The Dirichlet Problem for the Helmholtz Equation

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Contents

1. Introduction

In 1897 Lord RAYLEIGH [*16*] considered the relationship between potential problems (boundary value problems for the Laplace equation) and scattering problems (boundary value problems for the Helmholtz equation). In a typically virtuoso performance, RAYLEIGH considered two dimensional as well as three dimensional problems in the electromagnetic (vector) as well as acoustic (scalar) case. In particular, he showed that the potential of an obstacle in a uniform field not only was the near field limit of the solution of the corresponding scattering problem but also could yield the first term of an expansion of the far field. He gave explicit results for a general ellipsoidal scatterer including many limiting cases of interest such as the sphere, spheroid and disc.

Since that time considerable effort has been spent in deriving higher order terms in the expansion for these as well as other shapes. Long sought in this work is the development of a systematic procedure which will generate the solution of the Helmholtz equation, satisfying a particular boundary condition, from the solution of Laplace's equation which satisfies the same boundary condition. It is toward the achievement of this goal that the present work is directed.

The major drawback in most of the methods proposed heretofore is their intrinsic dependence on a particular geometry. That is, the techniques result from the (often adroit) exploitation of the geometric properties of the surface on which the boundary conditions are specified. Thus, restricting attention to three dimensional scalar problems, we find a variety of methods for obtaining the low frequency expansion for a disc (and an aperture in a plane screen); see BOUWKAMP [*4*] and NOBLE [*15*] for an extensive bibliography to which we may add HEINS [*7*], DE HOOP [*9*], SENIOR [*18*], and WILLIAMS [*28*]. However, success in generalizing these techniques has been limited to a class of axially symmetric problems (COLLINS [*5*], HEINS [*8*], and WILLIAMS [*29*]), and explicit results have been obtained only for a spherical cap (COLLINS [*5*] and THOMAS [*23*]). For those shapes where the Helmholtz equation is separable, of course, the low frequency expansion may always be obtained from the series solution provided sufficient knowledge of the special functions involved is available. A method for obtaining low frequency expansions for bodies which are intersections of such "separable" shapes has been proposed by DARLING [6] though as yet has been applied only to a spherically capped cone (SENIOR & DARLING [19]).

Most low frequency techniques, however, have as their starting point the formulation of scattering problems as integral equations using the Helmholtz representation of the solution in terms of its properties on the boundary and the free space Green's function, *e.g.* BAKER & COPSON [2]. This formulation is also vital to the proof of the existence of solutions for a general boundary given by WEYL [25] and MÜLLER [14] as well as that of WERNER [24]. NOBLE [15] shows how this integral formulation may be used to obtain a representation of the solution of a scattering problem for a general boundary as a perturbation of the solution of the corresponding potential problem. Each term in the low frequency expansion is the solution of an integral equation which differs only in its inhomogeneous part from term to term. However, this formulation does not yield an explicit representation for successive terms in general except as the formal inverse of an operator.

The present work describes a method whereby the solution of the general Dirichlet problem for the three dimensional Helmholtz equation is explicitly expressed in terms of the Green's function for the corresponding potential problem. A new integral equation for the scattered field is derived whose kernel is the potential Green's function for the surface instead of the free space Green's function for the Helmholtz equation. Despite the fact that the integral operates over all space, rather than just the scattering surface, and is really an integro differential operator, it is still possible to solve the equation iteratively in a standard Neumann expansion which has a nonzero radius of convergence and may be interpreted as a partial summation of the low frequency expansion. The results are valid for complex as well as real values of wave number, k, with no restriction on the sign of the imaginary part provided k is sufficiently small in absolute value. The present work also provides a constructive proof of the existence and uniqueness of solutions of the Dirichlet problem for the Helmholtz equation based on the existence and uniqueness of the potential Green's function.

The results stem from an integral representation of functions which are regular at infinity in the sense of KELLOGG [10]. This representation, which is a direct consequence of Green's theorem, is derived in Section 2. Wave functions, *i.e.* solutions of the Helmholtz equation which satisfy a radiation condition, are not regular. However, it is possible, using an expansion theorem (WILCOX [26]), to modify them so that the representation theorem applies. This is done in Section 3 where a new integral equation for the Green's function for the Helmholtz equation is derived. In Section 4 it is shown that this equation may be solved iteratively as a Neumann series and that the series converges for small enough values of the wave number. The relation between this series and the Rayleigh expansion is given in Section 5. As an illustration and a check, the methods are applied to the classic problem of scattering by a sphere in Section 6. This example serves not only to corroborate the analysis but also provides further insight into the manner in which the truncated Neumann series, *i.e.* the N^{th} iterate, approximates the solution.

2. A Representation Theorem

In this section we first adopt notation and record some definitions, then state and prove an important representation theorem.

Let B be the boundary of a smooth, closed, bounded surface in E^3 (or the union of a finite number of such surfaces provided they are disjoint), and let V be the volume exterior to B. Erect a spherical polar coordinate system with origin interior to B, and denote by p a point (r, ϑ, φ) in V and p_B a point $(r_B, \vartheta_B, \varphi_B)$ on B. The distance between any two points $p, p_1 \in \bar{V} = V \cup B$ will be denoted by $R(p, p_1)$. Explicitly

$$R(p, p_1) = \sqrt{r^2 + r_1^2 - 2 r r_1 [\cos\vartheta \cos\vartheta_1 + \sin\vartheta \sin\vartheta_1 \cos(\varphi - \varphi_1)]}. \qquad (2.1)$$

The assumption that B is bounded means that r_B has a finite maximum value, c, thus

$$c \equiv \max_{p \in B} r = \max r_B. \qquad (2.2)$$

The entire surface B may therefore be enclosed in a sphere of radius c*.

Furthermore, adopting the convention that the normal to B at any point is inward (directed out of V), the smoothness requirements on B are such as to ensure the existence of a unique normal at each point on B. The geometry is illustrated in Fig. 1.

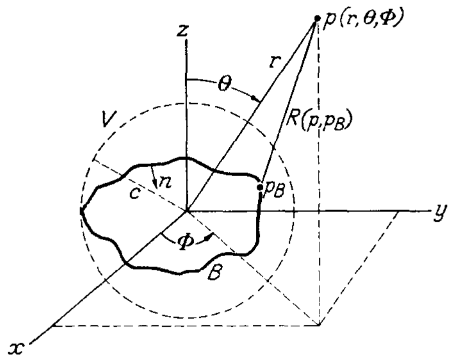

Fig. 1

Following KELLOGG [*10*, p. 217] a real valued function, $f: V \to$ Reals, is defined to be regular at infinity if

$$\lim_{r\to\infty} |r f(p)| < \infty \quad \text{and} \quad \lim_{r\to\infty} \left| r^2 \frac{\partial f(p)}{\partial r} \right| < \infty, \quad \begin{matrix} 0 \leqq \vartheta \leqq \pi. \\ 0 \leqq \varphi \leqq 2\pi. \end{matrix} \qquad (2.3)$$

Implicit in this definition of regularity is the fact that $f(p)$ must be differentiable for r sufficiently large. A complex valued function will be regular at infinity if both real and imaginary parts are regular.

Next we define the static or potential Green's function of the first kind (Dirichlet boundary condition) for the surface B to be a function $G_0(p, p_0)$ of two points such that

a) $\nabla^2 G_0(p, p_0) = \delta[R(p, p_0)], \qquad p, p_0 \in V,$

b) $G_0(p_B, p_0) = 0, \qquad (2.4)$

c) $G_0(p, p_0)$ is regular at infinity, *i.e.*, satisfies (2.3).

The existence and uniqueness of $G_0(p, p_0)$ for any surface B as restricted above is proven by KELLOGG. The normalization of $\delta(R)$ is such that the free space Green's function (when B is the null surface) is $-\frac{1}{4\pi R(p, p_0)}$. Thus G_0 admits of the following decomposition into "singular" and "regular" parts:

$$G_0(p, p_0) = -\frac{1}{4\pi R(p, p_0)} + U_0(p, p_0) \qquad (2.5)$$

* Since the center is required to be interior to B for many concave surfaces, c will be larger than the radius of the smallest sphere entirely enclosing B.

where U_0 is a solution of the homogeneous Laplace equation, *i.e.*, U_0 has no singularities in $\bar{V}$, and is regular at infinity. (The notation G_0 is consistent with the subsequent use of G_k to denote the Green's function for the Helmholtz equation).

With these definitions and conventions established we may state and prove the following representation theorem.

Theorem 2.1. *Any function $\omega(p)$, defined for all $p \in \bar{V}$, which is twice differentiable, $\omega \in c^2(V)$, and regular at infinity satisfies the integral equation*

$$\omega(p) = \int_V G_0(p, p_1)\, \nabla^2 \omega(p_1)\, dv + \int_B \omega(p_B) \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma \tag{2.6}$$

where dv is the volume element and ∇^2 the Laplacian expressed in coordinates $(r_1, \vartheta_1, \varphi_1)$; $d\sigma$ is the surface element and $\partial/\partial n$ the inward normal derivative (out of V) expressed in coordinates $(r_B, \vartheta_B, \varphi_B)$.

Proof. We state Green's second identity in the following form: if V' is the volume exterior to B and interior to a large sphere, B_1, entirely enclosing B, then

$$\begin{aligned}\int_{V'} [\omega(p_1)\, \nabla^2 G_0(p, p_1) - G_0(p, p_1)\, \nabla^2 \omega(p_1)]\, dv \\ = \int_{B+B_1} \left[\omega(p_B) \frac{\partial}{\partial n} G_0(p, p_B) - G_0(p, p_B) \frac{\partial}{\partial n} \omega(p_B)\right] d\sigma.\end{aligned} \tag{2.7}$$

With (2.4a) we see that

$$\omega(p) = \int_{V'} \omega(p_1)\, \nabla^2 G_0(p, p_1)\, dv. \tag{2.8}$$

Note that this result may be obtained without using the sifting property of the δ-function by excluding from V' a small sphere around the point p, explicitly evaluating the resulting additional surface integral, and then letting the radius go to zero.

Using this result, together with the boundary condition (2.4b) satisfied by $G_0(p, p_B)$, we put (2.7) into the form

$$\begin{aligned}\omega(p) = \int_{V'} G_0(p, p_1)\, \nabla^2 \omega(p_1)\, dv + \int_B \omega(p_B) \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma + \\ + \int_{B_1} \left[\omega(p_{B_1}) \frac{\partial}{\partial n} G_0(p, p_B) - G_0(p, p_{B_1}) \frac{\partial \omega}{\partial n}(p_{B_1})\right] d\sigma.\end{aligned} \tag{2.9}$$

Since both ω and G_0 are regular, the last term on the right in (2.9) vanishes as $r_{B_1} \to \infty$. Explicitly,

$$\begin{aligned}&\lim_{r_{B_1} \to \infty} \left| \int_{B_1} \left[\omega \frac{\partial G_0}{\partial n} - G_0 \frac{\partial \omega}{\partial n}\right] d\sigma \right| \\ &\leqq \lim_{r_{B_1} \to \infty} \int_0^{2\pi} d\varphi_{B_1} \int_0^{\pi} d\vartheta_{B_1}\, r_{B_1}^2 \sin\vartheta_{B_1} \times \left\{\left|\omega \frac{\partial G_0}{\partial r_{B_1}}\right| + \left|G_0 \frac{\partial \omega}{\partial r_{B_1}}\right|\right\} \\ &\leqq \lim_{r_{B_1} \to \infty} \int_0^{2\pi} d\varphi_{B_1} \int_0^{\pi} d\vartheta_{B_1} \sup_{\vartheta_{B_1}, \varphi_{B_1}} r_{B_1}^2 \sin\vartheta_{B_1} \times \left\{\left|\omega \frac{\partial G_0}{\partial r_{B_1}}\right| + \left|G_0 \frac{\partial \omega}{\partial r_{B_1}}\right|\right\} \\ &\leqq \lim_{r_{B_1} \to \infty} \sup_{\vartheta_{B_1}, \varphi_{B_1}} 2\pi^2 \left\{|\omega| \left|r_{B_1}^2 \frac{\partial G_0}{\partial r_{B_1}}\right| + |G_0| \left|r_{B_1}^2 \frac{\partial \omega}{\partial r_{B_1}}\right|\right\}.\end{aligned} \tag{2.10}$$

Rewriting (2.10) in the following way,

$$\begin{aligned}\lim_{r_{B_1}\to\infty}\left|\int\limits_{B_1}\left[\omega\frac{\partial G_0}{\partial n}-G_0\frac{\partial\omega}{\partial n}\right]d\sigma\right|\\ \leqq \lim_{r_{B_1}\to\infty}\sup_{\vartheta_{B_1},\varphi_{B_1}}\frac{2\pi^2}{r_{B_1}}\left\{\left|r_{B_1}\omega\right|\left|r_{B_1}^2\frac{\partial G_0}{\partial r_{B_1}}\right|+\left|r_{B_1}G_0\right|\left|r_{B_1}^2\frac{\partial\omega}{\partial r_{B_1}}\right|\right\},\end{aligned} \tag{2.11}$$

makes it clear that the bracketed quantity on the right is bounded since both ω and G_0 satisfy (2.3); thus it follows that

$$\lim_{r_{B_1}\to\infty}\left|\int\limits_{B_1}\left[\omega\frac{\partial G_0}{\partial n}-G_0\frac{\partial\omega}{\partial n}\right]d\sigma\right|=0. \tag{2.12}$$

Furthermore, in this limit $V'\to V$. Thus by letting $r_{B_1}\to\infty$ in equation (2.9), Theorem 2.1 is established.

3. Representation of Wave Functions

In this section we define the Green's function of the first kind for the Helmholtz equation and show how it may be written using the representation theorem of the previous section.

First we define the Green's function of the first kind for the surface B (restricted as before) to be a function $G_k(p, p_0)$ of two points such that

$$\begin{aligned}&\text{a)}\quad (\nabla^2+k^2)\,G_k(p,p_0)=\delta[R(p,p_0)], \qquad p,p_0\in V,\\ &\text{b)}\quad G_k(p_B,p_0)=0,\\ &\text{c)}\quad \lim_{r\to\infty} r\left(\frac{\partial G_k}{\partial r}-ikG_k\right)=0, \quad \text{uniformly in } \vartheta,\varphi.\end{aligned} \tag{3.1}$$

The δ-function normalization and radiation condition are such that the free space Green's function is $-\dfrac{e^{ikR(p,p_0)}}{4\pi R(p,p_0)}$. Thus G_k admits of a decomposition into singular (incident) and regular (scattered) parts

$$G_k(p,p_0)=-\frac{e^{ikR(p,p_0)}}{4\pi R(p,p_0)}+U_k(p,p_0) \tag{3.2}$$

where $U_k(p, p_0)$ is a solution of the homogeneous Helmholtz equation, *i.e.*, U_k has no singularities in $\bar{V}$ and is in fact twice differentiable, and satisfies the radiation condition (3.1c) at infinity.

If the Helmholtz equation is considered as derived from the wave equation, reduced by assuming a harmonic time dependence, which is then suppressed, then the present case corresponds to a multiplicative time factor $e^{-i\omega t}$. The comparable expressions for a time dependence $e^{i\omega t}$ are found by replacing k by $-k$ throughout.

The radiation condition is given in the form suggested by SOMMERFELD [*20*]. This statement is stronger than necessary (*cf.* RELLICH [*17*], WILCOX [*26*]), but is sufficient for our purpose and, in fact, has been shown to be equivalent to the weaker statements (WILCOX [*27*]).

The uniqueness of the Green's function was proven by SOMMERFELD [*20*] with additional assumptions which were subsequently removed by MAGNUS [*11, 12*], RELLICH [*17*], ATKINSON [*1*] and WILCOX [*26*]. The more difficult question of existence was resolved only recently by WEYL [25] and MÜLLER [14]. The present work offers an alternate proof of existence and uniqueness of the Green's function $G_k(p, p_0)$, predicated on the existence and uniqueness of the static Green's function $G_0(p, p_0)$ which is hesitantly termed a simplification. Whether it is indeed a simplification is a minor matter, however, since the major new result of the present work is the explicit representation of the Green's function $G_k(p, p_0)$ in terms of the static Green's function. Existence of $G_k(p, p_0)$ is an extra dividend, proven by the rather compelling argument of actually producing it.

It is our intent to represent the regular part of the Green's function, which we henceforth refer to as the scattered field, using the theorem of the previous section. However, as it stands this function, $U_k(p, p_0)$, does not satisfy the requirements of the theorem since it is not regular at infinity in the proper sense, although it does satisfy the radiation condition. The behavior is evident from an expansion theorem which also provides an obvious means of modifying the scattered field so as to satisfy the requirements of Theorem 2.1. The expansion theorem, given in its most general form by WILCOX [26] was first proven in a more restrictive case by ATKINSON [1] and later extended by BARRAR & KAY [3] (see also SOMMERFELD [21, p. 191]). It may be stated for our present purposes as follows:

Theorem 3.1 (ATKINSON, BARRAR & KAY, WILCOX). *The field scattered from the surface B, restricted as before, may be written in the form*

$$U_k(p, p_0) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^n}, \tag{3.3}$$

where the series converges absolutely and uniformly for $r > c + \varepsilon$, $\varepsilon > 0$ and c defined in (2.2). Furthermore the series may be differentiated term by term with respect to r, ϑ, or φ any number of times and the resulting series all converge absolutely and uniformly. The functions $f_n(\vartheta, \varphi)$ are understood to depend on the parameters k and $r_0, \vartheta_0, \varphi_0$, the coordinates of p_0.

Note that outside the sphere of radius r_0, the entire Green's function has a convergent expansion of the form (3.3) and also that the theorem provides an equally valid representation of the scattered field for plane wave, as well as point source, incidence.

It is evident from this expansion that for large r, $U_k \sim \frac{e^{ikr}}{r} f_0(\vartheta, \varphi)$ and while

$$\lim_{r\to\infty} |r U_k| = \lim_{r\to\infty} |f_0| < \infty, \tag{3.4}$$

the other requirement for regularity at infinity fails to hold, *i.e.*

$$\lim_{r\to\infty} \left| r^2 \frac{\partial U_k}{\partial r} \right| = \lim_{r\to\infty} |k r f_0| \tag{3.5}$$

and this limit does not exist. However, also evident from this expansion is the following

Corollary: If $U_k(p, p_0)$ is the field scattered from the surface B, then $e^{-ikr} U_k(p, p_0)$ is regular at infinity.

We introduce the notation $\widetilde{U}(p, p_0)$ with the defining equation

$$\widetilde{U}(p, p_0) = e^{-ikr} U_k(p, p_0), \tag{3.6}$$

and, clearly, in addition to being regular at infinity, $\widetilde{U}(p, p_0)$ is twice differentiable in V since $U_k(p, p_0)$ has this property. Therefore $\widetilde{U}(p, p_0)$ satisfies the requirements of Theorem 2.1 and may be written

$$\widetilde{U}(p, p_0) = \int_V G_0(p, p_1)\, \nabla^2 \widetilde{U}(p_1, p_0)\, dv + \int_B \widetilde{U}(p_B, p_0) \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma. \tag{3.7}$$

It is a simple task to establish the following

Lemma 3.1. *If a)* $\widetilde{U} = e^{-ikr} U_k$ *and b)* $(\nabla^2 + k^2) U_k = 0$, *then*

$$\nabla^2 \widetilde{U} = -\frac{2ik}{r} \frac{\partial}{\partial r} (r \widetilde{U}). \tag{3.8}$$

Incorporating this result in (3.7) yields

$$\begin{aligned} \widetilde{U}(p, p_0) = -2ik \int_V \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1} [r_1 \widetilde{U}(p_1, p_0)]\, dv\, + \\ + \int_B \widetilde{U}(p_B, p_0) \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma. \end{aligned} \tag{3.9}$$

The boundary values, $\widetilde{U}(p_B, p_0)$, are found from (3.1b), (3.2) and (3.6) to be

$$\widetilde{U}(p_B, p_0) = \frac{e^{-ikr_B + ikR(p_B, p_0)}}{4\pi R(p_B, p_0)}. \tag{3.10}$$

We may summarize the results of the present section in the following:

Theorem 3.2. *If*

(a) *V is the volume exterior to B, the union of a finite number of smooth, closed, bounded, disjoint surfaces,*

(b) $G_0(p, p_0) = -\dfrac{1}{4\pi R(p, p_0)} + U_0(p, p_0)$ *is the potential Green's function of the first kind for this surface* $(G_0(p_B, p_0) = 0)$,

and

(c) $G_k(p, p_0) = -\dfrac{e^{ikR(p, p_0)}}{4\pi R(p, p_0)} + U_k(p, p_0)$ *is the Green's function for the Helmholtz equation, also satisfying a Dirichlet condition on B,*

then the scattered field $U_k(p, p_0)$ satisfies the integral equation

$$\begin{aligned} U_k(p, p_0) = -2ik\, e^{ikr} \int_V \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1} [r_1 e^{-ikr_1} U_k(p_1, p_0)]\, dv\, + \\ + \frac{e^{ikr}}{4\pi} \int_B \frac{e^{-ikr_B + ikR(p_B, p_0)}}{R(p_B, p_0)} \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma \end{aligned} \tag{3.11}$$

where dv is a volume element in coordinates p_1 and $d\sigma$ is a surface element and $\partial/\partial n$ the normal derivative directed out of V expressed in coordinates p_B.

In addition, there is a comparable representation for the scattered field when the incident field is a plane wave or a superposition of point sources and/or plane waves. Specifically

Theorem 3.3. *If V is the exterior of a surface B restricted as before and*

a) $U(p) = U^{\mathrm{inc}}(p) + U^{\mathrm{scat}}(p), \qquad p \in \overline{V},$

b) $(\nabla^2 + k^2)\, U^{\mathrm{scat}} = 0, \qquad p \in V,$

c) $U(p_B) = 0,$

d) $\lim\limits_{r\to\infty} r\left(\dfrac{\partial U^{\mathrm{scat}}}{\partial r} - i\,k\,U^{\mathrm{scat}}\right) = 0,$

then

$$\begin{aligned} U^{\mathrm{scat}}(p) = &- 2ik\,e^{ikr} \int\limits_V \frac{G_0(p,p_1)}{r_1} \frac{\partial}{\partial r_1}\left[r_1\, e^{-ikr_1}\, U^{\mathrm{scat}}(p_1)\right] dv - \\ &- e^{ikr} \int\limits_B e^{-ikr_B}\, U^{\mathrm{inc}}(p_B) \frac{\partial}{\partial n} G_0(p, p_B)\, d\sigma. \end{aligned} \tag{3.12}$$

4. A Neumann Series for the Green's Function

In this section we find an explicit representation of the Green's function for the Helmholtz equation in terms of an iterative solution of the integral equation derived in the previous section.

The iteration scheme is clearly indicated by rewriting equation (3.9) in more suggestive form, namely:

$$\tilde{U} = K \circ \tilde{U} + U^{(0)} \tag{4.1}$$

where

$\tilde{U} = \tilde{U}(p)$ is the function we seek,

$K = -2ik \int\limits_V dv \dfrac{G_0(p,p_1)}{r_1} \dfrac{\partial[r_1}{\partial r_1}$ is an integro-differential operator,

and

$U^{(0)} \equiv U^{(0)}(p)$ is a known function (3.10).

The dependence on the point p_0 is understood and not explicitly shown. The form of (4.1) suggests that the solution may be found using the Liouville-Neumann series of Fredholm theory. That is, we rewrite (4.1) as

$$\tilde{U} = (I - K)^{-1} \circ U^{(0)} \tag{4.2}$$

and formally expand the inverse, obtaining

$$\tilde{U} = \sum_{n=0}^{\infty} K^n \circ U^{(0)}. \tag{4.3}$$

Denote by $U^{(N)}$ the partial sums

$$U^{(N)} = \sum_{n=0}^{N} K^n \circ U^{(0)}. \tag{4.4}$$

With this definition it follows immediately that for $N \geqq 1$, $U^{(N)}$ also satisfies the recursive relation

$$U^{(N)} = K \circ U^{(N-1)} + U^{(0)}. \tag{4.5}$$

The solution, $\tilde{U}$, is given by

$$\tilde{U} = \lim_{N\to\infty} U^{(N)} \tag{4.6}$$

where either (4.4) or (4.5) may be taken as defining $U^{(N)}$.

Thus far these results are purely formal. We must show that they are meaningful in a rigorous sense. In particular, we must demonstrate that the Neumann series, (4.3), converges and establish the sense in which it converges. Also we must show that it converges to the solution we seek. The remainder of this section is devoted to this necessary but tedious analysis. What we shall do is show that $U^{(0)}$, $\tilde{U}$, and all the iterates $U^{(N)}$ are elements of a normed vector space which is mapped into itself by the operator K. Further we shall show that for k sufficiently small, this operator has norm less than unity. The convergence of the Neumann series, in this norm, then follows as does the uniqueness of the solution.

First we record some properties of spherical harmonics and known expansions of the static Green's function which will prove useful.

Denote by $Y_n(\vartheta, \varphi)$ an n^{th} order spherical harmonic

$$Y_n(\vartheta, \varphi) = \sum_{m=-n}^{n} A_{mn} P_n^m(\cos\vartheta)\, e^{im\varphi} \tag{4.7}$$

and by $Y_n(\vartheta, \varphi; \vartheta_1, \varphi_1)$ a symmetric n^{th} order spherical harmonic

$$Y_n(\vartheta, \varphi; \vartheta_1, \varphi_1) = \sum_{m=0}^{n} A_{mn} P_n^m(\cos\vartheta)\, P_n^m(\cos\vartheta_1)\cos m(\varphi - \varphi_1). \tag{4.8}$$

These functions enjoy the orthogonality property

$$\begin{aligned}\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta\, Y_m(\vartheta, \varphi)\, Y_n(\vartheta, \varphi; \vartheta_1, \varphi_1) &= 0, \qquad m \neq n \\ &= Y_n(\vartheta_1, \varphi_1), \qquad m = n.\end{aligned} \tag{4.9}$$

Here it must be kept in mind that $Y_n(\vartheta, \varphi)$ and $Y_n(\vartheta_1, \varphi_1)$ occurring in (4.9) are not necessarily the same function but are elements of the same equivalence class. That is, they both may be written in the form (4.7) but the constant coefficients A_{mn} may differ. In what follows, it is often unnecessary to distinguish between spherical harmonics of the same order; thus we denote them all with the same symbol. This should not be overlooked in any specific calculation of the coefficients where a more precise specification is required.

It is well known (*e.g.* KELLOGG [*10*, p. 143]) that potential functions may be expanded in spherical harmonics. In particular the static Green's function for the surface B may be written

$$G_0(p, p_1) = -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \varphi)}{r^{n+1}}, \qquad r \geqq a, \tag{4.10}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\vartheta_1, \varphi_1)}{r_1^{n+1}}, \quad r_1 \geqq a, \tag{4.11}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \varphi; \vartheta_1, \varphi_1)}{(r r_1)^{n+1}}, \quad r, r_1 \geqq a \tag{4.12}$$

where the series are uniformly and absolutely convergent and may be differentiated or integrated any number of times with respect to r, ϑ, or φ; $a = c + \varepsilon$, $\varepsilon > 0$; and c, equation (2.2), is the radius of the sphere enclosing B. The reciprocity relation is explicitly exhibited. It is useful to note that the source term may also be expanded in spherical harmonics

$$\frac{1}{R(p,p_1)} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n[\cos\vartheta\cos\vartheta_1 + \sin\vartheta\sin\vartheta_1\cos(\varphi-\varphi_1)] \tag{4.13}$$

where

$$r_> = \max(r, r_1), \qquad r_< = \min(r, r_1).$$

Note that the expansion has the same convergence properties as the series in (4.10)—(4.12) provided $r \neq r_1$.

In addition to the orthogonality of spherical harmonics, it will be useful to define a related property.

Definition. A function $f(\vartheta,\varphi)$ will be called a "pseudo spherical harmonic of order n" if

$$\begin{aligned}\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta\, f(\vartheta,\varphi)\, Y_m(\vartheta,\varphi;\vartheta_1,\varphi_1) &= 0, && m<n \\ &= Y_m(\vartheta_1,\varphi_1), && m \geqq n.\end{aligned} \tag{4.14}$$

With the understanding that zero may be considered a spherical harmonic of any order (all coefficients in (4.7) are zero) it follows that any spherical harmonic of order n is also a pseudo spherical harmonic of order n.

Now we are in a position to define a particular function space in which we will establish the convergence of the iterations. Recalling that V is the volume exterior to the surface B and a is the radius of a sphere entirely containing B in its interior, we define $\mathcal{U}$ as follows:

$$\mathcal{U} = \left\{ U \,\middle|\, \begin{array}{l} \text{a) } U \in c^2(V), \\ \text{b) } U = \frac{1}{r}\sum_{n=0}^{\infty} \frac{f_n(\vartheta,\varphi)}{r^n},\ r \geqq a \text{ and the series is uniformly and} \\ \quad \text{absolutely convergent, term by term differentiable, with} \\ \quad \text{respect to } r,\ \vartheta, \text{ or } \varphi \text{ and the resulting series are uniformly} \\ \quad \text{and absolutely convergent,} \\ \text{c) } f_n(\vartheta,\varphi) \text{ are pseudo spherical harmonics, } i.e., \text{ satisfy (4.14).} \end{array} \right\} \tag{4.15}$$

Further we specify the following norm, implied both by the pointwise convergence of the series, (4.15b), and the fact that elements of $\mathcal{U}$ are twice differentiable everywhere in V,

$$\|U\| = \max_{p \in V} |U(p)|. \tag{4.16}$$

It is clear that much more could be said of $\mathcal{U}$ than that it is a linear normed vector space; however, rather than investigate this space in general, we confine our attention to those properties necessary for our present purpose. These are established in the following lemmas, which are then used to prove the main result of the paper.

Lemma 4.1. $U^{(0)} \in \mathcal{U}$.

Proof. Referring to equations (3.9) and (3.10), we see that

$$U^{(0)} = U^{(0)}(p) = \int_B \frac{e^{-ikr_B + ikR(p_B,p_0)}}{4\pi R(p_B,p_0)} \frac{\partial}{\partial n} G_0(p,p_B)\, d\sigma. \tag{4.17}$$

Since the integrand in (4.17) is infinitely differentiable for p_0, $p \notin B$ and B is not infinite, it follows that $U^{(0)} \in c^2(V)$ and in fact is infinitely differentiable as long as p and p_0 are not on the boundary. Actually KELLOGG [*10*, p. 172] established that the potential due to a double layer, with twice differentiable moment, of which (4.17) is an example, is also continuously differentiable for p on B, i.e. $U^{(0)} \in c^1(\overline{V})$. Furthermore when $r \geqq a$, we utilize (4.10) and (4.13) to obtain

$$\begin{aligned} G_0(p, p_B) = &- \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{r_B^n}{r^{n+1}} \times \\ &\times P_n[\cos\vartheta\cos\vartheta_B + \sin\vartheta\sin\vartheta_B\cos(\varphi - \varphi_B)] + \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \varphi)}{r^{n+1}}, \end{aligned} \tag{4.18}$$

or, since P_n is an n^{th} order spherical harmonic,

$$G_0(p, p_B) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} A_{mn}(p_B)\, P_n^m(\cos\vartheta)\, e^{\mathrm{i}m\varphi}. \tag{4.19}$$

This series converges uniformly, as does the derived series; therefore, we may rewrite (4.17) as

$$U^{(0)} = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\vartheta)\, e^{\mathrm{i}m\varphi} \int_B \frac{e^{-ikr_B + ikR(p_B, p_0)}}{4\pi R(p_B, p_0)} \frac{\partial}{\partial n} A_{mn}(p_B)\, d\sigma, \tag{4.20}$$

which is again of the form

$$U^{(0)} = \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \varphi)}{r^{n+1}}. \tag{4.21}$$

Hence conditions (4.15b, c) are satisfied as well as (4.15a), and the lemma is proven.

Lemma 4.2. *If* $U \in \mathscr{U}$, *then* $K \circ U \in \mathscr{U}$.

Proof. With the definition of K, equation (4.1), we write

$$K \circ U = -2ik \int_V dv \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1}[r_1 U(p_1)]. \tag{4.22}$$

We separate the volume over which the integration is performed into an infinite volume, V_e, where $r_1 \geqq a$ and the expansion Theorem 3.1 holds, and a finite volume, V_i, where it does not. V_i thus is the volume interior to the sphere of radius a and exterior the surface B. Thus we define two functions

$$U_{\substack{e\\i}}(p) = -2ik \int_{V_{\substack{e\\i}}} dv \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1}[r_1 U(p_1)]. \tag{4.23}$$

Clearly $U_e(p) + U_i(p) = K \circ U$, and if we can demonstrate that U_e and U_i are elements of $\mathscr{U}$, then, since the space is linear, it follows that $K \circ U$ is also in $\mathscr{U}$.

Consider first the finite volume. $U_i(p)$ is the potential of a volume distribution which KELLOGG has shown to be twice differentiable, for finite volumes, provided the density is piecewise continuous [*10*, p. 156]. This is certainly satisfied in the present case since $U \in \mathscr{U}$ which implies that the density $\frac{1}{r_1} \frac{\partial}{\partial r_1}[r_1 U(p_1)]$

is continuously differentiable. Therefore, $U_i(p) \in c^2(V)$. When $r \geqq a$, the expansion of the Green's function (4.19) is valid, with p_1 replacing p_B, since $r_1 < a$. The uniform convergence of the expansion and the fact that the integration is carried out over finite limits permits interchange of order, yielding

$$U_i(p) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\vartheta)\, e^{\mathrm{i}m\varphi} \int_{V_i} dv\,(-2ik)\, \frac{A_{mn}(p_1)}{r_1} \frac{\partial}{\partial r_1}[r_1 U(p_1)], \quad r \geqq a, \tag{4.24}$$

which is of the form

$$U_i(p) = \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \varphi)}{r^{n+1}}, \qquad r \geqq a. \tag{4.25}$$

Hence

$$U_i(p) \in \mathscr{U}. \tag{4.26}$$

Turning now to $U_e(p)$, we see that if V_e is replaced by any large but finite volume then the fact that $U_e \in c^2(V)$ again follows from KELLOGG's work. It is only necessary to show that U_e remains well defined when V_e becomes infinite. Explicitly

$$U_e(p) = \lim_{\varrho\to\infty} -2ik \int_a^{\varrho} dr_1 \int_0^{2\pi} d\varphi_1 \int_0^{\pi} d\vartheta_1\, r_1^2 \sin\vartheta_1\, \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1}[r_1 U(p_1)], \tag{4.27}$$

and it is sufficient to show that the integrand is $O(1/r_1^2)$ for large r_1. Since $U(p_1) \in \mathscr{U}$, it follows that

$$U(p_1) = \sum_{n=0}^{\infty} \frac{f_n(\vartheta_1, \varphi_1)}{r_1^{n+1}}, \qquad r_1 \geqq a, \tag{4.28}$$

and therefore that

$$\frac{\partial}{\partial r_1}[r_1 U(p_1)] = -\sum_{n=1}^{\infty} \frac{n f_n(\vartheta_1, \varphi_1)}{r_1^{n+1}}. \tag{4.29}$$

Thus for large r_1, $\frac{\partial}{\partial r_1}[r_1 U(p_1)] = O(1/r_1^2)$. Furthermore, the expansions of $G_0(p, p_1)$ given in (4.11) and (4.13) show that for r_1 sufficiently large, $\frac{G_0(p, p_1)}{r_1} = O(1/r_1^2)$. Hence, despite the factor r_1^2 in the volume element, the integrand is indeed $O(1/r_1^2)$ and it makes sense to let the r_1 integration extend to ∞. This calculation may be pursued more carefully to show that in addition, $U_e(p)$ satisfies the expansion properties required of elements of $\mathscr{U}$. Thus we rewrite (4.27) for $r, r_1 \geqq a$ as

$$\begin{aligned} U_e(p) = \int_a^{\infty} dr_1 \int_0^{2\pi} d\varphi_1 \int_0^{\pi} d\vartheta_1\, r_1 \sin\vartheta_1 \times \\ \times \left\{ -\frac{1}{4\pi R(p, p_1)} + \sum_{m=0}^{\infty} \frac{Y_m(\vartheta, \varphi; \vartheta_1 \varphi_1)}{(r r_1)^{m+1}} \right\} \sum_{n=1}^{\infty} \frac{f_n(\vartheta_1, \varphi_1)}{r_1^{n+1}}, \end{aligned} \tag{4.30}$$

where we have absorbed the factor $2ikn$ in the functions $f_n(\vartheta_1, \varphi_1)$. Now consider separately the integrals involving the regular and singular parts of the static Green's function, treating the regular part, U_e^{reg}, first. In this case both series are uniformly convergent, and the integral has been shown to exist; thus we may interchange order of integration and summation and perform the integration using the pseudo orthogonality condition (4.14) to obtain

$$\begin{aligned} U_e^{\text{reg}}(p) &= \int\limits_a^\infty dr_1 \int\limits_0^{2\pi} d\varphi_1 \int\limits_0^\pi d\vartheta_1 \sin\vartheta_1 \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{Y_m(\vartheta,\varphi;\vartheta_1,\varphi_1)\, f_n(\vartheta_1,\varphi_1)}{r^{m+1}\, r_1^{m+n+1}} \\ &= \sum_{n=1}^\infty \sum_{m=n}^\infty \frac{Y_m(\vartheta,\varphi)}{(m+n)\, a^{m+n}\, r^{m+1}}\,. \end{aligned} \tag{4.31}$$

Absorbing the constants factors in the spherical harmonics and renaming the second summation index yields

$$U_e^{\text{reg}}(p) = \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{Y_{m+n}(\vartheta,\varphi)}{r^{m+n+1}}\,. \tag{4.32}$$

Using Cauchy's formula to rearrange terms, which is allowed since the convergence is absolute, we obtain

$$U_e^{\text{reg}}(p) = \sum_{n=0}^\infty \sum_{m=0}^n \frac{Y_{n+1}(\vartheta,\varphi)}{r^{n+2}}\,. \tag{4.33}$$

While the coefficients in $Y_{n+1}(\vartheta,\varphi)$ may depend on m, the summation over m is still a spherical harmonic of order $n+1$ hence (4.33) is of the form

$$U_e^{\text{reg}}(p) = \sum_{n=0}^\infty \frac{Y_{n+1}(\vartheta,\varphi)}{r^{n+2}}\,. \tag{4.34}$$

The analysis involving the singular part of the Green's function is slightly more involved since the expansion of $1/R$, (4.13), is not uniformly convergent at $r=r_1$. From (4.30) we see that

$$U_e^{\text{sing}}(p) = -\frac{1}{4\pi} \int\limits_a^\infty dr_1 \int\limits_0^{2\pi} d\varphi_1 \int\limits_0^\pi d\vartheta_1 \frac{\sin\vartheta_1}{R(p,p_1)} \sum_{n=1}^\infty \frac{f_n(\vartheta_1,\varphi_1)}{r_1^n}\,. \tag{4.35}$$

Since the series occurring in (4.35) is uniformly convergent and the infinite integral has been shown to exist, we may interchange order of summation and integration, and absorb the factor $\left(-\frac{1}{4\pi}\right)$ into f_n, obtaining

$$U_e^{\text{sing}}(p) = \sum_{n=1}^\infty \int\limits_a^\infty dr_1 \int\limits_0^{2\pi} d\varphi_1 \int\limits_0^\pi d\vartheta_1 \frac{\sin\vartheta_1}{R(p,p_1)} \frac{f_n(\vartheta_1,\varphi_1)}{r_1^n}\,. \tag{4.36}$$

Now we employ the expansion (4.13) to obtain

$$\begin{aligned} U_e^{\text{sing}}(p) = \sum_{n=1}^\infty \Bigg\{ &\int\limits_a^r dr_1 \int\limits_0^{2\pi} d\varphi_1 \int\limits_0^\pi d\vartheta_1 \frac{\sin\vartheta_1}{r_1^n} f_n(\vartheta_1,\varphi_1) \sum_{m=0}^\infty \frac{r_1^m}{r^{m+1}} Y_m(\vartheta,\varphi,\vartheta_1,\varphi_1) + \\ &+ \int\limits_r^\infty dr_1 \int\limits_0^{2\pi} d\varphi_1 \int\limits_0^\pi d\vartheta_1 \frac{\sin\vartheta_1}{r_1^n} f_n(\vartheta_1,\varphi_1) \sum_{m=0}^\infty \frac{r^m}{r_1^{m+1}} Y_m(\vartheta,\varphi,\vartheta_1,\varphi_1) \Bigg\}\,. \end{aligned} \tag{4.37}$$

Although the inner summation is singular at $r=r_1$, $\vartheta=\vartheta_1$, $\varphi=\varphi_1$, it is a straightforward matter to exclude a small neighborhood of (r,ϑ,φ) from the integral in which case the interchange of summation and integration is legitimate and then show that the integral over the excluded neighborhood may be made as small as we wish by making the neighborhood sufficiently small. Thus we find, again using the pseudo orthogonality property (4.14)*,

$$U_{\mathrm{e}}^{\mathrm{sing}}(p) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \left\{ \frac{Y_m(\vartheta,\varphi)\,(r^{m-n+1} - a^{m-n+1})}{r^{m+1}(m-n+1)} + \frac{Y_m(\vartheta,\varphi)}{(n+m)\,r^n} \right\}. \tag{4.38}$$

Again absorbing the constants in the spherical harmonics, we obtain

$$U_{\mathrm{e}}^{\mathrm{sing}}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\vartheta,\varphi) + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{Y_m(\vartheta,\varphi)}{r^{m+1}}. \tag{4.39}$$

The second sum in (4.39) is of precisely the same form as (4.32). Hence the same argument allows us to write

$$U_{\mathrm{e}}^{\mathrm{sing}}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\vartheta,\varphi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\vartheta,\varphi)}{r^{n+2}}, \tag{4.40}$$

and, with (4.34), we find that $U_{\mathrm{e}}(p)$ is also of this form, *i.e.*,

$$U_{\mathrm{e}}(p) = U_{\mathrm{e}}^{\mathrm{reg}}(p) + U_{\mathrm{e}}^{\mathrm{sing}}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\vartheta,\varphi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\vartheta,\varphi)}{r^{n+2}}, \tag{4.41}$$

or by a trivial change of notation

$$U_{\mathrm{e}}(p) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=n+1}^{\infty} Y_m(\vartheta,\varphi) + \sum_{n=1}^{\infty} \frac{Y_n(\vartheta,\varphi)}{r^{n+1}}. \tag{4.42}$$

This is precisely the form required for $U_{\mathrm{e}}(p)$ to be in $\mathscr{U}$, *i.e.*,

$$U_{\mathrm{e}}(p) = \sum_{n=0}^{\infty} \frac{f_n(\vartheta,\varphi)}{r^{n+1}}, \qquad r \geqq a \tag{4.43}$$

where

$$\begin{aligned} f_0 &= \sum_{m=1}^{\infty} Y_m(\vartheta,\varphi), \\ f_n &= \sum_{m=n}^{\infty} Y_m(\vartheta,\varphi), \qquad n \geqq 1. \end{aligned} \tag{4.44}$$

Since $U_{\mathrm{e}}(p)$ has been shown to satisfy (4.15b), it remains only to demonstrate that $f_n(\vartheta,\varphi)$ defined in (4.44) are pseudo spherical harmonics. This follows immediately from the orthogonality of spherical harmonics, (4.9), since

$$\begin{aligned} &\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\, f_n(\vartheta,\varphi)\, Y_l(\vartheta,\varphi;\vartheta_1,\varphi_1) \sin\vartheta \\ &\quad = \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sum_{m=n}^{\infty} Y_m(\vartheta,\varphi)\, Y_l(\vartheta,\varphi;\vartheta_1,\varphi_1) \sin\vartheta = 0, \qquad l < n; \\ &\quad = Y_l(\vartheta_1,\varphi_1), \qquad l \geqq n. \end{aligned} \tag{4.45}$$

Therefore we may conclude that

$$U_{\mathrm{e}}(p) \in \mathscr{U}, \tag{4.46}$$

which, with (4.26), proves the lemma.

* The justification for requiring this apparently artificial restriction on the space $\mathscr{U}$ is found here since without this property, terms involving $\log r$ would occur.

This proof made no use of the factor k in the definition of K. In fact by explicitly exhibiting the k-dependence

$$K = kO,\ O = -2i \int_V dv \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1} [r_1 \qquad (4.47)$$

where O is independent of k, the same proof suffices to establish

Lemma 4.3. *If* $U \in \mathcal{U}$, *then* $O \circ U \in \mathcal{U}$.

Lemma 4.4. O *is bounded*, i.e., $\exists M < \infty \ni \|O\| \leq M$.

Proof:* Since $\|O\|$ is defined as $\sup\limits_{U \in \mathcal{U}} \frac{\|O \circ U\|}{\|U\|}$, it is sufficient to show that for any $U \in \mathcal{U}$

$$\|O \circ U\| \leq M \|U\|.$$

To accomplish this, it is necessary to establish the following estimate:

$$\left| U(p) - \frac{f_0(\vartheta, \varphi)}{r} \right| \leq \frac{2a^2}{r^2} \|U\|, \qquad p \in V \qquad (4.48)$$

where f_0 is the first coefficient in the expansion

$$U = \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^{n+1}}, \qquad r \geq a. \qquad (4.15\text{b})$$

Again it is convenient to separate V into two parts, V_e $(r \geq a)$ and V_i $(r < a)$. Recall that

$$\|U\| = \max_{p \in V} |U|;$$

hence

$$\max_{p \in V_e} |U| \leq \|U\|, \qquad \max_{p \in V_i} |U| \leq \|U\|. \qquad (4.49)$$

Consider first the volume V_e. Since U is analytic in $1/r$ for $r \geq a$, the maximum modulus theorem states that the maximum of the absolute value is achieved when $r = a$. Thus

$$|U| = \left| \frac{1}{r} \sum_{n=0}^{\infty} \frac{f_n}{r^n} \right| \leq \left| \frac{1}{a} \sum_{n=0}^{\infty} \frac{f_n}{a^n} \right| \leq \max_{p \in V_e} |U|.$$

With (4.49) it follows that

$$\left| \sum_{n=0}^{\infty} \frac{f_n}{a^n} \right| \leq a \|U\|. \qquad (4.50)$$

Similarly rU is analytic in $1/r$ for $r \geq a$; thus

$$|rU| = \left| \sum_{n=0}^{\infty} \frac{f_n}{r^n} \right| \leq \left| \sum_{n=0}^{\infty} \frac{f_n}{a^n} \right|.$$

Letting $r \to \infty$ and making use of (4.50) establishes that

$$|f_0| \leq a \|U\|. \qquad (4.51)$$

Furthermore, $\sum\limits_{n=1}^{\infty} \frac{f_n}{r^{n-1}}$ is also analytic for $r \geq a$; hence

$$\left| U - \frac{f_0}{r} \right| = \frac{1}{r^2} \left| \sum_{n=1}^{\infty} \frac{f_n}{r^{n-1}} \right| \leq \frac{1}{r^2} \left| \sum_{n=1}^{\infty} \frac{f_n}{a^{n-1}} \right|. \qquad (4.52)$$

* The author is indebted to Professor F. URSELL for pointing out an error in an earlier proof of this lemma and to Mr. E. AR for helping eliminate it.

On the other hand

$$\left|\sum_{n=1}^{\infty}\frac{f_n}{r^{n+1}}\right| = \left|U-\frac{f_0}{r}\right| \leqq |U| + \frac{|f_0|}{r}, \qquad r \geqq a.$$

With (4.49), (4.51), and the fact that $a/r \leqq 1$, it follows that

$$\left|\sum_{n=1}^{\infty}\frac{f_n}{r^{n+1}}\right| \leqq 2\|U\|, \qquad r \geqq a.$$

In particular, at $r=a$

$$\left|\sum_{n=1}^{\infty}\frac{f_n}{a^{n-1}}\right| \leqq 2a^2\|U\|. \tag{4.53}$$

Substituting (4.53) in (4.52) establishes the desired estimate for $p \in V_e$. For $p \in V_i$ the task is much simpler. Again

$$\left|U-\frac{f_0}{r}\right| \leqq |U| + \frac{|f_0|}{r},$$

and (4.51) is still valid; hence

$$\left|U-\frac{f_0}{r}\right| \leqq \|U\| + \frac{a}{r}\|U\|.$$

But $a/r \geqq 1$, thus the inequality is strengthened by writing

$$\left|U-\frac{f_0}{r}\right| \leqq \frac{2a^2}{r^2}\|U\|;$$

hence the estimate (4.48) is valid for all $p \in V$.

Proceeding now to the task of bounding $O \circ U$, note that $f_0(\vartheta_1, \varphi_1)$, the first coefficient in the series expansion of $U(p_1)$ is independent of r_1, therefore

$$O \circ U = -2i \int_V dv\, \frac{G_0(p, p_1)}{r_1}\, \frac{\partial}{\partial r_1}\left[r_1 U(p_1) - f_0(\vartheta_1, \varphi_1)\right]. \tag{4.54}$$

Integrating once by parts with respect to r_1 yields

$$O \circ U = 2i \int_V \left(U - \frac{f_0}{r_1}\right)\frac{\partial}{\partial r_1}(r_1 G_0)\, dv. \tag{4.55}$$

The integrated terms vanish at the lower limit ($p_1 \in B$) since $G_0(p, p_B) = 0$, and vanish at the upper limit ($r_1 \to \infty$) since $(r_1 U - f_0) r_1 G_0 = O(1/r_1)$ ((4.15b) shows that $r_1 U - f_0 = O(1/r_1)$ and (2.4c) states that $r_1 G_0$ remains bounded). Thus

$$|O \circ U| = 2\left|\int_V \left(U - \frac{f_0}{r}\right)\frac{\partial}{\partial r_1}(r_1 G_0)\, dv\right| \leqq 2\int_V \left|U - \frac{f_0}{r_1}\right| \left|\frac{\partial}{\partial r_1}(r_1 G_0)\right| dv. \tag{4.56}$$

Making use of the estimate (4.48), where now the variables are $r_1, \vartheta_1, \varphi_1$, it follows that

$$|O \circ U| \leqq 4a^2\|U\| \int_V \frac{1}{r_1^2}\left|\frac{\partial}{\partial r_1}(r_1 G_0)\right| dv. \tag{4.57}$$

At the singularity of the Green's function the integrand behaves as the inverse of the square of the distance,

$$\frac{1}{r_1^2}\left|\frac{\partial}{\partial r_1}(r_1 G_0)\right| = O(1/R^2) \quad \text{as} \quad R \to 0,$$

and is therefore integrable over a finite volume containing the singularity (KELLOGG [*10*], p. 148). In addition, with (4.11) and (4.13) it follows that

$$\frac{1}{r_1^2}\left|\frac{\partial}{\partial r_1}(r_1 G_0)\right| = O(1/r_1^4) \quad \text{as} \quad r_1 \to \infty$$

hence the integral over the infinite volume exists as well, *i.e.*,

$$\int\limits_V \frac{1}{r_1^2}\left|\frac{\partial}{\partial r_1}(r_1 G_0)\right| dv = I < \infty. \tag{4.58}$$

Thus (4.57) becomes

$$|O \circ U| \leqq 4a^2 I \|U\|. \tag{4.59}$$

This inequality is established for all values of $|O \circ U|$ including its maximum. Therefore, setting $M = 4a^2 I$, we have

$$\max |O \circ U| = \|O \circ U\| \leqq M \|U\|, \tag{4.60}$$

and the lemma is established.

Lemma 4.5. $\|K\|$ *is bounded with norm less than unity for k sufficiently small* i.e.,

$$\exists\, \alpha > 0 \ni \|K\| < 1 \quad \text{if} \quad 0 < |k| < \alpha.$$

Proof. Since $K = kO$, (4.47), and since both K and O are linear, it follows that

$$\|K\| = |k|\ \|O\|. \tag{4.61}$$

With Lemma 4.4 we obtain

$$\|K\| \leqq |k|\, M. \tag{4.62}$$

Therefore by choosing $|k| < 1/M$ or, equivalently, letting $\alpha = 1/M$, we prove the lemma.

Lemma 4.6. $\widetilde{U}(p) \in \mathscr{U}$.

Proof. The definition of $\widetilde{U}$ in terms of U_k (3.6) together with the fact that $U_k \in c^2(V)$ imply that $\widetilde{U} \in c^2(V)$. Furthermore, the expansion in Theorem 3.1 guarantees that we may write

$$\widetilde{U}(p) = \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^{n+1}}, \qquad r \geqq a. \tag{4.63}$$

It remains only to demonstrate that these $f_n(\vartheta, \varphi)$ are "pseudo spherical harmonics". To accomplish this, we employ the well known expansion of wave functions in spherical harmonics, *e.g.*, SOMMERFELD [*21*, p. 143],

$$U_k(p) = \sum_{n=0}^{\infty} h_n(kr)\, Y_n(\vartheta, \varphi), \qquad r \geqq a, \tag{4.64}$$

where $h_n(kr)$ are spherical Hankel functions of the first kind,

$$h_n(kr) = \frac{e^{ikr} i^{-n-1}}{r} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\, m!} \left(\frac{-1}{2ikr}\right)^m. \tag{4.65}$$

With this expression together with (4.64) we find that

$$\widetilde{U}(p) = e^{-ikr} U_k(p) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Y_n(\vartheta, \varphi)\, i^{-n-1}(n+m)!}{r^{m+1}(n-m)!\, m!} \frac{1}{(2ik)^m}, \tag{4.66}$$

or, upon rearranging terms and absorbing the multiplicative factors in the spherical harmonics,

$$\widetilde{U}(p) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Y_{n+m}(\vartheta, \varphi)}{r^{m+1}}. \tag{4.67}$$

This can be rewritten, with the obvious changes in notation so as to correspond to (4.63), as

$$\widetilde{U}(p) = \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^{n+1}}$$

where

$$f_n(\vartheta, \varphi) = \sum_{l=0}^{\infty} Y_{n+l}(\vartheta, \varphi). \tag{4.68}$$

The functions $f_n(\vartheta, \varphi)$ thus obviously satisfy the pseudo orthogonality condition, (4.14), and the lemma is proven. Note that this proof essentially duplicates SOMMERFELD's derivation of the expansion theorem but that, as is clear from the above, his statement [*21*, p. 191] that the $f_n(\vartheta, \varphi)$ are *finite* sums of spherical harmonics is in error.

We now, at last, are in a position to prove that the Neumann series (4.3) converges to the solution we seek. We state this in the form of a theorem.

Theorem 4.1. *There exists $\alpha > 0$ such that when $|k| < \alpha$, the Green's function for the Helmholtz equation, $(\nabla^2 + k^2) G_k = \delta$, which vanishes on the smooth closed bounded surface B in E^3, exists uniquely in V, the exterior to B, and is given explicitly by*

$$G_k(p, p_0) = -\frac{e^{ikR(p,p_0)}}{4\pi R(p, p_0)} + e^{ikr} \sum_{n=0}^{\infty} K^n \circ U^{(0)}$$

where

$$K \circ U^{(0)} = -2ik \int_V dv \frac{G_0(p, p_1)}{r_1} \frac{\partial}{\partial r_1} [r_1 U^{(0)}],$$

$$U^{(0)} = U^{(0)}(p_1, p_0) = \int_B \frac{e^{-ikr_B + ikR(p_B, p_0)}}{4\pi R(p_B, p_0)} \frac{\partial}{\partial n} G_0(p_1, p_B)\, d\sigma,$$

$G_0(p_1, p_B)$ is the static Green's function which vanishes on B, the normal is taken out of V, and the volume integration is over p_1, the surface integration over p_B.

Proof. With Theorem 3.2, it is sufficient to prove that $\sum_{n=0}^{\infty} K^n \circ U^{(0)}$ is the unique solution of the integral equation (4.1).

First we show that the series converges to the solution; that is, for any $\varepsilon > 0$

$$\exists N_0 \ni \|\widetilde{U} - U^{(N)}\| < \varepsilon \quad \text{if} \quad N > N_0.$$

Lemmas 4.1, 4.2 and 4.6 establish that $\widetilde{U}$, $U^{(0)}$, and all the iterates $U^{(N)}$ are in the space $\mathcal{U}$. Hence it makes sense to write $\|\widetilde{U} - U^{(N)}\|$ for any N. With equations (4.1) and (4.5) it follows that

$$\begin{aligned}
&\tilde{U} - U^{(0)} = K \circ \tilde{U},\\
&\tilde{U} - U^{(1)} = \tilde{U} - K \circ U^{(0)} - U^{(0)} = K \circ (\tilde{U} - U^{(0)}) = K^2 \circ \tilde{U},\\
&\tilde{U} - U^{(2)} = \tilde{U} - K \circ U^{(1)} - U^{(0)} = K \circ (\tilde{U} - U^{(1)}) = K^3 \circ \tilde{U},\\
&\quad\vdots\\
&\tilde{U} - U^{(N)} = K^{N+1} \circ \tilde{U};
\end{aligned}$$

hence

$$\|\tilde{U} - U^{(N)}\| \leqq \|K\|^{N+1} \|\tilde{U}\|. \tag{4.69}$$

But Lemma 4.5 states that $\|K\| < 1$ if $|k| < \alpha$ and $\|\tilde{U}\|$ is bounded since $\tilde{U} \in \mathcal{U}$ (Lemma 4.6); hence it is always possible to find N large enough so that

$$\|K\|^{N+1} \|\tilde{U}\| < \varepsilon. \tag{4.70}$$

Specifically, since $\log \|K\| < 0$, we find that (4.70) is satisfied if

$$N + 1 > \frac{\log \frac{\varepsilon}{\|\tilde{U}\|}}{\log \|K\|}. \tag{4.71}$$

We have thus established that for any $\varepsilon > 0$, $\|\tilde{U} - U^{(N)}\| < \varepsilon$ if $N > N_0$ and $|k| < \alpha$ where N_0 is the greatest integer in $\left[\frac{\log \frac{\varepsilon}{\|\tilde{U}\|}}{\log \|K\|} - 1\right]$ and α exists by Lemma 4.5.

To prove uniqueness we assume the existence of two solutions of (4.1), U_1 and U_2, such that $U_{\substack{1\\2}} = K \circ U_{\substack{1\\2}} + U^{(0)}$ and $\|U_1 - U_2\| \neq 0$. Then

$$U_1 - U_2 = K \circ U_1 - K \circ U_2. \tag{4.72}$$

Taking the norms, we obtain

$$\|U_1 - U_2\| \leqq \|K\| \, \|U_1 - U_2\|. \tag{4.73}$$

By assumption, $\|U_1 - U_2\| \neq 0$ hence we may divide, obtaining

$$1 \leqq \|K\|, \tag{4.74}$$

which violates Lemma 4.5.

5. A Low Frequency Expansion

In this section, the relation between the Neumann series and the low frequency expansion is derived, and it is shown how the Neumann series may be interpreted as a partial summation of the low frequency expansion.

In the previous section we proved that the solution of the integral equation (4.1) was given by equation (4.3) or, in the notation introduced in (4.47),

$$\tilde{U} = \sum_{n=0}^{\infty} k^n O^n \circ U^{(0)} \tag{5.1}$$

where the operator O is independent of k. This has the appearance of a power series in k; however, the function $U^{(0)}$, regardless of whether the incident field is a plane wave or point source, is not independent of k. It is true, nevertheless, that in either case $U^{(0)}$ is an entire function of k, *e.g.* equations (3.11) and (3.12), and hence has an absolutely convergent power series representation

$$U^{(0)} = \sum_{m=0}^{\infty} a_m k^m. \tag{5.2}$$

Substituting (5.2) in (5.1) and employing Cauchy's form for the product of two series, which is valid since they both converge absolutely, we obtain

$$\widetilde{U} = \sum_{m=0}^{\infty} k^m \sum_{n=0}^{m} O^n \circ a_{m-n}, \tag{5.3}$$

or

$$\widetilde{U} = \sum_{m=0}^{\infty} U_m k^m, \tag{5.4}$$

where

$$U_m = \sum_{n=0}^{m} O^n \circ a_{m-n}. \tag{5.5}$$

It is easily seen from this definition of U_m that

$$\begin{aligned} U_0 &= a_0, \\ U_m &= a_m + O \circ U_{m-1}. \end{aligned} \tag{5.6}$$

Equations (5.4) and (5.5) thus represent a low frequency expansion of $\widetilde{U}$. The functions U_m in (5.5) are precisely what we should have obtained had we assumed the expansion (5.4), substituted it, together with (5.2) in the equation $\widetilde{U} = kO \circ \widetilde{U} + U^{(0)}$ and equated coefficients of k. The corresponding expansion of the wave function U_k simply involves another Cauchy product, *viz,*

$$\begin{aligned} U_k = e^{ikr}\, \widetilde{U} &= \sum_{n=0}^{\infty} \frac{(i k r)^n}{n!} \sum_{m=0}^{\infty} U_m k^m \\ &= \sum_{n=0}^{\infty} k^n \sum_{m=0}^{n} \frac{(i r)^{n-m}}{(n-m)!} U_m. \end{aligned} \tag{5.7}$$

For many purposes, certainly any involving far field calculations, it is more convenient to leave out the phase factor e^{ikr} and work with the function $\widetilde{U}$. Note that the radius of convergence of these expansions is limited to the circle of convergence of the series (5.1), *cf.* Theorem 4.1.

There is of course a relation between the N^{th} iterate of the Neumann series (4.4) and the first N terms of the series (5.4). Specifically

$$U^{(N)} = \sum_{n=0}^{N} K^n \circ U^{(0)} = \sum_{n=0}^{N} k^n O^n \circ \sum_{m=0}^{\infty} a_m k^m \tag{5.8}$$

which upon rearranging terms is found to be

$$U^{(N)} = \sum_{m=0}^{N} \sum_{n=0}^{m} k^m O^n \circ a_{m-n} + \sum_{m=N+1}^{\infty} \sum_{n=0}^{N} k^m O^n \circ a_{m-n}. \tag{5.9}$$

With (5.5) we see that the first sum on the right represents the first N terms of the series (5.4). Thus the N^{th} iterate is seen to contain terms of all order in k, the first N of which correspond exactly to the first N terms of the low frequency expansion.

6. An Example: Scattering of a Plane Wave by a Sphere

To illustrate the methods derived in the previous sections, this section is devoted to consideration of a specific example, scattering of a plane wave by a sphere. In this case the exact result is well known, and by calculating the first few iterates of the Neumann series we are able to show not only how the techniques are to be employed but also the sense in which the N^{th} interate approximates the exact result.

The surface B is now a sphere of radius a, whose center is taken as the origin of the coordinate system. The static Green's function for the sphere is (Stratton [*22*, p. 201])

$$G_0(p,p_1) = -\frac{1}{4\pi\sqrt{r^2+r_1^2-2rr_1\cos\gamma}} + \frac{1}{4\pi\sqrt{a^2+(rr_1/a)^2-2rr_1\cos\gamma}} \tag{6.1}$$

where $\cos\gamma = \cos\vartheta\cos\vartheta_1 + \sin\vartheta\sin\vartheta_1\cos(\varphi-\varphi_1)$.

This may be expanded $\left(cf.\ (4.12),\ (4.13)\right)$ as

$$G_0(p,p_1) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\left(-\frac{r_<^n}{r_>^{n+1}} + \frac{a^{2n+1}}{(rr_1)^{n+1}}\right)P_n(\cos\gamma). \tag{6.2}$$

The incident field is a plane wave which, without loss of generality is chosen as propagating down the z-axis, *i.e.*

$$U^{\text{inc}} = e^{-ikz} = e^{-ikr\cos\vartheta}. \tag{6.3}$$

The scattered field then satisfies the equation $\left(cf.\ (3.12)\right)$

$$\begin{aligned}\tilde{U} = e^{-ikr}U^{\text{scat}} = -2ik\int\limits_V \frac{G_0(p,p_1)}{r_1}\frac{\partial}{\partial r_1}[r_1\tilde{U}(p_1)]\,dv - \\ -\int\limits_B e^{-ikr_B}U^{\text{inc}}(p_B)\frac{\partial}{\partial n}G_0(p,p_B)\,d\sigma.\end{aligned} \tag{6.4}$$

The volume V is now the exterior of the sphere $r=a$, B is the surface of this sphere, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r_B}$, G_0 is given by (6.1) or (6.2) and U^{inc} is given in (6.3).

With (6.2) we find that

$$\frac{\partial G_0(p,p_B)}{\partial n} = -\left.\frac{\partial G_0(p,p_B)}{\partial r}\right|_{r_B=a} = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(2n+1)\,a^{n-1}}{r^{n+1}}P_n(\cos\gamma). \tag{6.5}$$

The explicit forms of the iterates (4.5) are in this case

$$\begin{aligned}
U^{(0)}(p) &= -\frac{1}{4\pi}\int\limits_B d\sigma\sum_{n=0}^{\infty}\frac{(2n+1)\,a^{n-1}}{r^{n+1}}P_n(\cos\gamma)\,e^{-ika-ika\cos\vartheta_1},\\
d\sigma &= a^2\sin\vartheta_1\,d\vartheta_1\,d\varphi_1,\\
U^{(N)}(p) &= \frac{ik}{2\pi}\int\limits_V dv\sum_{n=0}^{\infty}\left(-\frac{r_<^n}{r_>^{n+1}} + \frac{a^{2n+1}}{(rr_1)^{n+1}}\right)\frac{P_n(\cos\gamma)}{r_1}\frac{\partial}{\partial r_1}[r_1U^{(N-1)}(p_1)],\\
dv &= r_1^2\sin\vartheta_1\,dr_1\,d\vartheta_1\,d\varphi_1.
\end{aligned} \tag{6.6}$$

By use of the orthogonality of spherical harmonics together with the plane wave expansion,

$$e^{-ika\cos\vartheta_1} = \sum_{m=0}^{\infty} (-i)^m (2m+1) j_m(ka)\, P_m(\cos\vartheta_1), \tag{6.7}$$

the first few iterates are found to be

$$\begin{aligned}
U^{(0)}(p) &= -e^{-ika}\sum_{n=1}^{\infty}(2n+1)(-i)^n (a/r)^{n+1} j_n(ka)\, P_n(\cos\vartheta) - e^{-ika} j_0(ka)\, a/r,\\
U^{(1)}(p) &= -e^{-ika}\sum_{n=1}^{\infty}(2n+1)(-i)^n (a/r)^{n+1} j_n(ka)\, P_n(\cos\vartheta)\,[1 - ik(r-a)] -\\
&\quad - e^{-ika} j_0(ka)\, a/r,\\
U^{(2)}(p) &= -e^{-ika}\sum_{n=1}^{\infty}(2n+1)(-i)^n (a/r)^{n+1} j_n(ka)\, P_n(\cos\vartheta)\Big[1 - ik(r-a) +\\
&\quad + k^2\left(\frac{(1-n)}{2n-1}(r^2-a^2) + a(r-a)\right)\Big] - e^{-ika} j_0(ka)\, a/r,\\
&\ \ \vdots
\end{aligned} \tag{6.8}$$

The exact expression of $U^{\text{scat}}(p)$ is (*e.g.* MORSE & FESHBACH [*13*, p. 1483])

$$U^{\text{scat}} = -\sum_{n=0}^{\infty} (-i)^n (2n+1) j_n(ka)\, P_n(\cos\vartheta)\, \frac{h_n(kr)}{h_n(ka)}. \tag{6.9}$$

With the definition of the spherical Hankel function (4.56) it is easily verified that

$$\frac{h_n(kr)}{h_n(ka)} = e^{ik(r-a)} (a/r)^{n+1} \frac{\sum\limits_{m=0}^{n} \frac{(-n)_m (-2ikr)^m}{(-2n)_m m!}}{\sum\limits_{m=0}^{n} \frac{(-n)_m (-2ika)^m}{(-2n)_m m!}}; \tag{6.10}$$

hence the exact expression for $e^{-ikr} U^{\text{scat}}$ may be written as

$$\begin{aligned}
\tilde{U}(p) &= -e^{-ika} j_0(ka)\, a/r -\\
&\quad - e^{-ika} \sum_{n=1}^{\infty} (-i)^n (2n+1)(a/r)^{n+1} j_n(ka)\, P_n(\cos\vartheta) \frac{\sum\limits_{m=0}^{n} \frac{(-n)_m (-2ikr)^m}{(-2n)_m m!}}{\sum\limits_{m=0}^{n} \frac{(-n)_m (-2ika)^m}{(-2n)_m m!}}.
\end{aligned} \tag{6.11}$$

Here we employ the notation $(x)_n = x(x+1)\ldots(x+n-1)$. If

$$\left|\sum_{m=1}^{n} \frac{(-n)_m}{(-2n)_m} \frac{(-2ika)^m}{m!}\right| < 1, \tag{6.12}$$

then the sum in the denominator in (6.11) may be expanded as a geometric series, and (6.11) becomes

$$\begin{aligned}
\tilde{U}(p) &= -e^{-ika} j_0(ka)\, a/r -\\
&\quad - e^{-ika} \sum_{n=1}^{\infty} (-i)^n (2n+1)(a/r)^{n+1} j_n(ka)\, P_n(\cos\vartheta)\, S(kr, ka)
\end{aligned} \tag{6.13}$$

where

$$S(kr, ka) = \sum_{m=0}^{n} \frac{(-n)_m(-2ikr)^m}{(-2n)_m m!} \sum_{l=0}^{\infty} (-1)^l \left[\sum_{t=1}^{n} \frac{(-n)_t(-2ika)^t}{(-2n)_t t!}\right]^l. \quad (6.14)$$

Clearly S is a power series in k, and if we denote by S_N the partial sums

$$S_N = \sum_{q=0}^{N} \alpha_q(r, a)\, k^q,$$

then with (6.14) we find that

$$\begin{aligned} S_0 &= 1, \\ S_1 &= 1 + ik(a-r), \\ S_2 &= 1 + ik(a-r) + k^2\left\{\frac{(n-1)}{2n-1}(a^2-r^2) + a(r-a)\right\}, \\ &\vdots \end{aligned} \quad (6.15)$$

Comparison of (6.15) with (6.8) enables us to induce that the N^{th} iterate is

$$\begin{aligned} U^{(N)}(p) = &- e^{-ika} j_0(ka)\, a/r - \\ &- e^{-ika} \sum_{n=1}^{\infty} (-i)^n (2n+1)(a/r)^{n+1} j_n(ka) P_n(\cos\vartheta) S_N. \end{aligned} \quad (6.16)$$

Each term of the N^{th} iterate approximates the exact result precisely as the first N terms of a power series expansion of a quotient of two polynomials (6.10) approximates the quotient. A rough estimate of the radius of convergence is obtained from (6.12) by noting that

$$\frac{(-n)_m}{(-2n)_m} \leqq \left(\frac{1}{2}\right)^m. \quad (6.17)$$

Thus

$$\begin{aligned} \left|\sum_{m=1}^{n} \frac{(-n)_m}{(-2n)_m} \frac{(-2ika)^m}{m!}\right| &\leqq \sum_{m=1}^{n} \left|\frac{(-n)_m}{(-2n)_m}\right| \frac{|2ka|^m}{m!} \\ &\leqq \sum_{m=1}^{n} \frac{|ka|^m}{m!} < e^{|ka|} - 1, \end{aligned} \quad (6.18)$$

and a condition sufficient to guarantee that $e^{|ka|} - 1 \leqq 1$ is $|ka| \leqq \log 2$.

If we expand the factor $e^{-ika} j_n(ka)$ in (6.16) in a power series in k (convergent everywhere since it is entire in k) and retain only the first N powers of k in the product on this expansion with S_N, then we have the first N terms of the low frequency series (5.4). These are in precise agreement, for $N = 0, 1, 2$ with the terms of the low frequency expansion derived from (5.6) with the known functions a_m defined in the present case as

$$a_m = -\frac{(-ia)^m}{4\pi m!} \int_0^{\pi} d\vartheta_1 \int_0^{2\pi} d\varphi_1 \sin\vartheta_1 (1+\cos\vartheta_1)^m \sum_{n=0}^{m} (2n+1)!\,(a/r)^{n+1} P_n(\cos\gamma). \quad (6.19)$$

Although this calculation was carried out independently as a check, it is clear that to this order agreement with the exact result is implied by the agreement between the Neumann series and the exact result, and we shall spare the reader the agonizing details.

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