On the Validity of the Geometrical Theory of Diffraction by Convex Cylinders

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Abstract

In this paper we consider the scattering of a wave from an infinite line source by an infinitely long cylinder C. The line source is parallel to the axis of C, and the cross section C of this cylinder is smooth, closed and convex. C is formed by joining a pair of smooth convex arcs to a circle C_0, one on the illuminated side, and one on the dark side, so that C is circular near the points of diffraction. By a rigorous argument we establish the asymptotic behavior of the field at high frequencies, in a certain portion of the shadow S that is determined by the geometry of C in S. The leading term of our asymptotic expansion is the field predicted by the geometrical theory of diffraction.

Previous authors have derived asymptotic expansions in the shadow regions of convex bodies in special cases where separation of variables is possible. Others, who have considered more general shapes, have only been able to obtain bounds on the field in the shadow. In contrast our result is believed to be the first rigorous asymptotic solution in the shadow of a non-separable boundary, whose shape is frequency independent.

Introduction

Scattering problems for the reduced wave equation occur frequently in mathematical physics. It is required to find the function $U(\vec{r}_o;k)$ that satisfies $(\Delta+k^2)U=f(\vec{r}_o)$ in an infinite domain $\mathscr{D}$ with a boundary B. On B a linear combination of $U(\vec{r}_o;k)$ and its normal derivative is prescribed. The solution $U(\vec{r}_o;k)$ can be written as $U^{(i)}(\vec{r}_o;k)+U^{(s)}(\vec{r}_o;k)$ where $U^{(i)}(\vec{r}_o;k)$ represents the incident field, and $U^{(s)}(\vec{r}_o;k)$ is the scattered field. $U^{(s)}(\vec{r}_o;k)$ satisfies $(\Delta+k^2)U=0$ in $\mathscr{D}$ and the Sommerfeld radiation condition at infinity.

Since exact solutions of such problems can only be obtained in special cases, methods for constructing approximate solutions are of great interest. If the wave number k is large ($k=2\pi/\lambda$, $\lambda=$wave length), the modern geometrical theories of optics and diffraction [13, 16, 15, 27, 28] provide what are conjectured to be asymptotic approximations in the illuminated and dark (shadow) portions of $\mathscr{D}$.

The geometrical theory of optics has been verified in situations where the wave equation is separable, and explicit solutions are available. Asymptotic expansions of these explicit solutions are precisely those postulated by the geometrical theory (see *e.g.* [6, 9]).

In the case of scattering by an arbitrary convex cylinder of finite cross section, it has been proved that the field given by the geometrical theory of optics is an asymptotic solution in the illuminated region. URSELL [26] established this result for the case where the normal derivative of the total field is prescribed on the cylindrical boundary. BABICH [1] considered the Neumann problem with a line source not restricted to lie on the cylinder. Using Ursell's method, he derived a two term asymptotic expansion of the exact solution, in agreement with the geometrical theory. GRIMSHAW [8] extended this result, establishing the formal series of geometrical optics as an asymptotic expansion of the exact solution. Using different methods, MORAWETZ & LUDWIG [21] have done this for the Dirichlet problem. In addition they have established the validity of the geometrical optics formalism for the case of a point source radiating outside a closed, convex surface, on which the Dirichlet condition is imposed. These appear to be the most general results to date.

The geometrical theory of diffraction has only been confirmed in special cases. In the case of scattering by a circular cylinder such confirmation is provided by the work of FRANZ [6], IMAI [9], and URSELL [26]. LEVY [17], and also KAZARINOFF & RITT [11] have confirmed the theory for the case of scattering by an elliptic cylinder. Similar results for other kinds of separable boundaries have been obtained in [20, 10, 18, 25, 2]. (It should be noted that the arguments of all of the above authors, with the exception of URSELL, are incomplete in that no consideration is given to error estimates.) In each of the cases mentioned the wave equation is separable, and asymptotic expansions are derived from exact solutions. CLEMMOW & WESTON [3] have verified the geometrical theory of diffraction for a perturbed circular cylinder. However, as $k \to \infty$ the cylinder becomes circular.

BABICH [1] has shown that the total field is exponentially small on the dark side of a convex cylinder, and algebraically small in the rest of the shadow. OLIMPIEV [22] and GRIMSHAW [8] have sharpened this result by obtaining the exponential bound in the whole shadow region. However, in none of these analyses have asymptotic representations of the diffracted field been obtained.

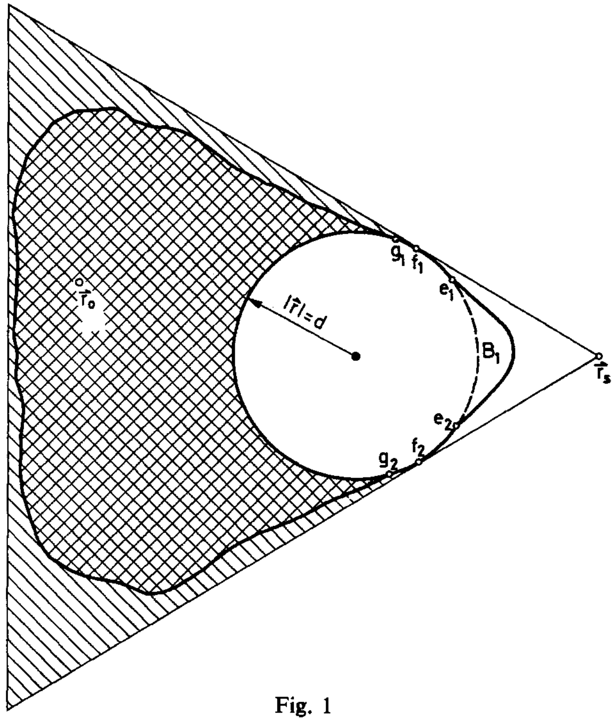
In this paper we consider the scattering of a wave from an infinite line source by an infinitely long cylinder $\boldsymbol{C}$. The line source is parallel to the axis of $\boldsymbol{C}$, and the cross section C of this cylinder is smooth, closed and convex. We require that the normal derivative of the total field vanish on $\boldsymbol{C}$. Mathematically this is equivalent to the two dimensional problem of the scattering of a circular wave by C, with the normal derivative of the total field required to vanish on C.

In Part 1 we consider the scattering of a circular wave emanating from $\vec{r}_s$, by a smooth convex curve C_1, which is circular on its dark side, and also near the points of "diffraction". These are the points on C_1 where the tangents pass through $\vec{r}_s$. C_1 may be thought of as formed by "pasting" a convex "bump" B_1 to the illuminated side of a circle C_0. (*Cf.* Fig. 1.)

We prove that if $\vec{r}_o$ is a point in the "deep" shadow of C_1, and k is large, then the total field $U_1(\vec{r}_o, \vec{r}_s; k)$ is given asymptotically by $U_0(\vec{r}_o, \vec{r}_s; k)$. The letter function represents the field at $\vec{r}_o$ due to the scattering of the circular wave by C_0.

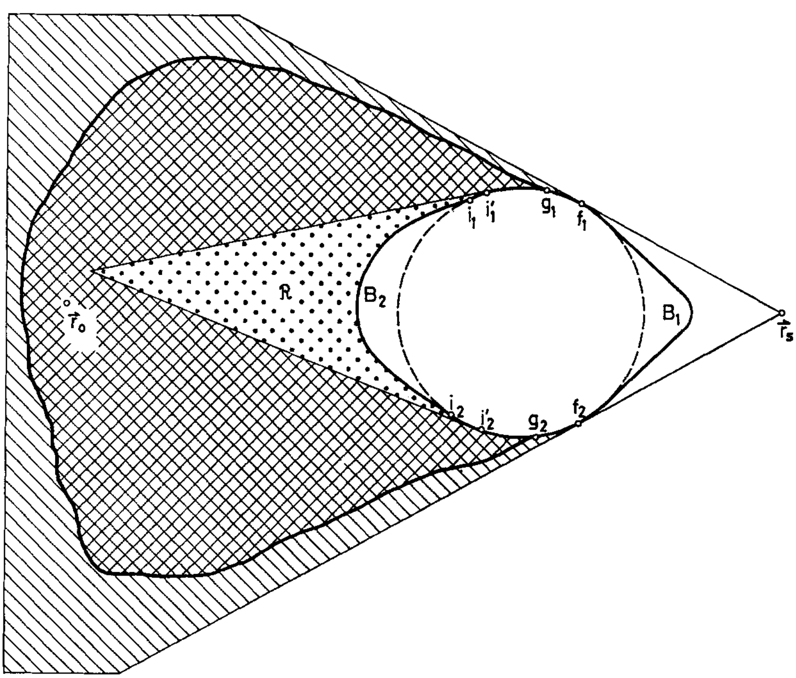
We then obtain a uniform asymptotic expansion of the field $U_1(\vec{r}_o, \vec{r}_s; k)$ in the form predicted by the formal theory of LEWIS, BLEISTEIN & LUDWIG [15]. From this we derive the non-uniform series expansions of the extended geometrical theory of diffraction [27].

The insensitivity of the field $U_1(\vec{r}_o, \vec{r}_s; k)$ to the geometry of B_1 as $k \to \infty$ is predicted by the geometrical theory of diffraction. The fact that we get a uniform



g_1
f_1
e_1
$|\vec{r}|=d$
$\vec{r}_0$
B_1
$\vec{r}_s$
e_2
f_2
g_2

Fig. 1



i_1
i'_1
g_1
f_1
R
B_2
B_1
$\vec{r}_0$
$\vec{r}_s$
i_2
i'_2
g_2
f_2

Fig. 2

asymptotic expansion for $U_1(\vec{r}_o, \vec{r}_s; k)$, which is also a uniform asymptotic expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$, corroborates this prediction.

In Part 2 we consider the more general case where a circular wave, emanating from $\vec{r}_s$, is scattered by a smooth convex curve C_2, which coincides with C_0 only near the points of diffraction. C_2 may be thought of as formed by "pasting" a convex "bump" B_2 to the dark side of C_1 (*cf.* Fig. 2).

Using the results of Part 1, we prove that if $\vec{r}_o$ lies in a certain subregion of the "deep" shadow of C_2 the total field $U_2(\vec{r}_o, \vec{r}_s; k)$ is also asymptotic to the field $U_0(\vec{r}_o, \vec{r}_s; k)$ associated with C_0 as $k \to \infty$. This subregion excludes from the shadow the range of influence $\mathscr{R}$ of the "bump" B_2 as defined below. We remark that $\mathscr{R}$ includes that part of the shadow where the geometrical theory of diffraction predicts that the geometry of B_2 should significantly influence the field (*cf.* Fig. 2).

As in Part 1 we obtain a uniform asymptotic expansion of $U_2(\vec{r}_o, \vec{r}_s; k)$ as $k \to \infty$ in the form predicted by LEWIS, BLEISTEIN & LUDWIG. From this we derive the non-uniform expansions of the extended geometrical theory of diffraction.

The fact that the uniform asymptotic expansion we get for $U_2(\vec{r}_o, \vec{r}_s; k)$ is also a uniform expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ confirms the prediction of the geometrical theory of diffraction, that the field should be insensitive to the geometry of B_2 and B_1, in the deep shadow outside $\mathscr{R}$.

Part 1

Consider the solution $U_1(\vec{r}_o, \vec{r}_s; k)$ of the scattering problem P_1:

$$(1.1)\qquad \begin{aligned} &\text{(i)} && \Delta U + k^2 U = \delta(\vec{r}_o, \vec{r}_s), && \vec{r}_o, \vec{r}_s \in \mathscr{D}_1; \\ &\text{(ii)} && \partial_n^{(1)} U = 0, && \vec{r}_o \in C_1; \\ &\text{(iii)} && \lim_{r_0 \to \infty} \rho^{\frac{1}{2}} |U_\rho - i k U| = 0, && \rho = |\vec{r}_o - \vec{r}|, \quad \vec{r} \in \mathscr{D}_1. \end{aligned}$$

Here $\mathscr{D}_1$ is the exterior of a smooth convex curve C_1 formed by "pasting" the ends e_1 and e_2 of a convex arc B_1 to the part of the circle $|\vec{r}| = a$ "illuminated" by a point source at $\vec{r}_s$, as shown in Fig. 1. $\partial_n^{(1)}$ denotes differentiation in the direction of the outward normal to C_1. $\vec{r}_s$ and $\vec{r}_o$ are the "source" and "observation" points respectively.

We shall establish the following result on the asymptotic behavior of $U_1(\vec{r}_o, \vec{r}_s; k)$ as $k \to \infty$.

Theorem 1: *As* $k \to \infty$

$$(1.2)\qquad U_1(\vec{r}_o, \vec{r}_s; k) = U_0(\vec{r}_o, \vec{r}_s; k)\left[1 + O(\exp\{-k^{\frac{1}{3}}\sigma\})\right],$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^{<}(\vec{r}_s)$.

Here σ is positive, independent of k, and of $\vec{r}_o$. The function $U_0(\vec{r}_o, \vec{r}_s; k)$ is the solution of the scattering problem P_0:

$$(1.3)\qquad \begin{aligned} &\text{(i)} && \Delta U + k^2 U = \delta(\vec{r}_o, \vec{r}_s), && r_0, r_s > a; \\ &\text{(ii)} && \frac{\partial U}{\partial r_0} = 0, && r_0 = a; \\ &\text{(iii)} && \lim_{r_0 \to \infty} \rho^{\frac{1}{2}} |U_\rho - i k U| = 0, && \rho = |\vec{r}_o - \vec{r}|, \quad r > a. \end{aligned}$$

$\mathscr{S}_1(\vec{r}_s)$ is the "shadow" of C_1: $\vec{r}_o \in \mathscr{S}_1(\vec{r}_s)$ if and only if $\vec{r}_o \in \mathscr{D}_1 \cup C_1$, and the straight line through $\vec{r}_o$ and $\vec{r}_s$ cuts C_1 at two distinct points. $\mathscr{S}_1^<(\vec{r}_s)$ is any closed bounded subset of $\mathscr{S}_1(\vec{r}_s)$. If $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$ we say that $\vec{r}_o$ lies in the "deep" shadow of C_1. (Note that the shadow $\mathscr{S}_0(\vec{r}_s)$ of the circle $C_0 = \{\vec{r}: |\vec{r}| = a\}$ is identical to $\mathscr{S}_1(\vec{r}_s)$ so that $\mathscr{S}_1^<(\vec{r}_s) = \mathscr{S}_0^<(\vec{r}_s)$.) (*Cf.* Fig. 1.)

In order to compare the assertion of Theorem 1 with the predictions of the geometrical theory of diffraction [12], the extended geometrical theory of diffraction [27, 28], and also with the "creeping wave" theory of LEWIS, BLEISTEIN & LUDWIG [15], it is necessary to expand $U_0(\vec{r}_o, \vec{r}_s; k)$ asymptotically for large k. This expansion will also be needed in the proof of Theorem 1.

The solution $U_0(\vec{r}_o, \vec{r}_s; k)$ of the problem P_0 can be represented as a Fourier series with respect to the "radial" eigenfunctions $(kr)^{\frac{1}{2}} H_{\nu_n}^{(1)}(kr)$, $n = 1, 2, 3, \ldots$, where $H_\nu^{(1)}(kr)$ is the Hankel function of the first kind of order ν, and $\nu_n = \nu_n(k)$ is the n^{th} zero of the derivative $H_\nu^{(1)\prime}(ka)$. (The "radial" expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ is a Watson transform of the Fourier expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ in the "angular" eigenfunctions $e^{in\theta}$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ (*cf.* [19]). The arguments necessary to carry out the Watson transform have been given in [7] and [24].)

It follows from the analysis of URSELL [26] that the leading term $T_1(\vec{r}_o, \vec{r}_s; k)$ of the radial expansion is an asymptotic representation of $U_0(\vec{r}_o, \vec{r}_s; k)$ as $k \to \infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_0^<(\vec{r}_s)$:

$$(1.4) \qquad U_0(\vec{r}_o, \vec{r}_s; k) = T_1(\vec{r}_o, \vec{r}_s; k)\left[1 + O\left(\exp\{-k^{\frac{1}{3}} \gamma \lambda_<(r_0, \theta_0; r_s)\}\right)\right].$$

Here $\theta_0 = \arg \vec{r}_o$, and with no loss of generality we have set $\arg \vec{r}_s = 0$. The positive constant γ is equal to $2^{-\frac{1}{3}} \sin(2\pi/3)[|q_2| - |q_1|] a^{-\frac{2}{3}}$ where q_1 and q_2 are the first and second zeros (in order of increasing magnitude) of the derivative $Ai'(q)$ of the Airy function $Ai(q)$.

If $\theta_0 \leqq \pi$, then $\lambda_<(r_0, \theta_0; r_s) = \lambda(r_0, \theta_0; r_s)$ where

$$(1.5) \qquad \lambda(r_0, \theta_0; r_s) = a\left[\theta_0 - \arccos\left(\frac{a}{r_0}\right) - \arccos\left(\frac{a}{r_s}\right)\right].$$

If $\theta > \pi$, then $\lambda_<(r_0, \theta_0; r_s) = \lambda(r_0, 2\pi - \theta_0; r_s)$ where

$$(1.6) \qquad \lambda(r_0, 2\pi - \theta_0; r_0) = a\left[2\pi - \theta_0 - \arccos\left(\frac{a}{r_0}\right) - \arccos\left(\frac{a}{r_s}\right)\right].$$

$\lambda_<(r_0, \theta_0; r_s) + \sqrt{r_0^2 - a^2} + \sqrt{r_s^2 - a^2}$ is the minimum length of paths in the region $|\vec{r}_o| \geqq a$ from $\vec{r}_s$ to $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_0(\vec{r}_s)$. Clearly, $\lambda_<(r_0, \theta_0; r_s) + \sqrt{r_0^2 - a^2} + \sqrt{r_s^2 - a^2}$ is also the minimum length of paths in the region $\mathscr{D}_1 \cup C_1$ from $\vec{r}_s$ to $\vec{r}_o$. (*Cf.* Fig. 3.)

Since $\lambda_<(r_0, \theta_0; r_s)$ is positive and continuous in r_0, θ_0 for all $\vec{r}_o$ in the closed bounded set $\mathscr{S}_1^<(\vec{r}_s)$, we have

$$(1.7) \qquad \lambda_<^*(r_s) = \underset{\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)}{\text{Minimum}} \, \lambda_<(r_0, \theta_0; r_s) > 0.$$

It follows from (1.4) and (1.7) that as $k \to \infty$

$$(1.8) \qquad U_0(\vec{r}_o, \vec{r}_s; k) = T_1(\vec{r}_o, \vec{r}_s; k)\left[1 + O\left(\exp\{-k^{\frac{1}{3}} \gamma \lambda_<^*(r_s)\}\right)\right],$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$.

The function $T_1(\vec{r}_o, \vec{r}_s; k)$ is a linear combination of the functions $\exp\{i\nu_1(k)\theta_0\}$ and $\exp\{-i\nu_1(k)\theta_0\}$:

$$T_1(\vec{r}_o, \vec{r}_s; k) = \sum_{m=1}^{2} \exp\{i\nu_1(k)\phi_m(\theta_0)\} L(r_0, r_s; k) \tag{1.9}$$

where $\phi_1(\theta)=\theta$, $\phi_2(\theta)=2\pi-\theta$, and

$$L(r_0, r_s; k) = \frac{1}{ika} \frac{[1-\exp\{2\pi i\nu_1\}]^{-1}}{H^{(1)}_{\nu_1}(ka)\cdot \frac{\partial}{\partial\nu} H^{(1)}_{\nu_1}(ka)} H^{(1)}_{\nu_1}(kr_0) H^{(1)}_{\nu_1}(kr_s). \tag{1.10}$$

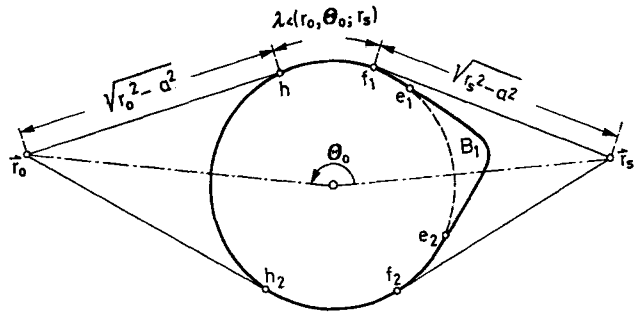


Fig. 3

As $k\to\infty$ we have for every positive integer M and N, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$, (*cf.* [23, 4, 6])

$$\exp\{i\nu_1(k)\phi_m(\theta_0)\} L(r_0, r_s; k) = \frac{\exp\{\Delta(\phi_m(\theta_0); k)+\chi(r_s; k)\}}{[1-\exp\{\Delta(2\pi; k)\}]}$$

$$\cdot\left[\frac{Ai(\Psi(r_0;k))}{(ka)^{\frac{1}{2}}}\left\{\sum_{n=0}^{M}\frac{P_n^{(1)}(r_0, \phi_m(\theta_0); r_s)}{(ka)^{n/3}} + O\left(\frac{1}{k^{(M+1)/3}}\right)\right\}\right. \tag{1.11}$$

$$\left. + \frac{Ai'(\Psi(r_0;k))}{(ka)^{\frac{7}{6}}}\left\{\sum_{n=0}^{N}\frac{P_n^{(2)}(r_0, \phi_m(\theta_0); r_s)}{(ka)^{n/3}} + O\left(\frac{1}{k^{(N+1)/3}}\right)\right\}\right].$$

Here

(i) $\Delta(\phi; k) = i[(ka)+(ka)^{\frac{1}{3}}\tau_1]\phi$,

(ii) $\chi(r_s; k) = (ka)\frac{2}{3}\zeta^{\frac{3}{2}}\left(\frac{r_s}{a}\right) - i(ka)^{\frac{1}{3}}\tau_1 \cdot \arccos\left(\frac{a}{r_s}\right)$

(iii) $\Psi(r_0; k) = \exp\left\{-\frac{2\pi}{3}\right\}\left[(ka)^{\frac{2}{3}}\zeta\left(\frac{r_0}{a}\right) - i\tau_1\cdot\arccos\left(\frac{a}{r_0}\right)\cdot\zeta^{-\frac{1}{2}}\left(\frac{r_0}{a}\right)\right]$,

(iv) $P_0^{(j)}(r_0, \phi_m(\theta_0); r_s)$

$$= \frac{\exp\{i\pi/12\}}{2^{\frac{1}{3}} i\sqrt{\pi}} \frac{1}{\tau_1[Ai(q_1)]^2} \frac{1}{\left[\left(\frac{r_s}{a}\right)^2 - 1\right]^{\frac{1}{4}}} \left(\frac{a}{r_0}\right)^{\frac{1}{2}} \frac{V^{(j)}_{(r_0)}}{\sqrt{\zeta^{(1)}\left(\frac{r_0}{a}\right)}}, \qquad j=1,2,$$

(1.12) (v) $V^{(1)}(r_0)=1$,

(vi) $V^{(2)}(r_0)=\left[-\frac{1}{6}\zeta\left(\frac{r_0}{a}\right)+\frac{1}{3}\frac{r_0}{a}\zeta^{(1)}\left(\frac{r_0}{a}\right)+\frac{1}{2}\left(\frac{r_0}{a}\right)^2\zeta^{(2)}\left(\frac{r_0}{a}\right)\right]\tau_1$

$$+\left[\frac{2}{3}\zeta\left(\frac{r_0}{a}\right)-\frac{r_0}{a}\zeta^{(1)}\left(\frac{r_0}{a}\right)\right]\tau_1^{(1)},$$

(vii) $\frac{2}{3}\zeta^{\frac{3}{2}}\left(\frac{r}{a}\right)=i\left[\sqrt{\left(\frac{r}{a}\right)^2-1}-\arccos\left(\frac{a}{r}\right)\right],$

(viii) $\tau_1=2^{-\frac{1}{3}}q_1\cdot\exp\left\{\frac{-2\pi i}{3}\right\},$

$$\tau_1^{(1)}=\frac{1}{30}2^{-\frac{2}{3}}q_1^2\cdot\exp\left\{\frac{-4\pi i}{3}\right\}.$$

$\zeta^{(m)}(x)=d^m\zeta(x)/dx^m$ for $m=1, 2$ in (iv) and (vi).

For $j=1, 2$, and $n\geqq 0$ the functions $P_n^{(j)}(r_0, \theta_0; r_s)$, $P_n^{(j)}(r_0, 2\pi-\theta_0; r_s)$ are independent of k, and uniformly continuous in $r_0, \theta_0, \vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$.

It follows from (1.8), (1.9), and (1.11) that as $k\to\infty$, for every positive integer M and N, uniformly in $\vec{r}_o, \vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$,

$$U_0(\vec{r}_o, \vec{r}_s, k)=\exp\{\chi(r_s; k)\}\sum_{m=1}^{2}\exp\{\Delta(\phi_m(\theta_0), k)\}$$

$$\text{(1.13)}\qquad \cdot\left[\frac{Ai(\Psi(r_0; k))}{(ka)^{\frac{1}{3}}}\left\{\sum_{n=0}^{M}\frac{P_n^{(1)}(r_0, \phi_m(\theta_0); r_s)}{(ka)^{\frac{n}{3}}}+O\left(\frac{1}{k^{(M+1)/3}}\right)\right\}\right.$$

$$\left.+\frac{Ai'(\Psi(r_0; k))}{(ka)^{\frac{2}{3}}}\left\{\sum_{n=0}^{N}\frac{P_n^{(2)}(r_0, \phi_m(\theta_0); r_s)}{(ka)^{\frac{n}{3}}}+O\left(\frac{1}{k^{(N+1)/3}}\right)\right\}\right].$$

Before proving Theorem 1 we first use the asymptotic expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ given by (1.13) to compare the assertion of the Theorem with the predictions of (12), (27), (28) and (15). Thus, if we assume the truth of (1.2), and use (1.13), it follows that as $k\to\infty$

$$\text{(1.14)}\qquad U_1(\vec{r}_o, \vec{r}_s; k)=F_{M,N}(r_0, \theta_0; r_s:k),$$

uniformly in $\vec{r}_o, \vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$, for every positive integer M and N. Here $F_{M,N}(r_0, \theta_0; r_s:k)$ denotes the right hand side of (1.13).

The uniform asymptotic expansion of $U_1(\vec{r}_o, \vec{r}_s; k)$ given by (1.14) has the structure of the formal asymptotic solution of the problem P_1 that is obtained by the procedure of Lewis, Bleistein & Ludwig.

If $\vec{r}_o$ lies in $\mathscr{S}_1^<(\vec{r}_s)$, but not on C_1, then the Airy function and its derivative in (1.14) can be expanded asymptotically for large argument as $k\to\infty$. Then (1.14) reduces to

$$\text{(1.15)}\qquad U_1(\vec{r}_o, \vec{r}_s; k)\sim\sum_{m=1}^{2}D(r_0, \phi_m(\theta_0); r_s:k)\left\{1+\sum_{n=0}^{\infty}\frac{Q_n(r_0, \phi_m(\theta_0); r_s)}{(ka)^{n/3}}\right\}.$$

Here

$$D(r_0, \phi_m(\theta_0); r_s:k)=\left\{\frac{e^{\frac{\pi i}{3}}}{4\pi i}\frac{1}{|q_1|\,Ai^2(q_1)}\right\}\frac{1}{k^{\frac{1}{3}}}\frac{a^{\frac{1}{6}}}{[r_0^2-a^2]^{\frac{1}{4}}}\frac{a^{\frac{1}{6}}}{[r_s^2-a^2]^{\frac{1}{4}}}$$
$$\cdot\exp\{ik[\lambda(r_0, \phi_m(\theta_0); r_s)+\sqrt{r_0^2-a^2}+\sqrt{r_s^2-a^2}]\}$$
$$\cdot\exp\{i\tau_1 k^{\frac{1}{3}} a^{-\frac{2}{3}}\lambda(r_0, \phi_m(\theta_0); r_s)\},$$

with $\lambda(r_0, \phi_m(\theta_0); r_s)$ defined by (1.5) for $m=1$, and by (1.6) for $m=2$.

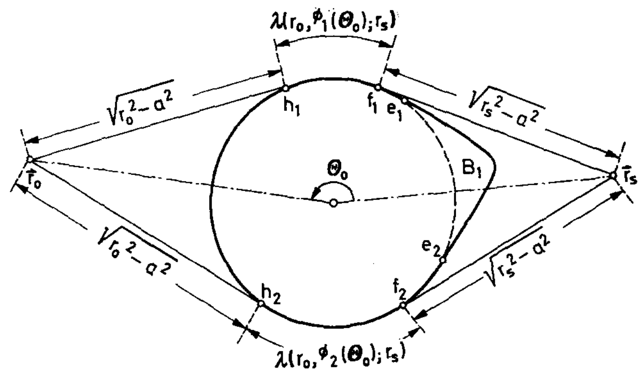


Fig. 4

The quantity $\sqrt{r_0^2-a^2}+\sqrt{r_s^2-a^2}+\lambda(r_0, \phi_m(\theta_0); r_s)$ is the length of the "optical path" $\vec{r}_s f_m h_m \vec{r}_o$ in $\mathscr{D}_1\cup C_1$ from $\vec{r}_s$ to $\vec{r}_o$ as depicted in Fig. 4. $\sqrt{r_s^2-a^2}$ is the linear distance from the source $\vec{r}_s$ to the point of "diffraction" f_m on C_1. $\lambda(r_0, \phi_m(\theta_0); r_s)$ is the distance along C_1 from f_m to the point of "shedding" h_m, and $\sqrt{r_0^2-a^2}$ is the linear distance from h_m to the point of observation $\vec{r}_o$.

The functions $Q_n(r_0, \phi_m(\theta_0); r_s)$, $n=1, 2, 3, \ldots$, $m=1, 2$, are independent of k and continuous in r_0, θ_0 for all $\vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)-C_1$. As $\vec{r}_o$ approaches the "caustic" C_1 each $Q_n(r_0, \phi_m(\theta_0); r_s)$ becomes infinite together with $D(r_0, \phi_m(\theta_0); r_s:k)$.

The functions $D(r_0, \phi_m(\theta_0); r_s:k)$, $m=1, 2$, are the "diffraction modes" of KELLER's geometrical theory of diffraction. The asymptotic expansion of $U_1(\vec{r}_o, \vec{r}_s; k)$ given by (1.15) is of the form predicted by the extended geometrical theory of diffraction.

Similarly, if $\vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)\cap C_1$, then $r_0=a$, and $\Psi(a;k)=q_1$. Since $Ai'(q_1)=0$, the expansion (1.14) reduces to

$$(1.16)\qquad U_1(\vec{r}_o, \vec{r}_s; k)\sim\sum_{m=1}^{2} D(\phi_m(\theta_0); r_s:k)\left[1+\sum_{n=1}^{\infty}\frac{Q_n(\phi_m(\theta_0); r_s)}{(ka)^{n/3}}\right].$$

Here

$$D(\phi_m(\theta_0); r_s:k)=\left\{\frac{e^{\frac{3\pi i}{4}}}{2^{\frac{5}{6}}}\frac{1}{|q_1|\,Ai(q_1)}\right\}\frac{1}{k^{\frac{1}{6}}a^{\frac{1}{6}}}\frac{a^{\frac{1}{6}}}{[r_s^2-a^2]^{\frac{1}{4}}}$$
$$\cdot\exp\{ik[\lambda(a, \phi_m(\theta_0); r_s)+\sqrt{r_s^2-a^2}]\}$$
$$\cdot\exp\{i\tau_1 k^{\frac{1}{3}} a^{-\frac{2}{3}}\lambda(a, \phi_m(\theta_0); r_s)\}.$$

The functions $Q_n(\phi_m(\theta_0); r_s)$, $n=1, 2, 3, \ldots$, $m=1, 2$, are independent of k and continuous in θ_0 for $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s) \cap C_1$.

The functions $D(\phi_m(\theta_0); r_s:k)$, $m=1, 2$, are the highest order "creeping wave modes" as discussed by FRANZ & DEPPERMAN [6], and KELLER [12].

Using (1.15) and (1.16) in (1.2), we find, as predicted by the geometrical theory, that if k is large, and $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s) - C_1$ or $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s) \cap C_1$, the geometry of B_1 affects only terms of order

$$\text{maximum}\left[\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\mu a^{-\frac{2}{3}}\lambda(r_0, \theta_0; r_s)\},\right.$$
$$\left.\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\mu a^{-\frac{2}{3}}\lambda(r_0, 2\pi-\theta_0; r_s)\}\right],$$

where μ is any constant greater than 1.

We turn now to the proof of Theorem 1. The nature of the result to be proved suggests reformulation of the problem P_1 as an integral equation expressing $U_1(\vec{r}_o, \vec{r}_s; k)$ in terms of $U_0(\vec{r}_o, \vec{r}_s; k)$. Applying Green's second identity [14] to these functions on the region $\mathscr{D}_1 \cup C_1$, we obtain

$$(1.17)\qquad U_1(\vec{r}_o, \vec{r}_s; k) = U_0(\vec{r}_o, \vec{r}_s; k) + I_1(\vec{r}_o, \vec{r}_s; k),$$

$$(1.18)\qquad I_1(\vec{r}_o, \vec{r}_s; k) = \int_0^{\theta_1} + \int_{\theta_2}^{2\pi} \partial_n^{(1)} U_0(\vec{r}_o, \vec{r}_{B_1}(\theta); k)\, U_1(\vec{r}_{B_1}(\theta), \vec{r}_s; k)\, |d\vec{r}_{B_1}(\theta)|.$$

Here the integration is over the convex arc B_1, which is represented in polar coordinates by $\vec{r}_{B_1}(\theta)$ $(\vec{r}_{B_1}(2\pi) = \vec{r}_{B_1}(0))$. θ_1 and θ_2 are the values of θ at the points e_1 and e_2 where B_1 is joined to C_1. $\partial_n^{(1)}$ denotes differentiation in the direction of the outward normal to C_1.

To prove Theorem 1, we show that as $k \to \infty$

$$(1.19)\qquad I_1(\vec{r}_o, \vec{r}_s; k) = O(\exp\{-k^{\frac{1}{3}}\sigma\})\, U_0(\vec{r}_o, \vec{r}_s; k),$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$.

We make use of the following estimates, which hold as $k \to \infty$:

$$(1.20)\qquad U_1(\vec{r}_1(\theta), \vec{r}_s; k) = O\left(\frac{1}{k^{\frac{1}{6}}|\vec{r}_1(\theta) - \vec{r}_s|}\right) = o(1),$$

uniformly in θ, $0 \leqq \theta \leqq 2\pi$;

$$(1.21)\qquad U_0^{-1}(\vec{r}_o, \vec{r}_s; k) = O(k^{\frac{2}{3}} \exp\{k^{\frac{1}{3}}(\operatorname{Im}\tau_1) a^{-\frac{2}{3}} \lambda_<(r_0, \theta_0; r_s)\}),$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$;

$$(1.22)\qquad \begin{aligned} &\partial_n^{(1)} U_0(\vec{r}_o, \vec{r}_{B_1}(\theta); k) \\ &\quad = \sum_{m=1}^{2} O(k^{\frac{1}{3}} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1) a^{-\frac{2}{3}} \lambda_<(r_0, \phi_m(|\theta_0 - \theta|); r_{B_1}(\theta))\}), \end{aligned}$$

uniformly in $\vec{r}_o$ and θ, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$, $0 \leqq \theta \leqq \theta_2$, $\theta_2 \leqq \theta \leqq 2\pi$.

The function $\vec{r}_1(\theta)$ in (1.20) is the polar representation of C_1. In (1.21) and (1.22) the constant $\operatorname{Im}\tau_1$ is positive since $\tau_1 = 2^{-\frac{1}{3}} e^{-2\pi i/3} q_1$. The positive function $\lambda_<(r_0, \theta_0; r_s)$ is given by (1.5) and (1.6). If $0 \leqq \theta \leqq \theta_1$, then $\theta_0 - \theta > 0$, and

$\lambda(r_0, \phi_m(|\theta_0-\theta|); r_{B_1}(\theta))$ in (1.22) is the length of that part of the optical path $\vec{r}_{B_1}(\theta) f_m(\theta) h_m \vec{r}_o$ in $|\vec{r}| \geqq a$, which coincides with C_0, as shown in Fig. 5a. If $\theta_2 \leqq \theta \leqq 2\pi$, then $\theta_0-\theta<0$, and $\lambda(r_0, \phi_1(|\theta_0-\theta|); r_{B_1}(\theta))$ is the length of that part of the optical path $\vec{r}_{B_1}(\theta) f_2(\theta) h_2 \vec{r}_o$ in $|\vec{r}| \geqq a$, which coincides with C_0. Similarly, $\lambda(r_0, \phi_2(|\theta_0-\theta|); r_{B_1}(\theta))$ is the length of that part of the optical path $\vec{r}_{B_1}(\theta) f_1(\theta) h_1 \vec{r}_o$ in $|\vec{r}| \geqq a$, which coincides with C_0 (*cf.* Fig. 5b).

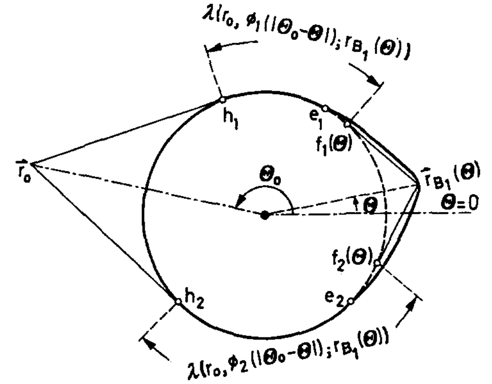


Fig. 5a

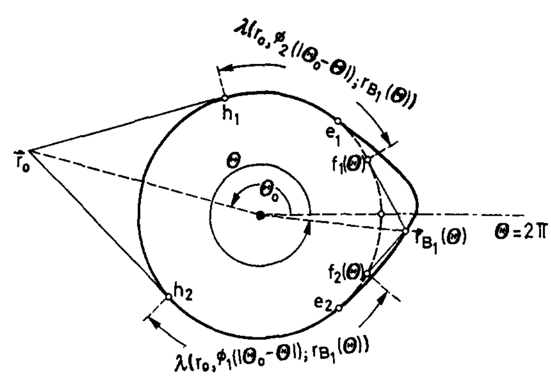


Fig. 5b

(1.20), which bounds the field $U_1(\vec{r}_o, \vec{r}_s; k)$ on C_1 as $k \to \infty$, is derived by Grimshaw in [8]. (1.21) follows from the uniform expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ given by (1.13) (*cf.* Appendix I).

The argument leading to (1.22) is similar to that used to establish (1.4), and subsequently to derive (1.13) from (1.4) (*cf.* Appendix II).

Using (1.20) and (1.21), we obtain the following estimate for the integral (1.18). As $k \to \infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^{<}(\vec{r}_s)$,

$$I_1(\vec{r}_o, \vec{r}_s; k) \tag{1.23}$$
$$= \sum_{m=1}^{2} O\left(k^{\frac{1}{6}} \int_{\theta_2}^{2\pi} + \int_{0}^{\theta_1} \exp\{-k^{\frac{1}{3}} (\operatorname{Im} \tau_1) a^{-\frac{2}{3}} \lambda(r_0, \phi_m(|\theta_0-\theta|); r_{B_1}(\theta))\}\right) d\theta.$$

If we recall the geometric interpretation of the functions $\lambda(r_0, \phi_m(|\theta_0-\theta|); r_{B_1}(\theta))$ given after (1.22) it is clear that (i) as θ increases from θ_2 to 2π the function $\lambda(r_0, \phi_1(|\theta_0-\theta|); r_{B_1}(\theta))$ $(=\lambda(r_0, \theta-\theta_0; r_{B_1}(\theta)))$ increases monotonically from $\lambda(r_0, \theta_2-\theta_0; r_{B_1}(\theta_2))$ $(=\lambda(r_0, \theta_2-\theta_0; a))$, while (ii) $\lambda(r_0, \phi_2(|\theta_0-\theta|); r_{B_1}(\theta))$ $(=\lambda(r_0, 2\pi-(\theta-\theta_0); r_{B_1}(\theta)))$ decreases monotonically to $\lambda(r_0, \theta_0; r_{B_1}(2\pi))$.

Furthermore, it is clear that as θ increases from 0 to θ_1 (iii) the function $\lambda(r_0, \phi_1(|\theta_0-\theta|); r_{B_1}(\theta))$ $(=\lambda(r_0, \theta_0-\theta; r_{B_1}(\theta)))$ decreases monotonically to $\lambda(r_0, \theta_0-\theta_1; r_{B_1}(\theta_1))$ $(=\lambda(r_0, \theta_0-\theta_1; a))$, while (iv) $\lambda(r_0, \phi_2(|\theta_0-\theta|); r_{B_1}(\theta))$ $(=\lambda(r_0, 2\pi-(\theta_0-\theta); r_{B_1}(\theta)))$ increases from $\lambda(r_0, 2\pi-\theta_0; r_{B_1}(0))$ $(=\lambda(r_0, 2\pi-\theta_0; r_{B_1}(2\pi)))$.

Consequently, we have for $m=1, 2$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$,

$$
(1.24)\quad
\begin{aligned}
&\text{(i)} && \int_{\theta_2}^{2\pi} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \phi_2(|\theta_0-\theta|); r_{B_1}(\theta))\}\,d\theta \\
& && \qquad \leqq (2\pi-\theta_2)\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \theta_0; r_{B_1}(0))\}, \\
&\text{(ii)} && \int_{\theta_2}^{2\pi} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \phi_1(|\theta_0-\theta|); r_{B_1}(\theta))\}\,d\theta \\
& && \qquad \leqq (2\pi-\theta_2)\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \theta_2-\theta_0; a)\}, \\
&\text{(iii)} && \int_{0}^{\theta_1} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \phi_2(|\theta_0-\theta|); r_{B_1}(\theta))\}\,d\theta \\
& && \qquad \leqq \theta_1\cdot\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, 2\pi-\theta_0; r_{B_1}(2\pi))\}, \\
&\text{(iv)} && \int_{0}^{\theta_1} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \phi_1(|\theta_0-\theta|); r_{B_1}(\theta))\}\,d\theta \\
& && \qquad \leqq \theta_1\cdot\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda(r_0, \theta_0-\theta_1; a)\}.
\end{aligned}
$$

Using the inequalities (1.24) in (1.23), and noting that $\lambda(r_0, \theta_0; r_{B_1}(0))$, $\lambda(r_0, 2\pi-\theta_0; r_{B_1}(2\pi)) > \max[\lambda(r_0, \theta_2-\theta_0; a), \lambda(r_0, \theta_0-\theta_1; a)]$, we find that as $k\to\infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$,

$$
(1.25)\qquad I_1(\vec{r}_o, \vec{r}_s; k) = O\left(k^{\frac{1}{6}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda_<(r_0, \theta_0)\}\right).
$$

Here $\lambda_<(r_0, \theta_0) = \min\limits_{m=1,2} \lambda(r_0, |\theta_m-\theta_0|; a)$.

We now re-write (1.25) as

$$
(1.26)\qquad I_1(\vec{r}_o, \vec{r}_s; k) = O\left(\frac{k^{\frac{1}{6}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}\lambda_<(r_0, \theta_0)\}}{U_0(\vec{r}_o, \vec{r}_s; k)}\right) U_0(\vec{r}_o, \vec{r}_s; k),
$$

and make use of (1.21) in (1.26) to obtain the estimate

$$
(1.27)\qquad
\begin{aligned}
&I_1(\vec{r}_o, \vec{r}_s; k) \\
&\quad = O\left(k^{\frac{5}{6}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}[\lambda_<(r_0, \theta_0) - \lambda_<(r_0, \theta_0; r_s)]\}\right) U_0(\vec{r}_o, \vec{r}_s; k)
\end{aligned}
$$

as $k\to\infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_1^<(\vec{r}_s)$.

Our requirement that B_1 lie entirely in the illuminated region assures that the distances $|e_1 f_1|$, and $|e_2 f_2|$ from the ends of B_1 to the points of diffraction are positive, and consequently that $\lambda_<(r_0, \theta_0) - \lambda_<(r_0, \theta_0; r_s) \geqq \min_{m=1,2} |e_m f_m| > 0$. Since $\operatorname{Im} \tau_1 > 0$, the factor multiplying $k^{\frac{1}{3}}$ in the exponential of (1.27) is negative, independent of k, and of $\vec{r}_o$. Therefore, the entire expression in the order symbol of (1.27) is $O(\exp\{-k^{\frac{1}{3}}\sigma\})$, where σ is positive, independent of k, and of $\vec{r}_o$. This completes the proof of Theorem 1.

Part 2

In this part we consider the solution $U_2(\vec{r}_o, \vec{r}_s; k)$ of the more general scattering problem P_2:

$$(2.1) \qquad \begin{aligned} &\text{(i)} \quad \Delta U + k^2 U = \delta(\vec{r}_o, \vec{r}_s), && \vec{r}_o, \vec{r}_s \in \mathscr{D}_2; \\ &\text{(ii)} \quad \partial_n^{(2)} U = 0, && \vec{r}_o \in C_2; \\ &\text{(iii)} \quad \lim_{r_0 \to \infty} \rho^{\frac{1}{2}} |U_\rho - i k U| = 0, \quad \rho = |\vec{r}_o - \vec{r}|, && \vec{r} \in \mathscr{D}_2. \end{aligned}$$

Here $\mathscr{D}_2$ is the exterior of the smooth convex curve C_2 formed by "pasting" the ends i_1 and i_2 of a smooth convex arc B_2 to the "dark" side of the convex curve C_1 (*cf.* Fig. 2). $\partial_n^{(2)}$ denotes differentiation in the direction of the outward normal to C_2. $\vec{r}_s$ and $\vec{r}_o$ again denote the "source" and "observation" points, respectively.

We shall establish the following result on the asymptotic behavior of $U_2(\vec{r}_o, \vec{r}_s; k)$ for large k.

Theorem 2. *As* $k \to \infty$

$$(2.2) \qquad U_2(\vec{r}_o, \vec{r}_s; k) = U_0(\vec{r}_o, \vec{r}_s; k)[1 + O(\exp\{-k^{\frac{1}{3}}\beta\})],$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s) = \mathscr{S}_2^<(\vec{r}_s) - \mathscr{R}$, *where* β *is positive, independent of* k, *and of* $\vec{r}_o$.

$\mathscr{S}_2(\vec{r}_s)$ is the "shadow" of C_2: $\vec{r}_o \in \mathscr{S}_2(\vec{r}_s)$ if and only if $\vec{r}_o \in \mathscr{D}_2 \cup C_2$, and the straight line through $\vec{r}_o$ and $\vec{r}_s$ cuts C_2 at two distinct points. $\mathscr{S}_2^<(\vec{r}_s)$ is any closed bounded subset of $\mathscr{S}_2(\vec{r}_s)$. If $\vec{r}_o \in \mathscr{S}_2^<(\vec{r}_s)$ we say $\vec{r}_o$ lies in the "deep" shadow of C_2 (*cf.* Fig. 2).

$\mathscr{R}$ is the region of influence of B_2, constructed as follows. Determine the smaller of the two circular arcs $f_1 i_1$ and $f_2 i_2$ of $C_2 \cap \mathscr{S}_2(\vec{r}_s)$, say it is $f_1 i_1$, as depicted in Fig. 2. Let i'_1 be a point on $f_1 i_1$, arbitrarily close to i_1. Next, let i'_2 be the point on (the circular part of) $C_2 \cap \mathscr{S}_2(\vec{r}_s)$ such that $|f_1 i'_1| = |f_2 i'_2|$. $\mathscr{R}$ is the region bounded by the tangents to C_2 at i'_1, i'_2, and the arc $i'_1 i_1 i_2 i'_2$ of $C_2 \cap \mathscr{S}_2(\vec{r}_s)$.

Since $\mathscr{S}_2^-(\vec{r}_s) \subseteq \mathscr{S}_1^<(\vec{r}_s)$, an immediate consequence of Theorem 2 is that the uniform asymptotic expansion of $U_0(\vec{r}_o, \vec{r}_s; k)$ given by the right side of (1.13) is also asymptotic expansion of $U_2(\vec{r}_o, \vec{r}_s; k)$ as $k \to \infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s)$. This expansion of $U_2(\vec{r}_o, \vec{r}_s; k)$ is of the form predicted by the formal theory of LEWIS, BLEISTEIN & LUDWIG.

If $\vec{r}_o$ lies in $\mathscr{S}_2^-(\vec{r}_s)$, but not on C_2, then the Airy functions in this uniform expansion of $U_2(\vec{r}_o, \vec{r}_s; k)$ can be expanded asymptotically for large argument, leading to the result predicted by the extended geometrical theory of diffraction [27, 28], *viz* the right hand side of (1.15).

If $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s) \cap C_2$, then this uniform expansion of $U_2(\vec{r}_o, \vec{r}_s; k)$ reduces to the "creeping wave" expansion [6, 15], *viz* the right hand side of (1.16).

Using the non-uniform asymptotic expansions of $U_2(\vec{r}_o, \vec{r}_s; k)$ just mentioned in (2.2), we find, as predicted by the geometrical theory, that if k is large, and $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s) - C_2$, or $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s) \cap C_2$, the geometry of B_1 and B_2 affects only terms of the order of the maximum of $\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,\mu\, a^{-\frac{2}{3}}\lambda(r_0, \theta_0; r_s)\}$ and $\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,\mu\, a^{-\frac{2}{3}}\lambda(r_0, 2\pi-\theta_0; r_s)\}$ where μ is any constant greater than 1.

We remark that the region $\mathscr{S}_2^-(\vec{r}_s)$ is not in general the maximal region where Theorem 2 would be expected to hold on the basis of the geometrical theory of diffraction. This maximal region consists of $\mathscr{S}_2^-(\vec{r}_s)$, and those points $\vec{r}_o$ of $\mathscr{R}$ that satisfy the following conditions:

(i) There is an optical path P in $\mathscr{D}_2 \cup C_2$ from $\vec{r}_s$ to $\vec{r}_o$ that does not intersect B_2,

(ii) $$\int_{C_2 \cap P} \kappa^{\frac{2}{3}}(\vec{r})\,|d\vec{r}| < \int_{C_2 \cap P'} \kappa^{\frac{2}{3}}(\vec{r})\,|d\vec{r}|$$

where $\kappa(\vec{r})$ is the curvature of C_2 at $\vec{r}$, and P' is the shortest of the other optical paths in $\mathscr{D}_2 \cup C_2$ from $\vec{r}_s$ to $\vec{r}_o$.

It is our intention to consider, in a sequel to this paper, the problem of extending our result to this maximal region, and also to determine the effect of B_2 on the field $U_2(\vec{r}_o, \vec{r}_s; k)$ inside $\mathscr{R}$.

To prove Theorem 2, we establish the following Lemma:

As $k \to \infty$ uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s)$,

$$U_2(\vec{r}_o, \vec{r}_s; k) = U_1(\vec{r}_o, \vec{r}_s; k)\left[1 + O\left(\exp\{-k^{\frac{1}{3}}\alpha\}\right)\right], \tag{2.3}$$

where α is positive, independent of k, and of $\vec{r}_o$.

Theorem 2 then follows immediately from (2.3), and Theorem 1 since $\mathscr{S}_2^-(\vec{r}_s) \subseteq \mathscr{S}_1^<(\vec{r}_s)$.

Proceeding as in Part 1, we reformulate P_2 as an integral equation expressing $U_2(\vec{r}_o, \vec{r}_s; k)$ in terms of $U_1(\vec{r}_o, \vec{r}_s; k)$. Applying Green's second identity to these functions on the region $\mathscr{D}_2 \cup C_2$, we obtain the equation

$$U_2(\vec{r}_o, \vec{r}_s; k) = U_1(\vec{r}_o, \vec{r}_s; k) + I_2(\vec{r}_o, \vec{r}_s; k), \tag{2.4}$$

where

$$I_2(\vec{r}_o, \vec{r}_s; k) = \int_{\theta_1'}^{\theta_2'} \partial_n^{(2)} U_1(\vec{r}_{B_2}(\theta), \vec{r}_s; k) \cdot U_2(\vec{r}_o, \vec{r}_{B_2}(\theta); k)\,|d\vec{r}_{B_2}(\theta)|. \tag{2.5}$$

Here the integration is over the convex arc B_2, which is represented in polar coordinates by $\vec{r}_{B_2}(\theta)$. θ_1' and θ_2' are the values of θ at the points i_1 and i_2 where B_2 is joined to C_1. $\partial_n^{(2)}$ indicates differentiation in the direction of the outward normal to B_2.

To prove (2.3), we show that as $k \to \infty$ uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s)$,

$$I_2(\vec{r}_o, \vec{r}_s; k) = O(\exp\{-k^{\frac{1}{3}}\alpha\})\, U_1(\vec{r}_o, \vec{r}_s; k). \tag{2.6}$$

We make use of the following estimates, which hold as $k \to \infty$:

$$U_2(\vec{r}_o, \vec{r}_2(\theta); k) = O\left(\frac{1}{k^{\frac{1}{6}}\,|\vec{r}_o - \vec{r}_2(\theta)|}\right) = o(1), \tag{2.7}$$

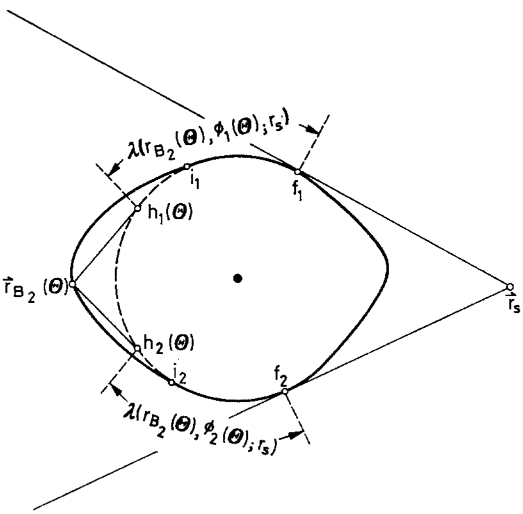


Fig. 6

uniformly in θ, $0 \leqq \theta \leqq 2\pi$;

$$\begin{aligned} U_1^{-1}(\vec{r}_o, \vec{r}_s; k) &= O(U_0^{-1}(\vec{r}_o, \vec{r}_s; k)) \\ &= O(k^{\frac{2}{3}} \exp\{k^{\frac{1}{3}} (\operatorname{Im} \tau_1)\, a^{-\frac{2}{3}} \lambda_<(r_0, \theta_0; r_s)\}), \end{aligned} \tag{2.8}$$

uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathscr{S}_2^-(\vec{r}_s)$;

$$\begin{aligned} \partial_n^{(2)} U_1(\vec{r}_{B_2}(\theta), \vec{r}_s; k) &= O(\partial_n^{(2)} U_0(\vec{r}_{B_2}(\theta), \vec{r}_s; k)) \\ &= \sum_{m=1}^{2} O(k^{\frac{1}{3}} \exp\{-k^{\frac{1}{3}} (\operatorname{Im} \tau_1)\, a^{-\frac{2}{3}} \lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)\}), \end{aligned} \tag{2.9}$$

uniformly in θ, $\theta_1' \leqq \theta \leqq \theta_2'$. Here $\vec{r}_2(\theta)$ is the polar representation of C_2, and $\vec{r}_{B_2}(\theta) = \vec{r}_2(\theta)$ for $\theta_1' \leqq \theta \leqq \theta_2'$. The definitions of τ_1, $\lambda_<(r_0, \theta_0; r_s)$ and $\phi_m(\theta)$ are those given in Part 1. For $\theta_1' \leqq \theta \leqq \theta_2'$ the function $\lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)$ in (2.9) is the length of part of the optical path $\vec{r}_s f_m h_m(\theta) \vec{r}_{B_2}(\theta)$ in $\mathscr{D}_1 \cup C_1$ that coincides with C_1 (*cf.* Fig. 6).

(2.7) is the estimate of Grimshaw obtained in [8]. (2.8) follows directly from Theorem 1, and the estimate (1.21). (2.9) is derived by essentially the same argument used to establish Theorem 1 (*cf.* Appendix III).

Using (2.7) and (2.9) in (2.5), we find that as $k \to \infty$

$$(2.10)\quad I_2(\vec{r}_o, \vec{r}_s; k) = \sum_{m=1}^{2} O\left(k^{\frac{1}{6}} \int_{\theta_1'}^{\theta_2'} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)\}\, d\theta\right).$$

As in Part 1 we remark at this point that the function $\lambda(r_{B_2}(\theta), \phi_1(\theta); r_s)$ $(=\lambda(r_{B_2}(\theta), \theta; r_s))$ in (2.10) increases monotonically from $\lambda(r_{B_2}(\theta_1'), \theta_1'; r_s)$ as θ increases from θ_1' to θ_2'. Also the function $\lambda(r_{B_2}(\theta), \phi_2(\theta); r_s)$ $(=\lambda(r_{B_2}(\theta), 2\pi-\theta; r_s))$ in (2.10) decreases monotonically to $\lambda(r_{B_2}(\theta_2'), 2\pi-\theta_2'; r_s)$ as θ increases from θ_1' to θ_2'.

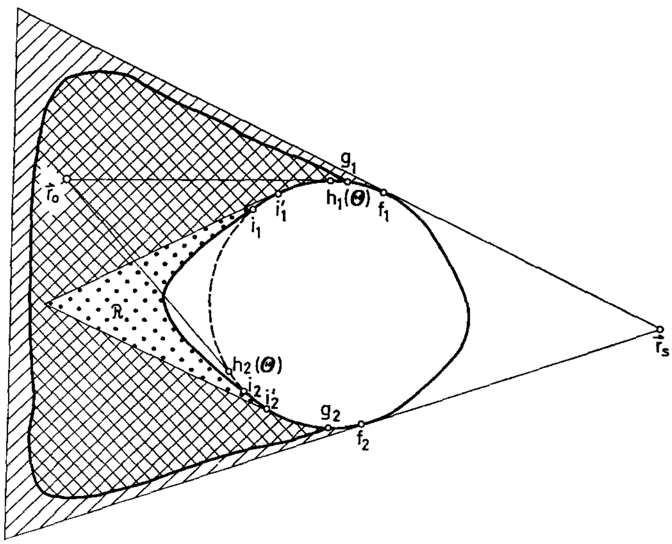


Fig. 7

Consequently, for $m=1, 2$,

$$\begin{aligned} &\int_{\theta_1'}^{\theta_2'} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)\}\, d\theta \\ (2.11)\qquad &\leqq (\theta_2'-\theta_1') \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda(r_{B_2}(\theta_m'), \phi_m(\theta_m'); r_s)\} \\ &= (\theta_2'-\theta_1') \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda(a, \phi_m(\theta_m'); r_s)\}. \end{aligned}$$

From (2.10) and (2.11) we conclude that as $k \to \infty$, uniformly in $\vec{r}_o$, $\vec{r}_o \in \mathcal{S}_2^-(\vec{r}_s)$,

$$(2.12)\qquad I_2(\vec{r}_o, \vec{r}_s; k) = O\left(k^{\frac{1}{6}} \exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda_<(r_s)\}\right)$$

where $\lambda_<(r_s) = \min_{m=1,2} \lambda(a, \phi_m(\theta_m'); r_s)$.

We now rewrite (2.12) in the form

$$(2.13)\qquad I_2(\vec{r}_o, \vec{r}_s; k) = O\left(\frac{\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\, a^{-\frac{2}{3}}\, \lambda_<(r_s)\}}{U_1(\vec{r}_o, \vec{r}_s; k)}\right) U_1(\vec{r}_o, \vec{r}_s; k).$$

Finally, using (2.8) in (2.13) we obtain the result that as $k\to\infty$, uniformly in $\vec{r}_o$, $\vec{r}_o\in\mathscr{S}_2^-(\vec{r}_s)$,

$$I_2(\vec{r}_o,\vec{r}_s;k) = O\left(k^{\frac{2}{3}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{2}{3}}[\lambda_<(r_s)-\lambda_<(r_0,\theta_0;r_s)]\}\right)U_1(\vec{r}_o,\vec{r}_s;k). \tag{2.14}$$

Our requirement that $\vec{r}_o\in\mathscr{S}_2^-(\vec{r}_s)$ assures that $\min|f_m h_m(\theta_0)|$ is less than $|i_1 f_1|$. In fact we have (*cf.* Fig. 7)

$$|i_1 f_1|-\max_{\vec{r}_0\in\mathscr{S}_2^-(\vec{r}_s)}\min_{m=1,2}|f_m h_m(\theta_0)|\geqq|i_1 f_1|-|i_1' f_1|=|i_1 i_1'|>0. \tag{2.15}$$

Consequently,

$$\lambda_<(r_s)-\lambda_<(r_0,\theta_0;r_s)=|i_1 f_1|-\min_{m=1,2}|f_m h_m(\theta_0)|\geqq|i_1 i_1'|>0. \tag{2.16}$$

Since $\operatorname{Im}\tau_1>0$, the factor multiplying $-k^{\frac{1}{3}}$ in the exponential of (2.14) is positive, independent of k, and of $\vec{r}_o$. This concludes the proof of Theorem 2.

Appendix I

To derive (1.21), we first set M, $N=1$ in (1.13) and then rewrite (1.13) as

$$\begin{aligned}\frac{1}{U_0(\vec{r}_o,\vec{r}_s;k)}=&\frac{(ka)^{\frac{1}{3}}}{P_0^{(1)}(r_0,r_s)}\,\frac{\exp\{-\chi(r_s;k)-\Delta_<(\theta_0;k)\}}{Ai(\Psi(r_0;k))}\\&\cdot\left[\left\{1+O\left(\frac{1}{(ka)^{\frac{1}{3}}P_0^{(1)}(r_0,r_s)}\right)\right\}\right.\\&\left.+\frac{Ai'(\Psi(r_0;k))}{Ai(\Psi(r_0;k))}\,O\left(\frac{1}{(ka)^{\frac{2}{3}}P_0^{(1)}(r_0,r_s)}\right)\right]^{-1}\\&\cdot[1+\exp\{\Delta_>(\theta_0;k)-\Delta_<(\theta_0;k)\}]^{-1}\end{aligned} \tag{I.1}$$

where

$$\Delta_>(\theta_0;k)=\begin{cases}\Delta(2\pi-\theta_0;k) & \text{if } \theta_0\leqq\pi\\ \Delta(\theta_0;k) & \text{if } \theta_0\geqq\pi,\end{cases}$$

$$\Delta_<(\theta_0;k)=\Delta_>(2\pi-\theta_0;k),$$

and $P_0^{(1)}(r_0,r_s)=P_0^{(1)}(r_0,\phi_m(\theta_0);r_s)$ for $m=1,2$.

As $k\to\infty$ the following inequalities hold uniformly in r_0, $r_0\geqq a$ (*cf.* [23, 5]):

$$\begin{aligned}[1+|\Psi(r_0;k)|^{\frac{1}{4}}]^{-1}\frac{1}{Ai(\Psi(r_0;k))}&=O\left(\exp\{\tfrac{2}{3}\Psi^{\frac{3}{2}}(r_0;k)\}\right)\\&=O\left(\exp\{-\chi(r_0;k)\}\right),\end{aligned} \tag{I.2}$$

$$[1+|\Psi(r_0;k)|^{\frac{1}{4}}]^{-1}Ai'(\Psi(r_0;k))=O\left(\exp\{-\tfrac{2}{3}\Psi^{\frac{3}{2}}(r_0;k)\}\right), \tag{I.3}$$

$$\frac{1}{P_0^{(1)}(r_0,r_s)}=O(1). \tag{I.4}$$

Furthermore, we have as $k\to\infty$, uniformly in r_0 and θ_0, $\vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$,

$$[1+\exp\{\Delta_>(\theta_0;k)-\Delta_<(\theta_0;k)\}]^{-1}=O(1). \tag{I.5}$$

To obtain (1.21), we make use of (I.2)–(I.5) in (I.1) and notice that

$$\exp\{-\chi(r_0;k)-\chi(r_s;k)-\Delta_<(\theta_0;k)\}=O(\exp\{k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{-\frac{1}{3}}\lambda_<(r_0,\theta_0;r_s)\}).$$

Appendix II

To obtain (1.22) we first apply the operator $\partial_n^{(1)}$ to the radial expansion of $U_0(\vec{r}_o,\vec{r}_s;k)$ to obtain a radial expansion of $\partial_n^{(1)}U_0(\vec{r}_o,\vec{r}_s;k)$. Then, using essentially the same arguments that lead to (1.8), we obtain the result that as $k\to\infty$

$$\partial_n^{(1)}U_0(\vec{r}_o,\vec{r}_{B_1}(\theta);k)=\partial_n^{(1)}T_1(\vec{r}_o,\vec{r}_{B_1}(\theta);k)[1+O(\exp\{-k^{\frac{1}{3}}\gamma\lambda_<^{**}\})], \tag{II.1}$$

uniformly in θ and $\vec{r}_o$, $\theta_2\leqq\theta\leqq 2\pi$, $0\leqq\theta\leqq\theta_1$, $\vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$. Here

$$\begin{aligned}\partial_n^{(1)}T_1(\vec{r}_o,\vec{r}_{B_1}(\theta);k)=&\frac{L_r(r_0,r_{B_1}(\theta);k)}{r_{B_1}(\theta)[r_{B_1}'^2(\theta)+r_{B_1}^2(\theta)]^{\frac{1}{2}}}\\&\cdot[\exp\{i\nu_1[2\pi-|\theta_0-\theta|]\}+\exp\{i\nu_1|\theta_0-\theta|\}]\\&+\frac{i\nu_1 r_{B_1}'(\theta)L(r_0,r_{B_1}(\theta);k)}{r_{B_1}(\theta)[r_{B_1}'^2(\theta)+r_{B_1}^2(\theta)]^{\frac{1}{2}}}\\&\cdot[-\exp\{i\nu_1[2\pi-|\theta_0-\theta|]\}+\exp\{i\nu_1|\theta_0-\theta|\}],\end{aligned} \tag{II.2}$$

$$\lambda_<^{**}=\min_{\vec{r}\in\mathscr{S}_1^<(\vec{r}_s)}\ \min_{\substack{0\leqq\theta\leqq\theta_1\\ \theta_2\leqq\theta\leqq 2\pi}}\lambda_<(r_0,|\theta_0-\theta|;r_{B_1}(\theta)), \tag{II.3}$$

with

$$\begin{aligned}\lambda_<(r_0,|\theta_0-\theta|;r_{B_1}(\theta))&=\min_{m=1,2}\lambda(r_0,\phi_m(|\theta_0-\theta|);r_{B_1}(\theta))\\&=\begin{cases}\lambda(r_0,|\theta_0-\theta|;r_{B_1}(\theta)) & \text{if } |\theta_0-\theta|\leqq\pi\\ \lambda(r_0,2\pi-|\theta_0-\theta|;r_{B_1}(\theta)) & \text{if } |\theta_0-\theta|>\pi.\end{cases}\end{aligned}$$

γ is the positive constant that appears in (1.8).

As $k\to\infty$ we have for $m=1,2$, uniformly in θ and $\vec{r}_o$, $\theta_2\leqq\theta\leqq 2\pi$, $0\leqq\theta\leqq\theta_1$, $\vec{r}_o\in\mathscr{S}_1^<(\vec{r}_s)$,

$$\begin{aligned}&i\nu_1(k)\exp\{i\nu_1(k)\phi_m(|\theta_0-\theta|)\}L(r_0,r_{B_1}(\theta);k)\\&=\frac{Ai(\Psi(r_0;k))Ai(\Psi(r_{B_1}(\theta);k))}{\left[\frac{r_0}{a}\zeta^{(1)}\left(\frac{r_0}{a}\right)\right]^{\frac{1}{4}}\left[\frac{r_{B_1}(\theta)}{a}\zeta^{(1)}\left(\frac{r_{B_1}(\theta)}{a}\right)\right]^{\frac{1}{4}}}\\&\quad\cdot O(k^{\frac{1}{3}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)a^{\frac{1}{3}}\phi_m(|\theta_0-\theta|)\}),\end{aligned} \tag{II.4}$$

$$\exp\{i\nu_1(k)\,\phi_m(|\theta_0-\theta|)\}\,L_r(r_0,\, r_{B_1}(\theta);\, k)$$

(II.5)
$$= \left[\frac{a}{r_{B_1}(\theta)}\,\zeta^{(1)}\left(\frac{r_{B_1}(\theta)}{a}\right)\right]^{\frac{1}{2}} \left[\frac{r_0}{a}\,\zeta^{(1)}\left(\frac{r_0}{a}\right)\right]^{-\frac{1}{2}}$$
$$\cdot Ai(\Psi(r_0;k))\,Ai'(\Psi(r_{B_1}(\theta);k))$$
$$\cdot O(\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a_{\frac{1}{3}}\,\phi_m(|\theta_0-\theta|)\}).$$

Furthermore, as $k\to\infty$, uniformly in r, $r\geqq a$ (*cf.* [23, 5]), we have

(II.6)
$$[1+|\Psi(r;k)|^{\frac{1}{4}}]\,Ai(\Psi(r;k)) = O(\exp\{-\tfrac{2}{3}\Psi^{\frac{3}{2}}(r;k)\})$$
$$= O(\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a^{-\frac{2}{3}}\,\lambda(r,0;a)\}),$$

(II.7)
$$[1+|\Psi(r;k)|^{\frac{1}{4}}]^{-1}Ai'(\Psi(r;k)) = O(\exp\{-\tfrac{2}{3}\Psi^{\frac{3}{2}}(r;k)\})$$
$$= O(\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a^{-\frac{2}{3}}\,\lambda(a,0;r)\}),$$

(II.8)
$$\frac{a}{r}\cdot\frac{1}{\zeta^{(1)}\left(\frac{r}{a}\right)},\ \frac{a}{r}\,\zeta^{(1)}\left(\frac{r}{a}\right) = O(1).$$

(1.22) follows from (II.1)–(II.8), and the remark that

$$\lambda(r_0,\phi_m(|\theta_0-\theta|);\, r_{B_1}(\theta)) = \lambda(r_0,0;a) + \lambda(a,0;r_{B_1}(\theta)) + a\,\phi_m(|\theta_0-\theta|).$$

Appendix III

To derive (2.9), we start with the following integral equation for $\partial_n^{(2)} U_1(\vec r_{B1}(\theta), \vec r_s; k)$, derived from (1.17):

(III.1)
$$\partial_n^{(2)} U_1(\vec r_{B_2}(\theta), \vec r_s; k) = \partial_n^{(2)} U_0(\vec r_{B_2}(\theta), \vec r_s; k) + I_3(\vec r_{B_2}(\theta), \vec r_s; k),$$

where

(III.2)
$$I_3(\vec r_{B_2}(\theta), \vec r_s; k) = \int_{\theta_2}^{2\pi} + \int_0^{\theta_1} \partial_n^{(2)}\partial_n^{(1)} U_0(\vec r_{B_2}(\theta), \vec r_{B_1}(\tilde\theta); k)$$
$$\cdot U_1(\vec r_{B_1}(\tilde\theta), \vec r_s; k)\,|d\vec r_{B_1}(\tilde\theta)|.$$

We then make use of (1.20), and the following estimates in (III.1) and (III.2) to obtain (2.9).

As $k\to\infty$

(III.3)
$$\partial_n^{(2)} U_0(\vec r_{B_2}(\theta), \vec r_s; k) = \sum_{m=1}^{2} O(k^{\frac{1}{3}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a^{-\frac{2}{3}}\,\lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)\})$$
$$= O(k^{\frac{1}{3}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a^{-\frac{2}{3}}\,\lambda_<(r_{B_2}(\theta), \theta; r_s)\}),$$

uniformly in θ, $\theta'_1\leqq\theta\leqq\theta'_2$;

(III.4)
$$\partial_n^{(2)}\partial_n^{(1)} U_0(\vec r_{B_2}(\theta), \vec r_{B_1}(\tilde\theta); k)$$
$$= \sum_{m=1}^{2} O(k^{\frac{4}{3}}\exp\{-k^{\frac{1}{3}}(\operatorname{Im}\tau_1)\,a^{-\frac{2}{3}}\,\lambda(r_{B_2}(\theta), \phi_m(|\theta-\tilde\theta|); r_{B_1}(\tilde\theta))\}),$$

uniformly in θ and $\tilde{\theta}$, $\theta'_1 \leqq \theta \leqq \theta'_2$, $\theta_2 \leqq \tilde{\theta} \leqq 2\pi$, $0 \leqq \tilde{\theta} \leqq \theta_1$. In (III.3) the function $\lambda_<(r_{B_2}(\theta), \theta; r_s) = \min_{m=1,2} \lambda(r_{B_2}(\theta), \phi_m(\theta); r_s)$. Note that in (III.3) we have $\vec{r}_{B_2}(\theta) \in \mathscr{S}_0^<(\vec{r}_s)$ if $\theta'_1 \leqq \theta \leqq \theta'_2$, while in (III.4) we have $\vec{r}_{B_2}(\theta) \in \mathscr{S}_0^<(\vec{r}_{B_1}(\tilde{\theta})) \supset \mathscr{S}_0^<(\vec{r}_s)$ if $\theta'_1 \leqq \theta \leqq \theta'_2$, $\theta_2 \leqq \tilde{\theta} \leqq 2\pi$, $0 \leqq \tilde{\theta} \leqq \theta_1$.

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