

Existence Theorems in Multidimensional Problems of Optimization with Distributed and Boundary Controls

LAMBERTO CESARI & DAVID E. COWLES

Contents

1. Introduction	321
2. Preliminaries	322
3. A Lower Closure Theorem	324
4. An Existence Theorem for Optimization Problems with State Equations in the Strong Form	326
5. Another Existence Theorem for Optimization Problems with State Equations in the Strong Form	337
6. An Existence Theorem for Optimization Problems with State Equations in the Weak Form	343
7. Application to Problems of Optimization with an Evolution Equation in Weak Form	351
8. References	354

1. Introduction

In this paper we consider multidimensional nonlinear problems of optimization of the Lagrange type involving a cost functional expressed by means of integrals on a fixed domain G in Euclidian space E^v , $v \geq 1$, and on its boundary ∂G , and also involving state equations, which usually are partial differential equations, in G and on ∂G , and controls both in G (distributed controls) and on ∂G (boundary controls), while our state variable x is an element of a topological space S . The state equations, both in G and on ∂G , are written in terms of abstract functional analysis and hence may represent partial differential equations or more general functional relations. The state equations, both in G and on ∂G , may be written in either "strong" or "weak" form, the latter being customary in the theory of partial differential equations. This paper extends to the present situation the method and ideas of previous papers by CESARI [3abc], and particularly of [3e].

Let G be a fixed bounded open set in E^v , $v \geq 1$, and let Γ be a given closed subset of ∂G on which we have a hyperarea measure μ . To simplify our exposition, let S be a Banach space of elements x , and let \mathcal{L} , \mathcal{M} , \mathcal{I} , \mathcal{K} be operators on S , not necessarily linear, with values in the following spaces:

$$\begin{aligned} \mathcal{L}: S \rightarrow (L_1(G))^r, & \quad \mathcal{I}: S \rightarrow (L_1(\Gamma))^r, \\ \mathcal{M}: S \rightarrow (L_1(G))^s, & \quad \mathcal{K}: S \rightarrow (L_1(\Gamma))^s, \end{aligned}$$

where r , s , r' and s' are given positive integers. (See Remark 4 of § 4, and analogous remarks in §§ 5 and 6 for a more general situation.)

For every $t=(t^1, \dots, t^v)$ in the closure of G , let $A(t)$ be a nonempty closed subset of the y -space E^s . Let A be the set of all (t, y) such that $t \in \text{cl}(G)$ and $y=(y^1, \dots, y^s) \in A(t)$. For every $(t, y) \in A$, let $U(t, y)$ be a nonempty subset of the u -space E^m , $u=(u^1, \dots, u^m)$. We define analogous sets on a closed subset Γ of ∂G as follows. For every $t \in \Gamma$, let $B(t)$ be a nonempty closed subset of the \hat{y} -space $E^{s'}$, $\hat{y}=(\hat{y}^1, \dots, \hat{y}^{s'})$. Let B be the set of all (t, \hat{y}) with $t \in \Gamma$ and $\hat{y} \in B(t)$. For every $(t, \hat{y}) \in B$, let $V(t, \hat{y})$ be a nonempty subset of the v -space $E^{m'}$, $v=(v^1, \dots, v^{m'})$.

We consider the problem of finding an element x of S , a measurable control vector $u(t)=(u^1, \dots, u^m)$, $t \in G$, and a μ -measurable control vector $v(t)=(v^1, \dots, v^{m'})$, $t \in \Gamma$, that minimize the cost functional

$$I[x, u, v] = \int_G f_0(t, (\mathcal{M}x)(t), u(t)) dt + \int_\Gamma g_0(t, (\mathcal{K}x)(t), v(t)) d\mu, \tag{1.1}$$

subject to the state equations

$$(\mathcal{L}x)(t) = f(t, (\mathcal{M}x)(t), u(t)) \quad \text{a.e. in } G, \tag{1.2}$$

$$(\mathcal{J}x)(t) = g(t, (\mathcal{K}x)(t), v(t)) \quad \mu\text{-a.e. on } \Gamma, \tag{1.3}$$

and the constraints

$$(\mathcal{M}x)(t) \in A(t), \quad u(t) \in U(t, (\mathcal{M}x)(t)) \quad \text{a.e. in } G, \tag{1.4}$$

$$(\mathcal{K}x)(t) \in B(t), \quad v(t) \in V(t, (\mathcal{K}x)(t)) \quad \mu\text{-a.e. on } \Gamma. \tag{1.5}$$

Here $u(t)$ is said to be a distributed control, and $v(t)$ a boundary control. The state equation (1.2) usually represents a system of partial differential equations, and (1.3) usually represents boundary data, or boundary controls but may just as well be a system of partial differential equations and related constraints and controls on the boundary. The state equations (1.2) and (1.3) are said to be written in the *strong* form. We shall consider in § 6 also the problem of minimizing the cost functional (1.1) when (1.2) and (1.3) are written in the corresponding *weak* form, as is customary in the theory of partial differential equations. The corresponding results are framed in the present general theory with no extra effort.

2. Preliminaries

In order to state our lower closure and existence theorems, we will use C. B. MORREY's definition of a regular transformation of class K from his paper [8a].

Let X and Y be subsets of a Euclidean space E^v , $v \geq 1$. A transformation $x=x(y)$ of Y onto X is said to be of *class* K provided it is one-to-one and continuous, and the functions $x=x(y)$ and $y=y(x)$ satisfy uniform Lipschitz conditions on each compact subset of X and Y , respectively. In addition, the transformation is said to be *regular* if the functions $x(y)$ and $y(x)$ satisfy uniform Lipschitz conditions on the whole of X and Y , respectively. For the concept of bounded open subset of E^v of class K in the sense of C. B. MORREY, we refer to his paper [8a]. The closure of such a set is often called a region of class K in E^v . Briefly, a region of class K , or K_1 , in E^v is a compact manifold with boundary with respect to regular transformations of class K , or K_1 . Analogous definitions hold for transformations and regions of class K_l , $l \geq 1$.

In stating our theorems we shall use the notations of § 1 and the properties of set-valued functions. Also, given a point $(t_0, y_0) \in A$ and a number $\delta > 0$, we denote by $N_\delta(t_0, y_0)$ the set of all points $(t, y) \in A$ at a distance $\leq \delta$ from (t_0, y_0) .

For every $(t, y) \in A$ let $Q(t, y)$ be a subset of the z -space E^{r+1} , $z = (z^0, \dots, z^r)$. We say that the sets $Q(t, y)$ have KURATOWSKI'S upper semicontinuity property [6], or property (U) , at the point $(t_0, y_0) \in A$ provided

$$Q(t_0, y_0) = \bigcap_{\varepsilon > 0} \text{cl } Q(t_0, y_0, \varepsilon)$$

where

$$Q(t_0, y_0, \varepsilon) = \bigcup_{(t, y) \in N_\varepsilon(t_0, y_0)} Q(t, y).$$

We say that the sets $Q(t, y)$ have property (Q) , or the modified upper semicontinuity property [3a], at $(t_0, y_0) \in A$, provided

$$Q(t_0, y_0) = \bigcap_{\varepsilon > 0} \text{cl co } Q(t_0, y_0, \varepsilon).$$

We say that the sets $Q(t, y)$ have property (U) or (Q) in A if they have that property at every point $(t_0, y_0) \in A$. Sets having property (U) are closed, and sets having property (Q) are closed and convex. It has been found useful to introduce also intermediate properties $Q(\rho)$, $0 \leq \rho \leq r+1$, of variable sets (D. E. COWLES [4a]).

Let ρ be any integer, $0 \leq \rho \leq r+1$. We say that the sets $Q(t, y)$ have property $Q(\rho)$ at the point $(t_0, y_0) \in A$ provided for every

$$\begin{aligned} z_0 &= (z_0^0, z_0^1, \dots, z_0^{r+1}) \in E^{r+1}, \\ Q(t_0, y_0) \cap \{z = (z^0, \dots, z^r) \in E^{r+1} \mid z^i &= z_0^i, i = \rho, \dots, r\} \\ &= \bigcap_{\varepsilon > 0} \bigcap_{\beta > 0} \text{cl co } (Q(t_0, y_0, \varepsilon) \cap \{z \in E^{r+1} \mid |z^i - z_0^i| \leq \beta, i = \rho, \dots, r\}). \end{aligned}$$

For $\rho = r+1$ we understand that the sets in braces in the first and second members of this relation coincide with E^r . We note also that if the sets $Q(t, y)$ have property $Q(\rho)$ at the point $(t_0, y_0) \in A$, then for every

$$z_0 = (z_0^0, z_0^1, \dots, z_0^{r+1}) \in E^{r+1}$$

the set

$$Q(t_0, y_0) \cap \{z \in E^{r+1} \mid z^i = z_0^i, i = \rho, \dots, r\}$$

is closed and convex, since it is the intersection of sets having the same property. Thus, for any two points

$$\begin{aligned} z_1 &= (z_1^0, \dots, z_1^{\rho-1}, z_0^\rho, \dots, z_0^r) \in Q(t_0, y_0), \\ z_2 &= (z_2^0, \dots, z_2^{\rho-1}, z_0^\rho, \dots, z_0^r) \in Q(t_0, y_0), \end{aligned}$$

the points $\alpha z_1 + (1-\alpha) z_2$ also belong to $Q(t_0, y_0)$, $0 \leq \alpha \leq 1$. Sets possessing this (partial) convexity property will be said to be ρ -convex.

(2.i) For any integer ρ , $0 \leq \rho \leq r$, property $Q(\rho+1)$ implies property $Q(\rho)$. Also, property $Q(r+1)$ holds if and only if property (Q) holds, and property $Q(0)$ holds if and only if property (U) holds.

For a proof of this statement, and of statement (2.ii) below, see COWLES [4a].

For every point $(t, y) \in A$, let $Q(t, y)$ be a subset of E^{r+1} , $r \geq 0$. We say that the sets $Q(t, y)$ have the "upper set property" on A provided $(t, y) \in A$, $z_0 = (z_0^0, z_0^1, \dots, z_0^r) \in Q(t, y)$, $\bar{z}_0 = (\bar{z}_0^0, \bar{z}_0^1, \dots, \bar{z}_0^r) \in E^{r+1}$ with $\bar{z}_0^0 \geq z_0^0$ implies $\bar{z}_0 \in Q(t, y)$.

(2.ii) *If the sets $Q(t, y)$ have the upper set property on A and also the property (U) , then the same sets have property $Q(1)$ on A .*

It was proved in [3fi] that property (Q) is essentially an extension to Lagrange problems of the seminormality condition often used for free problems. This condition (Q) , and variants, will be used in the lower closure and existence theorems below. In a number of these theorems property (Q) and variants can be relaxed or dropped. In particular this occur for linear problems, and for problems with f_0, f, g_0, g possessing suitable bounds. These modifications can be readily obtained within the present approach. We shall discuss these cases in detail in subsequent papers.

3. A Lower Closure Theorem

Let G be a bounded measurable subset of E^v , $v \geq 1$, and let ∂G denote the boundary of G .

Let $\Gamma_j, j=1, \dots, N$, be subsets of ∂G , each of which is the image under a regular transformation t_j of class K of a bounded open interval R'_j of E^{v-1} . Let Γ be a closed subset of $\bigcup_{j=1}^N \Gamma_j$, and let μ be a measure defined on $\bigcup_{j=1}^N \Gamma_j$. For each $j=1, \dots, N$, we assume that if e is a subset of Γ_j , measurable with respect to μ , then $E = t_j^{-1}(e)$ is measurable with respect to Lebesgue $(v-1)$ -dimensional measure $|\cdot|$ on R'_j . Also, we assume that the converse is true, so that measurable sets on Γ_j and R'_j correspond under t_j . Finally, we assume that there is a constant $K > 1$ such that if $e = t_j(E)$ is μ -measurable, then

$$K^{-1} |E| \leq \mu(e) \leq K |E|, \tag{3.1}$$

independently of $j=1, \dots, N$, and e . Since μ induces a measure on each set R'_j via the transformation t_j , we may define $J_j(\tau)$, $\tau \in R'_j$, as the function in $L_1(R'_j)$ which satisfies the relations

$$\mu(t_j(E)) = \int_E J_j(\tau) d\tau \tag{3.2}$$

for every measurable subset E of $R'_j, j=1, \dots, N$. Thus $J_j(\tau)$ is defined almost everywhere in R'_j and $K^{-1} \leq J_j(\tau) \leq K$ a.e. in R'_j . This situation actually occurs when G is an open bounded set of class K in E^v (see § 2) and μ is the usual hyper-area measure defined on $\Gamma = \partial G$. This is the situation we shall consider in all examples below. Nevertheless, μ could be a different measure with the properties set forth above. Actually, for the general theorems of § 3, 4, 5, 6, Γ need not even be a subset of ∂G , but only the union of parts Γ_j in a Euclidean space, each Γ_j being the image of an interval under a regular transformation of class K , and μ may be any measure on Γ with the properties stated above. Also, we could consider functionals (1.1) which are sums of integrals on parts of different dimensions, as indeed indicated in (1.1).

As mentioned in the introduction, for every $t \in \text{cl } G$ we denote by $A(t)$ a nonempty closed subset of the y -space $E^s, y = (y^1, \dots, y^s)$. Let A be the set of all points (t, y) with $t \in \text{cl } G$ and $y \in A(t)$. For every $(t, y) \in A$ let $U(t, y)$ be a nonempty subset of the u -space $E^m, u = (u^1, \dots, u^m)$. Let M be the set of all $(t, y, u) \in E^v \times E^s \times E^m$ such that $(t, y) \in A$ and $u \in U(t, y)$. For every $t \in \Gamma$, let $B(t)$ be a nonempty closed subset of the \hat{y} -space $E^{s'}, \hat{y} = (\hat{y}^1, \dots, \hat{y}^{s'})$. Let B be the set of all (t, \hat{y}) with $t \in \Gamma$ and $\hat{y} \in B(t)$. For every $(t, \hat{y}) \in B$, let $V(t, \hat{y})$ be a nonempty closed subset of the v -space $E^{m'}, v = (v^1, \dots, v^{m'})$. Let \hat{M} be the set of all $(t, \hat{y}, v) \in E^v \times E^{s'} \times E^{m'}$ with $(t, \hat{y}) \in B$ and $v \in V(t, \hat{y})$.

Let $f(t, y, u) = (f_0, f) = (f_0, f_1, \dots, f_r)$ be a continuous $(r + 1)$ -vector function on M , and for every $(t, y) \in A$ let $\tilde{Q}(t, y)$ denote the set

$$\tilde{Q}(t, y) = \{ \tilde{z} = (z^0, z) = (z^0, z^1, \dots, z^r) \in E^{r+1} \mid z^0 \geq f_0(t, y, u), z = f(t, y, u), u \in U(t, y) \}.$$

Let $\tilde{g}(t, \hat{y}, v) = (g_0, g) = (g_0, g_1, \dots, g_{r'})$ be a continuous $(r' + 1)$ -vector function on \hat{M} , and, for every $(t, \hat{y}) \in B$, let $\tilde{R}(t, \hat{y})$ denote the set

$$\tilde{R}(t, \hat{y}) = \{ \tilde{z} = (z^0, z) = (z^0, z^1, \dots, z^{r'}) \in E^{r'+1} \mid z^0 \geq g_0(t, \hat{y}, v), z = g(t, \hat{y}, v), v \in V(t, \hat{y}) \}.$$

We consider here the functional

$$I[y, \hat{y}, u, v] = \int_G f_0(t, y(t), u(t)) dt + \int_\Gamma g_0(t, \hat{y}(t), v(t)) d\mu. \tag{3.3}$$

In the lower closure theorem below we shall deal with sequences of functions all defined on G and Γ :

$$\begin{aligned} z(t) &= (z^1, \dots, z^r), & z_k(t) &= (z_k^1, \dots, z_k^r), \\ y(t) &= (y^1, \dots, y^s), & y_k(t) &= (y_k^1, \dots, y_k^s), \\ u_k(t) &= (u_k^1, \dots, u_k^m), & t \in G, k = 1, 2, \dots, \\ \hat{z}(t) &= (\hat{z}^1, \dots, \hat{z}^{r'}), & \hat{z}_k(t) &= (\hat{z}_k^1, \dots, \hat{z}_k^{r'}), \\ \hat{y}(t) &= (\hat{y}^1, \dots, \hat{y}^{s'}), & \hat{y}_k(t) &= (\hat{y}_k^1, \dots, \hat{y}_k^{s'}), \\ v_k(t) &= (v_k^1, \dots, v_k^{m'}), & t \in \Gamma, k = 1, 2, \dots \end{aligned} \tag{3.4}$$

(3.i) Lower Closure Theorem. *Let G be bounded and measurable, A, B, M, \hat{M} closed, $f_0(t, y, u), f(t, y, u) = (f_1, \dots, f_r)$ continuous on $M, g_0(t, \hat{y}, v), g(t, \hat{y}, v) = (g_1, \dots, g_{r'})$ continuous on \hat{M} , and assume that for some integers $\rho, \rho', 0 \leq \rho \leq r, 0 \leq \rho' \leq r'$, the sets $\tilde{Q}(t, y)$ have property $Q(\rho + 1)$ on A and the sets $\tilde{R}(t, \hat{y})$ have property $Q(\rho' + 1)$ on B . Let us assume that there are functions $\psi(t) \geq 0, t \in G, \psi \in L_1(G)$ and $\hat{\psi}(t) \geq 0, t \in \Gamma, \hat{\psi} \in L_1(\Gamma)$, such that $f_0(t, y, u) \geq -\psi(t)$ for all $(t, y, u) \in M$, and $g_0(t, \hat{y}, v) \geq -\hat{\psi}(t)$ for all $(t, \hat{y}, v) \in \hat{M}$. Let us assume that the functions $z^i(t), z_k^i(t), y^j(t), y_k^j(t), i = 1, \dots, r, j = 1, \dots, s$, are in $L_1(G)$, that the functions $u_k^l(t)$ are measurable on $G, l = 1, \dots, m$, that $f_0(t, y_k(t), u_k(t)) \in L_1(G)$, and that*

$$y_k(t) \in A(t), \quad u_k(t) \in U(t, y_k(t)), \quad z_k^i(t) = f_i(t, y_k(t), u_k(t)) \tag{3.5}$$

a.e. in $G, k = 1, 2, \dots$

Let us assume that the functions $\overset{\circ}{z}^i(t), \overset{\circ}{z}_k^i(t), \overset{\circ}{y}^j(t), \overset{\circ}{y}_k^j(t), i=1, \dots, r', j=1, \dots, s',$ are in $L_1(\Gamma)$, that the functions $v_k^l(t)$ are measurable in $\Gamma, l=1, \dots, m',$ that $g_0(t, \overset{\circ}{y}_k(t), v_k(t)) \in L_1(\Gamma)$, and that

$$\overset{\circ}{y}_k(t) \in B(t), \quad v_k(t) \in V(t, \overset{\circ}{y}_k(t)), \quad \overset{\circ}{z}_k^i(t) = g_i(t, \overset{\circ}{y}_k(t), v_k(t)) \quad (3.6)$$

μ -a.e. on $\Gamma, k=1, 2, \dots$

Finally, let us assume that as $k \rightarrow \infty$ we have

$$\begin{aligned} z_k^i(t) &\rightarrow z^i(t) \text{ weakly in } L_1(G), & i=1, \dots, \rho, \\ z_k^i(t) &\rightarrow z^i(t) \text{ strongly in } L_1(G), & i=\rho+1, \dots, r, \\ y_k^j(t) &\rightarrow y^j(t) \text{ strongly in } L_1(G), & j=1, \dots, s, \\ \overset{\circ}{z}_k^i(t) &\rightarrow \overset{\circ}{z}^i(t) \text{ weakly in } L_1(\Gamma), & i=1, \dots, \rho', \\ \overset{\circ}{z}_k^i(t) &\rightarrow \overset{\circ}{z}^i(t) \text{ strongly in } L_1(\Gamma), & i=\rho'+1, \dots, r', \\ \overset{\circ}{y}_k^j(t) &\rightarrow \overset{\circ}{y}^j(t) \text{ strongly in } L_1(\Gamma), & j=1, \dots, s', \end{aligned}$$

and

$$\lim_{k \rightarrow \infty} I[y_k, \overset{\circ}{y}_k, u_k, v_k] = a_0 < +\infty.$$

Then $y(t) \in A(t)$ a.e. in $G, \overset{\circ}{y}(t) \in B(t)$ μ -a.e. on Γ , and there are measurable functions $u(t) = (u^1, \dots, u^m), t \in G,$ and μ -measurable functions $v(t) = (v^1, \dots, v^{m'}), t \in \Gamma,$ such that

$$f_0(t, y(t), u(t)) \in L_1(G), \quad g_0(t, \overset{\circ}{y}(t), v(t)) \in L_1(\Gamma),$$

and such that

$$\begin{aligned} u(t) &\in U(t, y(t)), \quad z^i(t) = f_i(t, y(t), u(t)), & i=1, \dots, r, \quad \text{a.e. on } G, \\ v(t) &\in V(t, \overset{\circ}{y}(t)), \quad \overset{\circ}{z}^i(t) = g_i(t, \overset{\circ}{y}(t), v(t)), & i=1, \dots, r', \quad \mu\text{-a.e. on } \Gamma, \end{aligned}$$

and

$$I[y, \overset{\circ}{y}, u, v] \leq a_0.$$

The proof of this lower closure theorem has been given by CESARI [3cde] for $g_0 = g = 0,$ and by COWLES [4b] along the same lines in the situation above.

Remark. In applications it often occurs that the sets U and V are fixed and compact, or, alternatively, that $U(t, y), V(t, \overset{\circ}{y})$ are compact, equibounded, and have property (U) in A and $B,$ respectively. If f_0, f, g_0, g are continuous, the sets $\bar{Q}(t, y), \bar{R}(t, \overset{\circ}{y})$ certainly are compact and have property (U) in A and $B,$ respectively; if convex, also property (Q) (see [3a]); and if ρ -convex, property $Q(\rho)$ (see [4a]). On the other hand, if the closed sets $U(t, y), V(t, \overset{\circ}{y})$ are unbounded, and $f_0, f,$ as well as $g_0, g,$ are continuous and satisfy suitable growth conditions on the closed sets $M, \bar{M},$ then the sets $\bar{Q}(t, \overset{\circ}{y}), \bar{R}(t, y),$ if convex, necessarily satisfy condition (Q) (see [3b]).

4. An Existence Theorem for Optimization Problems with State Equations in the Strong Form

In this section we shall use mainly the notations of § 3. For the sake of simplicity we shall denote by T the family of all measurable m -vector functions $u(t) =$

(u^1, \dots, u^m) , $t \in G$, and by \mathring{T} the family of all μ -measurable m' -vector functions $v(t) = (v^1, \dots, v^{m'})$, $t \in \Gamma$.

Again for the sake of simplicity, let S be a Banach space of elements x , and let \mathcal{L} , \mathcal{M} , \mathcal{J} , \mathcal{K} be operators, not necessarily linear, as described in § 1, that is, $\mathcal{L}: S \rightarrow (L_1(G))^r$, $\mathcal{M}: S \rightarrow (L_1(G))^s$, $\mathcal{J}: S \rightarrow (L_1(\Gamma))^{r'}$, $\mathcal{K}: S \rightarrow (L_1(\Gamma))^{s'}$. We shall discuss here the problem of optimization (1.1-5) of § 1.

A triple (x, u, v) is said to be *admissible* (for the problem (1.1-5)) provided $x \in S$, $u \in T$, $v \in \mathring{T}$, $f_0(t, (\mathcal{M}x)(t), u(t)) \in L_1(G)$, $g_0(t, (\mathcal{K}x)(t), v(t)) \in L_1(\Gamma)$ and relations (1.2-5) hold.

A class Ω of admissible triples is said to be *closed* if the following occurs: if $(x_k, u_k, v_k) \in \Omega$, $k = 1, 2, \dots$, $x_k \rightarrow x$ weakly in S as $k \rightarrow \infty$,

$$\lim_{k \rightarrow \infty} I[x_k, u_k, v_k] = a < +\infty,$$

and there are admissible triples (x, u, v) such that $I[x, u, v] \leq a$, then there is also some triple $(x, \bar{u}, \bar{v}) \in \Omega$, with $I[x, \bar{u}, \bar{v}] \leq a$.

For a class Ω of admissible triples we denote by $\{x\}_\Omega$ the subset of S defined by

$$\{x\}_\Omega = \{x \in S \mid (x, u, v) \in \Omega \text{ for some } u \in T, v \in \mathring{T}\}.$$

Note that for

$$x \in S, \text{ then } z(t) = (z^1, \dots, z^r) = (\mathcal{L}x)(t) \in (L_1(G))^r;$$

we shall denote by $z^i(t) = (\mathcal{L}x)^i(t)$, $t \in G$, the i^{th} component of $\mathcal{L}x$. Analogously,

$$\begin{aligned} y(t) &= (y^1, \dots, y^s) = (\mathcal{M}x)(t) \in (L_1(G))^s, \\ \mathring{z}(t) &= (\mathring{z}^1, \dots, \mathring{z}^{r'}) = (\mathcal{J}x)(t) \in (L_1(\Gamma))^{r'}, \\ \mathring{y}(t) &= (\mathring{y}^1, \dots, \mathring{y}^{s'}) = (\mathcal{K}x)(t) \in (L_1(\Gamma))^{s'}, \end{aligned}$$

and we set

$$\begin{aligned} y^j(t) &= (\mathcal{M}x)^j(t), & t \in G, & j = 1, \dots, s; \\ \mathring{z}^i(t) &= (\mathcal{J}x)^i(t), & t \in \Gamma, & i = 1, \dots, r'; \\ \mathring{y}^j(t) &= (\mathcal{K}x)^j(t), & t \in \Gamma, & j = 1, \dots, s'. \end{aligned}$$

If ρ, ρ' are any two integers, $0 \leq \rho \leq r$, $0 \leq \rho' \leq r'$, we shall denote by $(C_{\rho\rho'})$ the following closure property of the operators $\mathcal{L}, \mathcal{M}, \mathcal{J}, \mathcal{K}$:

$(C_{\rho\rho'})$ For every sequence x, x_k , $k = 1, 2, \dots$, of elements $x \in S$, $x_k \in \{x\}_\Omega \subset S$, with $x_k \rightarrow x$ weakly in S , there is some subsequence $[k_\lambda]$ such that, as $\lambda \rightarrow \infty$, we have

$$\begin{aligned} (\mathcal{L}x_{k_\lambda})^i &\rightarrow (\mathcal{L}x)^i \text{ weakly in } L_1(G), & i = 1, \dots, \rho, \\ (\mathcal{L}x_{k_\lambda})^i &\rightarrow (\mathcal{L}x)^i \text{ strongly in } L_1(G), & i = \rho + 1, \dots, r, \\ (\mathcal{M}x_{k_\lambda})^j &\rightarrow (\mathcal{M}x)^j \text{ strongly in } L_1(G), & j = 1, \dots, s, \\ (\mathcal{J}x_{k_\lambda})^i &\rightarrow (\mathcal{J}x)^i \text{ weakly in } L_1(\Gamma), & i = 1, \dots, \rho', \\ (\mathcal{J}x_{k_\lambda})^i &\rightarrow (\mathcal{J}x)^i \text{ strongly in } L_1(\Gamma), & i = \rho' + 1, \dots, r', \\ (\mathcal{K}x_{k_\lambda})^j &\rightarrow (\mathcal{K}x)^j \text{ strongly in } L_1(\Gamma), & j = 1, \dots, s', \end{aligned} \tag{4.1}$$

(4.i) **Existence Theorem.** Let G be bounded and measurable, $A, B, M, \overset{\circ}{M}$ closed, $f_0(t, y, u), f(t, y, u) = (f_1, \dots, f_r)$ continuous on $M, g_0(t, \overset{\circ}{y}, v), g(t, \overset{\circ}{y}, v) = (g_1, \dots, g_r)$ continuous on $\overset{\circ}{M}$, and assume that, for given integers $\rho, \rho', 0 \leq \rho \leq r, 0 \leq \rho' \leq r'$, the sets $\overset{\circ}{Q}(t, y)$ have property $Q(\rho+1)$ on A , and the sets $\overset{\circ}{R}(t, \overset{\circ}{y})$ have property $Q(\rho'+1)$ on B . Let us assume that there are functions $\psi(t) \geq 0, t \in G, \psi \in L_1(G)$, and $\overset{\circ}{\psi}(t) \geq 0, t \in \Gamma, \overset{\circ}{\psi} \in L_1(\Gamma)$, such that $f_0(t, y, u) \geq -\psi(t)$ for all $(t, y, u) \in M$, and $g_0(t, \overset{\circ}{y}, v) \geq -\overset{\circ}{\psi}(t)$ for all $(t, \overset{\circ}{y}, v) \in \overset{\circ}{M}$. Let Ω be a nonempty closed class of admissible triples (x, u, v) such that the set $\{x\}_\Omega$ is weakly sequentially relatively compact, and let us assume that the operators $\mathcal{L}, \mathcal{M}, \mathcal{J}, \mathcal{K}$ satisfy the closure property $(C_{\rho, \rho'})$. Then the functional (1.1), or $I[x, u, v]$, has an absolute minimum in Ω .

In view of statements (2.i) and (2.ii), note that for $\rho=r$ we actually require above that the sets $\overset{\circ}{Q}(t, y)$ have property (Q) , and for $\rho=0$ we actually require that the sets $\overset{\circ}{Q}(t, y)$ have property (U) . Analogously, for $\rho'=r'$ we actually require that the sets $\overset{\circ}{R}(t, \overset{\circ}{y})$ have property (Q) ; for $\rho'=0$ we require that the sets $R(t, \overset{\circ}{y})$ have property (U) . In general, for $0 \leq \rho \leq r, 0 \leq \rho' \leq r'$, properties $Q(\rho+1)$ and $Q(\rho'+1)$ represent intermediate requirements.

Proof. Let i be the infimum of $I[x, u, v]$ in the class Ω . Then i is finite, and we consider a minimizing sequence of I in Ω , that is, a sequence

$$(x_k, u_k, v_k), \quad k=1, 2, \dots,$$

of admissible triples, all in Ω , with

$$I[x_k, u_k, v_k] \rightarrow i \quad \text{as } k \rightarrow \infty.$$

Since the set $\{x\}_\Omega$ is weakly sequentially relatively compact, there is some element $x \in S$ and some subsequence of $[x_k]$ which is weakly convergent to x . For the sake of simplicity we denote such a sequence by $[k]$, and thus $x_k \rightarrow x$ weakly in S . As a consequence, there is a subsequence $[k_\lambda]$ for which the convergence relations (4.1) hold. We shall denote this subsequence again by $[k]$. By using the notations

$$\begin{aligned} z_k(t) &= (\mathcal{L} x_k)(t), & y_k(t) &= (\mathcal{M} x_k)(t), & z(t) &= (\mathcal{L} x)(t), & y(t) &= (\mathcal{M} x)(t), & t \in G, \\ \overset{\circ}{z}_k(t) &= (\mathcal{J} x_k)(t), & \overset{\circ}{y}_k(t) &= (\mathcal{K} x_k)(t), & \overset{\circ}{z}(t) &= (\mathcal{J} x)(t), & \overset{\circ}{y}(t) &= (\mathcal{K} x)(t), & t \in \Gamma, \end{aligned}$$

we see that relations (1.2-5) imply

$$y_k(t) \in A(t), \quad u_k(t) \in U(t, y_k(t)), \quad z_k^i(t) = f_i(t, y_k(t), u_k(t))$$

a.e. in $G, i=1, \dots, r$, and

$$\overset{\circ}{y}_k(t) \in B(t), \quad v_k(t) \in V(t, \overset{\circ}{y}_k(t)), \quad z_k^i(t) = g_i(t, \overset{\circ}{y}_k(t), v_k(t))$$

μ -a.e. on $\Gamma, i=1, \dots, r', k=1, 2, \dots$. In addition, $z_k^i \rightarrow z^i$ weakly in $L_1(G), i=1, \dots, \rho, z_k^i \rightarrow z^i$ strongly in $L_1(G), i=\rho+1, \dots, r, y_k^j \rightarrow y^j$ strongly in $L_1(G), j=1, \dots, s, \overset{\circ}{z}_k^i \rightarrow \overset{\circ}{z}^i$ weakly in $L_1(\Gamma), i=1, \dots, \rho', \overset{\circ}{z}_k^i \rightarrow \overset{\circ}{z}^i$ strongly in $L_1(\Gamma), i=\rho'+1, \dots, r', \overset{\circ}{y}_k^j \rightarrow \overset{\circ}{y}^j$ strongly in $L_1(\Gamma), j=1, \dots, s'$. Finally, the sets $\overset{\circ}{Q}(t, y)$ satisfy property $Q(\rho+1)$ on A , and the sets $\overset{\circ}{R}(t, \overset{\circ}{y})$ satisfy property $Q(\rho'+1)$ on B . We can now apply the lower closure theorem (3.i). Then, $y(t) = (\mathcal{M} x)(t) \in A(t)$ a.e. in $G, \overset{\circ}{y}(t) = (\mathcal{K} x)(t) \in B(t)$ μ -a.e. on Γ , and there are elements $u \in T$ and $v \in \overset{\circ}{T}$

such that

$$\begin{aligned}
 u(t) \in U(t, y(t)), \quad z^i(t) = f_i(t, y(t), u(t)), \quad i = 1, \dots, r, \quad \text{a.e. in } G, \\
 v(t) \in V(t, \dot{y}(t)), \quad \dot{z}^i(t) = g_i(t, \dot{y}(t), v(t)), \quad i = 1, \dots, r', \quad \mu\text{-a.e. on } \Gamma, \quad (4.2) \\
 f_0(t, y(t), u(t)) \in L_1(G), \quad g_0(t, \dot{y}(t), v(t)) \in L_1(\Gamma), \quad I[x, u, v] \leq i;
 \end{aligned}$$

that is, the triple (x, u, v) is admissible. Since Ω is a closed class of admissible triples, in Ω there are some admissible triples (x, \bar{u}, \bar{v}) such that $I[x, \bar{u}, \bar{v}] \leq i$, and relations (4.2) hold also for u, v replaced by \bar{u}, \bar{v} . Since i is the infimum of I in Ω , we have $I[x, \bar{u}, \bar{v}] \geq i$, and finally $I[x, \bar{u}, \bar{v}] = i$.

Remark 1. Theorem (4.i) holds even if we replace the cost functional (1.1) by another analogous one with an added term $J[x]$, provided we know that $J[x]$ is lower semicontinuous functional on S with respect to weak convergence on S . That is, we need only require of J that $x_k \rightarrow x$ weakly in S implies

$$\lim_{k \rightarrow \infty} J[x_k] \geq J[x].$$

Remark 2. In both Theorems (3.i) and (4.i) we could have assumed that G and Γ are each made up of a finite number of components on each of which there is a distinct system of state equations.

Remark 3. In the existence theorem (4.i), in verifying that the closure hypothesis $(C_{\rho\rho'})$ is satisfied, it is often convenient to restrict Ω to the subclass Ω_0 of all triples $(x, u, v) \in \Omega$ such that $I[x, u, v] \leq M$ for M sufficiently large. For instance, if i denotes the infimum of $I[x, u, v]$ in Ω , we may take $M = i + 1$.

Remark 4. In the existence theorem (4.i) we have assumed, for the sake of simplicity, that S is a Banach space, and we have used the weak topology in S . This is indeed the most common situation in applications. More generally, we could consider instead any topological space (S, σ) , that is, any space S with a chosen topology σ . In particular, S need not be linear. Accordingly, then, we should have to require in the context of the closure property $(C_{\rho\rho'})$ that $x \in S, x_k \in \{x\}_\Omega, x_k \rightarrow x$ in (S, σ) implies the convergences in L_1 stated in $(C_{\rho\rho'})$. Also, accordingly, we should have to require in the Existence Theorem (4.i) that the set $\{x\}_\Omega$ is sequentially relatively compact in (S, σ) ; that is, any sequence of elements of $\{x\}_\Omega$ contains a subsequence which is convergent in the topology σ of S . Actually, (S, σ) does not need to be even a topological space, but only a set S with a definition σ of convergence of sequences (a Fréchet space L) (see, e.g., [10], p. 16). Examples where the underlying spaces are not Banach spaces and not even linear ones will occur in § 6. See also Remark 7 below.

Remark 5. Of particular interest is the case where the element x of an admissible triple (x, u, v) uniquely determines the controls u and v . That is, $(x, u, v), (x, \bar{u}, \bar{v})$ admissible implies $u = \bar{u}$ a.e. in G , and $v = \bar{v}$ μ -a.e. on Γ . In this situation, the lower closure theorem (3.i) reduces to a lower semicontinuity theorem, and corresponding particular existence theorems could have been obtained by a standard lower semicontinuity argument. This holds, for instance, for free problems

of the calculus of variations and other problems. For instance, let us consider the free problem of the minimum of the multiple integral

$$I[x] = \int_G f_0(t, x(t), \nabla x(t)) dt,$$

with

$$x(t) = (x^1, \dots, x^n), \quad t \in G \subset E^v, \quad \nabla x(t) = (\partial x^i / \partial t^j, i = 1, \dots, n, j = 1, \dots, v),$$

given Dirichlet boundary data on ∂G , or some suitable part Γ of ∂G . Here G is a bounded open connected subset of E^v of class K . We may take for S the Sobolev space $S = W_p^1(G)$, $1 < p \leq \infty$, and then $\nabla x(t) = u(t)$, that is, u is uniquely determined by x . We assume that f_0 is a continuous function in $M = \text{cl } G \times E^n \times E^{nv}$, with $f_0 \geq -\psi(t)$ for some $\psi \geq 0$, $\psi \in L_1(G)$. Note that

$$\begin{aligned} \mathcal{M}x &= x, \quad \mathcal{M} : S \rightarrow (L_p(G))^n, \quad \mathcal{L}x = \nabla x, \quad \mathcal{L} : S \rightarrow (L_p(G))^{nv}, \\ r &= m = nv, \quad s = n, \quad f = u, \quad U = E^{nv} \quad (g_0 = 0, \mathcal{K} = 0). \end{aligned}$$

Also, Ω is now a closed class of elements $x \in S$ (or pairs (x, u) with $x \in S, u = \nabla x$), with x satisfying the given boundary data, and

$$f_0(t, x(t), \nabla x(t)) \in L_1(G).$$

If

$$x \in S, \quad x_k \in \{x\}_\Omega \subset S, \quad k = 1, 2, \dots, \quad x_k \rightarrow x \text{ weakly in } S,$$

$$\lim_{k \rightarrow \infty} I[x_k] = a,$$

then $\mathcal{L}x_k \rightarrow \mathcal{L}x$ weakly in $(L_p(G))^{nv}$, $\mathcal{M}x_k \rightarrow \mathcal{M}x$ strongly in $(L_p(G))^n$, and the lower closure theorem guarantees, under the needed requirements, that $I[x] \leq a$.

Remark 6. As mentioned in the remark at the end of § 3, if U and V are fixed compact sets (or $U(t, y), V(t, \hat{y})$ are compact, equibounded, and have property (U)), then the sets $\bar{Q}(t, y), \bar{R}(t, \hat{y})$ certainly have property (U) and, if convex, have property (Q) (see [3a]). Also, an analogous statement holds for the intermediate properties $Q(\rho)$ in the sense that, if ρ -convex, then they have property $Q(\rho)$ (see [4a]).

Remark 7. The case $v \geq 1, g_0 = 0, g = 0$, has been considered by CESARI in [3e]. The case $v = 1$ has been considered by CESARI in [3a]. For $v = 1$, the underlying space in [3a] is the metric space S_0 of all continuous vector functions

$$x(t) = (x^1, \dots, x^n)$$

on arbitrary finite intervals $a \leq t \leq b$. If $x(t), a \leq t \leq b$, and $y(t), c \leq t \leq d$, are any two elements of S_0 , the distance function $\rho(x, y)$ is defined by

$$\rho = |a - c| + |b - d| + \max |x(t) - y(t)|,$$

where \max is taken in $-\infty < t < +\infty$, and x and y are defined by continuity and constancy outside of their original intervals. The actual space S is then a subset of S_0 , namely, the set of all $x \in S_0, x(t), t_1 \leq t \leq t_2$, which are absolutely continuous, and we take in S the topology induced by the one in S_0 ; in other words, S is now

a metric space with metric ρ , and S is not linear (see Remark 4 above). We are now concerned with the problem of the minimum of an integral

$$I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t)) dt,$$

with state equations and constraints

$$\begin{aligned} x'(t) &= f(t, x(t), u(t)), \\ x(t) &\in A(t), \quad u(t) \in U(t, x(t)), \end{aligned}$$

in a closed class Ω of pairs

$$x(t), u(t), t_1 \leq t \leq t_2, \quad x \in S, u \in T,$$

satisfying these relations, x satisfying also given boundary conditions, and such that

$$f_0(t, x(t), u(t)) \in L_1([t_1, t_2]).$$

Also, we have here $\{x\}_\Omega \subset S \subset S_0$,

$$\mathcal{M}x = x, \quad \mathcal{M}: S \rightarrow L_1, \quad \mathcal{L}x = x', \quad \mathcal{L}: S \rightarrow L_1.$$

If

$$x \in S, \quad x_k \in \{x\}_\Omega, \quad k=1, 2, \dots, \quad x_k \rightarrow x \text{ in the metric } \rho,$$

then $\mathcal{L}x_k \rightarrow \mathcal{L}x$ in the sense that

$$\int_I (\mathcal{L}x_k) dt \rightarrow \int_I (\mathcal{L}x) dt$$

as $k \rightarrow \infty$ for every interval I . A lower closure theorem analogous to (3.i) was proved in [3a] for these particular modes of convergence. Alternatively, we could take in S the metric

$$\rho = |a - c| + |b - d| + \int_{-\infty}^{+\infty} |x'(t) - y'(t)| dt,$$

take $\mathcal{M}x = x$ with $\mathcal{M}: S \rightarrow L_1$, $\mathcal{L}x = x'$ with $\mathcal{L}: S \rightarrow L_1$, and then the lower closure theorem (3.i) would apply, and consequently the existence theorem (4.i) would also apply with the Remarks 3 and 4.

Example 1. The following example, mentioned by FICHERA [5], illustrates the existence theorem (4.i). In this example the particular situation depicted in Remark 5 occurs, and therefore our lower closure theorem (3.i) reduces to a lower semicontinuity theorem (which includes Fichera's lower semicontinuity theorem). Let G be a bounded open subset of E^v of class K , $v \geq 1$ (see § 2). Let μ be the hyperarea measure defined on the boundary $\Gamma = \partial G$ of G . Let $W_p^l(G)$ be a Sobolev space on G for real p , $1 \leq p \leq +\infty$, and integral l , $1 \leq l < +\infty$, with the usual norm

$$\|x\| = \|x\|_{W_p^l(G)} = \sum_{|\alpha| \leq l} \|D^\alpha x\|_p = \sum_{|\alpha| \leq l} \left(\int_G |D^\alpha x(t)|^p dt \right)^{1/p},$$

where $D^\alpha x$ denotes the generalized partial derivative of x in G of order

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad \text{and} \quad |\alpha| = \alpha_1 + \dots + \alpha_n.$$

Let X_0 be the linear subset of $W_p^l(G)$ made up of all functions x which are continuous on $G \cup \partial G$, together with their partial derivatives of all orders. If $t \in \partial G$, let $(\mathcal{K}x)(t)$ denote the vector

$$(\mathcal{K}x)(t) = (\gamma x, \gamma \nabla x, \dots, \gamma \nabla x^{l-1})$$

where γf denotes the boundary values of f and ∇x^j , $0 \leq j \leq l-1$, denotes the vector of all partial derivatives $D^\alpha x$ of order $|\alpha| = j$. Let s' denote the total number of components of the vector $(\mathcal{K}x)(t)$. For given real-valued functions

$$a_\alpha(t), \quad t \in \partial G, \quad \text{with} \quad |\alpha| = l, \quad a_\alpha \in L_\infty(\partial G),$$

let $(\mathcal{J}x)(t)$ denote the real-valued function

$$(\mathcal{J}x)(t) = \sum_{|\alpha|=l} a_\alpha(t) \gamma D^\alpha x(t), \quad t \in \partial G.$$

Finally, let S be the completion of X_0 with respect to the norm

$$\|x\| = \|x\|_{W_p^l(G)} + \|\mathcal{J}x\|_{L_p(\partial G)}.$$

From Sobolev space theory we know that $\mathcal{K}x$ is defined on S . From the fact that S is the completion of X_0 with respect to the norm above we conclude that $\mathcal{J}x$ also is defined on S .

We are concerned with the problem of the minimum in S of the functional

$$I[x] = \int_{\partial G} g_0(t, (\mathcal{K}x)(t), (\mathcal{J}x)(t)) \, d\mu, \tag{4.3}$$

where g_0 is a given continuous function on the closed set $\overset{\circ}{M} = \partial G \times E^{s'} \times E^1$.

This problem is immediately reducible to the form (1.1-5) by taking

$$\mathcal{L}x = 0, \quad \mathcal{M}x = 0, \quad f = 0,$$

by taking

$$B(t) = E^{s'} \quad \text{for every } t \in \Gamma = \partial G,$$

and

$$V(t, \overset{\circ}{y}) = E^1 \quad \text{for every } (t, \overset{\circ}{y}) \in B = \partial G \times E^{s'}.$$

Thus there are no constraints on the control variable v , or $v \in V = E^1$. We now have a problem of the type (1.1-5) in which the functional to be minimized is

$$I[x, v] = \int_{\Gamma} g_0(t, (\mathcal{K}x)(t), v(t)) \, d\mu,$$

and the state equations (on the boundary) are

$$(\mathcal{J}x)(t) = v(t), \quad \mu\text{-a. e. on } \Gamma = \partial G.$$

In the present situation, and in terms of the notations of §§ 3 and 4, we have $r=s=m=0$, s' as above, $r'=m'=1$. The sets \tilde{R} are

$$\begin{aligned} \tilde{R}(t, \hat{y}) &= \{ \tilde{z} = (\hat{z}, z) \in E^2 \mid \hat{z} \geq g_0(t, \hat{y}, v), z = v \in E^1 \} \\ &= \{ (\hat{z}, v) \in E^2 \mid \hat{z} \geq g_0(t, \hat{y}, v), v \in E^1 \}, \end{aligned}$$

and thus they are convex if and only if $g_0(t, \hat{y}, v)$ is convex in v for every

$$(t, \hat{y}) \in B = \partial G \times E^{s'}.$$

Finally, if we assume that there is some real-valued continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, with

$$\begin{aligned} g_0(t, \hat{y}, v) &\geq \Phi(|v|) \quad \text{for all } (t, \hat{y}, v) \in \tilde{M}, \\ \Phi(\zeta)/\zeta &\rightarrow +\infty \quad \text{as } \zeta \rightarrow +\infty, \end{aligned} \tag{4.4}$$

then we know (CESARI [3b]) that the sets $\tilde{R}(t, \hat{y})$ satisfy property (Q) on B . Note that, if $\mu = \text{Min } \Phi(\zeta)$, $0 \leq \zeta < +\infty$, then relation $g_0(t, \hat{y}, v) \geq -\hat{\psi}(t)$ holds for all $(t, \hat{y}, v) \in \tilde{M}$ with $\hat{\psi} = -|\mu|$, a constant.

If $p > 1$, then for any sequence of elements $x, x_k, k=1, 2, \dots$, of S with $x_k \rightarrow x$ weakly in S as $k \rightarrow \infty$, certainly there is a subsequence $[k_\lambda]$ such that $\mathcal{J} x_{k_\lambda} \rightarrow \mathcal{J} x$ weakly in $L_p(\partial G)$ and $\mathcal{K} x_{k_\lambda} \rightarrow \mathcal{K} x$ strongly in $(L_p(\partial G))^{s'}$ as $\lambda \rightarrow \infty$. We may take

$$y(t) = (\mathcal{K} x)(t), \quad y_k(t) = (\mathcal{K} x_k)(t), \quad t \in G,$$

and

$$z(t) = v(t) = (\mathcal{J} x)(t), \quad z_k(t) = v_k(t) = (\mathcal{J} x_k)(t), \quad t \in \partial G, \quad k=1, 2, \dots$$

By lower closure theorem (3.i) with $\rho=r=0$, $\rho'=r'=1$, we derive now the following lower semicontinuity theorem concerning the integral $I[x]$:

(a) If $g_0(t, y, v)$ is continuous on $\tilde{M} = \partial G \times E^{s'} \times E^1$ and convex in v for every

$$(t, \hat{y}) \in B = \partial G \times E^{s'},$$

if there is a real-valued continuous function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$ such that (4.4) holds, then the functional $I[x]$ is lower semicontinuous in S .

By lower semicontinuous we mean here that, if $x, x_k \in S, k=1, 2, \dots, x_k \rightarrow x$ weakly in S as $k \rightarrow \infty$, and

$$\lim_{k \rightarrow \infty} I[x_k] = a < +\infty,$$

then $I[x] \leq a$.

The growth condition (4.4) can be disregarded in (a) if we know that the sets $\tilde{R}(t, \hat{y})$ satisfy property (Q) on B , and that

$$g_0(t, \hat{y}, v) \geq -\hat{\psi}(t) \quad \text{for all } (t, \hat{y}, v) \in \tilde{M} \quad \text{and some } \hat{\psi} \geq 0, \quad \hat{\psi} \in L_1(\partial G).$$

Also we note that under the assumed hypotheses $g_0(t, (\mathcal{K} x)(t), (\mathcal{J} x)(t))$ is certainly μ -measurable on ∂G and $\geq -\hat{\psi}(t)$ with $\hat{\psi} \in L(G)$. Thus the functional $I[x]$, or (4.3), is always defined in S , either finite, or $+\infty$.

Let Ω_N be any nonempty closed class of elements $x \in S$ with $\|x\| \leq N$ and $I[x]$ finite. If i denotes the infimum of $I[x]$ in Ω_N , then because $g_0 \geq -\hat{\psi}$, i is

finite, and in the search of the minimum of $I[x]$ in Ω_N , we can restrict ourselves to the subclass Ω_{N_0} of all $x \in \Omega_N$ with $I[x] \leq i + 1$. If $p > 1$, any such class Ω_{N_0} is obviously weakly compact in the topology of S . In particular, the class Ω_{N_0} of all $\|x\| \leq N$, $I[x] \leq i + 1$, is also weakly closed in the same topology. From (4.i) we may now derive the following assertion of existence:

(b) Under the conditions of (a) with $1 < p \leq +\infty$, the functional $I[x]$ has an absolute minimum in any nonempty closed class Ω_N .

As in (a), the growth condition (4.4) can be disregarded if we know that the sets $\tilde{R}(t, y)$ have property (Q) on B , and that

$$g_0(t, y, v) \geq -\dot{\psi}(t) \quad \text{for some } \dot{\psi} \geq 0, \dot{\psi} \in L_1(\partial G).$$

The following variants of the problem above may be of interest. Let r' denote the number of partial derivatives of order l , or $D^\alpha x$ with $|\alpha| = l$, and let $\mathcal{J}x$ now denote the operator (in X_0) $\mathcal{J}x = (D^\alpha x, |\alpha| = l)$. Let us consider the space S' obtained as the completion of X_0 with the norm

$$\|x\|' = \sum_{|\alpha| \leq l} \|D^\alpha x\|_{L_p(\partial G)}.$$

Now v is a r' -vector; and $g_0(t, y, v)$ denotes a continuous real-valued function on $\tilde{M} = \partial G \times E^{s'} \times E^{r'}$. Let us consider the corresponding integral $I[x]$, or (4.3), with the new definition of $\mathcal{J}x$ and g_0 . We still assume that a growth condition (4.4) holds. Again $\mathcal{K}x$ and $\mathcal{J}x$ are defined on S' , and statement (a) is valid without changes. If Ω'_N is any closed nonempty class of elements $x \in S'$ with $\|x\|' \leq N$, then also statement (b) holds without changes for $1 < p \leq +\infty$.

Let us retain now the last definition of $\mathcal{J}x$ and g_0 and assume that $p = 1$. Let i still denote the infimum of $I[x]$ in the class Ω'_N , and let Ω'_{N_0} be the class of all $x \in \Omega'_N$ with $I[x] \leq i + 1$. The growth condition (4.4) guarantees, by standard arguments (see, e.g., E. J. McSHANE, *Integration*, 1947, p. 176), that the r' components of $(\mathcal{J}x)(t)$, $t \in \partial G$, with $x \in \Omega'_{N_0}$ are equiabsolutely integrable on ∂G . Then, even for $p = 1$, statement (b) also holds, but the growth condition cannot be removed.

Finally, let us consider the case where g_0 satisfies a growth condition (4.4) and also a relation of the form

$$|\dot{y}| \leq c g_0(t, \dot{y}, v) + \psi_0(t) \quad \text{for all } (t, \dot{y}, v) \in \tilde{M},$$

some constant $c \geq 0$, and a function $\psi_0 \in L_1(\partial G)$. Now let Ω be any nonempty closed class of elements $x \in S'$ with $I[x] < +\infty$. If i denotes the infimum of $I[x]$ in Ω , let Ω_0 denote as usual the subclass of all $x \in \Omega$ with $I[x] \leq i + 1$. Then $\|x\|' \leq N$ for all elements $x \in \Omega_0$ and some constant N . Then existence theorem (b) holds in Ω for $p = 1$ provided the growth condition (4.4) holds and

$$|\dot{y}| \leq c g_0(t, \dot{y}, v) + \psi_0(t).$$

Example 2. Let G be a connected bounded open subset of the $\zeta\eta$ -plane E^2 . We take G to be of class K , so that the usual arc-length measure s is defined on $\Gamma = \partial G$. We are concerned with the minimum of the functional

$$I[x, u, v] = \iint_G f_0(\zeta, \eta, x, x_\zeta, x_\eta, u) d\zeta d\eta + \int_{\partial G} g_0(\zeta, \eta, \gamma x, \gamma x_\zeta, \gamma x_\eta, v) ds, \quad (4.5)$$

with state equations

$$\begin{aligned} x_{\zeta\zeta} + x_{\eta\eta} &= f(\zeta, \eta, x, x_{\zeta}, x_{\eta}, u) && \text{a. e. in } G, \\ a(\zeta, \eta)\gamma x + b(\zeta, \eta)\gamma x_{\zeta} + c(\zeta, \eta)\gamma x_{\eta} &= g(\zeta, \eta, \gamma x, \gamma x_{\zeta}, \gamma x_{\eta}, v) && \text{s-a. e. on } \partial G \end{aligned} \tag{4.6}$$

and constraints

$$\begin{aligned} u(\zeta, \eta) &\in U(\zeta, \eta, x, x_{\zeta}, x_{\eta}) && \text{a. e. in } G, \\ v(\zeta, \eta) &\in V(\zeta, \eta, \gamma x, \gamma x_{\zeta}, \gamma x_{\eta}) && \text{s-a. e. on } \partial G. \end{aligned} \tag{4.7}$$

Here γh denotes the boundary values of h . This problem is immediately written in the form (1.1–5) by taking

$$\begin{aligned} \mathcal{L}x &= x_{\zeta\zeta} + x_{\eta\eta}, & \mathcal{M}x &= (x, x_{\zeta}, x_{\eta}), \\ \mathcal{J}x &= a\gamma x + b\gamma x_{\zeta} + c\gamma x_{\eta}, & \mathcal{K}x &= (\gamma x, \gamma x_{\zeta}, \gamma x_{\eta}), \end{aligned}$$

$r=r'=1, s=s'=3$. We take for S the Sobolev space $S=W_p^2(G), p>1$, and we assume that the given functions a, b, c are of class $L_{\infty}(\partial G)$. Note that $(\mathcal{J}x)(\zeta, \eta)$ could be the normal derivative of x at $(\zeta, \eta) \in \partial G$ if only $a=0$, and b, c the direction cosines of the normal to ∂G at (ζ, η) (s-a.e. on ∂G). Also, we take $A(\zeta, \eta)=E^3, B(\zeta, \eta)=E^3$; hence $A=(\text{cl } G) \times E^3, B=(\partial G) \times E^3$.

For the sake of simplicity we assume $m=m'=1$, so that, if

$$y=(y^1, y^2, y^3) \in E^3, \quad \mathring{y}=(\mathring{y}^1, \mathring{y}^2, \mathring{y}^3) \in E^3,$$

then $U(\zeta, \eta, y)$ denotes a subset of E^1 for every $(\zeta, \eta, y) \in A$, and $V(\zeta, \eta, \mathring{y})$ a subset of E^1 for every $(\zeta, \eta, \mathring{y}) \in B$.

Finally, if M, \mathring{M} are the corresponding sets,

$$M \subset (\text{cl } G) \times E^3 \times E^1, \quad \mathring{M} \subset (\partial G) \times E^3 \times E^1,$$

then f_0, f are real-valued continuous functions on M , and g_0, g are real-valued continuous functions on \mathring{M} . We consider here the sets

$$\begin{aligned} \tilde{Q}(\zeta, \eta, y) &= \{(\mathring{z}, z) \in E^2 \mid \mathring{z} \geq f_0(\zeta, \eta, y, u), z = f(\zeta, \eta, y, u), u \in U(\zeta, \eta, y)\} \\ \tilde{R}(\zeta, \eta, \mathring{y}) &= \{(\mathring{z}, z) \in E^2 \mid \mathring{z} \geq g_0(\zeta, \eta, \mathring{y}, v), z = g(\zeta, \eta, \mathring{y}, v), v \in V(\zeta, \eta, \mathring{y})\} \end{aligned}$$

for every $(\zeta, \eta, y) \in A$ and for every $(\zeta, \eta, \mathring{y}) \in B$, respectively.

We note that if $x_k \rightarrow x$ weakly in $S=W_p^2(G)$, then there is certainly a subsequence $[k_{\lambda}]$ such that

$$\begin{aligned} \mathcal{L}x_{k_{\lambda}} &\rightarrow \mathcal{L}x && \text{weakly in } L_p(G), \\ \mathcal{M}x_{k_{\lambda}} &\rightarrow \mathcal{M}x && \text{strongly in } (L_p(G))^3, \\ \mathcal{J}x_{k_{\lambda}} &\rightarrow \mathcal{J}x && \text{strongly in } L_1(\partial G), \\ \mathcal{K}x_{k_{\lambda}} &\rightarrow \mathcal{K}x && \text{strongly in } L_1(\partial G). \end{aligned}$$

An admissible triple is now a triple (x, u, v) with $x \in W_p^2(G), u$ measurable on G, v s -measurable on ∂G , satisfying (4.6), (4.7), and such that

$$f_0(\zeta, \eta, x, x_{\zeta}, x_{\eta}, u) \in L_1(G) \quad \text{and} \quad g_0(\zeta, \eta, \gamma x, \gamma x_{\zeta}, \gamma x_{\eta}, v) \in L_1(\partial G).$$

From (4.i) with $\rho=r=1, \rho'=0$, we can now derive the following statement of existence:

(c) Let G be connected, bounded, open and of class K in E^2 , let M, \dot{M} be closed, let f_0, f be real-valued and continuous on M , and g_0, g real-valued and continuous on \dot{M} , and assume that the sets $\tilde{Q}(\zeta, \eta, y)$ satisfy property (Q) on $A=(cl\ G) \times E^3$ and the sets $\tilde{R}(\zeta, \eta, \dot{y})$ satisfy property (U) on $B=(\partial G) \times E^3$. Let us assume that there are functions

$$\psi(\zeta, \eta) \geq 0, \quad \psi \in L_1(G) \quad \text{and} \quad \dot{\psi}(\zeta, \eta) \geq 0, \quad \dot{\psi} \in L_1(\partial G)$$

such that

$$\begin{aligned} f_0(\zeta, \eta, y, u) &\geq -\psi(\zeta, \eta), \\ g_0(\zeta, \eta, \dot{y}, v) &\geq -\dot{\psi}(\zeta, \eta) \end{aligned}$$

for all $(\zeta, \eta, y, u) \in M$ and $(\zeta, \eta, \dot{y}, v) \in \dot{M}$, respectively. Let Ω be any closed non-empty class of admissible triples (x, u, v) for which the set $\{x\}_\Omega$ is norm bounded in $W_p^2(G), p > 1$; that is, there is a constant N such that $(x, u, v) \in \Omega$ implies $\|x\|_{W_p^2(G)} \leq N$. Then the cost functional (4.5) has an absolute minimum in Ω .

Remark 8. Many examples of optimization problems with distributed and boundary controls and state equations in the strong form are of the same general form of Example 2 above. The equation $(\mathcal{L}x)(t) = f(t, (\mathcal{M}x)(t), u(t)), t \in G$, is a partial differential equation (or a system), and the equation $(\mathcal{J}x)(t) = g(t, (\mathcal{N}x)(t), v(t)), t \in \partial G$, represents a certain set of constraints on the boundary values of the state variables. The conditions of theorem (4.i) are usually satisfied with $\rho=r$ and $\rho'=0$; that is, we require property (Q) on the sets Q and property (U) on the sets \tilde{R} . One more example is in ([4a], Section 5, Example 1). In this connection, Remark 6 may be relevant.

Example 3. In this example we wish to illustrate the use of the intermediate properties $Q(\rho)$. Let us consider the problem of minimizing the cost functional

$$I[x, u_1, u_2, v] = \iint_G (x^2 + x_\zeta^2 + x_\eta^2 + u_1^2 + u_2^2(1-u_2)^2) d\zeta d\eta + \int_\Gamma (\gamma x - 1)^2 ds,$$

with differential equations

$$\begin{aligned} x_{\zeta\zeta} + x_{\eta\eta} &= u_1, & x_\zeta + x_\eta &= u_2, & \text{a.e. in } G, \\ \gamma x_\zeta &= \cos v, & \gamma x_\eta &= \sin v, & \text{s-a.e. on } \Gamma = \partial G \end{aligned}$$

where $G = [(\zeta, \eta) | \zeta^2 + \eta^2 > 1]$, Γ is the boundary of G , where $\gamma x, \gamma x_\zeta, \gamma x_\eta$ denote the boundary values of x, x_ζ, x_η , and the control functions u_1, u_2, v have their values $(u_1, u_2) \in U = E^2, v \in V = E^1$. We wish to minimize I in a class Ω of systems (x, u_1, u_2, v) with u_1, u_2 measurable in G, v measurable on Γ, x any element of the Sobolev class $W_2^2(G)$ satisfying all relations above, satisfying an inequality $\|x_{\zeta\zeta}\|_2 + \|x_{\eta\eta}\|_2 + \|x_{\eta\eta}\|_2 \leq M$, and for which I is finite. Here the constant M is assumed sufficiently large so that Ω is not empty. We may well consider only those elements of Ω for which $I \leq N$ for some constant N . Here we have $f_0 \geq 0, g_0 \geq 0$,

and we can take $\psi=0, \dot{\psi}=0$. Also, we have

$$\begin{aligned} \mathcal{L}x &= (x_{\zeta\zeta} + x_{\eta\eta}, x_{\zeta} + x_{\eta}), & \mathcal{M}x &= (x, x_{\zeta}, x_{\eta}), & \mathcal{F}x &= (\gamma x_{\zeta}, \gamma x_{\eta}), & \mathcal{H}x &= \gamma x, \\ r &= 2, & s &= 3, & r' &= 2, & s' &= 1, & m &= 2, & m' &= 1. \end{aligned}$$

Then, for any sequence $[x_k]$ of elements $x \in \{x\}_{\Omega}$, certainly there is a subsequence, say still $[k]$ for the sake of simplicity, with $x_k \rightarrow x$ weakly in $S = W_2^2(G)$ for some $x \in S$, and

$$\begin{aligned} (\mathcal{L}x_k)^1 &\rightarrow (\mathcal{L}x)^1 && \text{weakly in } L_2(G), \\ (\mathcal{L}x_k)^2 &\rightarrow (\mathcal{L}x)^2 && \text{strongly in } L_2(G), \\ \mathcal{M}x_k &\rightarrow \mathcal{M}x && \text{strongly in } (L_2(G))^3, \\ \mathcal{F}x_k &\rightarrow \mathcal{F}x && \text{strongly in } (L_2(\Gamma))^2, \\ \mathcal{H}x_k &\rightarrow \mathcal{H}x && \text{strongly in } L_2(\Gamma). \end{aligned}$$

We consider here the sets

$$\begin{aligned} \tilde{Q}(y_1, y_2, y_3) &= [(z^0, z^1, z^2) \mid z^0 \geq y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2(1 - u_2)^2, \\ & \qquad \qquad \qquad z^1 = u_1, z^2 = u_2, (u_1, u_2) \in E^2], \\ \tilde{R}(y) &= [(z^0, z^1, z^2) \mid z^0 \geq (y - 1)^2, z^1 = \cos v, z^2 = \sin v, v \in E^1]. \end{aligned}$$

The sets $\tilde{Q} \subset E^3$ have property $Q(2)$, the sets $\tilde{R} \subset E^3$ have property $Q(1)$, and all have property (U) , or $Q(0)$. They are not convex, and do not have, therefore, property (Q) (precisely, the sets \tilde{Q} do not have property $Q(3)$, and the sets \tilde{R} have neither property $Q(2)$ nor property $Q(3)$). Nevertheless, the existence theorem (4.i) applies with $\rho=1, \rho'=0$, and the problem under consideration has an absolute minimum in Ω .

5. Another Existence Theorem for Optimization Problems with State Equations in the Strong Form

We now consider the case where the operators $\mathcal{L}, \mathcal{M}, \mathcal{F}, \mathcal{H}$ themselves depend on $x \in S$ and on suitable components of the controls, instead of depending on x alone as in § 4. Thus the theorem we shall prove here is, for practical purposes, more general than theorem (4.i). Nevertheless, we shall prove it as a corollary of theorem (4.i).

We shall consider here additional spaces of distributed and boundary controls, T and \tilde{T} , with elements $u \in T$ and $v \in \tilde{T}$, respectively, both T and \tilde{T} being given Banach spaces. (See Remark 1 after theorem (5.i) for a more general situation.)

It may happen that u and v are vector functions

$$u(t) = (u^{m+1}, \dots, u^{\bar{m}}), t \in G, \quad \text{and} \quad v(t) = (v^{m'+1}, \dots, v^{\bar{m}'}), t \in \Gamma,$$

and in this case the control $\tilde{u} = (u, u)$ is an \bar{m} -vector function on G , and $\tilde{v} = (v, v)$ is an \bar{m}' -vector function on Γ . In any case, we write our controls as $\tilde{u} = (u, u)$ and $\tilde{v} = (v, v)$.

We are concerned here with the problem of minimizing a functional

$$I[x, u, v] = \int_G f_0(t, (\mathcal{M}(x, u))(t), u(t)) dt + \int_\Gamma g_0(t, (\mathcal{K}(x, v))(t), v(t)) d\mu \quad (5.1)$$

subject to the state equations

$$(\mathcal{L}(x, u))(t) = f(t, (\mathcal{M}(x, u))(t), u(t)) \quad \text{a. e. in } G, \quad (5.2)$$

$$(\mathcal{J}(x, v))(t) = g(t, (\mathcal{K}(x, v))(t), v(t)) \quad \mu\text{-a. e. on } \Gamma, \quad (5.3)$$

and the constraints

$$(\mathcal{M}(x, u))(t) \in A(t), \quad u(t) \in U(t, (\mathcal{M}(x, u))(t)) \quad \text{a. e. in } G, \quad (5.4)$$

$$(\mathcal{K}(x, v))(t) \in B(t), \quad v(t) \in V(t, (\mathcal{K}(x, v))(t)) \quad \mu\text{-a. e. on } \Gamma. \quad (5.5)$$

As in § 1, G is a fixed bounded open set in E^n , $v \geq 1$, and Γ a given closed subset of ∂G on which we have a hyperarea measure μ . Let S, T, \mathring{T} be Banach spaces of elements x, u, v , and let $\mathcal{L}, \mathcal{M}, \mathcal{J}, \mathcal{K}$ be operators on $S \times T, S \times \mathring{T}$, not necessarily linear, with values in the following spaces:

$$\mathcal{L}: S \times T \rightarrow (L_1(G))^r, \quad \mathcal{J}: S \times \mathring{T} \rightarrow (L_1(\Gamma))^{r'}$$

$$\mathcal{M}: S \times T \rightarrow (L_1(G))^s, \quad \mathcal{K}: S \times \mathring{T} \rightarrow (L_1(\Gamma))^{s'}$$

where r, s, r', s' are given positive integers.

Let $A(t), A, U(t, y), M$ and $B(t), B, V(t, \mathring{y}), \mathring{M}$ be the sets defined in §§ 1 and 3,

$$M \subset (\text{cl } G) \times E^s \times E^m, \quad \mathring{M} \subset \Gamma \times E^{s'} \times E^{m'}, \quad U(t, y) \subset E^m, \quad V(t, \mathring{y}) \subset E^{m'};$$

let

$$f_0(t, y, u), \quad f(t, y, u) = (f_1, \dots, f_r)$$

be defined on M , and

$$g_0(t, \mathring{y}, v), \quad g(t, \mathring{y}, v) = (g_1, \dots, g_{r'})$$

be defined on \mathring{M} . Let $\tilde{Q}(t, y) \subset E^{r+1}$ be the sets defined in § 3 for every $(t, y) \in A$, and let $\tilde{R}(t, \mathring{y}) \subset E^{r'+1}$ be the analogous sets also defined in § 3 for every $(t, \mathring{y}) \in B$.

As in § 4, we denote by T the set of all measurable m -vector functions

$$u(t) = (u^1, \dots, u^m), \quad t \in G,$$

and by \mathring{T} the set of all measurable m' -vector functions

$$v(t) = (v^1, \dots, v^{m'}), \quad t \in \Gamma.$$

A triple $(x, \tilde{u}, \tilde{v})$, or system $(x, u, u; v, v)$, is said to be *admissible* (for the problem (5.1-5)) provided $x \in S, u \in T, v \in \mathring{T}, u \in T, v \in \mathring{T}$,

$$f_0(t, (\mathcal{M}(x, u))(t), u(t)) \in L_1(G), \quad g_0(t, (\mathcal{K}(x, v))(t), v(t)) \in L_1(\Gamma),$$

and relations (5.2-5) hold.

A class Ω of admissible systems is said to be *closed* if the following occurs: if

$$\begin{aligned} (x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k) &\in \Omega, \quad k=1, 2, \dots, \\ x_k &\rightarrow x \quad \text{weakly in } S, \\ u_k &\rightarrow u \quad \text{weakly in } T, \\ v_k &\rightarrow v \quad \text{weakly in } \overset{\circ}{T} \text{ as } k \rightarrow \infty, \\ \lim_{k \rightarrow \infty} I[x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k] &= a < +\infty, \end{aligned}$$

and there are admissible systems $(x, u, \mathbf{u}, v, \mathbf{v})$ such that $I[x, u, \mathbf{u}, v, \mathbf{v}] \leq a$, then there are also systems $(x, \bar{u}, \bar{\mathbf{u}}, \bar{v}, \bar{\mathbf{v}}) \in \Omega$ with $I[x, \bar{u}, \bar{\mathbf{u}}, \bar{v}, \bar{\mathbf{v}}] \leq a$.

For a class Ω of admissible system $(x, u, \mathbf{u}, v, \mathbf{v})$ we denote by $\{x\}_\Omega, \{\mathbf{u}\}_\Omega, \{v\}_\Omega$ the sets defined as $\{x\}_\Omega$ in § 4.

If ρ, ρ' are any two integers, $0 \leq \rho \leq r, 0 \leq \rho' \leq r'$, we denote by $(C'_{\rho\rho'})$ the following closure property of the operators $\mathcal{L}, \mathcal{M}, \mathcal{F}, \mathcal{H}$:

$(C'_{\rho\rho'})$ For every sequence $x, u, v, x_k, u_k, v_k, k=1, 2, \dots$, of elements $x \in S, u \in T, v \in \overset{\circ}{T}$,

$$x_k \in \{x\}_\Omega \subset S, \quad u_k \in \{\mathbf{u}\}_\Omega \subset T, \quad v_k \in \{v\}_\Omega \subset \overset{\circ}{T},$$

with

$$\begin{aligned} x_k &\rightarrow x \quad \text{weakly in } S, \\ u_k &\rightarrow u \quad \text{weakly in } T, \\ v_k &\rightarrow v \quad \text{weakly in } \overset{\circ}{T}, \end{aligned}$$

there is some subsequence $[k_\lambda]$ such that, as $\lambda \rightarrow \infty$, we have

$$\begin{aligned} (\mathcal{L}(x_{k_\lambda}, \mathbf{u}_{k_\lambda}))^i &\rightarrow (\mathcal{L}(x, \mathbf{u}))^i \quad \text{weakly in } L_1(G), \quad i=1, \dots, \rho, \\ (\mathcal{L}(x_{k_\lambda}, \mathbf{u}_{k_\lambda}))^i &\rightarrow (\mathcal{L}(x, \mathbf{u}))^i \quad \text{strongly in } L_1(G), \quad i=\rho+1, \dots, r, \\ (\mathcal{M}(x_{k_\lambda}, \mathbf{u}_{k_\lambda}))^j &\rightarrow (\mathcal{M}(x, \mathbf{u}))^j \quad \text{strongly in } L_1(G), \quad j=1, \dots, s, \\ (\mathcal{F}(x_{k_\lambda}, v_{k_\lambda}))^i &\rightarrow (\mathcal{F}(x, v))^i \quad \text{weakly in } L_1(\Gamma), \quad i=1, \dots, \rho', \\ (\mathcal{F}(x_{k_\lambda}, v_{k_\lambda}))^i &\rightarrow (\mathcal{F}(x, v))^i \quad \text{strongly in } L_1(\Gamma), \quad i=\rho'+1, \dots, r', \\ (\mathcal{H}(x_{k_\lambda}, v_{k_\lambda}))^j &\rightarrow (\mathcal{H}(x, v))^j \quad \text{strongly in } L_1(\Gamma), \quad j=1, \dots, s'. \end{aligned}$$

(5.i) **Existence Theorem.** Let G be bounded and measurable, $A, B, M, \overset{\circ}{M}$ closed,

$$f_0(t, y, u), f(t, y, u) = (f_1, \dots, f_r)$$

continuous on M ,

$$g_0(t, \overset{\circ}{y}, v), g(t, \overset{\circ}{y}, v) = (g_1, \dots, g_{r'})$$

continuous on $\overset{\circ}{M}$, and assume that for given integers

$$\rho, \rho', \quad 0 \leq \rho \leq r, \quad 0 \leq \rho' \leq r',$$

the sets $\overset{\circ}{Q}(t, y)$ have property $Q(\rho+1)$ on A , and the sets $\overset{\circ}{R}(t, \overset{\circ}{y})$ have property $Q(\rho'+1)$ on B . Let us assume that there are functions

$$\psi(t) \geq 0, \quad t \in G, \quad \psi \in L_1(G), \quad \text{and} \quad \overset{\circ}{\psi}(t) \geq 0, \quad t \in \Gamma, \quad \overset{\circ}{\psi} \in L_1(\Gamma),$$

such that

$$f_0(t, y, u) \geq -\psi(t) \quad \text{for all } (t, y, u) \in M,$$

and

$$g_0(t, \dot{y}, v) \geq -\dot{\psi}(t) \quad \text{for all } (t, \dot{y}, v) \in \dot{M}.$$

Let Ω be a nonempty closed class of admissible systems (x, u, v) such that $\{x\}_\Omega, \{u\}_\Omega, \{v\}_\Omega$ are weakly sequentially relatively compact, and let us assume that the operators $\mathcal{L}, \mathcal{M}, \mathcal{F}, \mathcal{K}$ satisfy the closure property $(C'_{\rho\rho'})$. Then the functional (5.1) has an absolute minimum in Ω .

Proof. Apply the existence theorem (4.i) with S replaced by $S \times T \times \dot{T}$ and x replaced by (x, u, v) .

Remark 1. Considerations similar to those of Remarks 1–6 after the existence theorem (4.i) apply here as well. In particular, $S, T,$ and \dot{T} need not be Banach spaces with the weak topology, but only topological spaces $(S, \sigma), (T, \tau), (\dot{T}, \dot{\tau})$, that is, spaces S, T, \dot{T} for which certain topologies $\sigma, \tau, \dot{\tau}$ have been chosen. Also, the spaces S, T, \dot{T} need not be linear spaces. Accordingly, in the existence theorem (5.i) we shall require that the convergence $x_k \rightarrow x$ in $(S, \sigma), u_k \rightarrow u$ in $(T, \tau),$ and $v_k \rightarrow v$ in $(\dot{T}, \dot{\tau})$ imply the convergence in $L_1(G)$ and $L_1(\Gamma)$ stated in $(C'_{\rho\rho'})$. Accordingly, we shall require in (5.i) that the sets $\{x\}_\Omega, \{u\}_\Omega, \{v\}_\Omega$ are sequentially relatively compact in $(S, \sigma), (T, \tau), (\dot{T}, \dot{\tau})$, respectively. Examples of these situations will occur in § 6.

Example 1 (a problem of evolution in strong form). Let G be a subset of the $t\tau$ -space $E^{v+1}, \tau = (\tau^1, \dots, \tau^v)$, of the form $G = (0, T) \times G'$, where G' is an open bounded connected subset of E^v of class K_l . Thus, $v+1$ replaces v , and $\Gamma = \partial G$ is made up of three parts

$$\Gamma_1 = \{0\} \times \text{cl } G', \quad \Gamma_2 = [0, T] \times \partial G', \quad \Gamma_3 = \{T\} \times \text{cl } G'.$$

On Γ_1 and Γ_3 we have the Lebesgue v -dimensional measure, or $|\cdot|_v$ (and we shall use the symbol $d\tau$ in integration). In Γ_2 we have the product measure $\sigma = |\cdot|_1 \times \mu$ of the one-dimensional measure on $[0, T]$ and of the hyperarea μ on the boundary $\partial G'$ of G' (and we shall use the symbol $dt d\mu$ in integration). Given a function x in G , we shall denote by γx the boundary values of x on $\Gamma = \partial G$, and specifically we shall denote by $\gamma_i x$ the boundary values of x on $\Gamma_i, i = 1, 2, 3$. We shall denote by T, \dot{T}_i the families of all measurable functions on $G, \Gamma_i, i = 1, 2, 3$, respectively.

We shall denote by $S_p^l(G), 1 < p \leq +\infty, l \geq 1$, the space of all real-valued functions $x(t, \tau), (t, \tau) \in G$ such that $\partial x / \partial t$ and all

$$D_\tau^\alpha x, \alpha = (\alpha_1, \dots, \alpha_v), \quad 0 \leq |\alpha| \leq l,$$

exist as generalized derivatives, and are all in $L_p(G)$. We shall make $S_p^l(G)$ a Banach space by means of the norm

$$\|x\| = \|x\|_{S_p^l(G)} = \left(\int_G |\partial x / \partial t|^p dt d\tau \right)^{1/p} + \sum_{0 \leq |\alpha| \leq l} \left(\int_G |D_\tau^\alpha x|^p dt d\tau \right)^{1/p}.$$

These spaces $S_p^l(G)$ have been studied by J. P. AUBIN [1], who has proved weak compactness properties similar to those for the Sobolev spaces. Each element

$x \in S_p^l(G)$ possesses boundary values

$$\begin{aligned} & \gamma_1 x \in L_p(\Gamma_1), \quad \gamma_3 x \in L_p(\Gamma_3) \quad \text{on } \Gamma_1, \Gamma_3 \\ \text{respectively, and all} & \\ & \gamma_2 D_\tau^\alpha x \in L_p(\Gamma_2), \quad 0 \leq |\alpha| \leq l-1, \quad \text{on } \Gamma_2. \end{aligned}$$

We are concerned with the minimum of a functional

$$\begin{aligned} I[x, u, \mathbf{u}, v_2, \mathbf{v}_2, v_3] = & \int_G f_0(t, \tau, (\mathcal{M}x)(t, \tau), u(t, \tau)) dt d\tau \\ & + \int_{\Gamma_2} g_0(t, \tau, (\mathcal{K}x)(t, \tau), v_2(t, \tau)) dt d\mu + \int_{\Gamma_3} g_0(\tau, (\mathcal{K}x)(T, \tau), v_3(\tau)) d\tau, \end{aligned}$$

with state equations (in the strong form)

$$\begin{aligned} \mathcal{L}(x, \mathbf{u})(t, \tau) = & f(t, \tau, (\mathcal{M}x)(t, \tau), u(t, \tau)) \quad \text{a.e. in } G, \\ \mathcal{J}(x, \mathbf{v}_2)(t, \tau) = & g(t, \tau, (\mathcal{K}x)(t, \tau), v_2(t, \tau)) \quad \sigma\text{-a.e. on } \Gamma_2, \end{aligned}$$

and constraints

$$\begin{aligned} (\mathcal{M}x)(t, \tau) \in & A(t, \tau), \quad u(t, \tau) \in U(t, \tau, (\mathcal{M}x)(t, \tau)) \quad \text{a.e. in } G, \\ (\mathcal{K}x)(t, \tau) \in & B_2(t, \tau), \quad v_2(t, \tau) \in V_2(t, \tau, (\mathcal{K}x)(t, \tau)) \quad \sigma\text{-a.e. on } \Gamma_2, \\ (\mathcal{K}x)(T, \tau) \in & B_3(\tau), \quad v_3(\tau) \in V_3(\tau, (\mathcal{K}x)(T, \tau)) \quad \text{a.e. on } \Gamma_3. \end{aligned}$$

On Γ_1 we have actually the further control $\gamma_1 x = v_1(\tau)$, with $v_1(\tau) \in L_p(\Gamma_1)$, that is, the initial values $\gamma_1 x$ are arbitrary. In other words, $\gamma_1 x$ is free (in $L_p(\Gamma_1)$). The optimal elements $x \in S = S_p^l(G)$ will determine the optimal initial values

$$v_1(\tau) = \gamma_1 x \in L_p(\Gamma_1).$$

We take now $x \in S = S_p^l(G)$, $u \in T$, $v_2 \in \mathring{T}_2$, $v_3 \in \mathring{T}_3$, and $\mathbf{u} \in \mathbf{T}$, $\mathbf{v}_2 \in \mathring{\mathbf{T}}_2$, where T and \mathring{T}_2 are weakly closed subsets of $L_q(G)$ and $L_q(\Gamma_2)$, respectively, and both are norm bounded (in the norms of $L_q(G)$ and $L_q(\Gamma_2)$). Above \mathcal{M} , \mathcal{K} , \mathcal{L} , \mathcal{J} are operators, not necessarily linear, say

$$\mathcal{M} : S \rightarrow (L_p(G))^s, \quad \mathcal{K} : S \rightarrow (L_p(\Gamma))^s, \quad \mathcal{L} : S \times T \rightarrow (L_p(G))^r, \quad \mathcal{J} : S \times \mathring{T}_2 \rightarrow (L_p(\Gamma_2))^r.$$

We assume that

$$\begin{aligned} x_k \rightarrow x & \quad \text{weakly in } S, \\ \mathbf{u}_k \rightarrow \mathbf{u} & \quad \text{weakly in } L_q(G), \\ v_{2k} \rightarrow v_2 & \quad \text{weakly in } L_q(\Gamma_2) \end{aligned}$$

implies that

$$\begin{aligned} \mathcal{M} x_k \rightarrow \mathcal{M} x & \quad \text{strongly in } (L_1(G))^s, \\ \mathcal{K} x_k \rightarrow \mathcal{K} x & \quad \text{strongly in } (L_1(\Gamma))^s, \\ \mathcal{L}(x_k, \mathbf{u}_k) \rightarrow \mathcal{L}(x, \mathbf{u}) & \quad \text{weakly in } (L_1(G))^r, \\ \mathcal{J}(x_k, v_{2k}) \rightarrow \mathcal{J}(x, v_2) & \quad \text{weakly in } L_1(\Gamma_2)^r. \end{aligned}$$

Theorem (5.1) now applies easily.

For instance, we can take $r=r'=1, m=m'=1, T=0, p=q=2, l=2$, and

$$\mathcal{L}x = \partial x / \partial t - \sum_{i=1}^v \partial^2 x / (\partial \tau^i)^2,$$

$$\mathcal{J}(x, v_2) = \sum_{i=1}^v a_i(t, \tau) \gamma_2(\partial x / \partial \tau^i) + v_2(t, \tau) \gamma_2 x(t, \tau).$$

Here $T_2 \subset L_2(\Gamma_2)$, and the coefficients a_i are given elements of $L_2(\Gamma_2)$.

For instance, we may also take

$$\mathcal{M}x = (x, \nabla_\tau x), \quad s = v + 1,$$

$$\mathcal{K}x = \partial x / \partial n, \quad s' = 1, \text{ on } \Gamma_2; \quad \mathcal{H}x = x, \quad s' = 1, \text{ on } \Gamma_3,$$

$$f_0 = t^2 + |\tau|^2 + x^2 + |\nabla_\tau x|^2 + |u| - 1 \geq -1,$$

$$f = t + \sum_i \tau^i + x + \sum_i \partial x / \partial \tau^i + u + 2^{-1} |u|,$$

$$g_0 = (\partial x / \partial n)^2 + v_2^2 \geq 0 \text{ on } \Gamma_2, \quad g_0 = x^2 - 1 \geq -1 \text{ on } \Gamma_3,$$

$$g = x(\partial x / \partial n) + v_2 \text{ on } \Gamma_2,$$

$v_3 = 0, A = E^{v+1}, B_2 = E^1, B_3 = E^1, U = E^1, v_2 = E^1, V_3 = \{0\}$, and choose for Ω the set of all admissible systems $x, u, \mathbf{u}, v_2, \mathbf{v}_2, v_3$ with $\|x\|_{S_2^2(G)} \leq N$ for a sufficiently large N .

Example 2. Using the same notations as in Example 1 above, we may take here $x \in S = S_2^2(G), u \in T, v_2 \in \overset{\circ}{T}_2, v_3 \in \overset{\circ}{T}_3$, and $\mathbf{u} \in T, \mathbf{v}_2 \in \overset{\circ}{T}_2$, where $T, \overset{\circ}{T}_2$ are weakly closed subsets of $L_2(G)$ and $L_2(\Gamma_2)$, respectively, and both are norm bounded (in the norms of $L_2(G)$ and $L_2(\Gamma_2)$). Here $\mathcal{M}, \mathcal{K}, \mathcal{L}, \mathcal{J}$ are operators, not necessarily linear, say

$$\mathcal{M}: S \rightarrow (L_1(G))^s, \quad \mathcal{K}: S \rightarrow (L_1(\Gamma_2))^{s'}, \quad \mathcal{L}: S \times T \rightarrow (L_1(G))^r, \quad \mathcal{J}: S \times \overset{\circ}{T}_2 \rightarrow (L_1(\Gamma_2))^{r'}.$$

We assume that

$$\begin{aligned} x_k &\rightarrow x && \text{weakly in } S, \\ \mathbf{u}_k &\rightarrow \mathbf{u} && \text{weakly in } L_2(G), \\ \mathbf{v}_{2k} &\rightarrow \mathbf{v}_2 && \text{weakly in } L_2(\Gamma_2) \end{aligned}$$

implies that

$$\begin{aligned} \mathcal{M}x_k &\rightarrow \mathcal{M}x && \text{strongly in } (L_1(G))^s, \\ \mathcal{K}x_k &\rightarrow \mathcal{K}x && \text{strongly in } (L_1(\Gamma_2))^{s'}, \\ \mathcal{L}(x_k, \mathbf{u}_k) &\rightarrow \mathcal{L}(x, \mathbf{u}) && \text{weakly in } (L_1(G))^r, \\ \mathcal{J}(x_k, \mathbf{v}_{2k}) &\rightarrow \mathcal{J}(x, \mathbf{v}_2) && \text{weakly in } (L_1(\Gamma_2))^{r'}. \end{aligned}$$

Theorem (5.i) now applies.

For instance, we may take $r=r'=1, m=m'=1, T=\{0\}$, and

$$\mathcal{L}x = \partial x / \partial t + x(\text{grad } x) - b \sum_{i=1}^v \partial^2 x / \partial \tau^{i2},$$

$$\mathcal{J}(x, v_2) = \sum_{i=1}^v a_i(t, \tau) \gamma_2(\partial x / \partial \tau^i) + v_2(t, \tau) \gamma_2 x(t, \tau).$$

Here $T_2 \subset L_2(\Gamma_2)$, the coefficients a_i are given elements of $L_2(\Gamma_2)$, and $b > 0$ is a given constant. We can take $\mathcal{M}, \mathcal{K}, f_0, f, g_0, g$ as in Example 1.

6. An Existence Theorem for Optimization Problems with State Equations in the Weak Form

We shall now consider the case, mentioned in § 1, where the equations of state (1.2), (1.3) are written in the weak form as is customary in partial differential equation theory.

We shall use the general notation of the previous sections. In addition, let W denote a normed space of test functions $w = (w_1, w_2)$, where

$$w_1 \in (L_q(G))^r, \quad w_2 \in (L_q(\Gamma))^r, \quad \text{and} \quad p^{-1} + q^{-1} = 1,$$

with $1 \leq p \leq +\infty, 1 \leq q \leq +\infty$, and let the usual conventions hold. We shall assume that the norms $\|w_1\|_q$ of w_1 in $(L_q(G))^r$ and $\|w_2\|_q$ of w_2 in $(L_q(\Gamma))^r$ are connected with the norm $\|w\|_W$ of $w = (w_1, w_2)$ in W by a relation of the form

$$\|w_1\|_q + \|w_2\|_q \leq K \|w\|_W \tag{6.1}$$

where K is a constant. We shall denote by W^* the dual space of W . We shall deal here with only three operators, \mathcal{M}, \mathcal{K} as in § 5, and \mathcal{J} replacing both \mathcal{L} and \mathcal{I} :

$$\mathcal{M}: S \times T \rightarrow (L_1(G))^s, \quad \mathcal{K}: S \times \overset{\circ}{T} \rightarrow (L_1(\Gamma))^s, \quad \mathcal{J}: S \times T \times \overset{\circ}{T} \rightarrow W^*.$$

For every $x \in S, u \in T, u \in \overset{\circ}{T}, v \in \overset{\circ}{T}$, we consider now the operator h , or $h(x, u, u, v, v), h: W \rightarrow E^1$, defined by

$$hw = \int_G f(t, (\mathcal{M}(x, u))(t), u(t)) w_1(t) dt + \int_\Gamma g(t, (\mathcal{K}(x, v))(t), v(t)) w_2(t) d\mu$$

where $f w_1$ and $g w_2$ denote inner products in E^r and $E^{r'}$, respectively.

Instead of state equations (5.2), (5.3) we shall now consider a single state equation in the weak form

$$\mathcal{J} = h, \tag{6.2}$$

that is,

$$\mathcal{J}w = hw \quad \text{for all } w \in W \tag{6.3}$$

or, specifically,

$$\begin{aligned} \mathcal{J}(x, u, v)(w_1, w_2) &= \int_G f(t, (\mathcal{M}(x, u))(t), u(t)) w_1(t) dt \\ &+ \int_\Gamma g(t, (\mathcal{K}(x, v))(t), v(t)) w_2(t) d\mu \end{aligned} \tag{6.4}$$

for all $(w_1, w_2) \in W$.

Note that relation (6.1) implies

$$\begin{aligned} (L_q(G))^r \times (L_q(\Gamma))^r &\supset W, \\ (L_p(G))^r \times (L_p(\Gamma))^r &\subset W^*, \end{aligned}$$

and, as mentioned, $\mathcal{J}: S \times T \times \overset{\circ}{T} \rightarrow W^*$. In most applications, however, we shall have

$$\mathcal{J}: S \times T \times \overset{\circ}{T} \rightarrow (L_p(G))^r \times (L_p(\Gamma))^r \subset W^*,$$

and the actual determination of W^* will be irrelevant.

We understand here that the present single equation of state (6.3) is the weak form of equations of state (5.2), (5.3). In other words, in any particular situation \mathcal{J} and W must be chosen so that any solution of the equations (5.2), (5.3) (strong form) is necessarily a solution of (6.3).

Thus we are interested here in the problem of minimizing of the functional

$$I[x, u, v] = \int_G f_0(t, (\mathcal{M}(x, u))(t), u(t)) dt + \int_\Gamma g_0(t, (\mathcal{K}(x, v))(t), v(t)) d\mu, \quad (6.5)$$

with state equation (in the weak form)

$$\mathcal{J}w = hw \quad \text{for all } w \in W, \quad (6.6)$$

and constraints

$$(\mathcal{M}(x, u))(t) \in A(t), \quad u(t) \in U(t, (\mathcal{M}(x, u))(t)) \quad \text{a. e. in } G, \quad (6.7)$$

$$(\mathcal{K}(x, v))(t) \in B(t), \quad v(t) \in V(t, (\mathcal{K}(x, v))(t)) \quad \mu\text{-a. e. on } \Gamma. \quad (6.8)$$

In the present situation we shall require a suitable growth condition, condition (H):

(H) For $p=1$ we assume that, given $\varepsilon > 0$, there are functions $\phi_\varepsilon \geq 0$, $\phi_\varepsilon \in L_1(G)$, and $\phi'_\varepsilon \geq 0$, $\phi'_\varepsilon \in L_1(\Gamma)$, such that

$$|f(t, y, u)| \leq \phi_\varepsilon(t) + \varepsilon f_0(t, y, u) \quad \text{for all } (t, y, u) \in M,$$

$$|g(t, \overset{\circ}{y}, v)| \leq \phi'_\varepsilon(t) + \varepsilon g_0(t, \overset{\circ}{y}, v) \quad \text{for all } (t, \overset{\circ}{y}, v) \in \overset{\circ}{M}.$$

If $p > 1$ we assume that there are functions $\phi_0 \geq 0$, $\phi_0 \in L_1(G)$, and $\phi'_0 \geq 0$, $\phi'_0 \in L_1(\Gamma)$, and constants $a > 0$, $b > 0$, such that

$$|f(t, y, u)|^p \leq \phi_0(t) + a f_0(t, y, u) \quad \text{for all } (t, y, u) \in M,$$

$$|g(t, \overset{\circ}{y}, v)|^p \leq \phi'_0(t) + b g_0(t, \overset{\circ}{y}, v) \quad \text{for all } (t, \overset{\circ}{y}, v) \in \overset{\circ}{M}.$$

This condition, for $p=1$, has been systematically used by CESARI [3be] as a suitable extension of previous more restrictive growth hypotheses used by TONELLI and MCSHANE.

A triple $(x, \tilde{u}, \tilde{v})$, or system (x, u, v) , is now said to be admissible provided $x \in S$, $u \in T$, $u \in \overset{\circ}{T}$, $v \in \overset{\circ}{T}$, $v \in \overset{\circ}{T}$, relations (6.6), (6.7), (6.8) hold,

$$f_0(t, (\mathcal{M}(x, u))(t), u(t)) \in L_1(G), \quad g_0(t, (\mathcal{K}(x, v))(t), v(t)) \in L_1(\Gamma).$$

Also, we require that

$$f(t, (\mathcal{M}(x, u))(t), u(t)) \in (L_p(G))^r$$

and that

$$g(t, (\mathcal{K}(x, v))(t), v(t)) \in (L_p(\Gamma))^r.$$

In the existence theorem below, however, this last requirement will be a consequence of property (H). We shall now consider nonempty closed classes Ω of admissible systems (x, u, v) , where the definition of closedness is analogous to the ones in §§ 4 and 5.

Finally, we shall need the following closure property (C) of the operators $\mathcal{M}, \mathcal{K}, \mathcal{J}$:

(C) For every sequence $x, u, v, x_k, u_k, v_k, k=1, 2, \dots$, of elements $x \in S, u \in T, v \in \overset{\circ}{T}$,

$$x_k \in \{x\}_\Omega \subset S, \quad u_k \in \{u\}_\Omega \subset T, \quad v_k \in \{v\}_\Omega \subset \overset{\circ}{T},$$

with

$$x_k \rightarrow x \text{ weakly in } S,$$

$$u_k \rightarrow u \text{ weakly in } T,$$

$$v_k \rightarrow v \text{ weakly in } \overset{\circ}{T},$$

there is some subsequence $[k_\lambda]$ such that, as $\lambda \rightarrow \infty$, we have

$$\mathcal{M}(x_{k_\lambda}, u_{k_\lambda}) \rightarrow \mathcal{M}(x, u) \text{ strongly in } (L_1(G))^s,$$

$$\mathcal{K}(x_{k_\lambda}, v_{k_\lambda}) \rightarrow \mathcal{K}(x, v) \text{ strongly in } (L_1(\Gamma))^s,$$

$$\mathcal{J}(x_{k_\lambda}, u_{k_\lambda}, v_{k_\lambda}) w \rightarrow \mathcal{J}(x, u, v) w \text{ for every } w \in W. \tag{6.9}$$

The hypothesis concerning \mathcal{J} above can be reworded by saying that

$$\mathcal{J}(x_{k_p}, u_{k_p}, v_{k_p}) \rightarrow \mathcal{J}(x, u, v)$$

in the weak star topology on W^* .

(6.i) **Existence Theorem.** Let G be bounded and measurable, $A, B, M, \overset{\circ}{M}$ closed,

$$f_0(t, y, u), f(t, y, u) = (f_1, \dots, f_r)$$

continuous on M ,

$$g_0(t, \overset{\circ}{y}, v), g(t, \overset{\circ}{y}, v) = (g_1, \dots, g_r)$$

continuous on $\overset{\circ}{M}$, and assume that the sets $\overset{\circ}{Q}(t, y)$ have property (Q) on A , and the sets $\overset{\circ}{R}(t, \overset{\circ}{y})$ have property (Q) on B . Let us assume that there are functions

$$\psi(t) \geq 0, t \in G, \psi \in L_1(G), \text{ and } \overset{\circ}{\psi}(t) \geq 0, t \in \Gamma, \overset{\circ}{\psi} \in L_1(\Gamma),$$

such that

$$f_0(t, y, u) \geq -\psi(t) \text{ for all } (t, y, u) \in M,$$

and

$$g_0(t, \overset{\circ}{y}, v) \geq -\overset{\circ}{\psi}(t) \text{ for all } (t, \overset{\circ}{y}, v) \in \overset{\circ}{M}.$$

Let us assume that relation (6.1) holds, and that growth condition (H) is satisfied. Let Ω be a nonempty closed class of admissible systems (x, u, v) such that $\{x\}_\Omega, \{u\}_\Omega, \{v\}_\Omega$ are weakly sequentially relatively compact, and let us assume that the operators $\mathcal{M}, \mathcal{K}, \mathcal{J}$ satisfy closure property (C) above. Then the functional (6.5), or $I[x, u, v]$, has an absolute minimum in Ω .

Proof. As usual let i be the infimum of $I[x, u, \mathbf{u}, v, \mathbf{v}]$ in the class Ω . Since $f_0 \geq -\psi, g_0 \geq -\dot{\psi}$ and Ω is nonempty, i is finite. Let

$$(x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k), \quad k=1, 2, \dots,$$

be a sequence with

$$I[x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k] \rightarrow i \quad \text{as } k \rightarrow \infty,$$

and we may well assume that

$$i \leq I[x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k] \leq i + k^{-1} \leq i + 1, \quad k=1, 2, \dots$$

Since we have assumed that the sets $\{x\}_\Omega, \{u\}_\Omega, \{\mathbf{u}\}_\Omega$ are weakly sequentially relatively compact, there is a subsequence, say still $[k]$ for the sake of simplicity, such that $x_k \rightarrow x$ weakly in $S, u_k \rightarrow u$ weakly in $T, v_k \rightarrow v$ weakly in \dot{T} as $k \rightarrow \infty$. We may even assume that the subsequence has been so chosen that limit relations (6.9) hold. Let

$$\begin{aligned} z_k(t) &= f(t, \mathcal{M}(x_k, \mathbf{u}_k)(t), u_k(t)), & t \in G, \\ \dot{z}_k(t) &= g(t, \mathcal{K}(x_k, \mathbf{v}_k)(t), v_k(t)), & t \in \Gamma, \quad k=1, 2, \dots \end{aligned}$$

By the growth condition (H) and

$$I[x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k] \leq i + 1 \quad \text{for all } k,$$

we see that, if $p > 1$, the functions $z_k(t)$ and $\dot{z}_k(t), k=1, 2, \dots$, are equibounded in the norms of $(L_p(G))^r$ and $(L_p(\Gamma))^r$, respectively. If $p=1$ it follows from an argument of CESARI [3be] that the same functions $z_k(t)$ and $\dot{z}_k(t)$ are equiabsolutely integrable in G and Γ , respectively. In any case, there exists a subsequence, say still $[k]$ for the sake of simplicity, and elements

$$z(t), t \in G, z \in (L_p(G))^r, \quad \text{and} \quad \dot{z}(t), t \in \Gamma, \dot{z} \in (L_p(\Gamma))^r,$$

such that

$$z_k \rightarrow z \quad \text{weakly in } (L_p(G))^r$$

and

$$\dot{z}_k \rightarrow \dot{z} \quad \text{weakly in } (L_p(\Gamma))^r.$$

In other words,

$$\begin{aligned} f(t, \mathcal{M}(x_k, \mathbf{u}_k)(t), u_k(t)) &\rightarrow z(t) \quad \text{weakly in } (L_p(G))^r, \\ g(t, \mathcal{K}(x_k, \mathbf{v}_k)(t), v_k(t)) &\rightarrow \dot{z}(t) \quad \text{weakly in } (L_p(\Gamma))^r, \end{aligned} \tag{6.10}$$

as $k \rightarrow \infty$, while

$$\lim_{k \rightarrow \infty} I[x_k, u_k, \mathbf{u}_k, v_k, \mathbf{v}_k] = i, \tag{6.11}$$

$$\begin{aligned} (\mathcal{M}(x_k, \mathbf{u}_k))(t) \in A(t), \quad u_k(t) \in U(t, (\mathcal{M}(x_k, \mathbf{u}_k))(t)) & \quad \text{a.e. in } G, \\ (\mathcal{K}(x_k, \mathbf{v}_k))(t) \in B(t), \quad v_k(t) \in V(t, (\mathcal{K}(x_k, \mathbf{v}_k))(t)) & \quad \mu\text{-a.e. on } \Gamma. \end{aligned} \tag{6.12}$$

If $w = (w_1, w_2)$ is any element of W then by relation (6.1) we know that

$$w_1 \in (L_q(G))^r \quad \text{and} \quad w_2 \in (L_q(\Gamma))^r;$$

hence

$$\int_G z_k(t) w_1(t) dt \rightarrow \int_G z(t) w_1(t) dt, \quad \int_\Gamma \dot{z}_k(t) w_2(t) d\mu \rightarrow \int_\Gamma \dot{z}(t) w_2(t) d\mu$$

as $k \rightarrow \infty$. Finally, by the definition of the operator h , we have

$$h(x_k, u_k, u_k, v_k, v_k) w \rightarrow \int_G z(t) w_1(t) dt + \int_\Gamma \overset{\circ}{z}(t) w_2(t) d\mu \tag{6.13}$$

as $k \rightarrow \infty$, for every $w = (w_1, w_2) \in W$. By hypothesis we have also

$$\mathcal{J}(x_k, u_k, v_k) w \rightarrow \mathcal{J}(x, u, v) w \tag{6.14}$$

as $k \rightarrow \infty$, again for every $w \in W$. Here each system

$$(x_k, u_k, u_k, v_k, v_k), \quad k = 1, 2, \dots,$$

is admissible; hence the first members of (6.13) and (6.14) are equal for every $w \in W$. From (6.13) and (6.14) by comparison we obtain

$$\mathcal{J}(x, u, v) w = \int_G z(t) w_1(t) dt + \int_\Gamma \overset{\circ}{z}(t) w_2(t) d\mu \tag{6.15}$$

for every $w \in W$.

Now relations (6.9), (6.10), (6.11), (6.12) show that we can apply the lower closure theorem (3.i) with S replaced by $S \times T \times \overset{\circ}{T}$ and with $\rho = r, \rho' = r'$. Here the sets $\tilde{Q}(t, y)$ have property (Q) on A , hence property $Q(r+1)$ by force of (1.i). Analogously, the sets $\tilde{R}(t, \overset{\circ}{y})$ have property (Q) on B , hence property $Q(r'+1)$. We conclude that

$$(\mathcal{M}(x, u))(t) \in A(t) \quad \text{a.e. in } G,$$

that

$$(\mathcal{N}(x, v))(t) \in B(t) \quad \mu\text{-a.e. on } \Gamma,$$

and that there are elements $u \in T, v \in \overset{\circ}{T}$ such that

$$u(t) \in U(t, (\mathcal{M}(x, u))(t)), \quad z(t) = f(t, (\mathcal{M}(x, u))(t), u(t)) \quad \text{a.e. in } G,$$

$$v(t) \in V(t, (\mathcal{N}(x, v))(t)), \quad \overset{\circ}{z}(t) = g(t, (\mathcal{N}(x, v))(t), v(t)) \quad \mu\text{-a.e. on } \Gamma,$$

$$f_0(t, (\mathcal{M}(x, u))(t), u(t)) \in L_1(G),$$

$$g_0(t, (\mathcal{N}(x, v))(t), v(t)) \in L_1(\Gamma),$$

$$I[x, u, u, v, v] \leq i. \tag{6.16}$$

Relations (6.15) and (6.16) show, by comparison, that

$$\mathcal{J}(x, u, v) w = h(x, u, u, v, v) w \quad \text{for all } w \in W.$$

Thus, the system (x, u, u, v, v) is admissible, and since Ω is closed, there is some admissible system

$$(x, \bar{u}, \bar{u}, \bar{v}, \bar{v}) \text{ in } \Omega \text{ with } I[x, \bar{u}, \bar{u}, \bar{v}, \bar{v}] \leq i.$$

The same system belongs to Ω , and

$$I[x, u, u, v, v] \geq i.$$

Thus, $I = i$, and the existence theorem (6.i) is thereby proved.

Remark 1. Remarks 1–6 of § 4 and Remark 1 of § 5 apply to the present theorem as well.

Example. Let G be an open bounded connected subset of E^v , $v \geq 1$, of class K . We are concerned with the minimum of a functional

$$I[x, u] = \int_G f_0(t, x(t), \nabla x(t), u(t)) dt, \tag{6.17}$$

with state equations which we wish to be a weak form of

$$\sum_{i=1}^v \partial^2 x / (\partial t^i)^2 = f(t, x(t), \nabla x(t), u(t)), \tag{6.18}$$

and with constraints

$$x(t) \in A(t), \quad u(t) \in U(t, x(t)). \tag{6.19}$$

Here x and u are functions on G . Thus, $g_0 = 0$, we have no boundary condition on x , we can take $g = 0$, $\mathcal{F} = 0$, $\mathcal{K} = 0$, and need make no references to Γ, B, V, \dot{M} .

By introducing the increased control $\tilde{u}(t) = (u^1, \dots, u^v, u)$, we have the equivalent problem of minimizing the integral

$$I[x, u] = \int_G f_0(t, x(t), \tilde{u}(t)) dt,$$

with differential equations (6.18) and

$$\partial x / \partial t^i = f_i = u^i, \quad i = 1, \dots, v,$$

and constraints

$$x(t) \in A(t), \quad \tilde{u}(t) \in E^v \times U(t, x(t)).$$

We shall think of W as simply being

$$(C_0^\infty(G))^{v+1} \quad \text{with} \quad w = (\tilde{\omega}, 0) \in W, \\ \tilde{\omega} = (\omega^1, \dots, \omega^v, \omega) \quad \text{and} \quad \omega^1, \dots, \omega^v, \omega \in C_0^\infty(G).$$

As a weak form of the present system of differential equations we now take

$$\sum_{i=1}^v \int_G (\partial x / \partial t^i)(\omega^i(t)) dt - \int_G \sum_{i=1}^v (\partial x / \partial t^i)(\partial \omega / \partial t^i) dt \\ = \int_G f(t, x(t), \tilde{u}(t)) \omega(t) dt + \sum_{i=1}^v \int_G u^i(t)(\omega^i(t)) dt \tag{6.20}$$

for all

$$\tilde{\omega} = (\omega^1, \dots, \omega^v, \omega) \in (C_0^\infty(G))^{v+1}.$$

It is easy to verify that any (strong) solution x, u of the original system of equations, say

$$x \in W_1^2(G), \quad u \in T,$$

is certainly a solution of (6.20). Instead, we take

$$x \in S = W_1^1(G), \quad u \in T, \quad p = 1,$$

or, equivalently, $x \in S = W_1^1(G)$ and \tilde{u} measurable in G . Note that in the present problem we have the further control $\gamma x = v(t)$ with $v(t) \in L_1(\Gamma)$ on the boundary $\Gamma = \partial G$ of G , that is, the boundary values γx of x are arbitrary. In other words,

γx is free (in $L_1(\Gamma)$). The optimal element $x \in S$ will determine the optimal boundary values $v(t) = \gamma x \in L_1(\Gamma)$. We shall take in $W = (C_0^\infty(G))^{\nu+1}$ the topology defined by the norm

$$\|\tilde{\omega}\|_W = \text{Max} |\omega(t)| + \sum_{i=1}^{\nu} \text{Max} |\partial\omega/\partial t^i| + \sum_{i=1}^{\nu} \text{Max} |\omega^i(t)|.$$

We have here $r = \nu + 1$, and if we denote by $\|\tilde{\omega}\|_\infty$ the norm of $\tilde{\omega}$ as an element of $(L_\infty(G) \times W_\infty^1(G))$, then $\|\tilde{\omega}\|_\infty = \|\tilde{\omega}\|_W$ for every element $\tilde{\omega} \in W$. Also

$$(L_\infty(G))^r \supset (L_\infty(G))^\nu \times W_\infty^1(G) \supset W, \quad (L_1(G))^r \subset W^*,$$

and relation (6.1) holds with $K = 1$ since

$$\|\tilde{\omega}\|_{(L_\infty(G))^r} \leq \|\tilde{\omega}\|_\infty = \|\tilde{\omega}\|_W.$$

Also, the operator \mathcal{J} defined by the first member of (6.20) has the expected property

$$\mathcal{J}: S \rightarrow (L_1(G))^r \subset W^*.$$

Now, if $x, x_k \in S = W_1^1(G)$, $k = 1, 2, \dots$, and

$$x_k \rightarrow x \text{ weakly in } S = W_1^1(G),$$

then

$$\partial x_k / \partial t^i \rightarrow \partial x / \partial t^i \text{ as } k \rightarrow \infty \text{ weakly in } L_1(G), \quad i = 1, \dots, \nu;$$

hence $(\mathcal{J} x_k) \tilde{\omega} \rightarrow (\mathcal{J} x) \tilde{\omega}$ for every $\tilde{\omega} \in (L_\infty(G))^\nu \times W_\infty^1(G)$, and then certainly for every $\tilde{\omega} \in W = (C_0^\infty(G))^{\nu+1}$. Thus, the hypothesis required on \mathcal{J} in (6.i) is satisfied. Also note that $\mathcal{M} x = x$, $\mathcal{M}: S \rightarrow L_1(G)$, $s = 1$. Here

$$A(t) \in E^1, \quad A \in E^{\nu+1}, \quad u \in U(t, y) \in E^1, \quad \tilde{u} \in \tilde{U}(t, y) = E^\nu \times U(t, y),$$

and thus $M \subset E^{2\nu+2}$ is the set of all (t, y, \tilde{u}) with

$$t \in \text{cl } G, \quad y \in A(t), \quad \tilde{u} \in \tilde{U}(t, y) = E^\nu \times U(t, y).$$

Let $f_0(t, y, \tilde{u}), f(t, y, u)$ be real-valued continuous on M . For

$$Z = (Z^1, \dots, Z^\nu), \quad \tilde{u} = (u^1, \dots, u^\nu, u),$$

let $\tilde{Q}(t, y)$ be the subsets of $E^{\nu+2}$ defined by

$$\begin{aligned} \tilde{Q}(t, y) &= [(z^0, z, Z) | z^0 \geq f_0(t, y, \tilde{u}), z = f(t, y, \tilde{u}), Z^i = u^i, \tilde{u} \in \tilde{U}(t, y)] \\ &= [(z^0, z, Z) | z^0 \geq f_0(t, y, Z, u), z = f(t, y, Z, u), u \in U(t, y), Z \in E^\nu]. \end{aligned}$$

We assume that the sets $\tilde{Q}(t, y)$ have property (Q) in A . We shall assume that there is some function $\psi \geq 0$, $\psi \in L_1(G)$ such that $f_0(t, y, u) \geq -\psi(t)$ for all $t \in G$. We shall also assume that growth condition (H) holds for f_0 and f, f_1, \dots, f_ν with $p = 1$, and $f_1 = u^1, \dots, f_\nu = u^\nu$.

A pair (x, u) is here admissible provided $x \in W_1^1(G)$, u is measurable in G ,

$$x(t) \in A(t), \quad u(t) \in U(t, x(t)) \quad \text{a.e. in } G,$$

$$f_0(t, x(t), \tilde{u}(t)) \in L_1(G), \quad u^i = \partial x / \partial t^i, \quad i = 1, \dots, \nu,$$

and relation (6.20) holds for every $\tilde{\omega} \in W$. Note that hypothesis (H) (for $p=1$) certainly assures that also $f(t, x(t), u(t)) \in L_1(G)$.

We shall take for Ω a nonempty closed class of admissible pairs (x, u) such that $\{x\}_\Omega$ is weakly sequentially compact in $W_1^1(G)$, and also such that if

$$x \in W_1^1(G), \quad x_k \in \{x\}_\Omega, \quad k=1, 2, \dots,$$

$$x_k \rightarrow x \quad \text{weakly in } W_1^1(G),$$

then

$$x_k \rightarrow x \quad \text{strongly in } L_1(G),$$

(that is, $\mathcal{M}x_k \rightarrow \mathcal{M}x$ in $L_1(G)$). Existence theorem (6.i) now guarantees that the functional (6.17) with state equations (6.20) (in the weak form) and constraints (6.19) has an absolute minimum in Ω .

For instance, we may take

$$A(t) = E^1, \quad \mathcal{M}x = x, \quad \mathcal{M}: S \rightarrow L_1(G), \quad U = E^1,$$

$$f_0 = |t|^\alpha(x^2 + |\nabla x|^2 + u^2), \quad f = -1 + u + 2^{-1}|u|,$$

and we may take for Ω the class of all admissible pairs (x, u) with

$$x \in S = W_1^1(G), \quad u \in T.$$

Here α is a fixed number, $0 < \alpha < v$.

First we have to prove that f, f_0 satisfy condition (H), that is, f, f_0 satisfy a growth condition (ε). Since G is bounded, there is a constant $c > 0$ such that $|t| \leq c$ for all $t \in G$. Now, for every $\varepsilon > 0$, the function

$$\psi_\varepsilon(t) = \varepsilon^{-1}|t|^{-\alpha} + 1$$

is L -integrable in G , and

$$2^{-1}|u| \leq |u + 2^{-1}|u|| \leq (3/2)|u|.$$

Either $(3/2)|u| \geq \varepsilon^{-1}|t|^{-\alpha}$, and then

$$|f| \leq (3/2)|u| + 1 = (3/2)|u|^{-1}u^2 + 1 \leq (9/4)\varepsilon|t|^\alpha u^2 + 1 \leq \psi_\varepsilon(t) + (9/4)\varepsilon f_0;$$

or

$$(3/2)|u| \leq \varepsilon^{-1}|t|^{-\alpha},$$

and then again

$$|f| \leq \psi_\varepsilon(t) + (9/4)\varepsilon f_0.$$

An analogous statement holds for each function $f_i = u^i, i=1, \dots, v$. We denote by Ω_0 the class of all admissible pairs x, u with $I[x, u] \leq M_0$ for sufficiently large M_0 so that Ω_0 is not empty.

Let us prove now that the class $\{x\}_{\Omega_0}$ is sequentially relatively compact in $W_1^1(G)$. It is enough to prove that for $(x, u) \in \Omega_0$ the functions

$$\Phi(t) = |x(t)| + \sum_{i=1}^v |\partial x / \partial t^i|, \quad t \in G,$$

are equiabsolutely integrable in G . Indeed, either $\Phi \geq \varepsilon^{-1} |t|^{-\alpha}$, and then

$$\Phi = \Phi^{-1} \Phi^2 \leq \varepsilon |t|^\alpha (\nu + 1) (x^2 + |\nabla x|^2) \leq \psi_\varepsilon(t) + \varepsilon(\nu + 1) f_0,$$

or $\Phi \leq \varepsilon^{-1} |t|^{-\alpha}$, and then again

$$\Phi \leq \psi_\varepsilon(t) + \varepsilon(\nu + 1) f_0.$$

Thus Φ, f_0 satisfy a growth condition (ε) , and again the functions Φ are equiabsolutely integrable in G .

Finally, the functions $\Phi(t)$ being equiabsolutely integrable in G , we conclude from Sobolev space theory that $x_k \rightarrow x$ weakly in $W_1^1(G)$ implies $x_{k_\lambda} \rightarrow x$ strongly in $L_1(G)$ for a suitable subsequence $[k_\lambda]$.

7. Application to Problems of Optimization with an Evolution Equation in Weak Form

In this section we apply Theorem (6.i) to problems of optimization with an evolution equation in the weak form.

Example. We are using here the notations of § 5. We are concerned with the problem of minimizing the functional

$$I[x, u, v_1, v_2, v_3] = \int_G f_0(t, \tau, x(t, \tau), (\nabla x)(t, \tau), u(t, \tau)) dt d\tau + \int_{\Gamma_3} g_0(\tau, (\gamma_3 x)(\tau), v_3(\tau)) d\tau, \tag{7.1}$$

with a state equation (concerning G and Γ_2) which will be a suitable weak form of the system of equations

$$\partial x / \partial t - \sum_{i=1}^{\nu} \partial^2 x / (\partial \tau^i)^2 = f(t, \tau, x(t, \tau), (\nabla x)(t, \tau), u(t, \tau)) \quad \text{in } G, \tag{7.2}$$

$$\partial x / \partial n + v_2(t, \tau) \gamma_2 x(t, \tau) = 0 \quad \text{on } \Gamma_2, \tag{7.3}$$

and constraints

$$x(t, \tau) \in A(t, \tau), \quad u(t, \tau) \in U(t, \tau, x(t, \tau)) \quad \text{a.e. in } G, \tag{7.4}$$

$$(\gamma_3 x)(\tau) \in B(\tau), \quad v_3(\tau) \in V(\tau, (\gamma_3 x)(\tau)) \quad \text{a.e. on } \Gamma_3. \tag{7.5}$$

Here $x, u: G \rightarrow E^1, v_2: \Gamma_2 \rightarrow E^1, v_3: \Gamma_3 \rightarrow E^1$, denote real-valued functions, x state variable, u, v_2, v_3 controls. In other words, we are interested in the determination of a function $x(t, \tau)$ in G (in particular, of its initial values, say $v_1(\tau) = x(0, \tau)$ on Γ_1), and of controls $u(t, \tau)$ in $G, v_2(t, \tau)$ on $\Gamma_2, v_3(\tau)$ on Γ_3 , such that the functional (7.1) has its minimum value, under constraints (7.4), (7.5), and a suitable weak form of state equations (7.2), (7.3).

By introducing the increased control $\tilde{u}(t, \tau) = (u^1, \dots, u^{\nu+1}, u)$, we have the equivalent problem of the minimum of the integral

$$I[x, \tilde{u}, v_1, v_2, v_3] = \int_G f_0(t, \tau, x(t, \tau), \tilde{u}(t, \tau)) dt d\tau + \int_{\Gamma_3} g_0(\tau, (\gamma_3 x)(\tau), v_3(\tau)) d\tau,$$

with differential equations

$$\begin{aligned} \partial x/\partial t - \sum_{i=1}^v \partial^2 x/(\partial \tau^i)^2 &= f(t, \tau, x(t, \tau), \tilde{u}(t, \tau)) \quad \text{in } G, \\ \partial x/\partial n + v_2(t, \tau) \gamma_2 x(t, \tau) &= 0 \quad \text{on } \Gamma_2, \\ \partial x/\partial \tau^i &= f_i = u^i, \quad i=1, \dots, v, \quad \partial x/\partial t = u^{v+1} \quad \text{in } G, \end{aligned} \tag{7.6}$$

with constraints (7.4), (7.5), and $u^i \in E^1, i=1, \dots, v+1$.

We take for W the space of all pairs $w = (\tilde{\omega}, \gamma \tilde{\omega})$, with

$$\tilde{\omega} = (\omega^1, \dots, \omega^{v+1}, \omega), \quad \omega^i \in C_0^\infty(G), \quad i=1, \dots, v+1, \quad \omega \in C^\infty(\text{cl } G).$$

Then, $\gamma \tilde{\omega} = (0, \dots, 0, \gamma \omega)$. As a weak form of (7.6) we take the equation

$$\begin{aligned} \int_G \sum_{i=1}^v (\partial x/\partial \tau^i) (\partial \omega/\partial \tau^i) dt d\tau + \int_G (\partial x/\partial t) \omega(t, \tau) dt d\tau \\ + \int_{\Gamma_2} v_2(t, \tau) \gamma_2 x(t, \tau) \gamma_2 \omega(t, \tau) dt d\mu \\ + \sum_{i=1}^v \int_G (\partial x/\partial \tau^i) \omega^i(t, \tau) dt d\tau + \int_G (\partial x/\partial t) \omega^{v+1}(t, \tau) dt d\tau \\ = \int_G f(t, \tau, x(t, \tau), \tilde{u}(t, \tau)) \omega(t, \tau) dt d\tau \\ + \sum_{i=1}^{v+1} \int_G u^i(t, \tau) \omega^i(t, \tau) dt d\tau \end{aligned} \tag{7.7}$$

for all

$$\tilde{\omega} = (\omega^1, \dots, \omega^{v+1}, \omega) \in (C_0^\infty(G))^{v+1} \times C^\infty(\text{cl } G).$$

Here $\mathcal{J} w$, or $\mathcal{J}(x, v_2) w$, that is, the operator \mathcal{J} , is defined by the first member of (7.7). It is easy to verify that any strong solution x, u, v_2 of (7.2), (7.3), say with

$$x' \in S_p^2, \quad u \in T, \quad v_2 \in L_q(\Gamma_2),$$

is certainly a solution of (7.7) for all $w \in W$.

We shall take here

$$x \in S = W_p^1(G), \quad p > 1, \quad u \in T, \quad v_2 \in \mathring{T}_2$$

with \mathring{T}_2 a weakly closed subset of $L_q(\Gamma_2)$, $1/p + 1/q = 1$, which is bounded in the norm of $L_q(\Gamma_2)$. We take in W the topology defined by the norm

$$\begin{aligned} \|w\|_W = \|(\tilde{\omega}, \gamma \tilde{\omega})\| = \text{Max } |\omega(t, \tau)| + \text{Max } |\partial \omega/\partial t| \\ + \sum_{i=1}^v \text{Max } |\partial \omega/\partial \tau^i| + \sum_{i=1}^v \text{Max } |\omega^i(t, \tau)| \end{aligned}$$

where all Max are taken in $\text{cl } G$. With $r = v + 2$ we have

$$\begin{aligned} \|\tilde{\omega}\|_{(L_q(G))^r} + \|\gamma \tilde{\omega}\|_{L_q(\Gamma)} \leq \|\omega\|_{W_p^1(G)} + \|\gamma \omega\|_{L_q(\Gamma)} + \sum_{i=1}^v \|\omega^i\|_{L_q(G)} \\ \leq (|G|^{1/q} + |\Gamma|^{1/q}) \|w\|_W. \end{aligned}$$

From this we deduce

$$W \subset W_q^1(G) \times L_q(\Gamma) \times (W_q(G))^{v+1} \subset (L_q(G))^r \times L_q(\Gamma),$$

and that relation (6.1) holds with $K = |G|^{1/q} + |\Gamma|^{1/q}$.

We have here $x \in S = W_p^1(G)$, $\mathcal{M}x = x$, $\mathcal{M} : S \rightarrow L_p(G)$, $\mathcal{K}x = \gamma_3 x$, $\mathcal{K} : S \rightarrow L_p(\Gamma_3)$, $s = s' = 1$. Also $x_k \rightarrow x$ weakly in S implies $\mathcal{M}x_k \rightarrow \mathcal{M}x$ strongly in $L_p(G)$, $\gamma_2 x_k \rightarrow \gamma_2 x$ strongly in $L_p(\Gamma_2)$, $\mathcal{K}x_k \rightarrow \mathcal{K}x$ strongly in $L_p(\Gamma_3)$, $\partial x_k / \partial t \rightarrow \partial x / \partial t$, $\partial x_k / \partial \tau^i \rightarrow \partial x / \partial \tau^i$ weakly in $L_p(G)$, $i = 1, \dots, v$. Furthermore, if $v_2, v_{2k}, k = 1, 2, \dots$, are elements of $\tilde{I}_2 \subset L_q(\Gamma_2)$, with $v_{2k} \rightarrow v_2$ weakly in $L_q(\Gamma_2)$, then the products $v_{2k}(\gamma_2 x_k)$ converge weakly in $L_1(\Gamma_2)$ to $v_2(\gamma_2 x)$. Finally, from the definition of $\mathcal{J}(x, v_2)$ we conclude that

$$\mathcal{J}(x_k, v_{2k})w \rightarrow \mathcal{J}(x, v_2)w \text{ as } k \rightarrow \infty$$

for every $w \in W$ as requested.

For any $(t, \tau) \in G$ we have

$$A(t, \tau) \subset E^1, \quad A \subset E^{v+2}, \quad \tilde{u} \in \tilde{U}(t, \tau, y) = E^{v+1} \times U(t, \tau, y) \subset E^{v+2}.$$

Now M is the set of all

$$(t, \tau, y, \tilde{u}) \in E^{2v+4}$$

with

$$(t, \tau) \in \text{cl } G = \text{cl } G' \times [0, T], \quad y \in A(t, \tau), \quad \tilde{u} \in \tilde{U}(t, \tau, y).$$

Let

$$f_0(t, \tau, y, \tilde{u}), f(t, \tau, y, \tilde{u})$$

be real-valued continuous functions on M . Let $\tilde{Q}(t, \tau, y)$ be the subsets of E^{v+3} defined by

$$\begin{aligned} \tilde{Q}(t, \tau, y) &= [(z^0, z, Z) \mid z^0 \geq f_0(t, \tau, y, \tilde{u}), z = f(t, \tau, y, \tilde{u}), Z^i = u^i, \tilde{u} \in \tilde{U}(t, \tau, y)] \\ &= [(z^0, z, Z) \mid z^0 \geq f_0(t, \tau, y, Z, u), z = f(t, \tau, y, Z, u), u \in U(t, \tau, y), Z \in E^{v+1}] \end{aligned}$$

where

$$Z = (Z^1, \dots, Z^{v+1}), \text{ and } \tilde{u} = (u^1, \dots, u^{v+1}, u),$$

and let us assume that these sets have property (Q) in A . Also, in harmony with (6.i), we assume that there is a function $\psi(t, \tau) \geq 0, \psi \in L_1(G)$ with

$$f_0(t, \tau, y, Z, u) \geq -\psi(t, \tau) \text{ for all } (t, \tau, y, Z, u) \in M,$$

and that there is a constant $a > 0$ and a function

$$\psi_0(t, \tau) \geq 0, \quad \psi_0 \in L_1(G)$$

such that

$$|f(t, \tau, y, Z, u)|^p + \sum_{i=1}^{v+1} |u^i|^p \leq \psi_0(t, \tau) + af_0$$

for all $(t, \tau, y, Z, u) \in M$ (condition (H)).

For any $(T, \tau) \in \Gamma_3$, that is, $\tau \in \text{cl } G'$, let $B(\tau) \subset E^1$ be a given set, let $B \subset E^{v+1}$ be the set of all (τ, \hat{y}) with $\tau \in \text{cl } G', \hat{y} \in B(\tau)$, and for every $(\tau, \hat{y}) \in B$ let $V(\tau, \hat{y})$ be a given subset of E^1 . Then, let \tilde{M} be the set of all $(\tau, \hat{y}, v_3) \in E^{v+2}$ with $(\tau, \hat{y}) \in B$,

$v_3 \in V(\tau, \hat{y})$, and let $g_0(\tau, \hat{y}, v_3)$ be real-valued and continuous on \hat{M} . We assume that there is a real-valued function $\psi_3(\tau) \geq 0, \psi_3 \in L_1(G')$ with $g_0(\tau, \hat{y}, v_3) \geq -\psi_3(\tau)$ for all $(\tau, \hat{y}, v_3) \in \hat{M}$. Here $g=0$, hence the corresponding sets R are half straight lines, and certainly convex. We assume that these sets $\hat{R}(\tau, \hat{y})$ satisfy property (U) on B . The corresponding condition (H) is trivially satisfied. On Γ_1 we have both $g=0, g_0=0$, and no further discussion is needed.

A system (x, \tilde{u}, v_2, v_3) is here admissible provided $x \in S = W_p^1(G)$, \tilde{u} is measurable in G , v_3 is measurable in Γ_3 , $v_2 \in \hat{T}_2 \subset L_q(\Gamma_2)$,

$$f_0(t, \tau, x(t, \tau), (Vx)(t, \tau), u(t, \tau)) \in L_1(G),$$

$$g_0(\tau, v_3, x(\tau), v_3(\tau)) \in L_1(\Gamma_3),$$

equations $\partial x^i / \partial \tau^i = u^i, i=1, \dots, \nu, \partial x^i / \partial t = u^{\nu+1}$ hold in G , relations (7.4), (7.5) hold, and (7.7) holds for all $w \in W$. Because of hypothesis (H), then certainly

$$f(t, \tau, x(t, \tau), (Vx)(t, \tau), u(t, \tau)) \in L_p(G).$$

We shall take for Ω a nonempty closed class of admissible systems, such that the set $\{x\}_\Omega$ is bounded in the norm of $W_p^1(G)$. Theorem (6.i) now guarantees that the functional (7.1) with state equations (7.7) (in the weak form) and constraints (7.4), (7.5) has an absolute minimum in Ω .

For instance, we may take $p=q=2$,

$$f_0 = t^2 + |\tau|^2 + x^2 + |Vx|^2 + u^2,$$

$$f = -1 + t + |\tau| + x + u + 2^{-1} |u|,$$

$$g_0 = (1 + t^2 |\tau|^2) x^2 + (1 + |x|) |v_3|,$$

$$U = E^1, \quad V_3 = E^1, \quad T_2 = [v_2 \in L_2(\Gamma_2) \mid \|v_2\|_2 \leq 1].$$

Then condition (H) (for $p=2$) is satisfied since $f^2 \leq 2 + 6f_0$. Also, we can take for Ω the class of all admissible systems x, u, v_2, v_3 . If i denotes the infimum of I in Ω , and Ω_0 the subclass of all $(x, u) \in \Omega$ with $I \leq i + 1$, then the set $\{x\}_{\Omega_0}$ is certainly bounded in the norm of $W_2^1(G)$.

This research was partially supported by AFOSR Research Project No. 69-1662.

8. References

1. AUBIN, J. P., Un théorème de compacité. C. R. Acad. Sci. Paris **256**, 5042–5044 (1963).
2. BUTKOVSKY, A. G., A. I. EGOROV, & K. A. LURIE, Optimal control of distributed systems (a survey of Soviet publications). SIAM J. Control **6**, 437–476 (1968).
3. CESARI, L., [a] Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints. I and II. Trans. Amer. Math. Soc. **124**, 369–412 and 413–429 (1966). [b] Existence theorems for optimal controls of the Mayer type. SIAM J. Control **4**, 517–552 (1968). [c] Existence theorems for multidimensional Lagrange problems. J. Optimization Theory and Appl. **1**, 87–112 (1967). [d] Multidimensional Lagrange problems of optimization in a fixed domain and an application to a problem of magnetohydrodynamics. Arch. Rational Mech. Anal. **29**, 81–104 (1968). [e] Existence theorems for abstract multidimensional control problems (Intern. Conference Optimal Control, Tbilisi, Georgia, USSR), J. Optimization Theory and Appl. **6**, 210–236 (1970). [f] Seminormality

- and upper semicontinuity in optimal control. *J. Optimization Theory and Appl.* **6**, 114–137 (1970). [g] Lagrange problems of optimal control and convex sets not containing any straight line. To appear. [h] Existence theorems for problems of optimization with distributed and boundary controls. *International Congress of Mathematicians, Nice 1970*, vol. **3**, 157–161. [i] Closure, lower closure, and semicontinuity theorems in optimal control. *SIAM J. Control* **9**, 287–315 (1971).
4. COWLES, D. E., [a] Upper semicontinuity properties of variable sets in optimal control. *J. Optimization Theory and Appl.*, **10**, No. 4 (1972). [b] Lower closure theorems for Lagrange problems of optimization with distributed and boundary controls. *J. Optimization Theory and Appl.*, **10**, No. 5 (1972).
 5. FICHERA, G., Semicontinuity of multiple integrals in ordinary form. *Arch. Rational Mech. Anal.* **17**, 339–352 (1964).
 6. KURATOWSKI, C., Les fonctions semi-continues dans l'espace des ensembles fermés. *Fund. Math.* **18**, 148–166 (1932).
 7. LIONS, J. L., [a] On some optimization problems for linear parabolic equations. *Functional Analysis and Optimization* (E. R. CAIANIELLO, editor), Academic Press, 115–131 (1966). [b] Optimisation pour certaines classes d'équations d'évolution nonlinéaire. *Annali Mat. pura e applicata*, **72**, 275–294 (1966). [c] *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Dunod 1968.
 8. MORREY, C. B., [a] Functions of several variables and absolute continuity. *Duke Math. J.* **6**, 187–285 (1940); **8**, 120–124 (1942). [b] Multiple integral problems in the calculus of variations. *Univ. of California Publ. Math. (N.S.)* **1**, 1–130 (1943).
 9. ROTHE, E. H., An existence theorem in the calculus of variations based on Sobolev's embedding theorems. *Arch. Rational Mech. Anal.* **21**, 151–162 (1966).
 10. VOLTERRA, V., & J. PÉRÈS, *Théorie de Fonctionnelles*, Gauthier-Villars 1936.

Department of Mathematics
University of Michigan
Ann Arbor

(Received May 4, 1971 — Revised August 1, 1971)