Uniqueness and Stability Theorem for the Generalized Solution of the Initial-Value Problem for a Class of Quasi-Linear Equations in Several Space Variables

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§ 1. Introduction

In a previous paper, [*1*], we proved the existence of global (weak) solutions of the initial-value problem

$$u_t+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}f_j(u)=0, \tag{1.1}$$

$$u(0,x)=u_0(x) \tag{1.2}$$

in the region $t\geqq 0$. Here $x=(x_1,x_2,\dots,x_n)$, $u=u(t,x)$, the f_j, $j=1,2,\dots,n$, are continuously differentiable functions of a single real variable, and $u_0(x)$ is in the class F; *i.e.*, $u_0(x)$ is a bounded function having locally bounded variation in the sense of TONELLI-CESARI. Moreover for each fixed t, $u(t,x)$ is in the same class F.

A function f is said to have bounded variation in the sense of TONELLI-CESARI over a compact set Q if there exists a set Z of measure zero in Q such that the functions

$$V(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=\operatorname*{var}_{Q-Z}f(x_1,\dots,x_{i-1},\cdot,x_{i+1},\dots,x_n),\qquad i=1,2,\dots,n,$$

are measurable and summable. This is equivalent to the statement that the gradient of f be a measure (in the sense of the theory of distributions) whose total variation is finite over the compact set Q. (See [*1*] for these definitions, or [7] and [*8*]. This equivalence was proved in [*8*].)

It is well known that the generalized solutions of the above problem are not uniquely determined by the initial data [*2*]. However, in the case of a single space variable ($n=1$), it has been shown that uniqueness can be achieved by imposing a further restriction upon the solution; *i.e.*, generalized solutions satisfying this additional condition (usually referred to as the "entropy" condition) are uniquely determined by the initial data ([*3*−*6*]). In this paper we obtain a result of this type for the case of several space variables.

In the case when $n=1$ our result is that of OLEINIK [*3*]. In that paper she proves that a generalized solution of the Cauchy problem

$$u_t+\frac{\partial}{\partial x}f(u)=0,\qquad u(0,x)=u_0(x) \tag{1.3}$$

satisfying the additional condition

$$\frac{u(t,\xi_1)-u(t,\xi_2)}{\xi_1-\xi_2} \leqq K(t,\xi_1,\xi_2), \tag{1.4}$$

where K is continuous in the half-space $t>0$, will be unique if

$$f''(u)>0 \tag{1.5}$$

for all real u. In the case of the Cauchy problem (1.1), (1.2), the same conclusion will follow if we impose the analogue of condition (1.4) on each space variable separately; *i.e.*,

$$\frac{u(t,x_1,\dots,x_{j-1},\xi_1,x_{j+1},\dots,x_n)-u(t,x_1,\dots,x_{j-1},\xi_2,x_{j+1},\dots,x_n)}{\xi_1-\xi_2} \tag{1.6}$$
$$\leqq K_j(t,\xi_1,\xi_2,x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$$

for $j=1,2,\dots,n$, and if we replace (1.5) by

$$f_j''=\alpha_j f_1''\geqq 0, \qquad \alpha_j\geqq 0, \; j=1,2,\dots,n. \tag{1.7}$$

Although these conditions are sufficient to insure uniqueness, with one exception it is not known that the property (1.6) is possessed by any weak solutions of an appreciable class of initial value problems in more than one space variable. In the exceptional case, all the K_j's can be replaced by zero. In fact, in [*1*] we have shown that if the initial data $u_0(x)$ is a monotonic function in each variable separately, then there exists a generalized solution $u(t,x)$ with the property that $u(t,\cdot)$, for each fixed t, is monotonic in the same sense in the same variables. Therefore, the desirability of presenting the proof of the theorem using the condition (1.6) in its full generality is questionable. Moreover, the proof of the general theorem differs from that given by OLEINIK for one space variable at only one point. But the additional argument is fully present in the proof of the special theorem using zero for the K_j's. For these reasons we shall be content to formulate and prove the following theorem.

Theorem. *Let f_j, $j=1,2,\dots,n$, be C^2 functions of a single real variable which satisfy* (1.7). *Let F_M be the class of functions $u(x)$, $x\in E^n$, which belong to F and are monotonically non-increasing in each variable separately. Then the generalized Cauchy problem* (1.1), (1.2) *is well-posed in F_M; i.e.*,

A. *For each $u_0(x)$ in F_M, there is a generalized solution $u(t,x)$ of* (1.1), (1.2) *such that $u(t,\cdot)\in F_M$ for each fixed value of t.*

B. *$u(t,x)$ is uniquely determined by $u_0(x)$.*

C. *For each fixed t, the mapping $u_0(\cdot)\to u(t,\cdot)$ of F_M into itself is continuous in the topology generated by L_1-convergence on compacta. In fact, if $u(t,x)$ and $v(t,x)$ are the solutions of* (1.1) *for the initial data $u_0(x)$ and $v_0(x)$ respectively, then the following estimate is valid:*

$$\int_{Q(L)} |u(t,x)-v(t,x)|\,dx \leqq \int_{Q(L+\delta t)} |u_0(x)-v_0(x)|\,dx$$

where δ is a constant and $Q(K)$ denotes a hypercube $|x_i|\leqq K$.

In this formulation part A is a special case of theorem 2 of our previous paper [1] and has been included for the sake of completeness.

Before proceeding to the proof of this theorem we remark that it might be necessary to perform a transformation of the coordinate system to satisfy the hypotheses. For example, if $u_0(x)$ is monotonically non-decreasing in x_k, then the theorem would still apply if $f_k'' \leqq 0$; that is, $f_k'' = \alpha_k f_1''$ where $\alpha_k \leqq 0$. In this case we merely make the transformation of coordinates $x_k \to -x_k'$, $x_i \to x_i'$, $i \neq k$.

§ 2. Proof of B (Uniqueness)

Just as in [1], we will avoid cumbersome notation by giving the details of the proof only for the case of two space variables. It will be clear that the arguments apply to the more general case modulo an unambiguous selection of subscripts and superscripts. Thus we consider the Cauchy problem

$$u_t + \frac{\partial}{\partial x} f(u) + \frac{\partial}{\partial y} g(u) = 0, \qquad u(0,x,y) = u_0(x,y) \tag{2.1}$$

where f and g are functions in the class C^2 satisfying

$$f'' = \alpha g'' \geqq 0 \tag{2.2}$$

for some real number $\alpha \geqq 0$. Let us recall that by a solution $u = u(t, x, y)$ of (2.1) we mean that u is a bounded measurable function which satisfies

$$\iiint\limits_{t \geqq 0} [u \varphi_t + f(u) \varphi_x + g(u) \varphi_y] \, dx \, dy \, dt + \iint\limits_{t=0} u_0(x,y) \varphi(0,x,y) \, dx \, dy = 0 \tag{2.3}$$

for every compactly supported $\varphi = \varphi(t, x, y)$ belonging to the class C^1. Furthermore, we shall assume that for each fixed $t \geqq 0$, the solution u is monotinically non-increasing in both x and y; *i.e.*, the conditions

$$\frac{u(t,\xi_1,y) - u(t,\xi_2,y)}{\xi_1 - \xi_2} \leqq 0, \tag{2.4a}$$

$$\frac{u(t,x,\eta_1) - u(t,x,\eta_2)}{\eta_1 - \eta_2} \leqq 0 \tag{2.4b}$$

are fulfilled for every $t \geqq 0$ and for all real numbers $x, y, \xi_1, \xi_2, \eta_1, \eta_2$.

(B′). *Let f and g be C^2 functions of a single real variable which satisfy* (2.2). *Then there is at most one solution of the initial-value problem* (2.1) *which satisfies the conditions* (2.4)a *and* (2.4)b.

Proof. Suppose that u and v are solutions of (2.1) which satisfy (2.4). We shall show that $u = v$ almost everywhere. For this it is sufficient to show that

$$\iiint\limits_{t \geqq 0} (u - v) \varphi \, dx \, dy \, dt = 0 \tag{2.5}$$

for all smooth (*i.e.*, C^1) functions φ which have compact support.

Let ω_r be the usual Gaussian averaging kernel having as its support the sphere of radius r centered at the origin, and let $u_r = u * \omega_r$, $v_r = v * \omega_r$, where the symbol $*$

denotes convolution product. We set

$$F_r(t,x,y)=\frac{f(u_r)-f(v_r)}{u_r-v_r}=\int_0^1 f'(\vartheta v_r+(1-\vartheta)u_r)\,d\vartheta,$$

$$G_r(t,x,y)=\frac{g(u_r)-g(v_r)}{u_r-v_r}=\int_0^1 g'(\vartheta v_r+(1-\vartheta)u_r)\,d\vartheta.$$

Then if φ is a smooth function vanishing outside of a compact set contained in $t\geqq 0$, we consider smooth solutions $\psi(t,x,y)$ having compact support of the first order linear equation

$$\psi_t+F_r\psi_x+G_r\psi_y=\varphi, \tag{2.6}$$

with $\psi=0$ on the plane $t=T$, where T is any number satisfying $\varphi(t,x,y)=0$ if $t\geqq T$. From (2.3) and (2.6), it follows that

$$\iiint_{t\geqq 0}(u-v)\varphi\,dx\,dy\,dt=\iiint_{t\geqq 0}(u-v)\left[(F_r-F)\psi_x+(G_r-G)\psi_y\right]dx\,dy\,dt, \tag{2.7}$$

where

$$F(t,x,y)=\frac{f(u)-f(v)}{u-v}$$

and

$$G(t,x,y)=\frac{g(u)-g(v)}{u-v}.$$

Now if we let

$$C_1=\max\{|f''(\eta)|:|\eta|\leqq\max[\|u\|_\infty,\|v\|_\infty]\}$$

and

$$C_2=\max\{|g''(\eta)|:|\eta|\leqq\max[\|u\|_\infty,\|v\|_\infty]\},$$

then

$$|F_r-F|\leqq C_1\int_0^1\{\vartheta|v_r-v|+(1-\vartheta)|u_r-u|\}\,d\vartheta=\tfrac{1}{2}C_1[|v_r-v|+|u_r-u|].$$

Similarly,

$$|G_r-G|\leqq\tfrac{1}{2}C_2[|v_r-v|+|u_r-u|]$$

so that

$$\lim_{r\to 0}\iiint_Q|F_r-F|\,dx\,dy\,dt=\lim_{r\to 0}\iiint_Q|G_r-G|\,dx\,dy\,dt=0$$

for every compact set Q. Therefore, if we can show that ψ_x and ψ_y are bounded uniformly in r, then from (2.7) we would obtain (2.5), and this would complete the proof. Thus we shall now turn our attention to solutions of (2.6).

The characteristics of (2.6) are curves which are solutions of the system

$$\frac{dx}{dt}=F_r(t,x,y),\qquad \frac{dy}{dt}=G_r(t,x,y). \tag{2.8}$$

If

$$x=x_r(t,\tau,\xi,\eta),\qquad y=y_r(t,\tau,\xi,\eta)$$

is the characteristic passing through the point (τ, ξ, η), then the solution of (2.6) which is equal to zero on $t=T$ is given by

$$\psi(\tau,\xi,\eta)=\int_T^\tau \varphi(s, x_r(s,\tau,\xi,\eta), y_r(s,\tau,\xi,\eta))\,ds. \tag{2.9}$$

We see that since f' and g' are continuous and u and v are bounded, u_r and v_r are bounded so that F_r and G_r are bounded. Therefore from (2.8) and (2.9) we see that ψ is equal to zero for sufficiently large $|\xi|+|\eta|$. Consequently ψ has compact support. Moreover, it is clear that ψ is also smooth.

We shall now show that the first derivatives of ψ are bounded uniformly in r. First note that in the region $t\geqq 0$, the conditions (2.4) together with the basic properties of the averaging kernels imply that

$$\frac{\partial u_r}{\partial x}\leqq 0,\quad \frac{\partial v_r}{\partial x}\leqq 0,\quad \frac{\partial u_r}{\partial y}\leqq 0,\quad \frac{\partial v_r}{\partial y}\leqq 0 \tag{2.10}$$

for each $r>0$. Next, from (2.9) we obtain

$$\frac{\partial\psi}{\partial\xi}=\int_T^\tau\left(\frac{\partial\varphi}{\partial x}\frac{\partial x_r}{\partial\xi}+\frac{\partial\varphi}{\partial y}\frac{\partial y_r}{\partial\xi}\right)ds \tag{2.11}$$

so that in order to prove that $\partial\psi/\partial\xi$ is bounded independently of r, it suffices to prove that $\partial x_r/\partial\xi$ and $\partial y_r/\partial\xi$ are bounded independently of r. We set

$$z_r=\frac{\partial x_r}{\partial\xi},\qquad w_r=\frac{\partial y_r}{\partial\xi},$$

and then using (2.8), we obtain the system

$$\begin{aligned}\frac{dz_r}{dt}&=\frac{\partial F_r}{\partial x}z_r+\frac{\partial F_r}{\partial y}w_r,\\ \frac{dw_r}{dt}&=\frac{\partial G_r}{\partial x}z_r+\frac{\partial G_r}{\partial y}w_r.\end{aligned} \tag{2.12}$$

Now using (2.2) and (2.10), we have

$$\begin{aligned}\frac{\partial F_r}{\partial x}&=\int_0^1 f''(\vartheta v_r+(1-\vartheta)u_r)\left[\vartheta\frac{\partial v_r}{\partial x}+(1-\vartheta)\frac{\partial u_r}{\partial x}\right]d\vartheta\\ &=\alpha\int_0^1 g''(\vartheta v_r+(1-\vartheta)u_r)\left[\vartheta\frac{\partial v_r}{\partial x}+(1-\vartheta)\frac{\partial u_r}{\partial x}\right]d\vartheta\\ &=\alpha\frac{\partial G_r}{\partial x}\leqq 0.\end{aligned}$$

Similarly

$$\frac{\partial F_r}{\partial y}=\alpha\frac{\partial G_r}{\partial y}\leqq 0.$$

Therefore (2.12) becomes

$$\begin{aligned}\frac{dz_r}{dt}&=\alpha\frac{\partial G_r}{\partial x}z_r+\alpha\frac{\partial G_r}{\partial y}w_r,\\ \frac{dw_r}{dt}&=\frac{\partial G_r}{\partial x}z_r+\frac{\partial G_r}{\partial y}w_r,\end{aligned}$$

so that

$$\frac{dz_r}{dt}=\alpha\frac{dw_r}{dt},$$

and since, at $t=\tau$, $z_r=1$ and $w_r=0$, we see that $z_r=\alpha w_r+1$. Then

$$\begin{aligned}\frac{dw_r}{dt}&=\frac{\partial G_r}{\partial x}(\alpha w_r+1)+\frac{\partial G_r}{\partial y}w_r\\&=\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)w_r+\frac{\partial G_r}{\partial x}.\end{aligned}$$

Suppose now that $\alpha>0$. Then

$$\begin{aligned}w_r(t)&=\exp\left[\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)ds\right]\int_\tau^t\frac{\partial G_r}{\partial x}\exp\left[-\int_\tau^s\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)d\sigma\right]ds\\&\geqq\frac{1}{\alpha}\exp\left[\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)ds\right]\times\\&\quad\times\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)\exp\left[-\int_\tau^s\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)d\sigma\right]ds\\&=\frac{1}{\alpha}\exp\left[\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)ds\right]\times\\&\quad\times\int_\tau^t\frac{d}{ds}\left(\exp\left[-\int_\tau^s\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)d\sigma\right]\right)ds\\&=\frac{1}{\alpha}\exp\left[\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)ds\right]\left\{-\exp\left[-\int_\tau^t\left(\alpha\frac{\partial G_r}{\partial x}+\frac{\partial G_r}{\partial y}\right)d\sigma\right]+1\right\}\\&\geqq-\frac{1}{\alpha}.\end{aligned}$$

Therefore

$$0\geqq w_r\geqq-\frac{1}{\alpha},$$

from which it follows that $1\geqq z_r\geqq 0$. Consequently, in view of our previous remarks, we see that ψ_x is bounded independently of r. In a similar manner, we obtain that ψ_y is also bounded independently of r. This proves the theorem in the case where $\alpha>0$.

Suppose now that $\alpha=0$. Then from (2.2) and the definitions of F and F_r, we see that $F=F_r=\text{const}$, so that if we show that ψ_y is bounded independently of r, we could again conclude from (2.7) that (2.5) is valid. To this end, we set

$$a_r=\frac{\partial y_r}{\partial\eta};$$

then using (2.8) and (2.9), we get

$$\frac{\partial\psi}{\partial\eta}=\int_T^\tau\left(\frac{\partial\varphi}{\partial x}\frac{\partial x_r}{\partial\eta}+\frac{\partial\varphi}{\partial y}\frac{\partial y_r}{\partial\eta}\right)ds=\int_T^\tau\left(\frac{\partial\varphi}{\partial y}\right)a_r\,ds.$$

Furthermore, using (2.8) again, we obtain

$$\frac{\partial a_r}{\partial t}=\frac{\partial G_r}{\partial y}a_r$$

so that

$$a_r=\exp\int_\tau^t \frac{\partial G_r}{\partial y}\,ds,$$

and since

$$\frac{\partial G_r}{\partial y}\leqq 0,$$

we conclude that a_r is bounded independently of r and hence the same is true for $\partial\psi/\partial\eta$. This completes the proof of part B.

§ 3. Proof of C (Stability)

In this section we shall show that under the given hypotheses, the solution of the Cauchy problem (1.1), (1.2) is a continuous function (in the topology of L_1 convergence on compacta) of the initial data. As before, we shall give the details only for $n=2$ space variables.

Let us first recall some facts from our paper [1]. The difference scheme that we used was defined by

$$\text{(3.1)}\qquad \frac{u_{n,m}^{k+1}-\frac{1}{4}\left[u_{n+1,m}^k+u_{n-1,m}^k+u_{n,m+1}^k+u_{n,m-1}^k\right]}{h}+ \\ +\frac{f(u_{n+1,m}^k)-f(u_{n-1,m}^k)}{2q}+\frac{g(u_{n,m+1}^k)-g(u_{n,m-1}^k)}{2p}=0,$$

for $n, m=0, \pm1, \pm2, \ldots$; $k=0, 1, 2, \ldots$. Here p, q, and h are fixed positive numbers satisfying $q\leqq\delta h$, $p\leqq\delta h$ for some $\delta>0$, and we are using the notation

$$u_{\beta,\gamma}^{\alpha}=u(\alpha h, \beta q, \gamma p).$$

Now if our initial data satisfies $|u_{n,m}^0|\leqq M$ for every n and m, we define A and B by

$$A=\max\{|f'(u)|:|u|\leqq M\},\qquad B=\max\{|g'(u)|:|u|\leqq M\},$$

and we choose our mesh lengths p, q, h such that $2Ah<q$ and $2Bh<p$. We extend the definition of u to the entire t, x, y space ($t\geqq 0$) by defining

$$U(t,x,y)=U_{p,q,h}(t,x,y)=u_{n,m}^k$$

for $kh\leqq t<(k+1)h$, $nq\leqq x<(n+1)q$, $mp\leqq y<(m+1)p$. Then if our initial data is in the class F, we have shown that the functions $\{U_{p,q,h}\}$ are compact in the topology of L_1 convergence on compacta.

We now make an important observation. Namely, under the hypotheses of our theorem (in § 1), the entire sequence of difference approximations $\{U_{p,q,h}\}$ converges to the solution u of (1.1) and (1.2) constructed in [1]. This is clear since the sequence of difference approximations forms a compact family and the solution u is unique.

Lemma. *Suppose the $u^k_{n,m}$ and $v^k_{n,m}$ are solutions of the difference scheme* (3.1) *corresponding to initial data $u^0_{n,m}$ and $v^0_{n,m}$ respectively, where $|u^0_{n,m}|\leqq M$ and $|v^0_{n,m}|\leqq M$ for all n and m. Then for arbitrary $L>0$,*

$$\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}} |u^k_{n,m}-v^k_{n,m}|\leqq \sum_{\substack{q|n|\leqq L+k\delta h\\ p|m|\leqq L+k\delta h}} |u^0_{n,m}-v^0_{n,m}|, \tag{3.2}$$

where $\alpha|a|\leqq X$ denotes summation over all integers a for which $\alpha|a|\leqq X$.

Proof. Set $w^k_{n,m}=u^k_{n,m}-v^k_{n,m}$; then from (3.1) and the mean value theorem,

$$\begin{aligned}
w^k_{n,m}&=\tfrac{1}{4}\left[w^{k-1}_{n+1,m}+w^{k-1}_{n-1,m}+w^{k-1}_{n,m+1}+w^{k-1}_{n,m-1}\right]\\
&\quad-\frac{h}{2q}\left[f(u^{k-1}_{n+1,m})-f(v^{k-1}_{n+1,m})+f(v^{k-1}_{n-1,m})-f(u^{k-1}_{n-1,m})\right]\\
&\quad-\frac{h}{2p}\left[g(u^{k-1}_{n,m+1})-g(v^{k-1}_{n,m+1})+g(v^{k-1}_{n,m-1})-g(u^{k-1}_{n,m-1})\right]\\
&=\tfrac{1}{4}\left[w^{k-1}_{n+1,m}+w^{k-1}_{n-1,m}+w^{k-1}_{n,m+1}+w^{k-1}_{n,m-1}\right]\\
&\quad-\frac{h}{2q}\left[f'(\alpha^{k-1}_{n+1,m})\,w^{k-1}_{n+1,m}-f'(\alpha^{k-1}_{n-1,m})\,w^{k-1}_{n-1,m}\right]\\
&\quad-\frac{h}{2p}\left[g'(\beta^{k-1}_{n,m+1})\,w^{k-1}_{n,m+1}-g'(\beta^{k-1}_{n,m-1})\,w^{k-1}_{n,m-1}\right],
\end{aligned}$$

where $\alpha^{k-1}_{i,j}$ and $\beta^{k-1}_{i,j}$ are intermediate values for $u^{k-1}_{i,j}$ and $v^{k-1}_{i,j}$. Thus

$$\begin{aligned}
w^k_{n,m}&=\left[\frac{1}{4}-\frac{h}{2q}f'(\alpha^{k-1}_{n+1,m})\right]w^{k-1}_{n+1,m}+\left[\frac{1}{4}+\frac{h}{2q}f'(\alpha^{k-1}_{n-1,m})\right]w^{k-1}_{n-1,m}\\
&\quad+\left[\frac{1}{4}-\frac{h}{2p}g'(\beta^{k-1}_{n,m+1})\right]w^{k-1}_{n,m+1}+\left[\frac{1}{4}+\frac{h}{2p}g'(\beta^{k-1}_{n,m-1})\right]w^{k-1}_{n,m-1},
\end{aligned}$$

and since $2Ah<q$ and $2Bh<p$, the coefficients of the $w^{k-1}_{i,j}$ are non-negative. Therefore

$$\begin{aligned}
\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}}|w^k_{n,m}|&\leqq\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}}\left\{\left[\frac{1}{4}-\frac{h}{2q}f'(\alpha^{k-1}_{n+1,m})\right]|w^{k-1}_{n+1,m}|\right.\\
&\qquad\left.+\left[\frac{1}{4}+\frac{h}{2q}f'(\alpha^{k-1}_{n-1,m})\right]|w^{k-1}_{n-1,m}|\right\}\\
&\quad+\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}}\left\{\left[\frac{1}{4}-\frac{h}{2p}g'(\beta^{k-1}_{n,m+1})\right]|w^{k-1}_{n,m+1}|\right.\\
&\qquad\left.+\left[\frac{1}{4}+\frac{h}{2p}g'(\beta^{k-1}_{n,m-1})\right]|w^{k-1}_{n,m-1}|\right\}.
\end{aligned}\tag{3.3}$$

Now

$$\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}}\left[\frac{1}{4}-\frac{h}{2q}f'(\alpha^{k-1}_{n+1,m})\right]|w^{k-1}_{n+1,m}|+\left[\frac{1}{4}+\frac{h}{2q}f'(\alpha^{k-1}_{n-1,m})\right]|w^{k-1}_{n-1,m}|$$

$$=\sum_{\substack{p|m|\leqq L\\ -(L/q)+1\leqq r\leqq (L/q)+1}}\left[\frac{1}{4}-\frac{h}{2q}f'(\alpha^{k-1}_{r,m})\right]|w^{k-1}_{r,m}|$$

$$+\sum_{\substack{p|m|\leqq L\\ -(L/q)-1\leqq r\leqq (L/q)-1}}\left[\frac{1}{4}+\frac{h}{2q}f'(\alpha^{k-1}_{r,m})\right]|w^{k-1}_{r,m}|$$

$$\leqq\sum_{\substack{p|m|\leqq L\\ -(L/q)-1\leqq r\leqq (L/q)+1}}\left[\frac{1}{4}-\frac{h}{2q}f'(\alpha^{k-1}_{r,m})\right]|w^{k}_{r,m}|$$

$$+\sum_{\substack{p|m|\leqq L\\ -(L/q)-1\leqq r\leqq (L/q)+1}}\left[\frac{1}{4}+\frac{h}{2q}f'(\alpha^{k-1}_{r,m})\right]|w^{k-1}_{r,m}|$$

$$=\sum_{\substack{p|m|\leqq L\\ q|n|\leqq L+q}}\tfrac{1}{2}|w^{k-1}_{r,m}|\leqq\tfrac{1}{2}\sum_{\substack{p|m|\leqq L+p\\ q|n|\leqq L+q}}|w^{k-1}_{r,m}|.$$

Similarly

$$\sum_{\substack{p|n|\leqq L\\ p|m|\leqq L}}\left[\frac{1}{4}-\frac{h}{2p}g'(\beta^{k-1}_{n,m+1})\right]|w^{k-1}_{n,m+1}|+\left[\frac{1}{4}+\frac{h}{2p}g'(\beta^{k-1}_{n,m-1})\right]|w^{k-1}_{n,m-1}|$$

$$\leqq\tfrac{1}{2}\sum_{\substack{p|m|\leqq L+p\\ q|n|\leqq L+q}}|w^{k-1}_{n,m}|,$$

so from (3.3) it follows that

$$\sum_{\substack{q|n|\leqq L\\ p|m|\leqq L}}|w^{k}_{n,m}|\leqq\sum_{\substack{p|m|\leqq L+p\\ q|n|\leqq L+q}}|w^{k-1}_{n,m}|\leqq\sum_{\substack{p|m|\leqq L+\delta h\\ q|n|\leqq L+\delta h}}|w^{k-1}_{n,m}|.$$

Thus continuing in this way, we obtain (3.2). This completes the proof of the lemma.

(C′) *Let $u(t,x,y)$ and $v(t,x,y)$ be solutions of the Cauchy problem* (2.1) *corresponding to the initial conditions $u_0(x,y)$ and $v_0(x,.y)$ respectively, where $u_0(x,y)$ and $v_0(x,y)$ are both in the class F. Then*

$$\iint_{Q(L)}|u(t,x,y)-v(t,x,y)|\,dx\,dy\leqq\iint_{Q(L+\delta t)}|u_0(x,y)-v_0(x,y)|\,dx\,dy,\tag{3.4}$$

where $Q(X)$ denotes a square whose sides have length X.

Proof. We know that if $\{U^i\}$ and $\{V^i\}$ are the functions constructed from the difference equations corresponding to the inital data u_0 and v_0 respectively, then by our earlier observation

$$U^i\to u \quad\text{and}\quad V^i\to v,$$

where the convergence is understood to be L_1 convergence on compacta. Then

$$\iint_{Q(L)}|u(t,x,y)-v(t,x,y)|\,dx\,dy\leqq\iint_{Q(L)}|u(t,x,y)-U^i(t,x,y)|\,dx\,dy+$$

$$+\iint_{Q(L)}|U^i(t,x,y)-V^i(t,x,y)|\,dx\,dy+\iint_{Q(L)}|V^i(t,x,y)-v(t,x,y)|\,dx\,dy.$$

By choosing i sufficiently large, we can make the first and third integrals less than any preassigned $\varepsilon>0$. Then if we denote by $[z]$ the largest integer not exceeding z, we obtain from the lemma

$$\begin{aligned}\iint_{Q(L)} |u(t,x,y)-v(t,x,y)|\,dx\,dy &\leqq 2\varepsilon + \iint_{Q(L)} |U^i(t,x,y)-V^i(t,x,y)|\,dx\,dy \\ &\leqq 2\varepsilon + \sum_{\substack{q\,|n|\leqq L \\ p\,|m|\leqq L}} |u_{n,m}^{[t/h]} - v_{n,m}^{[t/h]}|\,q\,p \\ &\leqq 2\varepsilon + \sum_{\substack{q\,|n|\leqq L+\delta k h \\ p\,|m|\leqq L+\delta k h}} |u_{n,m}^0 - v_{n,m}^0|\,q\,p \\ &= 2\varepsilon + \iint_{Q(L+\delta t)} |U_0^i(x,y)-V_0^i(x,y)|\,dx\,dy,\end{aligned}$$

where $U_0^i(x,y)$ and $V_0^i(x,y)$ are grid functions constructed from the initial data $u_{n,m}^0$ and $v_{n,m}^0$ respectively. Now from lemma 5 in [*1*], we have that $U_0^i \to u_0$ and $V_0^i \to v_0$ (again in the topology of L_1 convergence on compacta). Therefore we see from this that (3.4) is valid. This completes the proof of (C′).

We conclude this section by noting that the methods employed in this paper show that if one obtains a condition which implies uniqueness of the solution for the Cauchy problem (1.1) and (1.2), where the initial data is in the class F, then for this class of solutions, it is always true that the entire sequence of difference approximations converges to the solution and the solution is a continuous function of the initial data. The topology here is given by L_1 convergence on compacta.

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