

Eigenvalue Problems Associated with Korn's Inequalities

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1. Introduction

We shall discuss a class of problems associated with certain inequalities which apparently originated in the work of A. KORN [1, 2] on the linear theory of elasticity. To describe these inequalities we consider a vector field $\mathbf{u}(\mathbf{x})$, continuously differentiable on the closure $R+B$ of a bounded domain R with boundary B in two or three dimensions. Let x_i be coordinates in a given rectangular Cartesian coordinate system, and denote by u_i the components of \mathbf{u} in this system. Define the functionals D and S by

$$D(\mathbf{u}) = \int_R u_{i,j} u_{i,j} dV, \quad (1.1)$$

$$S(\mathbf{u}) = \frac{1}{4} \int_R (u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) dV, \quad (1.2)$$

where Latin subscripts have the range 1 to n , and n is the number of dimensions. Summation over repeated subscripts is implied, and subscripts preceded by a comma indicate differentiation with respect to the corresponding coordinate.

A Korn inequality asserts the existence of a number $K_1 > 0$, depending only on the domain R , such that

$$D(\mathbf{u}) \leq K_1 S(\mathbf{u}) \quad (1.3)$$

for all vector fields \mathbf{u} satisfying certain side conditions. Since equality holds in (1.3) for any K_1 if \mathbf{u} is a constant vector, we will agree henceforth to identify vector fields differing only by a constant.

The necessity for imposing some side conditions on \mathbf{u} in order that (1.3) should hold can be seen from the fact that if \mathbf{u} is a pure rotation, then $S(\mathbf{u}) = 0$, $D(\mathbf{u}) > 0$.

Let

$$R(\mathbf{u}) = \frac{1}{4} \int_R (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) dV. \quad (1.4)$$

Then

$$D(\mathbf{u}) = S(\mathbf{u}) + R(\mathbf{u}), \quad (1.5)$$

so that K_1 cannot be less than unity, and the inequality (1.3) is equivalent to either of the following inequalities:

$$R(\mathbf{u}) \leq \left(1 - \frac{1}{K_1}\right) D(\mathbf{u}), \quad R(\mathbf{u}) \leq (K_1 - 1) S(\mathbf{u}). \quad (1.6)$$

K. O. FRIEDRICHS [3] proved Korn's inequality in the form of the second of (1.6) under each of the following three side conditions on \mathbf{u} :

First Case: $\mathbf{u} = 0$ on boundary B , (1.7)

Second Case: $\int_R (u_{i,j} - u_{j,i}) dV = 0$, (1.8)

Main Case: $\int_R (u_{i,j} - u_{j,i}) dV = 0$, (1.9)
 $\mathbf{u} \in C^2(R)$, $u_{i,jj} + u_{j,ij} = 0$ in R . (1.10)

Either of the conditions (1.7) or (1.8) serves to eliminate pure rotations. The class of admissible domains R is specified in detail in [3]; we merely note here that this class includes bounded domains with corners or edges.

As shown in [3], Korn's inequality in the first case is particularly simple. For all \mathbf{u} satisfying (1.7) the identity

$$2S(\mathbf{u}) = D(\mathbf{u}) + \int_R (\operatorname{div} \mathbf{u})^2 dV \tag{1.11}$$

may be derived by a suitable application of the divergence theorem. From (1.11) it is clear that

$$D(\mathbf{u}) \leq 2S(\mathbf{u})$$

so that (1.3) holds with $K_1 = 2$. Moreover it also follows from (1.11) that equality holds for vector fields \mathbf{u} such that $\mathbf{u} = 0$ on B and $\operatorname{div} \mathbf{u} = 0$ in R . Thus $K_1 = 2$ is the best possible constant in (1.3).

FRIEDRICHS shows that (1.3) may be readily established in the second case once it is proved for the main case. The central argument in [3] is thus devoted to the proof of Korn's inequality in the main case.

We introduce another case of Korn's inequality to which we refer as the *Extended Main Case*: there exists a constant $K_1(\sigma) > 0$ such that

$$D(\mathbf{u}) \leq K_1(\sigma) S(\mathbf{u}) \tag{1.12}$$

for all $\mathbf{u} \in C^1(R+B) \cap C^2(R)$ and satisfying

$$\int_R (u_{i,j} - u_{j,i}) dV = 0, \tag{1.13}$$

and

$$u_{i,jj} + \frac{1}{1-2\sigma} u_{j,ij} = 0 \quad \text{in } R. \tag{1.14}$$

Here σ is a real parameter such that $-1 < \sigma < \frac{1}{2}$. When $\sigma = 0$, (1.12)–(1.14) reduce to the main case. The existence of $K_1(\sigma)$ for an admissible domain follows from the validity of (1.3) in the second case, since the class of vector fields \mathbf{u} satisfying (1.13) and (1.14) is a subclass of the admissible functions in the second case.

The side conditions (1.14) in the extended main case are the displacement equations of equilibrium, in the case of zero body force, for an isotropic, homogeneous elastic solid subject to an infinitesimal displacement \mathbf{u} ; σ is Poisson's ratio.

The strain energy in an elastic solid occupying the domain R and subject to the displacement field \mathbf{u} is proportional to

$$E(\mathbf{u}) = S(\mathbf{u}) + \frac{\sigma}{1-2\sigma} \int_R (\operatorname{div} \mathbf{u})^2 dV. \tag{1.15}$$

If (1.3) holds for \mathbf{u} , then it is easily shown that

$$D(\mathbf{u}) \leq K_0 E(\mathbf{u}), \tag{1.16}$$

where

$$K_0 = \begin{cases} K_1 & 0 \leq \sigma < \frac{1}{2}, \\ \frac{1-2\sigma}{1+(n-2)\sigma} K_1 & -1 < \sigma \leq 0, \end{cases} \tag{1.17}$$

and n is the number of dimensions.

As an illustration of the use of Korn’s inequality in the extended main case we consider the problem of finding a lower bound for the functional

$$J(\mathbf{u}) = \frac{E(\mathbf{u})}{\int_R |\mathbf{u}|^2 dV} \tag{1.18}$$

where the vector field \mathbf{u} satisfies the elasticity equations (1.14) and the constraints

$$\int_R u_i dV = \int_R (u_{i,j} - u_{j,i}) dV = 0. \tag{1.19}$$

This problem arises in attempting to estimate the “characteristic decay length” associated with Saint-Venant’s principle. In the analysis of this issue carried out by R. A. TOUPIN [4], a lower bound for $J(\mathbf{u})$ was given in terms of the smallest positive frequency of free elastic vibration of R . This estimate takes no account of the fact that \mathbf{u} satisfies (1.14).

To obtain an alternate estimate, we may apply Korn’s inequality in the extended main case to \mathbf{u} in (1.18) as follows. Assuming $\sigma \geq 0$ for simplicity, we have from (1.16), (1.17) and (1.18) that

$$J(\mathbf{u}) \geq \frac{1}{K_1(\sigma)} \frac{D(\mathbf{u})}{\int_R |\mathbf{u}|^2 dV} \tag{1.20}$$

where use has been made of the fact that \mathbf{u} satisfies (1.14) and the second of (1.19), so that (1.12) holds. Since \mathbf{u} also satisfies the first of (1.19) we have

$$D(\mathbf{u}) \geq \lambda_1 \int_R |\mathbf{u}|^2 dV, \tag{1.21}$$

where λ_1 is the smallest positive eigenvalue of the problem

$$\nabla^2 \varphi + \lambda \varphi = 0 \quad \text{in } R, \quad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } B, \tag{1.22}$$

corresponding to free acoustic vibrations of R . Thus we obtain the lower bound

$$J(\mathbf{u}) \geq \frac{\lambda_1}{K_1(\sigma)}. \tag{1.23}$$

It was the potential application of Korn's inequality in the analysis of Saint-Venant's principle which originally motivated the present investigation.

Most of the earlier applications of Korn's inequality have been related to existence theorems for the boundary value problems of linear elasticity. Thus FRIEDRICHS in [3] uses Korn's inequality in the first case to prove the existence of a solution to the displacement boundary value problem, and he also treats the free elastic vibration problem with the aid of Korn's inequality in the second case. S. G. MIKHLIN [5], making use of the work of D. M. EIDUS [6] which in turn is based on the approach in [3], proves Korn's inequality in the second case and employs it to establish the existence of a solution for the traction boundary value problem in anisotropic elastostatics. MIKHLIN's use of Korn's inequality is directed toward showing that the elasticity operator is positive definite for the boundary conditions considered. The work of G. FICHERA [7, 8] involves similar considerations; see also the recent papers of I. HLAVÁČEK & J. NEČAS [9, 10].

For a given admissible domain R , the set of numbers $D(\mathbf{u})/S(\mathbf{u})$ corresponding to various vector fields \mathbf{u} satisfying the appropriate side conditions has a least upper bound. We call this least upper bound Korn's constant for the domain R in the first case, second case, main case or extended main case, and we denote these numbers by $K_R^{(1)}$, $K_R^{(2)}$, $K_R^{(M)}$, or $K_R^{(EM)}$, respectively. The argument given following (1.11) shows that $K_R^{(1)} = 2$ for any domain R .

B. BERNSTEIN & R. A. TOUPIN [11] posed the question of obtaining numerical information concerning $K_R^{(2)}$ and $K_R^{(M)}$ for particular domains. They considered the problem of finding upper and lower bounds for these constants, paying particular attention to the cases of the sphere in three dimensions and the circle in two dimensions. At about the same time L. E. PAYNE & H. F. WEINBERGER [12], employing the procedures of the calculus of variations, derived the eigenvalue problem associated with the question of maximizing $D(\mathbf{u})/S(\mathbf{u})$ under the side condition (1.8) for the second case. By solving this eigenvalue problem explicitly for the sphere and the circle and establishing the appropriate completeness properties of the resulting eigenfunctions, PAYNE & WEINBERGER showed that for a sphere

$$K_R^{(2)} = \frac{56}{13}, \quad (1.24)$$

and for the circle

$$K_R^{(2)} = 4. \quad (1.25)$$

C. M. DAFERMOS [13], using the approach of PAYNE & WEINBERGER, determined $K_R^{(2)}$ for a circular ring in two dimensions. He also gave an upper bound for the Korn constant $K_R^{(2)}$ for the union of two domains in terms of the Korn constants for the separate domains.

The present work is concerned with the question of obtaining information about the Korn constants for general domains through consideration of the associated eigenvalue problems.

In the following section we describe the eigenvalue problems associated with Korn's inequalities in the first and second cases. It turns out that the eigenvalue problem corresponding to the first case coincides with the problem of finding the values of Poisson's ratio for which the displacement boundary value problem

of elastostatics fails to have a unique solution. In § 3 we show that, in two dimensions, the eigenvalue problem associated with Korn's inequality in the second case can be transformed to the problem of determining such values of Poisson's ratio for the elastostatic displacement boundary value problem. Section 4 consists of a summary of known results concerning this plane displacement boundary value problem; these results are applied in Section 5 to elucidate the spectrum of the Korn eigenvalue problem in the second case. Finally, in §§ 6 and 7 we obtain analogous results in two dimensions for the eigenvalue problem associated with Korn's inequality in the extended main case.

2. The Eigenvalue Problems in the First and Second Cases

We now record the eigenvalue problems associated with Korn's inequalities in the first and second cases. By formally applying the procedures of the calculus of variations to the problem of finding the maximum of the ratio $D(\mathbf{u})/S(\mathbf{u})$ over all nontrivial vector fields* satisfying the constraint (1.7) for the first case, we obtain the Euler equations

$$(2-K)u_{i,jj} - K u_{j,ij} = 0 \quad \text{in } R \quad (2.1)$$

where the parameter K arises as a Lagrange multiplier.** The boundary condition for (2.1) is

$$u_i = 0 \quad \text{on } B. \quad (2.2)$$

When the constraint (1.8) corresponding to the second case is considered, the variational procedure yields

$$(2-K)u_{i,jj} - K u_{j,ij} = 0 \quad \text{in } R \quad (2.3)$$

with the natural boundary conditions

$$[(2-K)u_{i,j} - K u_{j,i}] n_j = 0 \quad \text{on } B. \quad (2.4)$$

Here \mathbf{n} is the unit outward normal on B . We refer to the problems of finding the values of K for which either (2.1), (2.2) or (2.3), (2.4) has a nontrivial solution \mathbf{u} as the Korn eigenvalue problem in the first or second case, respectively. In the first case we require $\mathbf{u} \in C(R+B) \cap C^2(R)$, in the second case $\mathbf{u} \in C^1(R+B) \cap C^2(R)$.

It was shown in [12] that when B is a sphere, the Korn eigenvalue problem in the second case has an infinite number of real eigenvalues with supremum $56/13$. When B is a circle, the same problem has a finite spectrum consisting of the numbers $K=1, 2, 4$, each occurring with infinite multiplicity. Moreover PAYNE & WEINBERGER proved the completeness of the eigenfunctions (for the sphere and the circle) with respect to vector fields with finite Dirichlet integral $D(\mathbf{u})$. It follows from this completeness result that the largest eigenvalue of the Korn eigenvalue problem in the second case is in fact the Korn's constant $K_R^{(2)}$.

* A vector field \mathbf{u} is trivial on R if it is identically constant.

** For details see [14].

Since $K_k^{(1)}$ is known to have the value 2 for any domain, we shall not consider the Korn eigenvalue problem (2.1), (2.2) for the first case in any detail. The eigenvalue problem (2.3), (2.4) for the second case has the following properties:

1°) $K=1$ is an eigenvalue of infinite multiplicity for any R .

This follows from the fact that (2.3), (2.4) become

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{u}) &= 0 && \text{in } R \\ (\nabla \times \mathbf{u}) \times \mathbf{n} &= 0 && \text{on } B \end{aligned}$$

when $K=1$. Thus for any $\psi \in C^3(R) \cap C^2(R+B)$, $\mathbf{u} = \nabla \psi$ is an eigenfunction of (2.3), (2.4) when $K=1$. Note that for this \mathbf{u} , $D(\mathbf{u}) = S(\mathbf{u})$

2°) $K=2$ is an eigenvalue of infinite multiplicity for any R .

It can be shown [14] that any $\mathbf{u} \in C^2(R) \cap C^1(R+B)$ for which $\text{div } \mathbf{u} = 0$ in $R+B$, $\mathbf{u} = 0$ on B is an eigenfunction of (2.3), (2.4) when $K=2$. Note that for any such \mathbf{u} , $D(\mathbf{u}) = 2S(\mathbf{u})$, as follows from (1.11).

3°) $K = \infty$ is not an eigenvalue of (2.3), (2.4).

When $K = \infty$, (2.3), (2.4) become the differential equations and boundary conditions of the traction boundary value problem of linear elastostatics for the special case of zero Poisson's ratio.* By the classical Kirchhoff uniqueness theorem for this boundary value problem, it follows that \mathbf{u} is a rigid body displacement. The side condition (1.8) associated with the second case then shows that \mathbf{u} is a rigid body translation and is therefore trivial.

4°) Every solution \mathbf{u} of (2.3), (2.4) satisfies

$$\int_R (u_{i,j} - u_{j,i}) dV = 0. \tag{2.5}$$

This may be verified by multiplying (2.3) by $\varepsilon_{ikl} x_l$, where ε_{ikl} is the three-dimensional alternating tensor, and integrating over R .

5°) If \mathbf{u} , \mathbf{v} are solutions of (2.3), (2.4) corresponding to distinct values of K , then the following orthogonality conditions hold:

$$\mathcal{S}(\mathbf{u}, \mathbf{v}) \equiv \frac{1}{4} \int_R (u_{i,j} + u_{j,i})(v_{i,j} + v_{j,i}) dV = 0, \tag{2.6}$$

$$\mathcal{D}(\mathbf{u}, \mathbf{v}) \equiv \int_R u_{i,j} v_{i,j} dV = 0, \tag{2.7}$$

$$\mathcal{R}(\mathbf{u}, \mathbf{v}) \equiv \frac{1}{4} \int_R (u_{i,j} - u_{j,i})(v_{i,j} - v_{j,i}) dV = 0, \tag{2.8}$$

6°) The eigenvalues K of (2.3), (2.4) are all real and not less than unity.

Properties 5°) and 6°) are easily verified by standard calculations.

3. Transformation of the Korn Eigenvalue Problem in Two Dimensions

The eigenvalue problem (2.1), (2.2) associated with Korn's inequality in the *first case* is easily seen to be related to the displacement boundary value problem

* The case $K = \infty$ for (2.3), (2.4) is understood to mean the result obtained formally by dividing (2.3), (2.4) by K and letting $K \rightarrow \infty$. Analogous interpretations apply in similar situations arising in the sequel.

of linear elastostatics. If K in (2.1) is identified with $1/\nu$, where ν is Poisson's ratio,* Eqs. (2.1) are formally equivalent to the displacement equations of elastic equilibrium. This observation concerning the differential equations (2.1) was also made by DAFERMOS [13]. It follows that the problem of finding the eigenvalues K of (2.1), (2.2) is equivalent to the problem of finding the values of Poisson's ratio ν for which the displacement boundary value problem of elastostatics fails to have a unique solution.

The main result of the present section establishes an equivalence between the Korn eigenvalue problem in two dimensions in the *second case* (Eqs. (2.3), (2.4)) and the displacement boundary value problem of plane elastostatics. Here the relation between K in (2.3), (2.4) and the appropriate Poisson's ratio ν turns out to be $K=(1-\nu)^{-1}$.

In § 7 we show that the Korn eigenvalue problem in the extended main case—to be stated in § 6—is also related to the displacement boundary value problem of elastostatics when attention is restricted to two dimensions.

We now limit consideration to the case of two-dimensional simply connected domains R unless otherwise indicated, and we use the convention that Greek subscripts have the range 1, 2.

Theorem 3.1. *Suppose $K \neq 1$, and let $u \in C^1(R+B) \cap C^2(R)$ be a nontrivial solution of the Korn eigenvalue problem with eigenvalue K , so that*

$$(2-K)u_{\alpha,\beta\beta} - K u_{\beta,\alpha\beta} = 0 \quad \text{in } R, \tag{3.1}$$

$$[(2-K)u_{\alpha,\beta} - K u_{\beta,\alpha}] n_\beta = 0 \quad \text{on } B. \tag{3.2}$$

Then there exists a nontrivial vector field $\xi \in C^1(R+B) \cap C^2(R)$ such that

$$\xi_{\alpha,\beta\beta} + \frac{1}{1-2\nu} \xi_{\beta,\alpha\beta} = 0 \quad \text{in } R, \tag{3.3}$$

$$\xi_\alpha = 0 \quad \text{on } B, \tag{3.4}$$

where $\nu = 1 - \frac{1}{K}$. Moreover,

$$\xi_{\alpha,\beta} = K(u_{\beta,\alpha} - u_{\alpha,\beta}) + 2(1-K)u_{\beta,\alpha} - 2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma}. \tag{3.5}$$

We also have the following converse of Theorem 3.1.

Theorem 3.2. *Suppose $\nu \neq 1$, and let $\xi \in C^1(R+B) \cap C^2(R)$ be a nontrivial solution of the displacement boundary value problem (3.3), (3.4) with eigenvalue ν . Then there exists a nontrivial vector field $u \in C^1(R+B) \cap C^2(R)$ such that (3.1), (3.2) hold with $K=(1-\nu)^{-1}$; moreover u and ξ are related by (3.5).*

Proof of Theorem 3.1. We first show that there is a vector field ξ satisfying (3.5). Let

$$H_{\alpha\beta} = K(u_{\beta,\alpha} - u_{\alpha,\beta}) + 2(1-K)u_{\beta,\alpha} - 2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma}. \tag{3.6}$$

* In differential equations related to a Korn eigenvalue problem, we shall consistently use ν to represent a parameter playing the role of Poisson's ratio; ν as so used will often lie outside the interval $(-1, \frac{1}{2}]$ of physical interest. On the other hand, Poisson's ratio as it occurs in the physically relevant differential equations of elasticity (e.g. in the constraints (1.14)) will be denoted by σ ; we always assume $-1 < \sigma < \frac{1}{2}$.

Direct calculation shows that

$$\begin{aligned} \varepsilon_{\rho\mu} \varepsilon_{\beta\lambda} H_{\rho\beta, \lambda} &= H_{\beta\beta, \mu} - H_{\lambda\mu, \lambda} = -2(1-K)u_{\gamma, \gamma\mu} - (2-K)u_{\mu, \lambda\lambda} \\ &+ K u_{\lambda, \mu\lambda} + 2(1-K)\delta_{\lambda\mu} u_{\gamma, \gamma\lambda} = -(2-K)u_{\mu, \lambda\lambda} + K u_{\lambda, \lambda\mu}, \end{aligned} \tag{3.7}$$

where $\varepsilon_{\beta\lambda}$ is the two-dimensional alternator* and we have used the identity

$$\varepsilon_{\rho\mu} \varepsilon_{\beta\lambda} = \delta_{\rho\beta} \delta_{\mu\lambda} - \delta_{\rho\lambda} \delta_{\mu\beta}. \tag{3.8}$$

Since u satisfies (3.1), it follows from (3.7) that

$$\varepsilon_{\rho\mu} \varepsilon_{\beta\lambda} H_{\rho\beta, \lambda} = 0 \quad \text{in } R. \tag{3.9}$$

Multiplying (3.9) by $\varepsilon_{\alpha\mu}$ and using (3.8) contracted on μ, λ , we find that

$$\varepsilon_{\beta\lambda} H_{\alpha\beta, \lambda} = H_{\alpha 1, 2} - H_{\alpha 2, 1} = 0 \quad \text{in } R. \tag{3.10}$$

Equation (3.10) and the simple connectivity of R imply that there exists a function ξ_α with the requisite smoothness, unique to within an arbitrary additive constant, such that

$$\xi_{\alpha, \beta} = H_{\alpha\beta}. \tag{3.11}$$

In view of the definition (3.6) this is precisely (3.5).

To show that the vector field ξ of (3.11) satisfies the differential equation (3.3) with $v = 1 - \frac{1}{K}$, we proceed as follows. If $K \neq 2$, (3.11), (3.6) provide

$$\xi_{\alpha, \beta\beta} + \frac{1}{1-2\left(1-\frac{1}{K}\right)} \xi_{\beta, \beta\alpha} = -K u_{\alpha, \beta\beta} + \frac{K^2}{2-K} u_{\beta, \beta\alpha},$$

and hence, by (3.1),

$$\xi_{\alpha, \beta\beta} + \frac{1}{1-2\left(1-\frac{1}{K}\right)} \xi_{\beta, \beta\alpha} = 0 \quad \text{in } R.$$

Thus, (3.3) holds if $K \neq 2$. When $K = 2$, (3.11), (3.6) yield

$$\xi_{\beta, \beta\alpha} = 2u_{\beta, \beta\alpha},$$

and hence by (3.1) with $K = 2$,

$$\xi_{\beta, \beta\alpha} = 0.$$

Thus, (3.3) holds** if $K = 2$.

To establish the boundary condition (3.4) we observe that $\varepsilon_{\lambda\beta} n_\lambda$ is tangent to B , so that by (3.11)

$$\frac{d\xi_\rho}{ds} = H_{\rho\beta} \varepsilon_{\lambda\beta} n_\lambda \quad \text{on } B. \tag{3.12}$$

We thus have, with the aid of (3.6), (3.8), and (3.2),

$$\varepsilon_{\rho\mu} \frac{d\xi_\rho}{ds} = \varepsilon_{\rho\mu} \varepsilon_{\lambda\beta} H_{\rho\beta} n_\lambda = [(2-K)u_{\mu, \lambda} - K u_{\lambda, \mu}] n_\lambda = 0 \quad \text{on } B. \tag{3.13}$$

* $\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = 1$.

** See footnote, p. 390.

Multiplying (3.13) by $\varepsilon_{\alpha\mu}$ and using (3.8) contracted on λ, μ , we obtain

$$\frac{d\xi_\alpha}{ds} = 0 \quad \text{on } B. \tag{3.14}$$

Thus by choosing the arbitrary additive constant suitably in the construction of ξ from (3.11), we may arrange that (3.4) holds.

Finally, we remark that (3.5) may be uniquely inverted to give $u_{\alpha,\beta}$ in terms of $\xi_{\lambda,\mu}$. It follows that ξ is constant in R if and only if u is constant in R . Thus ξ is nontrivial. This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is similar and will be omitted.

Theorems 3.1 and 3.2 establish an equivalence between the two-dimensional Korn eigenvalue problem in the second case and the displacement boundary value problem of plane elastostatics. Excluding the exceptional values $K=1$ and $\nu=1$, the problem of determining the eigenvalues K of (3.1), (3.2) for simply connected plane domains reduces to finding the values of Poisson's ratio ν for which the displacement boundary value problem fails to have a unique solution.

We conclude this section by indicating the way in which the results of Theorems 3.1 and 3.2 were discovered. Since the differential equations (3.1) correspond to the displacement equations of equilibrium with Poisson's ratio $1/K$, it is possible in two dimensions to represent solutions of (3.1) in terms of a biharmonic Airy stress function φ and auxiliary harmonic functions. When solutions represented in this way are subjected to the boundary conditions (3.2), it turns out that φ satisfies the boundary conditions for the deflection in the problem of bending of a thin elastic plate with a free edge, when appropriate identification is made between K and Poisson's ratio for the plate. The bending problem for a plate with a free edge may, in turn, be transformed to the homogeneous displacement boundary value problem for elastostatic plane strain when an appropriate identification is made between the Poisson's ratios of the two problems. This was shown by S. G. LEKHNITSKII [15], whose results are presented in the notes of I. S. SOKOLNIKOFF [16]; see also A. E. GREEN & W. ZERNA's text [17]. In this way one is led to the transformation (3.5) between the Korn eigenvalue problem and the plane displacement boundary value problem.

The transformation analogous to (3.5) does not appear to provide a result corresponding to Theorems 3.1 and 3.2 in three dimensions.

4. Some Results on the Displacement Boundary Value Problem in Plane Elastostatics

We accumulate here some known results concerning the displacement boundary value problem in plane strain for homogeneous isotropic elastic solids. Throughout this section the two dimensional vector field $\xi \in C^1(R+B) \cap C^2(R)$ is assumed to satisfy

$$\nabla^2 \xi + \frac{1}{1-2\nu} \nabla(\nabla \cdot \xi) = 0 \quad \text{in } R, \tag{4.1}$$

$$\xi = 0 \quad \text{on } B, \tag{4.2}$$

for some constant ν , where R is a bounded, simply connected plane domain. If $\xi \neq 0$, we say that ν is an eigenvalue of (4.1), (4.2).

Theorem 4.1. $\xi \equiv 0$ if $\nu < \frac{1}{2}$ or $\nu > 1$.

Theorem 4.1 is the uniqueness theorem for the displacement boundary value problem. The proof given by M. E. GURTIN & E. STERNBERG [18] for three dimensional bounded domains is also valid in two dimensions.

Theorem 4.2. $\nu = 1$ and $\nu = \frac{1}{2}$ are eigenvalues of infinite multiplicity* for (4.1), (4.2) for any R .

It is easily shown that $\xi = \nabla\varphi$, for any scalar field φ whose first partial derivatives vanish on B , is an eigenfunction of (4.1), (4.2) when $\nu = 1$. When $\nu = \frac{1}{2}$, any ξ such that $\xi = 0$ on B , $\nabla \cdot \xi = 0$ in R is an eigenfunction. Theorem 4.2 remains valid in three dimensions (see MIKHLIN [19]).

Theorem 4.3. Suppose $\frac{1}{2} < \nu < 1$. Then there is a domain R for which ν is an eigenvalue of (4.1), (4.2).

This result was proved by J. L. ERICKSEN [20] for the n -dimensional case. In two dimensions the domain R may be taken as the interior of an ellipse whose axes are in the ratio $[(2\nu - 1)/2(1 - \nu)]^{\frac{1}{2}}$.

Theorem 4.4. If $\nu = \frac{3}{4} + \delta$, $0 < \delta \leq \frac{1}{4}$, is an eigenvalue of (4.1), (4.2) with eigenfunction $\xi = (\xi_1, \xi_2)$, then $\nu = \frac{3}{4} - \delta$ is also an eigenvalue with corresponding eigenfunction $\xi = (\xi_2, -\xi_1)$.

This result may be established by direct verification.

Further information on the spectrum of (4.1), (4.2) may be obtained by making use of the well known alternative formulation of the plane displacement boundary value problem in terms of regular functions of a complex variable. Let $z = x_1 + ix_2$ be a complex variable, and assume the origin $z = 0$ lies in R . Let $\varphi(z)$, $\psi(z)$ be regular functions in R , $\varphi(0) = 0$, and suppose that φ , φ' , φ'' and ψ , ψ' are continuous on $R + B$. Define

$$V(z) = (3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \quad \text{on } R + B, \quad (4.3)$$

for any real ν . By adapting the analysis given by N. I. MUSKHELISHVILI [21] (see [14] for details), the following result may be established.

Theorem 4.5. Suppose $\nu \neq \frac{1}{2}, 1$. Then ξ is a nontrivial solution of (4.1), (4.2) if and only if there exists a function $V(z)$ of the form (4.3) such that $V \neq 0$ on R ,

$$V = 0 \quad \text{on } B \quad (4.4)$$

and

$$2(\xi_1 + i\xi_2) = V \quad \text{on } R + B. \quad (4.5)$$

The problem of determining V of the form (4.3) satisfying (4.4) may be reduced to an integral equation. We now assume that the boundary B of R consists of a simple closed curve with curvature which is a continuously differentiable function

* Cf. Properties 1°, 2° of § 2.

of arc length. Let R be mapped conformally onto the unit disc $|\zeta| < 1$ by the mapping $z = \omega(\zeta)$, $\omega(0) = 0$. If γ denotes the unit circle $|\zeta| = 1$, the above assumptions ensure that the derivative $\omega'(\zeta)$ does not vanish inside or on γ (see § 47 of [21]). The analysis in [21] can be adapted to yield the following result (see also [22]).

Theorem 4.6. *Suppose $\nu \neq \frac{1}{2}, \frac{3}{4}, 1$. Then there exists a nontrivial function $V(z)$ of the form (4.3) satisfying (4.4) if and only if there exists a function $\vartheta(\zeta) \neq 0$ defined and continuous for $|\zeta| \leq 1$, regular for $|\zeta| < 1$, whose boundary values satisfy the integral equation*

$$\vartheta(\zeta) - \frac{\lambda}{2\pi i} \int_{\gamma} G(\zeta, \zeta') \overline{\vartheta(\zeta')} d\zeta' = 0, \quad \zeta \in \gamma, \tag{4.6}$$

where

$$\lambda = \frac{1}{3 - 4\nu} \tag{4.7}$$

and

$$G(\zeta, \zeta') = \frac{1}{\omega'(\zeta')} \frac{\partial}{\partial \zeta} \left[\frac{\omega(\zeta') - \omega(\zeta)}{\zeta' - \zeta} \right], \quad \zeta, \zeta' \in \gamma. \tag{4.8}$$

By taking real and imaginary parts, it is possible to write (4.6) as a system of two Fredholm integral equations, which may be further reduced to a single Fredholm equation of the second kind [22]. This leads to

Theorem 4.7. *The eigenvalues λ of the integral equation (4.6) are real and such that $|\lambda| > 1$, have finite multiplicity, and may accumulate only at infinity.*

The facts that the eigenvalues are real and satisfy $|\lambda| > 1$ are proved by MIKHLIN in § 47 of [22], where use is made of the work of D. I. SHERMAN [23]. The remainder of Theorem 4.7 is a consequence of results for general Fredholm equations (see [22]).

From Theorems 4.7, 4.6 and 4.5 we conclude that those eigenvalues ν of (4.1), (4.2) which differ from $\frac{1}{2}, \frac{3}{4}, 1$ have finite multiplicity and may accumulate only at $\nu = \frac{3}{4}$.

MIKHLIN in [19] and [24] examines the eigenvalue problem (4.1), (4.2) in three dimensions, referring to the early work of E. & F. COSSERAT [25, 26, 27]. By using methods of functional analysis, MIKHLIN proves several results, of which we note here only the following: the eigenvalues of (4.1), (4.2) are real, those which differ from $\frac{1}{2}, \frac{3}{4}, 1$ have finite multiplicity and may accumulate only at $\frac{1}{2}, \frac{3}{4}$, and 1. With respect to possible points of accumulation of the eigenvalues ν of (4.1), (4.2), Theorem 4.7 leads to a stronger result for two dimensions than that obtained for the three dimensional case by MIKHLIN.

The following theorem is an immediate consequence of the structure of the kernel G of (4.8); see § 44 of [22].

Theorem 4.8. *If the conformal mapping which takes R to the unit disc is rational, then the kernel G of (4.8) is degenerate and the spectrum of (4.6) is finite.*

The value $\nu = \frac{3}{4}$, which is clearly exceptional for problem (4.1), (4.2), may or may not be an eigenvalue. The situation is clarified by the following result.

Theorem 4.9. *Let R be a plane domain bounded by a simple analytic closed curve B . Then $\nu = \frac{3}{4}$ is an eigenvalue of (4.1), (4.2) if and only if R can be mapped conformally onto the unit disc by a rational function. If $\nu = \frac{3}{4}$ is an eigenvalue, it has infinite multiplicity.*

The result of Theorem 4.9 may be found in §9, Chapter IV of A. V. BITSADZE [28]. BITSADZE considers a system of differential equations identical with that obtained from (4.1) by setting $\nu = \frac{3}{4}$ and replacing ξ_2 by $-\xi_2$. The relevance of the system considered by BITSADZE to the displacement equations of elasticity was noted by ERICKSEN [20].

5. The Spectrum of the Korn Eigenvalue Problem

(i) General Properties

Using the results of the previous section and the equivalence between the two-dimensional Korn eigenvalue problem (3.1), (3.2) in the second case and the displacement boundary value problem of plane elastostatics established in §3, we may make the following assertions with regard to the eigenvalues K .

1°) For any bounded, simply connected plane domain R , $K=1, 2$ are eigenvalues of infinite multiplicity; all other eigenvalues are such that $K > 2$.

2°) Suppose $k > 2$. Then there exists an elliptical domain R for which $K=k$ is an eigenvalue. From this result we conclude that there is no upper bound on the K -spectrum which is valid for all plane domains R .

3°) If $K > 2$ is an eigenvalue for a bounded, simply connected plane domain R , then so is $2K(K-2)^{-1}$.

4°) Let R be a plane domain bounded by a simple closed curve with continuously differentiable curvature. Eigenvalues K for R which differ from 1, 2, 4 have finite multiplicity and may accumulate only at 4. Thus for a given R , there exists a largest eigenvalue K_{\max} .

5°) If the bounded plane domain R can be mapped conformally onto the unit disc by a rational function, then the K -spectrum is finite.

6°) $K=4$ is an eigenvalue for a plane domain R bounded by a simple analytic closed curve if and only if R can be mapped conformally onto the unit disc by a rational function. When $K=4$ is an eigenvalue, it has infinite multiplicity. For domains of this type we have the lower bound $K_{\max} \geq 4$.

(ii) The Circle

Let the domain R be a disc of radius r with its center at the origin. The mapping $\omega(\zeta) = r\zeta$, and the kernel G of (4.8) vanishes identically. Thus the integral equation (4.6) has no finite eigenvalues. It follows that the only possible eigenvalues ν are $\nu = 1, \frac{3}{4}, \frac{1}{2}$. To show directly that $\nu = \frac{3}{4}$ is an eigenvalue, we observe that the function

$$V(z) = (r^2 - z\bar{z}) \overline{g(z)},$$

where $g(z)$ is an arbitrary function regular in $|z| < r$ and g and g' are continuous for $|z| \leq r$, is of the form (4.3) for $\nu = \frac{3}{4}$ and vanishes for $|z| = r$. By Theorem 4.5, $\nu = \frac{3}{4}$ is therefore an eigenvalue of infinite multiplicity for the disc. The fact that

$\nu=1$ and $\nu=\frac{1}{2}$ are eigenvalues of infinite multiplicity follows from Theorem 4.2. Using Theorem 3.2 and property 1°) of the present section, we conclude that the K -spectrum for the circle consists of the eigenvalues $K=1, 2, 4$, each with infinite multiplicity. Thus we recover the result of PAYNE & WEINBERGER [12].

(iii) *Domains Mapped Conformally Onto the Unit Disc by a Rational Mapping*

By Theorem 4.8 we know that for such domains the kernel G of the integral equation (4.6) is degenerate and the spectrum of eigenvalues λ is finite. By using the theory of degenerate kernels, it is possible in principle to calculate the eigenvalues explicitly.

In [14] the mapping

$$z = \omega(\zeta) = \zeta + m\zeta^2, \quad 0 \leq m < \frac{1}{2}, \tag{5.1}$$

is treated in detail. When ζ in (5.1) describes the unit circle, z describes a curve called Pascal's limaçon. When $m=0$, the limaçon is a circle, while for $m=\frac{1}{2}$ the curve becomes a cardioid with a cusp of $\zeta = -1$, where $\omega'(\zeta) = 0$.

The eigenvalues λ of (4.6) for the domain bounded by Pascal's limaçon are

$$\lambda = \pm \frac{1}{2m^2}, \tag{5.2}$$

so that by (4.7), and Theorems 4.2 and 4.9, we find the ν -spectrum as

$$\nu = \frac{1}{2}, \frac{3}{4} - \frac{m^2}{2}, \frac{3}{4}, \frac{3}{4} + \frac{m^2}{2}, 1. \tag{5.3}$$

Finally we obtain for the K -spectrum

$$K = 1, 2, \frac{4}{1+2m^2}, 4, \frac{4}{1-2m^2}, \tag{5.4}$$

where $K=1, 2, 4$ occur with infinite multiplicity, the others with finite multiplicity.

(iv) *Korn's Constant*

If for a given region R the eigenfunctions of the Korn problem (3.1), (3.2) are complete in the norm $[S(\mathbf{u})]^\frac{1}{2}$ (see (1.2)) with respect to functions satisfying (1.8) and possessing square integrable first derivatives, it can be shown that $K_{\max} = K_R^{(2)}$ (see [14]). For the special cases of the sphere and the circle, PAYNE & WEINBERGER establish the corresponding completeness in [12]. It may be noted that MIKHLIN [24] considers the completeness question for the eigenfunctions of the displacement boundary value problem (4.1), (4.2) in three dimensions. By adapting his analysis it may be possible to establish the completeness of the eigenfunctions of the Korn problem.

(v) *The First Case*

The eigenvalue problem associated with Korn's inequality in the first case was described at the beginning of § 3. Since this problem coincides with the

homogeneous displacement boundary value problem when K and $1/\nu$ are identified (in two or three dimensions), the largest eigenvalue K corresponds to the smallest eigenvalue ν . Thus we find that $K=2$ (corresponding to $\nu=\frac{1}{2}$) is the largest eigenvalue of the Korn problem in the first case (see Theorems 4.1, 4.2). This is consistent with the fact that $K_k^{(1)}=2$ for any domain.

6. The Eigenvalue Problem in the Extended Main Case

In this section we consider the extended main case of Korn's inequality as formulated in § 1. Let R be an admissible domain in two or three dimensions. Then

$$D(\mathbf{u}) \leq K_1(\sigma) S(\mathbf{u}), \tag{6.1}$$

for all \mathbf{u} satisfying

$$\int_R (u_{i,j} - u_{j,i}) dV = 0, \tag{6.2}$$

$$u_{i,jj} + \frac{1}{1-2\sigma} u_{j,ij} = 0 \quad \text{in } R, \tag{6.3}$$

where σ is real, $-1 < \sigma < \frac{1}{2}$.

We pose the variational problem, analogous to that of § 2, of maximizing the ratio $D(\mathbf{u})/S(\mathbf{u})$ over all nontrivial vector fields \mathbf{u} satisfying the constraints (6.2), (6.3). It is shown in [14] that the Euler equations and natural boundary conditions formally associated with this problem are as follows.

$$(2-K) \nabla^2 \mathbf{u} - K \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \lambda - \frac{1}{1-2\sigma} \nabla(\nabla \cdot \lambda) = 0 \quad \text{in } R, \tag{6.4}$$

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\sigma} \nabla(\nabla \cdot \mathbf{u}) = 0 \quad \text{in } R, \tag{6.5}$$

$$\lambda = 0 \quad \text{on } B, \tag{6.6}$$

$$\left[(2-K) u_{i,j} - K u_{j,i} - \left(\lambda_{i,j} + \lambda_{j,i} + \frac{2\sigma}{1-2\sigma} \delta_{ij} \lambda_{k,k} \right) \right] n_j = 0 \quad \text{on } B. \tag{6.7}$$

Here K is a constant Lagrange multiplier, as in § 2; $\lambda(\mathbf{x})$ is a vector field whose components $\lambda_i(\mathbf{x})$ are Lagrange multipliers associated with the constraints (6.3).

By a nontrivial vector field \mathbf{u} in (6.4)–(6.7), we mean $\mathbf{u} \not\equiv \text{constant}$, and by a nontrivial vector field λ we mean $\lambda \not\equiv 0$. The eigenvalue problem in the extended main case consists of determining those values of K , for given σ , for which (6.4)–(6.7) has a solution $(\mathbf{u}, \lambda) \in C^1(R+B) \cap C^2(R)$ with \mathbf{u} nontrivial.

In the remainder of our work we confine our attention to bounded, two-dimensional, simply connected domains R . The eigenvalue problem (6.4)–(6.7) has the following properties.

1° The eigenvalues K of (6.4)–(6.7) are real and not less than unity; every solution (\mathbf{u}, λ) of (6.4)–(6.7) is such that \mathbf{u} satisfies (6.2).

2° If $K \neq 1, 1/\sigma, \infty$, then vector fields \mathbf{u}, λ satisfying (6.4)–(6.7) are either both trivial or both nontrivial.

To establish 2°, suppose first that \mathbf{u} is trivial. Then λ is a solution of the homogeneous displacement boundary value problem of elasticity with Poisson's

ratio σ , $-1 < \sigma < \frac{1}{2}$. By Theorem 4.1, $\lambda \equiv 0$. On the other hand if λ is trivial in (6.4)–(6.7), then it can be shown from (6.4), (6.5) that for $K \neq 1/\sigma$, u_α is harmonic and $u_{\alpha,\beta} - u_{\beta,\alpha}$ is constant on R . It then follows from (6.2) that $u_{\alpha,\beta} - u_{\beta,\alpha} \equiv 0$. For $K \neq 1$ the boundary conditions (6.7) then become $\partial u_\alpha / \partial n = 0$, so that $\mathbf{u} \equiv$ constant by the uniqueness theorem for the Neumann problem for harmonic functions.

3°) $K=1$ is an eigenvalue of infinite multiplicity of (6.4)–(6.7) for any R .

To show this, take $\lambda \equiv 0$, $\mathbf{u} = (f, -g)$ where f, g are conjugate harmonic functions. Direct calculation verifies that (6.4)–(6.7) with $K=1$ are identically satisfied for any such f, g .

4°) $K = \infty$ is not an eigenvalue of (6.4)–(6.7).

If $\sigma = 0$, the case $K = \infty$ in (6.4)–(6.7) reduces to the traction boundary value problem of plane elasticity for \mathbf{u} with zero Poisson's ratio. This, together with (6.2), implies that \mathbf{u} is trivial. If $\sigma \neq 0$, an argument similar to that used to establish property 2°) above shows that \mathbf{u} is trivial.

5°) If $-1 < \sigma \leq 0$, $K = 1/\sigma$ is not an eigenvalue of (6.4)–(6.7). If $0 < \sigma < \frac{1}{2}$, $K = 1/\sigma$ is an eigenvalue of (6.4)–(6.7) if and only if $K = 1/\sigma$ is an eigenvalue of the Korn problem in the second case (3.1), (3.2).

7. Transformation Theorem for the Extended Main Case

For the two-dimensional Korn eigenvalue problems in the first and second cases, it has been shown that the eigenvalues K are related to the eigenvalues ν of the displacement boundary value problem by the respective formulas $K = 1/\nu$ and $K = 1/(1 - \nu)$. We now show that for the extended main case in two dimensions, the eigenvalues K can be characterized in terms of the eigenvalues ν by the relation

$$K = \frac{\sigma^2 - 2\sigma\nu - \nu^2 + 3\nu - 1}{(1 - \nu)(2\sigma^2 - 2\sigma + \nu)}, \quad \nu \neq \frac{1}{2}, 1, \tag{7.1}$$

where σ is the given Poisson's ratio in the constraint (6.3).

Before stating the main result, it is convenient to introduce the notation

$$\begin{aligned} & \nu^{(1)}(K, \sigma) \\ &= \frac{2(1 - \sigma)(1 - K\sigma) + (1 - K) - [4(\sigma - 1)^2(1 - K\sigma)^2 + (K - 1)^2(1 - 2\sigma)^2]^{\frac{1}{2}}}{2(1 - K)}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} & \nu^{(2)}(K, \sigma) \\ &= \frac{2(1 - \sigma)(1 - K\sigma) + (1 - K) + [4(\sigma - 1)^2(1 - K\sigma)^2 + (K - 1)^2(1 - 2\sigma)^2]^{\frac{1}{2}}}{2(1 - K)}, \end{aligned} \tag{7.3}$$

for the two roots ν of (7.1) for given $K \neq 1$.

Theorem 7.1. *Let ν be an eigenvalue of the plane displacement boundary value problem*

$$\nabla^2 \mathbf{w} + \frac{1}{1 - 2\nu} \nabla(\nabla \cdot \mathbf{w}) = 0 \quad \text{in } R, \tag{7.4}$$

$$\mathbf{w} = 0 \quad \text{on } B, \tag{7.5}$$

with $\frac{1}{2} < v < 1$. Then

$$K(v, \sigma) = \frac{\sigma^2 - 2\sigma v - v^2 + 3v - 1}{(1-v)(2\sigma^2 - 2\sigma + v)}, \quad -1 < \sigma < \frac{1}{2}, \tag{7.6}$$

is an eigenvalue of the Korn problem (6.4)–(6.7) in the extended main case. Conversely let $K \neq 1$ be an eigenvalue of (6.4)–(6.7).

(a) If (i) $-1 < \sigma \leq 0$, or (ii) $0 < \sigma < \frac{1}{2}$, $1 < K \leq 1/\sigma$, then

$$v = v^{(1)}(K, \sigma) \tag{7.7}$$

is an eigenvalue of (7.4), (7.5).

(b) If $0 < \sigma < \frac{1}{2}$, $K > 1/\sigma$, then

$$v = v^{(2)}(K, \sigma) \tag{7.8}$$

is an eigenvalue of (7.4), (7.5).

Proof. We first show that u, λ and K satisfy (6.4)–(6.7) if and only if u and λ are expressible in terms of two vector fields $u^{(1)}, u^{(2)}$, each of which is a solution of a homogeneous displacement boundary value problem with suitable Poisson's ratio.

Lemma. If the plane vector fields $u, \lambda \in C^1(R+B) \cap C^2(R)$ and the constant $K \neq 1$ satisfy (6.4)–(6.7), then there exist plane vector fields $u^{(1)}, u^{(2)} \in C^1(R+B) \cap C^2(R)$ such that for $i=1, 2$,

$$\nabla^2 u^{(i)} + \frac{1}{1-2v^{(i)}} \nabla(\nabla \cdot u^{(i)}) = 0 \text{ in } R, \quad u^{(i)} = 0 \text{ on } B, \tag{7.9}$$

where $v^{(1)}, v^{(2)}$ are given by (7.2), (7.3). Moreover, $u^{(1)}, u^{(2)}$ are obtained from the formulas

$$2(1-\sigma)u_{\alpha,\beta}^{(i)} = (v^{(i)}-\sigma) \left\{ K(u_{\beta,\alpha} - u_{\alpha,\beta}) + 2(1-K)u_{\beta,\alpha} - 2(1-K)\delta_{\alpha\beta}u_{\gamma,\gamma} \right\} - 2(1-\sigma)\lambda_{\alpha,\beta} - (v^{(i)}-\sigma) \left\{ \lambda_{\beta,\alpha} - \lambda_{\alpha,\beta} - \frac{2(1-\sigma)}{1-2\sigma}\delta_{\alpha\beta}\lambda_{\gamma,\gamma} \right\}. \tag{7.10}$$

Conversely, if $u^{(1)}, u^{(2)}$ satisfy (7.9) for any constants $v^{(1)}, v^{(2)}$ such that

$$(1-2\sigma)^2 = -(1-2v^{(1)})(1-2v^{(2)}), \tag{7.11}$$

then there exist vector fields u, λ satisfying (6.4)–(6.7), (7.10), where K is given by (7.6) with $v = v^{(1)}$ or $v = v^{(2)}$.

The first part of the lemma is proved by first showing that the differential equations (6.4), (6.5) imply that for fixed α , the right side of (7.10) is indeed a gradient, thus establishing the existence of $u^{(1)}, u^{(2)}$. It is then verified directly that $u^{(1)}, u^{(2)}$ as so constructed satisfy (7.9). The proof of the converse part of the lemma is similar. We omit the details.

Returning to Theorem 7.1, we begin by supposing that v is an eigenvalue of (7.4), (7.5), and we apply the converse portion of the lemma as follows. Choose

$v^{(1)} = v$, $u^{(1)} = w$ and determine $v^{(2)}$ so that (7.11) holds. Since $\frac{1}{2} < v < 1$ and $-1 < \sigma < \frac{1}{2}$, it follows from (7.11) that $v^{(2)}$ is outside the interval $[\frac{1}{2}, 1]$ so that necessarily $u^{(2)} \equiv 0$. The lemma then provides the existence of u, λ satisfying (6.4)–(6.7), with K given by (7.6). To show that this value of K is indeed an eigenvalue of (6.4)–(6.7), we must prove that u is nontrivial. If the given v is different from $1 - \sigma$, then in (7.6) we shall have $K \neq 1, 1/\sigma, \infty$; by property 2° of § 6, it is then sufficient to show that u, λ are not both trivial. But if u, λ were both trivial, (7.10) with $i=1$ would imply $u^{(1)} = w \equiv \text{constant}$, contradicting the fact that w is an eigenfunction of (7.4), (7.5). In the special circumstance that $v^{(1)} = v = 1 - \sigma$, we have $v^{(2)} = \sigma$ from (7.11), so that (7.10) with $i=2$, $u^{(2)} \equiv 0$ yields $\lambda_{\alpha, \beta} \equiv 0$. But then if u were trivial, (7.10) with $i=1$ would again contradict the nontriviality of w . This completes the proof of the first part of the theorem.

To prove the converse statement in Theorem 7.1, we suppose that $u, \lambda, K \neq 1$ satisfy (6.4)–(6.7), and that u is nontrivial. Appealing to the first part of the lemma, we conclude that vector fields $u^{(1)}, u^{(2)}$ exist which satisfy (7.9) with $v^{(1)}, v^{(2)}$ given by (7.2), (7.3), respectively. Detailed consideration of (7.2), (7.3) shows that if either $-1 < \sigma \leq 0$ or $0 < \sigma < \frac{1}{2}, 1 < K \leq 1/\sigma$, then $v^{(1)}$ lies in the interval $(\frac{1}{2}, 1)$ while $v^{(2)}$ lies outside $[\frac{1}{2}, 1]$. Thus $u^{(2)} \equiv 0$. After inverting (7.10) to express $u_{\alpha, \beta}$ and $\lambda_{\alpha, \beta}$ in terms of $u^{(1)}$, it can be seen that $u^{(1)}$ is nontrivial. Thus $v^{(1)}$ as given by (7.2) is an eigenvalue of (7.4), (7.5) with eigenfunction $u^{(1)}$, provided that either $-1 < \sigma \leq 0$ or $0 < \sigma < \frac{1}{2}, 1 < K \leq 1/\sigma$. Finally, if $0 < \sigma < \frac{1}{2}$ and $K > 1/\sigma$, we find in a similar way that $u^{(1)} \equiv 0$, and $v^{(2)}$ is an eigenvalue of (7.4), (7.5) with eigenfunction $u^{(2)}$. This completes the proof of Theorem 7.1.

The equivalence established in this theorem between the Korn eigenvalue problem in the extended main case and the displacement boundary value problem can be used, with the aid of the results in § 5, to describe the general features of the K -spectrum. We shall not list the detailed results of such a description here. We remark, however, that the relation (7.6) between the K -eigenvalues in the extended main case and the v -eigenvalues of the displacement boundary value problem shows that the largest eigenvalue K is obtained by using in (7.6) the largest eigenvalue v in the interval $(\frac{1}{2}, 1)$.

Finally, it may be observed that for a circular region, the K -spectrum in the extended main case consists of the two values

$$K = 1, \quad K\left(\frac{3}{4}, \sigma\right) = \frac{16\sigma^2 - 24\sigma + 11}{8\sigma^2 - 8\sigma + 3}. \tag{7.12}$$

The largest eigenvalue is

$$K_{\max}(\sigma) = K\left(\frac{3}{4}, \sigma\right) = \frac{16\sigma^2 - 24\sigma + 11}{8\sigma^2 - 8\sigma + 3}. \tag{7.13}$$

As σ increases from -1 to $\frac{1}{2}$, K_{\max} rises from a minimum value of $51/19$ at $\sigma = -1$ to a maximum of 4 at $\sigma = \frac{1}{4}$, then decreases to a value of 3 at $\sigma = \frac{1}{2}$.

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References

1. KORN, A., Die Eigenschwingungen eines elastischen Körpers mit ruhender Oberfläche. Akad. der Wissensch., Munich, Math.-phys. Kl., Berichte **36**, 351–401 (1906).
2. KORN, A., Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen. Bulletin Internationale, Cracovie Akademie Umiejt, Classe des sciences mathématiques et naturelles, 705–724 (1909).
3. FRIEDRICHS, K. O., On the boundary-value problems of the theory of elasticity and Korn's inequality. *Annals of Mathematics* **48**, 441–471 (1947).
4. TOUPIN, R. A., Saint-Venant's principle. *Archive for Rational Mechanics and Analysis* **18**, 83–96 (1965).
5. MIKHLIN, S. G., The Problem of the Minimum of a Quadratic Functional. San Francisco: Holden-Day, Inc. 1965.
6. EIDUS, D. M., On a mixed problem of elasticity theory. *Dokl. Akad. Nauk SSSR* **76**, 181–184 (1951). (Russian.)
7. FICHERA, G., Sull'esistenza e sul calcolo delle soluzioni dei problemi al contorno relativi all'equilibrio di un corpo elastico. *Annali della Scuola Normale Superiori di Pisa, Ser. III*, **4**, 35–99 (1950).
8. FICHERA, G., *Linear Elliptic Differential Systems and Eigenvalue Problems*. Berlin-Heidelberg-New York: Springer 1965.
9. HLAVÁČEK, I., & J. NEČAS, On inequalities of Korn's type. I. Boundary-value problems for elliptic systems of partial differential equations. *Archive for Rational Mechanics and Analysis* **36**, 305–311 (1970).
10. HLAVÁČEK, I., & J. NEČAS, On inequalities of Korn's type. II. Applications to linear elasticity. *Archive for Rational Mechanics and Analysis* **36**, 312–334 (1970).
11. BERNSTEIN, B., & R. A. TOUPIN, Korn inequalities for the sphere and circle. *Archive for Rational Mechanics and Analysis* **6**, 51–64 (1960).
12. PAYNE, L. E., & H. F. WEINBERGER, On Korn's inequality. *Archive for Rational Mechanics and Analysis* **8**, 89–98 (1961).
13. DAFERMOS, C. M., Some remarks on Korn's inequality. *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* **19**, 913–920 (1968).
14. HORGAN, C. O., *Eigenvalue Problems Associated with Korn's Inequalities in the Theory of Elasticity*. Ph. D. thesis, California Institute of Technology, Pasadena, California, June 1970.
15. LEKHNITSKII, S. G., On some problems related to the theory of bending of thin strips. *Prikl. Mat. i Mech. (PMM)* **2**, 181–210 (1938). (Russian with German summary.)
16. SOKOLNIKOFF, I. S., *Mathematical Theory of Elasticity*. Mimeographed lecture notes, Brown University, pp. 382–386 (1941).
17. GREEN, A. E., & W. ZERNA, *Theoretical Elasticity*. Oxford: Oxford University Press 1954.
18. GURTIN, M. E., & E. STERNBERG, On the first boundary-value problem of linear elastostatics. *Archive for Rational Mechanics and Analysis* **6**, 177–187 (1960).
19. MIKHLIN, S. G., Further investigation of the Cosserat functions. *Vestnik Leningrad Univ.* **22**, no. 7, 96–102 (1967). (Russian with English summary.)
20. ERICKSEN, J. L., On the Dirichlet problem for linear differential equations. *Proc. American Math. Soc.* **8**, 521–522 (1957).
21. MUSKHELISHVILI, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity*. Fourth Edition. Groningen: P. Noordhoff 1963.
22. MIKHLIN, S. G., *Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology*. Second Edition. New York: Pergamon Press 1957.
23. SHERMAN, D. I., On the distribution of the eigenvalues of the integral equations of the plane theory of elasticity. *Trudy Seism. Inst. Akad. Nauk SSSR*, No. 82 (1938). (Works, Institute of Seismology, Ac. Sci. USSR.) (Russian.)
24. MIKHLIN, S. G., On Cosserat Functions. In: *Boundary Value Problems and Integral Equations*. (Problems in the Mathematical Analysis Series, Vol. 1.) Editor: V. I. SMIRNOV. Izdat. Leningrad. Univ., Leningrad, 1966. (English translation by Consultants Bureau, New York City, 1968 pp. 55–65.)
25. COSSERAT, E., & F., Sur les équations de la théorie de l'élasticité. *Comptes Rendus* **126**, 1089–1091 (1898).

26. COSSERAT, E., & F., Sur la déformation infiniment petite d'un ellipsoïde élastique. *Comptes Rendus* **127**, 315-318 (1898).
27. COSSERAT, E., & F., Sur la solution des équations de l'élasticité, dans le cas où les valeurs des inconnues à la frontière sont données. *Comptes Rendus* **133**, 145-147 (1901).
28. BITSADZE, A. V., *Boundary Value Problems for Second Order Elliptic Equations*. Amsterdam: North-Holland Publishing Company 1968.

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