Sobolev Mappings with Integrable Dilatations

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Abstract

We show that each quasi-light mapping f in the Sobolev space $W^{1,n}(\Omega, \mathbf{R}^n)$ satisfying $|Df(x)|^n \leq K(x, f) J(x, f)$ for almost every x and for some $K \in L^r(\Omega)$, $r > n - 1$, is open and discrete. The assumption that f be quasi-light can be dropped if, in addition, it is required that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ for some $p \geq n + 1/(n - 2)$. More generally, we consider mappings in the John Ball classes $\mathscr{A}_{p,q}(\Omega)$, and give conditions that guarantee their discreteness and openness.

1. Introduction

Let f be a mapping in the Sobolev space $W^{1,p}(\Omega, \mathbf{R}^n)$, where $\Omega \subset \mathbf{R}^n$ is open, $n \geq 2$, and $1 \leq p \leq \infty$. A fundamental problem in nonlinear elasticity is to determine what analytic data on f provide topological information on the deformation that f represents. For instance, under what conditions is f locally or globally invertible? Does f map open sets to open sets? These problems have been studied in a series of articles by Ball and others; see [B1, B2, Š, IŠ, MTY] and the references therein. The purpose of this paper is to give some further results in this direction. Before we formulate our objective more precisely, we introduce some terminology.

Throughout this paper $|Df(x)|$ designates the supremum norm of the differential $Df(x)$ with $J(x, f) = \det Df(x)$ its Jacobian determinant; they exist for almost every x in Ω.

Definition. A measurable function $K(\cdot, f)$ in Ω is called a *dilatation* of f if

(1.1) $\qquad 1 \leq K(x, f) < \infty \quad$ for almost every $x \in \Omega$,

(1.2) $\qquad |Df(x)|^n \leq K(x, f) J(x, f) \quad$ for almost every $x \in \Omega$.

We say that f is a mapping of *finite dilatation* if a function $K(\cdot, f)$ exists satisfying (1.1) and (1.2).

The above terminology is essentially the same as used by IWANIEC & ŠVERÁK [IŠ]. They assume that $J(x, f) \geqq 0$ a.e. and that $Df(x) = 0$ whenever $J(x, f) = 0$; they then define $K(x, f)$ to be the quotient $|Df(x)|^n / J(x, f)$, whenever this exists, and $K(x, f) = 1$ otherwise. The usual regularity assumption on f in this connection would be $f \in W^{1,n}(\Omega, \mathbf{R}^n)$, which trivially gurantees the integrability of the Jacobian $J(x, f)$. However, recent studies have shown that there are natural, weaker integrability hypotheses on Df guaranteeing the same conclusion; see, e.g., [IS, MTY]. Notice that for a mapping of finite dilatation we have $J(x, f) \geqq 0$ a.e. and that any mapping satisfying $J(x, f) > 0$ a.e. is a mapping of finite dilatation.

The following theorem lists some important properties of mappings of finite dilatation in $W^{1,n}(\Omega, \mathbf{R}^n)$:

1.3. Theorem. *Let $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ be a mapping of finite dilatation. Then*
  (1) *f has a continuous representative (also called f),*
  (2) *locally*

(1.4) $|f(x) - f(y)| \leqq C(n) \|Df\|_n \left(\log \dfrac{2}{|x - y|}\right)^{-1/n},$

  (3) *f is differentiable a.e.,*
  (4) *f is sense-preserving,*
  (5) *f satisfies condition (N), i.e., $|f(E)| = 0$ whenever $E \subset \Omega$ is such that $|E| = 0$, where $|\cdot|$ stands for Lebesgue n-measure.*

We recall that a continuous mapping $f: \Omega \to \mathbf{R}^n$ is *sense-preserving* if the topological index $\mu(y, f, D)$ is positive for all domains $D \subset\subset \Omega$ and $y \in f(D) \setminus f(\partial D)$. We refer to [R, p. 85] or [Ri, Chapter 1] for the precise definition and basic properties of the topological index.

The adverb "locally" in (2) can be made more precise: for each compact $F \subset \Omega$ there is a constant $\delta > 0$, depending only on $\mathrm{dist}(F, \partial\Omega)$, such that (1.4) holds for each pair of points $x, y \in F$ with $|x - y| < \delta$.

Proof. Assertion (1) is due to GOL'DSHTEIN & VODOP'YANOV [GV]; they show that f is monotone, and the proof in [R, p. 339] can be modified so as to yield the continuity together with (2) (and the more precise statement cited after the theorem). A more elementary proof of (1) was recently given by MANFREDI [M]. See also [Š].

Now (3) follows from the proof of Theorem 1.2 in [R, p. 84]; the only property of quasiregular mappings used in that proof is the equicontinuity of a family of maps, but this property is guaranteed by (2).

Next, by [Ri, Lemma 5.1] f is weakly sense-preserving, and by the proof of [Ri, I.4.5] f is actually sense-preserving (see also [R, p. 175]). The crucial point is that for mappings of finite dilatation the differential $Df(x)$ vanishes at almost every point x where $J(x, f) = 0$. (See also Section 5 below.)

Finally, (5) is a consequence of a result of GOL'DSHTEIN & VODOP'YANOV [GV, 2.4]. See also [MZ] and [MM] for recent studies on condition (N).

The reader familiar with quasiregular mappings notices that if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ and if $K(x, f) \in L^\infty(\Omega)$, then f is by definition quasiregular,

and hence either a constant or a discrete and open mapping in each component of Ω, by a fundamental result of RESHETNYAK [R, p. 183]. Recall that a continuous mapping is said to be *discrete* if the inverse image of each point consists of isolated points.

The question we wish to address is:

Problem. *Under which conditions is a mapping of finite dilatation discrete and open?*

Because this problem is of a local nature, we assume from now on that Ω is *bounded* and *connected*, and that all mappings in question are *non-constant*.

IWANIEC & ŠVERÁK proved in [IŠ] that if $n = 2$, $f \in W^{1,2}(\Omega, \mathbf{R}^2)$, and $K(x, f) \in L^1(\Omega)$, then f is not only discrete and open but admits a factorization $f = \varphi \circ h$, where h is a homeomorphism and φ is a holomorphic function. Their proof uses the powerful existence theorem for the two-dimensional Beltrami equation, not available in space. It was conjectured in [IŠ] that for all $n \geq 3$ and $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ the condition $K(x, f) \in L^{n-1}(\Omega)$ implies that f is a discrete and open mapping. In our first theorem (Theorem 1.7) we confirm this conjecture for quasi-light mappings which satisfy $K(x, f) \in L^{n-1+\varepsilon}$ for some $\varepsilon > 0$.

Definition. A continuous mapping $f: \Omega \to \mathbf{R}^n$ is *quasi-light* if for each $y \in \mathbf{R}^n$ the components of the set $f^{-1}(y)$ are compact.

The term *quasi-light* is due to VÄISÄLÄ [V1]. Notice that f is quasi-light, for instance, if it agrees with a discrete mapping in a neighborhood of $\partial\Omega$, in particular, if f is a homeomorphism near $\partial\Omega$. Thus (modulo ε) Theorem 1.7 extends some results of BALL [B2] and ŠVERÁK [Š]. We emphasize that we do not assume that $J(x, f) > 0$ a.e. as was done in [B2] and [Š]. It is in fact an interesting problem whether $J(x, f) > 0$ a.e. provided $K(x, f)$ is integrable to some power $r < \infty$ (see Section 6 below).

In an attempt to dispense with any *a priori* topological assumptions on f, we have to take a slightly different approach. It seems that for our method to work, the regularity requirement $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is too weak. If we assume instead that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ for some $p > n$, then we can always find $r = r(n, p) < \infty$ such that $K(x, f) \in L^r(\Omega)$ implies f is discrete and open; in fact, it turns out that for $p \geq n + 1/(n - 2)$, any $r > n - 1$ works. A further glance at the method (which is a refinement of RESHETNYAK's idea) reveals that what is essential is not the integrability of Df, but the integrability of the *adjugate* adj Df of Df, which we recall satisfies

$$(1.5) \qquad Df(x) \text{ adj } Df(x) = J(x, f) \text{ Id}$$

for almost every x. Since the entries of adj Df are polynomials of degree $n - 1$ of entries of Df, we have by Hölder's inequality that

$$(1.6) \qquad \text{adj } Df(x) \in L^{p/(n-1)}(\Omega)$$

whenever $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ and $p \geq n - 1$.

We consider the following classes of mappings:

$$\mathscr{A}_{p,q}(\Omega) = \{ f \in W^{1,p}(\Omega, \mathbf{R}^n) : \text{adj } Df \in L^q \},$$

where $p \geq n - 1$ and $q \geq p/(p - 1)$. It follows from the identity (1.5) that $J(x, f) \in L^1(\Omega)$ if $f \in \mathscr{A}_{p,q}(\Omega)$. The classes $\mathscr{A}_{p,q}(\Omega)$ were introduced by BALL and they have been studied by several people in nonlinear elasticity; see [B1, B2, Š, M, MTY].

Now we are ready to state the main results of this paper:

1.7. Theorem. *Suppose that $f \in \mathscr{A}_{p,q}(\Omega)$ is a continuous, quasi-light mapping of finite dilatation, where $p \geq n - 1$ and $q \geq p/(p - 1)$. If $K(x, f) \in L^{n-1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, then f is discrete and open.*

1.8. Corollary. *If $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is a quasi-light mapping of finite dilatation and if $K(x, f) \in L^{n-1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, then f is discrete and open.*

1.9. Theorem. *Suppose that $f \in \mathscr{A}_{p,q}(\Omega)$ is a continuous mapping of finite dilatation, where $p \geq n - 1$ and $q \geq p/(p - 1)$. If $K(x, f) \in L^r(\Omega)$ for some $r > n - 1$ such that*

$$r \geq \frac{q}{n(q - 1) - q},$$

then f is discrete and open.

If $p \geq n$ in these theorems, we do not need to assume that f is continuous, as this follows from Theorem 1.3. If $n - 1 \leq p < n$ in Theorem 1.9, it suffices to assume that $K(x, f)$ is in $L^r(\Omega)$ for some $r \geq n/(n - p)$, as can be seen from the condition $q \geq p/(p - 1)$. Moreover, since any mapping in $W^{1,p}(\Omega, \mathbf{R}^n)$ belongs to $\mathscr{A}_{p,q}(\Omega)$ for some $q \geq p/(n - 1)$, the requirement for r in terms of $p \geq n$ reads

$$r \geq \frac{p}{(n - 1)(p - n)},$$

which in the borderline case $p = n$ means $r = \infty$. In orther words, starting with mappings in $W^{1,n}(\Omega, \mathbf{R}^n)$ we do not find any integrability assumption on $K(x, f)$ (beyond $K(x, f) \in L^\infty$ which is RESHETNYAK's theorem) to guarantee the discreteness and openness of f, unless an extra condition on adj Df is imposed. However, if $p > n$, then adj Df automatically satisfies something which gives a finite value of r. Since

$$\frac{p}{(n - 1)(p - n)} = n - 1$$

for

$$p = n + \frac{1}{(n - 2)} \leq \frac{4n}{3}, \quad n > 2.$$

Theorem 1.9 permits us to deduce the following result:

1.10. Corollary. *Suppose that $n \geq 3$ and that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ for some $p \geq n + 1/(n-2)$. If $K(x, f) \in L^r(\Omega)$ for some $r > n - 1$, then f is discrete and open.*

The example in [B2, p. 317–318] shows that the integrability condition on $K(\cdot, f)$ in both Theorems 1.7 and 1.9 cannot be improved much. Indeed, BALL exhibits a mapping f in the Sobolev space $W^{1,\infty}(\Omega, \mathbf{R}^n)$ such that f is the identity on the boundary of Ω and $K(x, f) \in L^r(\Omega)$ for all $r < n - 1$, yet f takes a line segment to a point. Thus both of our results are quite sharp for mappings with sufficient regularity.

The borderline case $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ remains somewhat problematic, and in light of the above discussion we reformulate the conjecture advanced by IWANIEC & ŠVERÁK: Let $a(n)$ denote the infimum of exponents $p \geq n$ so that each Sobolev mapping $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ whose dilatation belongs to $L^r(\Omega)$ for some $r > n - 1$ is discrete and open. Corollary 1.10 states that $a(n) \leq n + 1/(n-2) \leq 4n/3$ for $n \geq 3$ and by [IŠ], $a(2) = 2$. Hence we are led to the conjecture $a(n) = n$ for all $n \geq 2$. This conjecture is slightly weaker than that in [IŠ]. Indeed, for $n = 2$ we only need $K(x, f) \in L^1(\Omega)$.

2. The Degenerate Equation Associated with f

Throughout this section we assume that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq n - 1$, is a continuous mapping with dilatation $K = K(\cdot, f)$. By Theorem 1.3 the continuity assumption is superfluous provided $p \geq n$. We also recall that Ω is assumed to be a bounded domain and f non-constant.

Define $\mathscr{A}: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$\mathscr{A}(x, \xi) = \langle G(x) \xi, \xi \rangle^{\frac{n-2}{2}} G(x) \xi,$$

where

$$G(x) = J_f(x)^{2/n} [Df(x)]^{-1} [Df(x)]^{-1^T}$$

if $x \in \Omega$ and $J_f(x) \neq 0$ exists, and $G(x) = \text{Id}$ otherwise; here T denotes transpose. We also write

$$F(x, \xi) = \langle G(x) \xi, \xi \rangle^{n/2}.$$

Then a calculation (see, e.g., [HKM, p. 269]) shows

(2.1) $F(x, \xi) = \mathscr{A}(x, \xi) \cdot \xi \geq K(x)^{-1} |\xi|^n$ for almost every x,

(2.2) $|\mathscr{A}(x, \xi)| \leq K(x)^{n-1} |\xi|^{n-1}$ for almost every x,

(2.3) $F(x, \xi) \leq |\xi|^n K(x)^{n-1}$ for almost every x.

Next, let $h(x) = -\log |f(x)|$. Then we have that $h \in W^{1,p}_{\text{loc}}(\Omega \setminus Z_f)$ and

$$\nabla h(x) = \frac{-Df(x)^T f(x)}{|f(x)|^2},$$

where $Z_f = f^{-1}(0)$ is the zero-set of f. Further,

$$(2.4) \qquad \mathscr{A}(x, \nabla h) = \frac{-\operatorname{adj} Df(x) \cdot f(x)}{|f(x)|^n} \in L^q_{\mathrm{loc}}(\Omega \setminus Z_f),$$

$$(2.5) \qquad \mathscr{A}(x, \nabla h) \cdot \nabla h = J(x, f) |f(x)|^{2-n} \in L^1_{\mathrm{loc}}(\Omega \setminus Z_f)$$

if $f \in \mathscr{A}_{p,q}(\Omega)$. Moreover, the above integrability conditions hold in any subset of Ω with positive distance from Z_f. It is important to observe that we have used the fact that f is of finite dilatation in an essential way in formulas (2.4) and (2.5), namely, if $J(x, f) = 0$, then $\mathscr{A}(x, \nabla h) = |\nabla h|^{n-2} \nabla h$, but then because $Df(x) = 0$, we also find that $\nabla h(x) = 0$ so that (2.4) and (2.5) always hold.

Next we invoke the important identity [BI, p. 271]:

2.6. Proposition. $\mathscr{A}(x, \nabla h)$ *is a divergence-free vector field in* $\Omega \setminus Z_f$; *that is,*

$$(2.7) \qquad \int \mathscr{A}(x, \nabla h) \cdot \nabla \phi \, dx = 0$$

for each $\phi \in C_0^\infty(\Omega \setminus Z_f)$.

Moreover, if $f \in \mathscr{A}_{p,q}(\Omega)$, then by approximation we see that

$$(2.8) \qquad \int \mathscr{A}(x, \nabla h) \cdot \nabla v \, dx = 0$$

for each $v \in W_0^{1,s}(\Omega \setminus Z_f)$, $s \geq q/(q-1)$, provided the support of ∇v has positive distance to Z_f.

Equation (2.7) means that $h(x) = -\log|f(x)|$ is a weak solution to the second-order partial differential equation $-\operatorname{div} \mathscr{A}(x, \nabla u) = 0$ outside Z_f. This equation is quasilinear elliptic with bounded coefficients if and only if f is a quasiregular mapping, and in the general case the difficulties arise from the degeneracy described in (2.1) and (2.2). In the bounded-coefficient case there is a well-understood connection between polar sets (*i.e.*, sets where a supersolution can take the value at infinity) and sets of zero capacity, and the latter concept provides a way to metric estimates; see [HKM, Chapters 2 and 7]. We shall mimic this approach as much as possible, and for that purpose we next introduce weighted capacities.

Let $1 < s \leq n$ and let ω be a nonnegative locally integrable function in Ω. For a compact set $C \subset \Omega$, define

$$(2.9) \qquad \operatorname{cap}_{s, \omega}(C, \Omega) = \inf_\psi \int_\Omega |\nabla \psi|^n \omega \, dx,$$

where the infimum is taken over all $\psi \in C_0^\infty(\Omega)$ with $\psi = 1$ in a neighborhood of C. If $\omega \equiv 1$, we have the standard variational s-capacity $\operatorname{cap}_s(C, \Omega)$. By Hölder's inequality,

$$\int_\Omega |\nabla \psi|^s \, dx \leq \left(\int_\Omega |\nabla \psi|^n \omega \, dx \right)^{s/n} \left(\int_\Omega \omega^{s/(s-n)} \, dx \right)^{(n-s)/n}$$

for $1 < s < n$, and hence

$$(2.10) \qquad \operatorname{cap}_{n, \omega}(C, \Omega) = 0 \Rightarrow \operatorname{cap}_s(C, \Omega) = 0$$

provided $\omega^{-1} \in L^{s/(n-s)}(\Omega)$. If ω is bounded, we can take the infimum in (2.9) over continuous functions $\psi \in W_0^{1,s}(\Omega)$ without affecting the value of $\mathrm{cap}_{s,\omega}(C, \Omega)$. Finally we recall the well-known Hausdorff dimension estimates:

$$(2.11) \qquad\qquad \mathrm{cap}_s(C, \Omega) = 0 \Rightarrow \dim_H C \le n - s$$

for $1 < s \le n$. See, *e.g.*, [HKM, 2.24].

3. Proof of Theorem 1.7

We first show that $C(y) = f^{-1}(y)$ has Hausdorff dimension less than 1 for each $y \in \mathbf{R}^n$; in particular, $C(y)$ is a totally disconnected set for each $y \in \mathbf{R}^n$, which means that f is a *light* mapping. To prove this, we assume, for simplicity, that $y = 0$. Denote $C(0) = Z_f$. Because f is quasi-light, we may further assume that Z_f is a compact and non-empty subset of Ω, and that f is continuous in a neighborhood of $\partial\Omega$ (cf. [V1, Theorem 3.1]).

Consider $h(x) = -\log|f(x)|$. Since $h(x) \to \infty$ as $x \to Z_f$, we may further assume that

$$\Omega = \{x : h(x) > 0\}$$

and that $h(x) \to 0$ as $x \to \partial\Omega$. For $M > 0$ write

$$E_M = \{x \in \Omega : h \ge M > 0\}.$$

Then E_M is compact and $Z_f \subset E_M$. Pick $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ in a fixed neighborhood U of Z_f. Then $E_M \subset U$ for all large M, and the support of the gradient of the function

$$v_M = \phi - \min\left\{\frac{h}{M}, 1\right\} \in W_0^{1,p}(\Omega \setminus Z_f)$$

stays away from Z_f. Because $p \ge q/(q-1)$ by assumption, (2.8) yields

$$0 = \int_{\Omega \setminus E_M} \mathscr{A}(x, \nabla h) \cdot \nabla v_M \, dx = \int_{\Omega \setminus E_M} \mathscr{A}(x, \nabla h) \cdot \left(\nabla\phi - \frac{\nabla h}{M}\right) dx$$

and hence (see (2.4) and (2.5))

$$M^{n-1} \int_{\Omega \setminus E_M} \mathscr{A}\left(x, \frac{\nabla h}{M}\right) \cdot \frac{\nabla h}{M} \, dx = \int_{\Omega \setminus E_M} \mathscr{A}(x, \nabla h) \cdot \frac{\nabla h}{M} \, dx$$

$$= \int_{\Omega \setminus E_M} \mathscr{A}(x, \nabla h) \cdot \nabla\phi \, dx = \int_{\Omega \setminus \bar{U}} \mathscr{A}(x, \nabla h) \cdot \nabla\phi \, dx = c < \infty.$$

Applying (2.1) we observe that

$$M^{n-1} \int_{\Omega \setminus E_M} \mathscr{A}\left(x, \frac{\nabla h}{M}\right) \cdot \frac{\nabla h}{M} \, dx \geqq M^{n-1} \int_{\Omega \setminus E_M} \left|\frac{\nabla h}{M}\right|^n K^{-1} \, dx$$

$$\geqq M^{n-1} \operatorname{cap}_{n, K^{-1}}(E_M, \Omega) \geqq M^{n-1} \operatorname{cap}_{n, K^{-1}}(Z_f, \Omega),$$

where we also used the fact that $K = K(x, f) \geqq 1$. Because c does not depend on M, we conclude by letting $M \to \infty$ that

$$\operatorname{cap}_{n, K^{-1}}(Z_f, \Omega) = 0.$$

Since $K \in L^r(\Omega)$ for some $r > n - 1$, we deduce that the variational s-capacity of Z_f is zero for $s = nr/(1 + r) > n - 1$, and hence that the Hausdorff dimension of Z_f is strictly less than 1; see (2.10) and (2.11).

Next, we invoke a theorem of TITUS & YOUNG [TY], [Ri, I.5.6], which asserts that every sense-preserving light mapping is discrete and open. Therefore we only need to verify that f is sense-preserving. For $p \geqq n$ this follows from Theorem 1.3, while the case $n - 1 \leqq p < n$ is established in Section 5 below. We thus conclude the proof of Theorem 1.7.

Remark. That the integrability degree of $K(x, f)$ is at least some $r > n - 1$ in Theorem 1.7 was needed only to ensure that the Hausdorff dimension of the inverse image of each point under f is less than 1. A glance at the above proof together with (2.11) implies the following result: Let $f \in \mathscr{A}_{p,q}(\Omega)$ be a continuous quasi-light mapping of finite dilatation. If $K(x, f) \in L^r(\Omega)$ for some $r > 1/(n - 1)$, then for each $y \in \mathbf{R}^n$ the Hausdorff dimension of $f^{-1}(y)$ is at most $n/(r + 1) < n - 1$.

The following simple example shows that this estimate for the Hausdorff dimension of the preimage of a point is essentially sharp. Define $f: B(0, 2) \to B(0, 1)$ by $f(x) = x(|x| - 1)$ if $1 < |x| < 2$ and $f(x) = 0$ otherwise. Then $f \in W^{1, \infty}(B(0, 2))$ and a straightforward calculation shows that $K(x, f) \in L^r(B(0, 2))$ for all $r < 1/(n - 1)$. However, $f^{-1}(0)$ is the closed unit ball.

Notice that the case $r = n - 1$ gives $\dim_H f^{-1}(y) \leqq 1$, and in particular $f^{-1}(y)$ may not be totally disconnected. Therefore our method cannot fully solve the conjecture of IWANIEC & ŠVERÁK.

4. Proof of Theorem 1.9

As in the proof of Theorem 1.7, we only need to show that $Z_f = f^{-1}(0)$ has Hausdorff dimension less than 1. The rest follows by invoking the theorem of TITUS & YOUNG and observing that f is sense-preserving (by Theorem 5.1 if $p < n$ and by Theorem 1.3 if $p \geqq n$).

Fix $x_0 \in \partial Z_f \cap \Omega$ and pick a ball $B = B(x_0, r) \subset\subset \Omega$; such an x_0 exists because f is non-constant. We shall write $\frac{1}{2} B = B(x_0, r/2)$.

Fix $n - 1 < s \leqq n$ such that

(4.1)
$$s \geqq \frac{q}{q - 1},$$

(4.2)
$$r \geqq \frac{s}{n - s}.$$

This selection is possible by the assumption of the theorem. We also understand that $s = n$ if and only if $q = n/(n - 1)$, which requires $K \in L^\infty(\Omega)$. This is the well-known case of quasiregular mappings which also are covered by our theorem, and it may be worth pointing out that the ensuing proof does not use the relatively deep, albeit well-known regularity theory of nonlinear elliptic equations.

Here and throughout this section q.e. means *quasieverywhere* with respect to the variational s-capacity meaning "everywhere except on a set of zero s-capacity". We say that $u \in W^{1,s}(B)$ is *quasicontinuous* if for each $\varepsilon > 0$ there exists an open set E of s-capacity less than ε such that the restriction of u to $B \setminus E$ is continuous. We remind the reader that each function in $W^{1,s}(B)$ has a quasicontinuous representative, and while discussing Sobolev functions we always assume that a quasicontinuous representative has been selected. See [HKM, Chapter 4] or [Z, Chapter 3].

Our strategy is to construct a quasicontinuous function $u \in W_0^{1,s}(B)$ such that $u = 1$ q.e. on $Z_f \cap \frac{1}{2} B$ and that $u = 0$ q.e. in $\frac{1}{2} B \setminus Z_f$. Because f is continuous and because $x_0 \in \partial Z_f \cap \Omega$, this clearly implies that $\mathrm{cap}_s Z_f = 0$ and hence, by (2.11), that the Hausdorff dimension of Z_f is less than 1.

We begin by introducing the function space

$$H = \left\{ u \in W_0^{1,1}(B) : \int_B F(x, \nabla u) \, dx < \infty \right\}$$

and putting

$$\|u\|_{F,s} = \left(\int_B |u|^s \, dx \right)^{1/s} + \left(\int_B F(x, \nabla u) \, dx \right)^{1/n}.$$

Here, as in Section 2,

$$F(x, \xi) = \langle G(x) \xi, \xi \rangle^{n/2} = |\Theta(x) \xi|^n,$$

$G(x) = \Theta(x)^T \Theta(x)$, where $\Theta(x) = J_f(x)^{1/n} [Df(x)]^{-1^T}$ if $J_f(x) \neq 0$ exists and $\Theta(x) = \mathrm{Id}$ otherwise. Now $F(x, \xi)$ is strictly convex and homogeneous of degree n in ξ, and we recall from (2.1) and (2.3) that

(4.3)
$$|\xi|^n K(x)^{-1} \leqq F(x, \xi) \leqq |\xi|^n K(x)^{n-1}$$

for almost every x, where $K(x) = K(x, f)$. In particular, $F(x, \nabla \psi) \in L^1(B)$ for all $\psi \in C_0^\infty(B)$; moreover, for all $u \in H$,

$$\int_B |\nabla u|^s \, dx = \int_B |\nabla u|^s \, K^{-s/n} K^{s/n} \, dx$$

$$\leq \left(\int_B |\nabla u|^n K^{-1} \, dx \right)^{s/n} \left(\int_B K^{s/(n-s)} \, dx \right)^{(n-s)/n}$$

$$\leq \|K\|_{s/(n-s)}^{s/n} \left(\int_B F(x, \nabla u) \, dx \right)^{s/n}.$$

Consequently, since

$$K \in L^{s/(n-s)}(\Omega)$$

by (4.2), we have the embedding $H \hookrightarrow W_0^{1,s}(B)$.

More precisely:

4.4. Lemma. *The space H is a reflexive Banach space under the norm $\|\cdot\|_{F,s}$, continuously embedded in $W_0^{1,s}(B)$.*

Proof. Since

$$\|u\|_{F,s} = \|u\|_s + \left(\int_B |\Theta \nabla u|^n \, dx \right)^{1/n},$$

it is clear that $\|\cdot\|_{F,s}$ defines a norm.

To show that H is complete, let (u_j) be a Cauchy sequence in H. Then, as we noticed above, (u_j) is a Cauchy sequence in $W_0^{1,s}(B)$, and thus

$$u_j \to u \in W_0^{1,s}(B)$$

in $W_0^{1,s}(B)$. Also

$$\int_B |\Theta \nabla u_j - \Theta \nabla u_i|^n \, dx = \int_B F(x, \nabla u_j - \nabla u_i) \, dx,$$

and so $(\Theta \nabla u_j)$ is a Cauchy sequence in $L^n(B; \mathbf{R}^n)$. Let $v \in L^n(B; \mathbf{R}^n)$ satisfy

$$\Theta \nabla u_j \to v \quad \text{in } L^n(B; \mathbf{R}^n)$$

and write $w = \Theta^{-1} v$; notice that $\Theta(x)^{-1}$ exists for all $x \in B$ by the definition of $\Theta(x)$. Now

(4.5) $$\int_B F(x, w) \, dx = \int_B |\Theta w|^n \, dx = \int_B |v|^n \, dx < \infty.$$

Moreover,

$$\int_B F(x, \nabla u_j - w) \, dx = \int_B |\Theta(\nabla u_j - w)|^n \, dx$$

$$= \int_B |\Theta \nabla u_j - v|^n \, dx \to 0$$

as $j \to \infty$. In particular, we conclude from the embedding $H \hookrightarrow W_0^{1,s}(B)$ that

$$\nabla u_j \to w \quad \text{in } L^s(B, \mathbf{R}^n).$$

Thus $w = \nabla u$, $u \in W_0^{1,s}(B)$, $\int_B F(x, \nabla u)\, dx < \infty$ by (4.5), and (u_j) converges to u in H. This shows that H is complete.

Finally, the norm $\|\cdot\|_{F,s}$ is clearly uniformly convex, whence H is reflexive; see, *e.g.*, [HS, p. 224–225]. The lemma follows.

For a positive integer k let

$$E_k = \{h \geq k\} \cap \tfrac{1}{2} \bar{B},$$

and set

$$I_k = \inf_{\mathcal{F}_k} \int_B F(x, \nabla u)\, dx,$$

where

$$\mathcal{F}_k = \{u \in H : u \geq 1 \text{ q.e. on } E_k\}.$$

By (4.3), the family $\mathcal{F}_k$ is non-empty. We next establish the existence of a unique minimizer in $\mathcal{F}_k$. This follows from the fact that H is reflexive; for completeness we provide the standard details.

Pick a sequence (u_j) of functions in $\mathcal{F}_k$ for which

$$I_k = \lim_{j \to \infty} \int_B F(x, \nabla u_j)\, dx.$$

Since $F(x, 0) = 0$ for almost every x, we may also assume that $0 \leq u_j \leq 1$ q.e. in B. Then (u_j) is a bounded sequence in the reflexive Banach space H, and so passing to a subsequence if necessary we find that

$$u_j \rightharpoonup v \in H.$$

By the weak lower semicontinuity of norms, we obtain

$$I_k = \liminf_{j \to \infty} \int_B F(x, \nabla u_j)\, dx \geq \int_B F(x, \nabla v)\, dx.$$

Mazur's lemma guarantees that a sequence of convex combinations (v_j) of u_j converges to v in $\|\cdot\|_{F,s}$ and by the embedding $H \hookrightarrow W_0^{1,s}(B)$ we deduce that $v \in W_0^{1,s}(B)$; moreover, $v_j \to v$ in $W_0^{1,s}(B)$.

Passing to a subsequence if necessary we may assume that the sequence (v_j) converges to v pointwise q.e. in B (see [HKM, Theorem 4.3]), and hence $v = 1$ q.e. on E_k. In consequence, $v \in \mathcal{F}_k$, and we rename it as u_k. We thus have

$$(4.6) \qquad I_k = \int_B F(x, \nabla u_k)\, dx.$$

where $u_k \in W_0^{1,s}(B)$ satisfies $u_k = 1$ q.e. on E_k.

For each $k = 1, 2, \ldots$ we have now obtained a minimizer u_k as in (4.6). By using the strict convexity of F, we easily obtain the uniqueness:

$$(4.7) \qquad \int_B F(x, \nabla u_k)\, dx < \int_B F(x, \nabla v)\, dx$$

for all $v \in \mathcal{F}_k$ different from u_k.

As earlier, write $h(x) = -\log|f(x)|$.

4.8. Lemma. $u_k \leq h/k$ *q.e. in* $B \setminus E_k$ *for each* $k \geq 1$.

Proof. Write $v = ku_k$ and put $U = \{v > h\}$. Recall that v is assumed to be quasicontinuous so that U is a measurable set which is well-defined up to zero s-capacity. We may assume that $h > 0$ on ∂B. Since $v = 0$ on ∂B and since $0 \leqq v \leqq k$, we conclude that $U \subset B \setminus E_k$. In particular, $U \subset\subset \Omega \setminus Z_f$. Define

$$w = \min\{v, h\}.$$

Then $w \in W_0^{1,s}(B)$, because

$$F(x, \nabla h) = \frac{J_f(x)}{|f(x)|^{n-2}}$$

by (2.5). In fact, $w = h$ q.e. on U, $w = v$ q.e. on $B \setminus U$, and

$$\int_B F(x, \nabla w) \, dx = \int_U F(x, \nabla h) \, dx + \int_{B \setminus U} F(x, \nabla v) \, dx < \infty.$$

Thus

$$(4.9) \qquad \int_B F(x, \nabla v) \, dx < \int_B F(x, \nabla w) \, dx$$

by homogeneity and the uniqueness (4.7), provided $|U| > 0$.

If $|U| = 0$, there is nothing to prove because h is continuous and v is quasicontinuous (see [HKM, Theorem 4.12]). Thus suppose that $|U| > 0$. Then

$$\int_B F(x, \nabla w) \, dx = \int_{B \setminus U} F(x, \nabla v) \, dx + \int_U F(x, \nabla h) \, dx$$

$$> \int_{B \setminus U} F(x, \nabla v) \, dx + \int_U F(x, \nabla v) \, dx$$

by (4.9), and hence

$$(4.10) \qquad \int_U F(x, \nabla v) \, dx < \int_U F(x, \nabla h) \, dx.$$

On the other hand,

$$(4.11) \qquad \int_U F(x, \nabla v) - F(x, \nabla h) \, dx \geqq \int_U \nabla_\xi F(x, \nabla h) \cdot (\nabla v - \nabla h) \, dx$$

$$= n \int_B \mathscr{A}(x, \nabla h) \cdot (\nabla v - \nabla h) \, dx$$

by convexity (see [HKM, p. 98]). Next, notice that

$$w' = \max\{v - h, 0\} \in W_0^{1,s}(B \setminus E_k);$$

indeed, w' can be extended to a quasicontinuous $W^{1,s}$-function in $\mathbf{R}^n$ by setting $w' = 0$ in $E_k \cup (\mathbf{R}^n \setminus B)$, and hence the assertion follows from [HKM, 4.5].

Because $s \geqq q/(q-1)$ by (4.1), it follows that

$$\int_B \mathscr{A}(x, \nabla h) \cdot \nabla w' \, dx = 0$$

by (2.8). From the definition of U we conclude that

$$\int_U \mathscr{A}(x, \nabla h) \cdot (\nabla v - \nabla h) \, dx = 0,$$

and combining this with (4.11) yields

$$\int_U F(x, \nabla h)\, dx \leqq \int_U F(x, \nabla v)\, dx,$$

which contradicts (4.10). We have thus proved that $|U| = 0$, and the lemma follows.

We have constructed a sequence (u_k) of functions in $W_0^{1,s}(B)$ such that

$$u_k = 1 \quad \text{q.e. on } E_k,$$

$$u_k \leqq h/k \quad \text{q.e. on } B \setminus E_k.$$

Moreover, this sequence is bounded in $W_0^{1,s}(B)$, and hence a sequence of its convex combinations converges to some $u_0 \in W_0^{1,s}(B)$. Passing to a subsequence allows us to assume that the convergence occurs pointwise q.e. in B and that u_0 is quasicontinuous. It follows that $u_0 = 1$ q.e. on $Z_f \cap \frac{1}{2} \bar{B}$ and $u_0 = 0$ q.e. on $\frac{1}{2} B \setminus Z_f$, as desired.

This completes the proof of Theorem 1.9.

5. Sense-Preserving Sobolev Mappings

Suppose that f is a continuous mapping in $W^{1,p}(\Omega, \mathbf{R}^n)$, where Ω is an arbitrary open subset of $\mathbf{R}^n$. For simplicity, we assume that f is not constant in any of the components of Ω.

If $p \geqq n$, relatively mild analytic conditions guarantee that f is a sense-preserving mapping; it suffices, for instance, to assume that f is of finite dilatation (see Theorem 1.3). For $p < n$ no simple criteria seem to be available, although it would be desirable to have some. In this section we provide a condition under which a mapping $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ of finite dilatation is sense-preserving. The considerations in [Š] most likely would yield more general results in this direction, but we prefer a more elementary argument sufficient to the purposes of this paper.

5.1. Theorem. *Suppose that $f \in \mathscr{A}_{p,q}(\Omega)$ is a continuous mapping of finite dilatation, where $p \geqq n - 1$ and $q \geqq p/(p - 1)$. If the 1-dimensional Hausdorff measure of $f^{-1}(y)$ is zero for each $y \in \mathbf{R}^n$, then f is sense-preserving.*

Proof. Fix a domain $D \subset\subset \Omega$ and $y \in f(D) \setminus f(\partial D)$. We must show that $\mu(y, f, D) > 0$. Let $E = f^{-1}(y) \cap D$; then E is compact and has zero 1-dimensional Hausdorff measure. In particular, projections of E into the coordinate axes all have zero 1-dimensional Hausdorff measure, and a simple argument then shows that E can be covered by a finite number of open n-cubes $Q_1, \ldots, Q_N$ in D with pairwise disjoint closures such that $\partial Q_i \cap E = \emptyset$ for all $i = 1, \ldots, N$. (See [G, Lemma 1] for a more precise argument.) Next, by precomposing f with an appropriate bi-Lipschitz mapping we can assume without loss of generality that E is covered by open balls $B_1, \ldots, B_N$ whose closures are pairwise disjoint and lie in D such that $\partial B_i \cap E = \emptyset$ for all

$i = 1, \ldots, N$. Moreover, by invoking [Š, Proposition 1 and Theorem 3] we can further assume that, for all $i = 1, \ldots, N$,

$$(5.2) \qquad \int_{B_i} u(f(x)) J(x, f)\, dx = \mu(y, f, B_i)$$

whenever u is a smooth function, supported in the y-component of $\mathbf{R}^n \setminus f(\partial B_i)$ with $\int u\, dy = 1$. Here we need the assumption that $f \in \mathscr{A}_{p,q}(\Omega)$.

Since $E \subset \bigcup_i B_i$ and since the balls are disjoint, by well-known properties of the topological index (see [Ri, I 4.4] or [R, p. 88]) we have that

$$\mu(y, f, D) = \sum_{i=1}^{N} \mu(y, f, B_i)\,.$$

Consequently, it suffices to show that, for all $i = 1, \ldots, N$,

$$\int_{B_i} u(f(x)) J(x, f)\, dx > 0$$

for some u as in (5.2).

To this end, fix i and consider an open ball $B_\varepsilon = B(y, \varepsilon)$, where $\varepsilon > 0$ is so small that $\bar{B}_\varepsilon$ belongs to the y-component of $\mathbf{R}^n \setminus f(\partial B_i)$. Let D_ε be a component of the open set $f^{-1}(B_\varepsilon) \cap B_i$ containing points from E (we may naturally assume that $B_i \cap E \neq \emptyset$). It clearly suffices to show that the measure of $\{z \in D_\varepsilon : J(z, f) > 0\}$ is positive. Suppose on the contrary that $J(z, f) = 0$ a.e. in D_ε. Then $Df = 0$ a.e. in D_ε since f has finite dilatation, and the absolute continuity on lines of Sobolev functions (see [Z, p. 44]) now implies that f is constant in D_ε; more precisely, $f(D_\varepsilon) = \{y\}$. An easy argument using the continuity of f then shows that D_ε must coincide with B_i, but this is a contradiction since $y \in \mathbf{R}^n \setminus f(\partial B_i)$. The theorem follows.

Remarks. (a) For the above proof it is essential that $f^{-1}(y)$ has zero 1-dimensional Hausdorff measure. In fact, the covering argument may fail if we only assume that $f^{-1}(y)$ is totally disconnected, as can be seen by considering the well-known Antoine's necklace (a compact, totally disconnected set in $\mathbf{R}^3$ whose complement is not simply-connected).

(b) The techniques in [R, p. 173] and [MZ] can be combined to obtain further conditions which guarantee that a Sobolev mapping $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is sense-preserving for $p < n$. For instance, consider the following *Jacobian approximation property*: $J(x, f) \in L^1(\Omega)$ and there exists a sequence of smooth mappings $f_i : \Omega \to \mathbf{R}^n$ such that $f_i \to f$ locally uniformly in Ω and $J(x, f_i) \to J(x, f)$ locally in $L^1(\Omega)$.

One can show that if $p \geq 1$ and if $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is of finite dilatation and has the Jacobian approximation property, then f is sense-preserving. It would be interesting to find some analytic conditions on f which are weaker than being in $W^{1,n}(\Omega, \mathbf{R}^n)$ and which would guarantee the Jacobian approximation property.

6. Homeomorphisms with Integrable Dilatation

In this section we assume that f is a mapping in $W^{1,n}(\Omega, \mathbf{R}^n)$ with finite dilatation $K(\cdot, f)$. For simplicity, we further assume that Ω is bounded and connected. Then Theorem 1.3 implies that f is continuous and satisfies condition (N), that is, f maps sets of measure zero into sets of measure zero.

Definition. We say that a continuous mapping $f: \Omega \to \mathbf{R}^n$ satisfies condition (N^{-1}) if $|f^{-1}(E)| = 0$ whenever $|E| = 0$ for a measurable set $E \subset \mathbf{R}^n$.

It is not difficult to give examples of Lipschitz homeomorphisms of finite dilatation which do not satisfy condition (N^{-1}). However, quasiregular mappings satisfy (N^{-1}) and therefore it is natural to ask whether some integrability condition on $K(\cdot, f)$ implies the same conclusion, at least when f satisfies certain natural topological assumptions such as discreteness and openness. In the quasiregular case this is usually done by first establishing that the *branch set* B_f of f has zero measure and then exploiting the fact that the inverse of a quasiconformal mapping (*i.e.*, a homeomorphic quasiregular mapping) is also quasiconformal. Recall that the branch set B_f is the set of all points where f does not define a local homeomorphism.

We have the following result:

6.1. Theorem. *Suppose that $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is a homeomorphism with dilatation $K(x, f) \in L^{n-1}(\Omega)$. Then f^{-1} belongs to $W^{1,n}(f(\Omega))$ and is of finite dilatation with $K(y, f^{-1}) = K(f^{-1}(y), f)^{n-1}$ for almost every $y \in f(\Omega)$. In particular, $J(x, f) > 0$ for almost every $x \in \Omega$ and f satisfies condition (N^{-1}).*

From this result one easily deduces:

6.2. Corollary. *If $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ satisfies $K(x, f) \in L^{n-1}(\Omega)$, then $|B_f| = 0$ if and only if $J(x, f) > 0$ for almost every $x \in \Omega$.*

We do not know whether $|B_f| = 0$ even if it is assumed that $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is discrete and open and $K(x, f) \in L^r(\Omega)$ for some $r < \infty$.

Proof. To prove the first assertion we use the reasoning in [BI, Theorem 9.1, p. 314]. In fact, we employ the change of variables formula:

$$(6.3) \qquad \int_\Omega u(f(x)) J(x, f)\, dx = \int_{f(\Omega)} u(y)\, dy$$

for all nonnegative measurable u in $f(\Omega)$ such that the integrals converge. This follows, *e.g.*, from [V2, Theorem 24.8] by standard arguments; note that we need both the fact that f satisfies condition (N) and that $J(x, f) = 0$ for almost every x in any set $E \subset \Omega$ such that $|f(E)| = 0$ (the latter fact follows from [MZ, Theorem 3.1]). Now with formula (6.3) in hand, we can apply the argument in [BI, pp. 314, 315] verbatim to conclude

$$(6.4) \qquad \int_{f(\Omega)} D\varphi(y)^T g(y)\, dy = -\int_\Omega \operatorname{adj} Df(x)^T \varphi(f(x))\, dx$$

for any mapping $\varphi \in C_0^\infty (f(\Omega), \mathbf{R}^n)$, where $g = f^{-1}$. By basic linear algebra, we have the inequality

$$|\operatorname{adj} Df(x)| = |Df(x)^{-1} J(x, f)| \leq K(x, f)^{(n-1)/n} J(x, f)^{(n-1)/n}$$

for almost every x (see, *e.g.*, [R, p. 59]); note that here again we use the fact that f is of finite dilatation, which implies that $Df(x) = \operatorname{adj} Df(x) = 0$ for almost every x such that $J(x, f) = 0$. Thus by (6.4) and (6.3)

$$\left| \int\limits_{f(\Omega)} D\varphi(y)^T g(y)\, dy \right| \leq \int\limits_{\Omega} K(x, f)^{(n-1)/n} J(x, f)^{(n-1)/n} |\varphi(f(x))|\, dx$$

$$\leq \left(\int\limits_{\Omega} K(x,f)^{n-1}\, dx \right)^{1/n} \left(\int\limits_{\Omega} |\varphi(f(x))|^{n/(n-1)} J(x,f)\, dx \right)^{(n-1)/n}$$

$$\leq \left(\int\limits_{\Omega} K(x,f)^{n-1}\, dx \right)^{1/n} \left(\int\limits_{f(\Omega)} |\varphi(y)|^{n/(n-1)}\, dy \right)^{(n-1)/n}$$

for all mappings $\varphi \in C_0^\infty (f(\Omega), \mathbf{R}^n)$. This implies (see [BI, (1.23)]) that $g = f^{-1} \in W^{1,n}(f(\Omega), \mathbf{R}^n)$ with

(6.5) $$\int\limits_{f(\Omega)} |Dg(y)|^n\, dy \leq \int\limits_{\Omega} K(x, f)^{n-1}\, dx.$$

A theorem of Reshetnyak [R, Corollary 1, p. 182] asserts that homeomorphisms in $W^{1,n}$ satisfy condition (N). Therefore the change of variables formula holds for f^{-1} as well, and we find

$$\int\limits_{\Omega} K(x, f)^{n-1}\, dx = \int\limits_{f(\Omega)} K(f^{-1}(y), f)^{n-1} J(y, f^{-1})\, dy.$$

Next, because (6.5) holds true for any subdomain of Ω, we arrive at the pointwise estimate

$$|Dg(y)|^n \leq K(f^{-1}(y), f)^{n-1} J(y, g)$$

for almost every $y \in f(\Omega)$, and it remains to observe that $K(y, g)$ is finite a.e. in $f(\Omega)$ because $K(x, f)$ is finite a.e. in Ω and f satisfies condition (N). This completes the proof of both the first assertion of the theorem and the assertion that f satisfies condition (N^{-1}).

Finally, that $J(x, f) > 0$ a.e. follows from the change of variables formula (6.3) because f satisfies condition (N^{-1}).

Remark. To illustrate the sharpness of Theorem 6.1, consider the mapping $f(x_1, x_2) = (x_1, x_2^\alpha)$, $\alpha > 1$, in the unit disk $\mathbf{B}^2$ in $\mathbf{R}^2$. A calculation shows that $K(x, f) \in L^r(\mathbf{B}^2)$ for all $r < 1/(\alpha - 1)$, but $f^{-1} \in W^{1,2}(\mathbf{B}^2)$ only if $\alpha < 2$.

Note added in proof. Recently J. Manfredi & E. Villamor have announced a positive answer to the conjecture stated at the end of the Introduction.

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