xi\eta} = \frac{(2T + 1)}{(2T + \frac{G^R_{xy}}{G_{\xi\eta}})} \sigma^T_{\xi\eta}$$

Recalling that σ_ξ, σ_η and $\sigma_{\xi\eta}$ are the stresses acting on the two angular plies due to external loads, Equation (12) clearly illustrates the fact that for usual carcass constructions the two angular plies absorb most of the loading effects in the ξ-direction while the third ply absorbs most of the loading effects in the η-direction. Also these equations show that the applied shear load is absorbed more by the two angular plies than the third ply. The basis for this reasoning lies in the fact that for most combinations of cord and rubber (E^R_y/E_ξ) is less than one, (E^R_x/E_η) is greater than one and $(G^R_{xy}/G_{\xi\eta})$ is less than one. Thus, in the first of Eqs. (12) the coefficient of σ^T_ξ is greater than one and hence σ_ξ is greater than σ_ξ^T, indicating that the load is not divided uniformly between the plies but being absorbed more by the two angular plies than by the third ply. In the second of Eqs. (12) the coefficient is less than one and hence σ_η is less than σ_η^T, and in the third equation the coefficient of $\sigma_{\xi\eta}^T$ is greater than one and hence $\sigma_{\xi\eta}$ greater than $\sigma_{\xi\eta}^T$.

It is of interest to note that the general expressions for E_ξ^T, E_η^T, $F_{\xi\eta}^T$ and $G_{\xi\eta}^T$ obtained from the strength of materials approach in the previous section can acutally be obtained from the general development presented in this section. For example, to determine $G_{\xi\eta}^T$ one applies a $\sigma_{\xi\eta}^T$ and finds the ratio between $\sigma_{\xi\eta}^T$ and $\epsilon_{\xi\eta}$. In so doing it is postulated that the shearing distortion takes place without extension and that the only components of interply stresses generated are σ_ξ' and σ_η'. If this is done, Eqs. (7)-(10) reduce to:

$$
\begin{aligned}
&(a) \quad 0 = a_{11}\,\sigma_\xi' + a_{12}\,\sigma_\eta' + a_{13}\,\sigma_{\xi\eta} \\
&(b) \quad 0 = a_{12}\,\sigma_\xi' + a_{22}\,\sigma_\eta' + a_{23}\,\sigma_{\xi\eta} \\
&(c) \quad \epsilon_{\xi\eta} = a_{13}\,\sigma_\xi' + a_{23}\,\sigma_\eta' + a_{33}\,\sigma_{\xi\eta} \\
&(d) \quad \epsilon_{\xi\eta} = \sigma_{\xi\eta}^R / G_{xy}^R \\
&(e) \quad \sigma_{\xi\eta}^T = (2t\,\sigma_{\xi\eta} + t_R\,\sigma_{\xi\eta}^R)/(2t + t_R)
\end{aligned}
\tag{13}
$$

Solving Eq. (13e) for $\sigma_{\xi\eta}^R$ and substituting into Eq. (13d) gives an expression for $\sigma_{\xi\eta}$ in terms of $\sigma_{\xi\eta}^T$ and $\epsilon_{\xi\eta}$:

$$
\sigma_{\xi\eta} = \frac{1}{2t}\left[(2t + t_R)\sigma_{\xi\eta}^T - t_R\,\epsilon_{\xi\eta}\,G_{xy}^R\right] \tag{14}
$$

Thus, substituting Eqs. (11) and (14) into Eq. (13c) and solving for the ratio $\sigma^T_{\xi\eta}/\epsilon_{\xi\eta}$ gives $G^T_{\xi\eta}$. Again omitting the intermediate algebra:

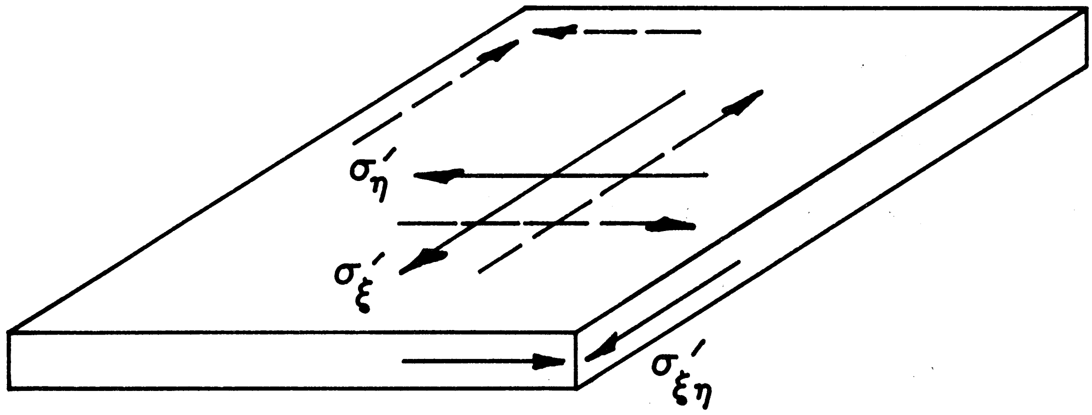
$$G^T_{\xi\eta} = (2G_{\xi\eta}T + G^R_{xy})/(2T + 1)$$

This agrees with the result previously obtained in Eq. (6).

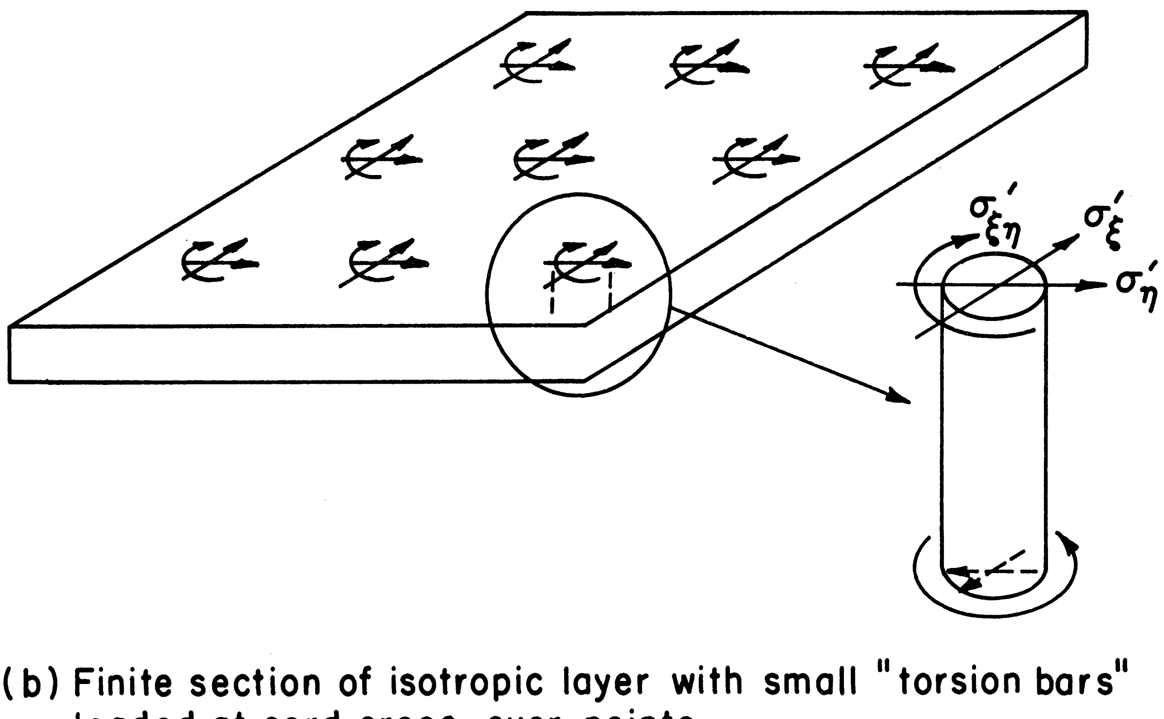
In a similar manner the other elastic properties can be checked.

An additional observation can be made about interply stresses by extending the present analysis to a three-ply structure of such a construction that it has two identical plies arranged at angles of plus and minus α, separated by a third ply of homogeneous isotropic material. This in essence is a two-ply structure since the third ply can represent the rubber layer separating the layers of cord. The analysis of the two-ply structure in this context is identical to that carried out in this section and the previous section except the properties of the third ply are different. However, the interply stresses between the two angular plies must now be transmitted through the third ply. This offers some interesting models as sources for the generation of interply stresses. First, from this approach it seems plausible that the layer between the cords might behave as an isotropic sheet which has been loaded by surface shear tractions of two types, extensional types represented by σ'_ξ and σ'_η and shearing type represented by $\sigma_{\xi\eta}'$. Thus, to actually analyze the effects of the interply stresses in the isotropic layer between the cords, as visualized by this mechanism, requires the analysis of a thin plate loaded by surface shear forces as well as in-plane shear forces (See Fig. 9(a)).

This three-ply approach also lends support to the previously conceived



(a) Increment of isotropic layer loaded by interply stresses.



(b) Finite section of isotropic layer with small "torsion bars" loaded at cord cross-over points.

Fig. 9. Models for interply stress generation.

idea of interply stress generation through small "torsion bars" located at the cross-over points between layers of cord. For, if the interply stress is generated only in the vicinity of the cross-over points, it can be thought of as the result of many small torsion bars, each loaded with two types of stresses, the twisting type represented by $\sigma_{\xi\eta}'$ and those represented by σ_{ξ}' and σ_{η}' (see Fig. 9(b)).

Although this analysis does not solve the problem of the nature of interply stresses, it presentes an alternate view point from which to discuss the ever vexing problem of these "extra" stresses.

V. EXAMPLE PROBLEMS

Example problems are given below to illustrate possible uses of the information of Figs. 4-7.

1. Given: A three-ply othotropic pressure vessel constructed from three plies of the same ply stock (see Fig. 10). The original material and construction is such that $N = N^* = 10^{-3}$, $N^{**} = 1$, $\alpha = 20°$, and $T = 1$. The tube is equipped with end plugs to allow an internal pressure of 5 psi. In addition an external torque of 25 in.-lb. is carried by the tube. The tube has been sufficiently preloaded so that all cords will remain in a state of tension.

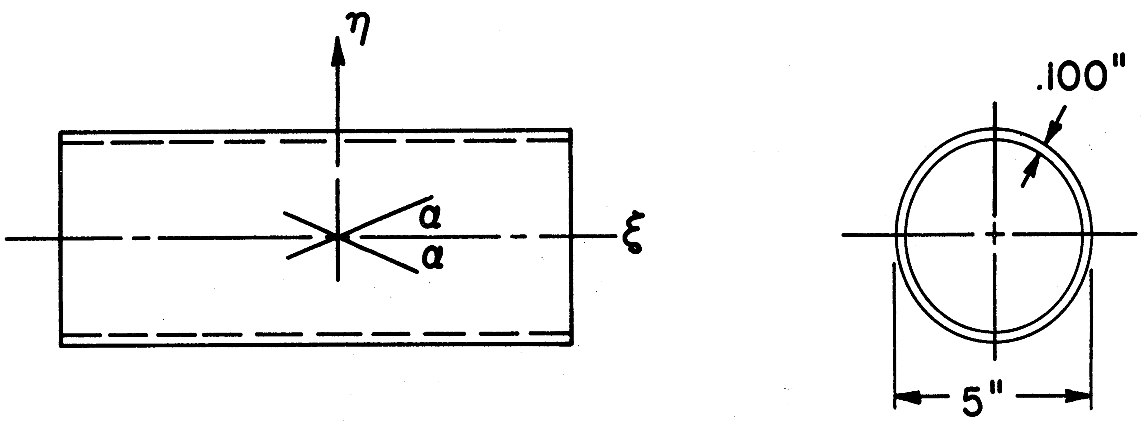


Fig. 10. Three-ply cylindrical tube.

To Find:

(a) Original strain state in the tube if $G_{xy} = 100$ psi.

(b) Strain state if cords are stiffened such that $N = N^* = 10^{-4}$. All other ratios and dimensions remain as in (a).

(c) Strain state if cord half-angle is reduced to 10°. All other ratios and dimensions remain as in (a).

(d) Strain state if the plies are changed such that $N^* = 10^{-3}$, $N = 10^{-4}$

and T = 2. All other ratios and dimensions as in (a). This change implies that the circumferential ply has cords of lower modulus than the other two plies. Also, the thickness of the circumferential ply is now .02 while the other two are .04.

Solution: The external stresses are calculated from simple pressure vessel expressions:

$$\sigma_\xi^T = \frac{5(2.5)}{(2)(.1)} = 62.5 \text{ psi}$$

$$\sigma_\eta^T = \frac{5(2.5)}{.1} = 125 \text{ psi}$$

$$\sigma_{\xi\eta}^T = \frac{25(2.5)}{\pi(2.5)^3(.1)} = 6.38 \text{ psi}$$

(a) The original strain is:

$$G_{xy}\,\epsilon_\xi^T = \sigma_\xi^T\left(\frac{G_{xy}}{E_\xi^T}\right) + \sigma_\eta^T\left(\frac{G_{xy}}{F_{\xi\eta}^T}\right)$$

$$G_{xy}\,\epsilon_\eta^T = \sigma_\eta^T\left(\frac{G_{xy}}{E_\eta^T}\right) + \sigma_\xi^T\left(\frac{G_{xy}}{F_{\xi\eta}^T}\right)$$

$$G_{xy}\,\epsilon_{\xi\eta}^T = \sigma_{\xi\eta}^T\left(\frac{G_{xy}}{G_{\xi\eta}^T}\right)$$

From Fig. 4,

$$\frac{G_{xy}}{E_\xi^T} = \frac{1}{105}, \quad \frac{G_{xy}}{E_\eta^T} = \frac{1}{330}, \quad \frac{G_{xy}}{F_{\xi\eta}^T} = \frac{-1}{680}, \quad \frac{G_{xy}}{G_{\xi\eta}} = \frac{1}{70}$$

Therefore:

$$G_{xy}\,\epsilon_\xi^T = \frac{62.5}{105} - \frac{125}{680} = 0.41$$

$$\epsilon_\xi^T = .0041$$

$$G_{xy}\ \epsilon_\eta^T = \frac{125}{330} - \frac{62.5}{680} = 0.29$$

$$\epsilon_\eta^T = .0029$$

$$G_{xy}\ \epsilon_{\xi\eta}^T = 6.38\left(\frac{1}{70}\right) = 0.091$$

$$\epsilon_{\xi\eta}^T = 0.00091$$

(b) For this case one refers again to Fig. 4.

$$\epsilon_\xi^T = \frac{1}{100}\left[\frac{62.5}{130} - \frac{125}{6700}\right] = 0.0046$$

$$\epsilon_\eta^T = \frac{1}{100}\left[\frac{125}{3300} - \frac{62.5}{6700}\right] = 0.00029$$

$$\epsilon_{\xi\eta}^T = \frac{1}{100}\left[\frac{6.38}{690}\right] = 0.000091$$

(c) Again referring to Fig. 4:

$$\epsilon_\xi^T \quad \frac{1}{100}\left[\frac{62.5}{500} - \frac{125}{740}\right] = -0.00044$$

$$\epsilon_\eta^T = \frac{1}{100}\left[\frac{125}{330} - \frac{62.5}{740}\right] = 0.0030$$

$$\epsilon_{\xi\eta}^T = \frac{1}{100}\left[\frac{6.38}{20.5}\right] = 0.0031$$

(d) For this case one refers to Fig. 6:

$$\epsilon_\xi^T = \frac{1}{100}\left[\frac{62.5}{158} - \frac{125}{420}\right] = 0.00098$$

$$\epsilon_\eta^T = \frac{1}{100}\left[\frac{125}{200} - \frac{62.5}{420}\right] = 0.0048$$

$$\epsilon^T_{\xi\eta} = \frac{1}{100}\left[\frac{6.38}{830}\right] = 0.000077$$

For easy comparison, the results of these different cases are included in the following table:

Case	ϵ^T_ξ	ϵ^T_η	$\epsilon^T_{\xi\eta}$	Nature of Change
a	.0041	.0029	.00091	Original Tube
b	.0046	.00029	.000091	Cord tensile strength increased by factor of 10
c	-.00044	.0030	.0031	Cord half-angle decreased by 10°
d	.00098	.0048	.000077	Circumferential ply weaker than two longitudinal plies

As can be seen from these results, dramatic changes in the strain state of a three-ply orthotropic structure can be accomplished with relatively few changes in constructional properties.

2. <u>Given</u>: Same cylindrical tube as in problem 1 except $N = N^* = 10^{-4}$. The tube is also loaded as in problem 1.

<u>To Find</u>: The interply stress components of the three-ply tube as well as for the same tube with the circumferential ply removed.

<u>Solution</u>: The external stress components are again calculated from pressure vessel expressions. For the three-ply tube the components are the same as in problem 1:

$$\sigma^T_\xi = 62.5 \text{ psi}\ ,\quad \sigma^T_\eta = 125 \text{ psi}\ ,\quad \sigma^T_{\xi\eta} = 6.38 \text{ psi}$$

For the same tube with the circumferential ply removed:

$\sigma_{\xi}^{T} = 62.5\ (\frac{.1}{.067}) = 93.75,\quad \sigma_{\eta}^{T} = 125\ (\frac{.1}{.067}) = 187.5$ psi, $\quad \sigma_{\xi\eta}^{T} = 6.38\ (\frac{.1}{.067}) = 9.57$ psi

From Eqs. (11), the interply stress components are:

$$\sigma'_{\eta} = K_1\ \sigma_{\xi\eta}$$

$$\sigma'_{\xi} = K_2\ \sigma_{\xi\eta}$$

$$\sigma'_{\xi\eta} = K_3\ \sigma_{\xi} + K_4\ \sigma_{\eta}$$

where K_1, K_2, K_3 and K_4 are constants throughout the problem since they are properties of the two angular plies only.

For the three-ply tube the stresses carried by the two angular plies, σ_{ξ}, σ_{η} and $\sigma_{\xi\eta}$ are found from Eqs. (12). Before working out these values, it is noted that these expressions require the quantities (E_y^R/E_{ξ}), (E_x^R/E_{η}) and $(G_{xy}^R/G_{\xi\eta})$. These ratios can be obtained from Fig. 5 if it is noted that $E_y^R = E_y$, $E_x^R = E_{\eta}$, and $G_{xy}^R = G_{xy}$. Also recall $E_y^R = 4N^{**}G_{xy}$, $E_x^R = G_{xy}/N^*$, and $G_{xy}^R = G_{xy}N^{**}$. Therefore

$$\frac{E_y^R}{E_{\xi}} = 4N^{**}\left(\frac{G_{xy}}{E_{\xi}}\right),\quad \frac{E_x^R}{E_{\eta}} = \frac{1}{N^*}\left(\frac{G_{xy}}{E_{\eta}}\right),\quad \frac{G_{xy}^R}{G_{\xi\eta}} = N^{**}\left(\frac{G_{xy}}{G_{\xi\eta}}\right)$$

$$\frac{E_y^R}{E_{\xi}} = 4(1)\ (\frac{1}{195}) = .020$$

$$\frac{E_x^R}{E_{\eta}} = \frac{1}{10^{-4}}\left(\frac{1}{3.55}\right) = 2817$$

$$\frac{G_{xy}^R}{G_{\xi\eta}} = 1\left(\frac{1}{1034}\right) = .00097$$

And from Eqs. (12), the stresses on the two angular plies of the three-ply tube are:

$$\sigma_{\xi} = \frac{[\ 2(1)+1\ ]}{[2(1)+.020]} (62.5) = 92.82 \text{ psi}$$

$$\sigma_{\eta} = \frac{[2(1)+1]}{[2(1)+2817]} (125) = 0.1330 \text{ psi}$$

$$\sigma_{\xi\eta} = \frac{[2(1)+1]}{[2(1)+.00097]} (6.38) = 9.56 \text{ psi}$$

And from Eqs. (11), the interply stress components are:

$$\sigma'_{\eta} = 9.56\ K_1$$

$$\sigma'_{\xi} = 9.56\ K_2$$

$$\sigma'_{\xi\eta} = 92.82\ K_3 + 0.133\ K_4$$

And for the two-ply tube:

$$\sigma'_{\eta} = 9.57\ K_1$$

$$\sigma'_{\xi} = 9.57\ K_1$$

$$\sigma'_{\xi\eta} = 93.75\ K_3 + 187.5\ K_4$$

As can be seen, the only significant change occurs in the shearing component, $\sigma'_{\xi\eta}$.

VI. ACKNOWLEDGMENTS

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