Qualitative Behavior of Dissipative Wave Equations on Bounded Domains

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Abstract

The qualitative behavior of solutions of the mixed problem $u_{tt}=\Delta u-a(x)u_t$ in $\mathbb{R}\times\Omega$, $u=0$ on $\mathbb{R}\times\partial\Omega$, is studied in the case when $a>0$ and $\Omega\subset\mathbb{R}^n$ is bounded. Roughly speaking, if $a\geqq a_{\min}>0$, then solutions decay at least as fast as $\exp t(\varepsilon-\frac{1}{2}a_{\min})$, with the possible exception of a finite dimensional set of smooth solutions whose existence is associated with a phenomenon of overdamping. If $a_{\max}$ is sufficiently small, depending on Ω, then no overdamping occurs.

Let $\Omega\subset\mathbb{R}^n$ be a bounded open set lying on one side of its smooth boundary $\partial\Omega$. We are interested in solutions of the dissipative wave equation

$$u_{tt}=\Delta u-a(x)u_t \quad \text{on} \quad \mathbb{R}\times\Omega, \tag{1}$$

where Δ is the Laplacian and where the coefficient of friction $a(x)$ is a smooth positive function on $\bar{\Omega}$. On the boundary the Dirichlet boundary condition

$$u=0 \quad \text{on} \ \mathbb{R}\times\partial\Omega \tag{2}$$

is imposed. The methods used generalize immediately to a wide class of equations and boundary conditions.

A feeling for the phenomena encountered can be obtained by considering the special case $a=$constant, which can be solved by an eigenfunction expansion. Let Φ_j, $j=1,2,\ldots$, be an orthonormal sequence of eigenfunctions of Δ with Dirichlet conditions. That is, $\Delta\Phi_j=\lambda_j\Phi_j$ in Ω, $\Phi_j=0$ on $\partial\Omega$ and $0>\lambda_1>\lambda_2\geqq\lambda_3\ldots$. Then

$$u=\sum u_j(t)\,\Phi_j, \tag{3}$$

where $u_j(t)$ satisfies the damped spring equation

$$\ddot{u}_j+a\dot{u}_j-\lambda_j u_j=0 \tag{4}$$

with $u_j(0)$, $\dot{u}_j(0)$ determined by the Cauchy data of u. The general solution of (4) is

$$\beta_+e^{r_+t}+\beta_-e^{r_-t}, \qquad \beta_\pm\in\mathbb{C} \tag{5}$$

$$r_\pm=\frac{-a\pm\sqrt{a^2+4\lambda_j}}{2}. \tag{6}$$

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Since $\lambda_j \to -\infty$ as $j \to \infty$ we see that for j large $\operatorname{Re} r_\pm = -a/2$ and the corresponding terms in (3) decay like $e^{-ta/2}$. The "slow modes" with $\operatorname{Re} r > -a/2$ occur when the spring equation (4) is overdamped. In summary, solutions consist of a finite dimensional part decaying exponentially, but slower than $e^{-ta/2}$, and a part decaying like $e^{-ta/2}$. Our goal is to prove similar results when a depends on x. To do this, an appropriate framework for discussing (1), (2) must be built.

The energy of a solution u at time t is defined as

$$E(t,u) = \int_\Omega (|u_t(t,x)|^2 + |\nabla u(t,x)|^2)\,dx.$$

The basic energy identity for smooth solutions of (1), (2) is then

$$\frac{d}{dt} E(t,u) = -2 \int_\Omega a(x)\,|u_t(t,x)|^2\,dx \leqq 0.$$

This relation would seem to indicate that if one increases a then the energy should decay more rapidly. It was pointed out in [5, pp. 365–367] however that this is not quite correct, since for large a the overdamped states actually decay slower. With the exception of this finite dimensional set of states, however, the idea is right.

We introduce the natural Hilbert space of states, $H = \mathring{H}_1(\Omega) \times L_2(\Omega)$ with norm

$$\|(\phi,\psi)\|_H^2 = \int_\Omega (|\nabla\phi|^2 + |\psi|^2)\,dx$$

and define the evolution operator $S(t)\colon H \to H$ by

$$S(t)(\phi,\psi) = (u(t), u_t(t)),$$

where u is the solution of the mixed problem (1), (2) with initial conditions

$$u(0) = \phi, \qquad u_t(0) = \psi. \tag{7}$$

(If one wants to avoid weak solutions one may consider functions $(\phi,\psi) \in C_0^\infty(\Omega)^2$, in which case the solution of (1), (2), (7) is smooth and $S(t)$ has a unique continuous linear extension to all of H.) The family $S(t)$, $-\infty < t < \infty$, is a C_0 one-parameter group of linear transformations on H with $\|S(t)\| \leqq 1$ for $t \geqq 0$. The generator of this group is

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix},$$

$$D(G) = (H_2(\Omega) \cap \mathring{H}_1(\Omega)) \times \mathring{H}_1(\Omega).$$

Notice that $\begin{pmatrix} 0 & 1 \\ \Delta & 1 \end{pmatrix}$, with the same domain, is skewadjoint (see [4, § V.1]) and that $\begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$ is a bounded dissipative operator on H so that G is maximal dissipative.

We are interested in $S(t)$ for t large. If Ω were a Riemannian manifold without boundary and Δ the Laplace-Beltrami operator, then the results of [7], see especially § 4, give a precise formula for $\|S(t)\|_{\mathscr{L}/\mathscr{K}}$ where $\mathscr{L}$ is the algebra of bounded operators on H and $\mathscr{K}$ the ideal of compact operators. In particular, $\|S(t)\|_{\mathscr{L}/\mathscr{K}}$ depends monotonically on a, in the sense that increasing a decreases the norm of $S(t)$. We have also the simple estimates

$$e^{-t a_{\max}/2} \leqq \|S(t)\|_{\mathscr{L}/\mathscr{K}} \leqq e^{-t a_{\min}/2}, \tag{8}$$

where

$$a_{\max} = \max_{\Omega} a, \qquad a_{\min} = \min_{\Omega} a. \tag{9}$$

The proof in [7] relies on a detailed construction using geometrical optics, not available in case Ω has a boundary because of the existence of glancing rays. We will give a direct and elementary proof of inequalities corresponding to (8) in the case of regions with boundary. The method is that of energy integrals with an indefinite energy form.

Definition. *Let* $\pi_1: H \to \mathring{H}_1(\Omega)$ be defined by $\pi_1((\phi, \psi)) = \phi$. If $h \in H$ then $u = \pi_1(S(t)\,h)$ is called a *finite energy solution* of (1), (2). Notice that

$$u \in C(\mathbb{R} \mid \mathring{H}_1(\Omega)) \cap C^1(\mathbb{R} \mid L_2(\Omega)).$$

The next result contains our basic estimate.

Theorem 1. *If u is a solution of* (1), (2) *with finite energy, then*

$$E(t, u) \leqq c_1^2\, e^{-a_{\min} t}\, E(0, u) + c_2\, \|u(t)\|_{L_2}^2, \tag{10}$$

where

$$c_1^2 = \max(4, a_{\min}^2/2\lambda),$$
$$c_2 = a_{\max}\, a_{\min} - \lambda$$

and λ is given by (11).

One consequence of (10) is that for solutions which decay slower than $e^{-t a_{\min}/2}$ the energy is dominated by $(c_2 + \varepsilon)$ times the L_2 norm, for large time. Thus the slowly decaying modes must be fairly smooth.

Proof. First assume that $u \in C^\infty(\mathbb{R} \times \bar{\Omega})$. Let $\alpha = a_{\min}/2$ and $v = e^{\alpha t} u$. Then

$$u_t = e^{-\alpha t}(v_t - \alpha v)$$
$$u_{tt} = e^{-\alpha t}(v_{tt} - 2\alpha\, v_t + \alpha^2 v)$$
$$\Delta v = e^{\alpha t} \Delta u = e^{\alpha t}(u_{tt} + a u_t).$$

Substituting the expressions for u_t, u_{tt} into the last equation yields

$$v_{tt} - \Delta v + (\alpha^2 - a\alpha)\, v = -(a - a_{\min})\, v_t$$

(notice that $a - a_{\min} \geqq 0$ and $\alpha^2 - a\alpha \leqq -\alpha^2 < 0$). Multiply this equation by $\bar{v}_t$, integrate over Ω, and form the real part. This shows that

$$\frac{d}{dt} \mathscr{I}(t) = -2 \int_{\Omega} (a - a_{\min})\, |v_t(t)|^2\, dx \leqq 0,$$

where

$$\mathscr{I}(t)=\int_\Omega \left(|v_t(t)|^2+|\nabla v(t)|^2+(\alpha^2-a\alpha)\,|v(t)|^2\right)dx.$$

Notice that the decreasing functional $\mathscr{I}(t)$ is indefinite since $\alpha^2-a\alpha<0$. To make use of the decrease of $\mathscr{I}$, observe that

$$\begin{aligned}
e^{2\alpha t}\,E(t,u)&=\int_\Omega (|v_t-\alpha v|^2+|\nabla v|^2)\,dx\\
&\leqq \int_\Omega (2\,|v_t|^2+2\alpha^2\,|v|^2+|\nabla v|^2)\,dx\\
&\leqq 2\mathscr{I}(t)+2a_{\max}\,\alpha\,\|v(t)\|_{L_2}^2-\|\nabla v(t)\|_{L_2}^2,
\end{aligned}$$

where we have used the relation $2a_{\max}\,\alpha+2(\alpha^2-a\alpha)\geqq 2\alpha^2$. Let λ be the smallest eigenvalue of $-\Delta$ with Dirichlet boundary conditions, that is

$$\lambda=\min_{\phi\in H_1(\Omega)\setminus\{0\}}\frac{\|\nabla\phi\|_{L_2}^2}{\|\phi\|_{L_2}^2}. \tag{11}$$

Then the decrease of $\mathscr{I}$ and the inequality for E yield

$$E(t,u)\leqq 2e^{-2\alpha t}\,\mathscr{I}(0)+(a_{\max}\,a_{\min}-\lambda)\,\|u(t)\|_{L_2}^2.$$

In addition,

$$\begin{aligned}
\mathscr{I}(0)&=\int_\Omega \left(|u_t(0)+\alpha u(0)|^2+|\nabla u(0)|^2+(\alpha^2-a\alpha)\,|u(0)|^2\right)dx\\
&\leqq E(0,u)+\int_\Omega \left(|u_t(0)|^2+(3\alpha^2-a\alpha)\,|u(0)|^2\right)dx\\
&\leqq \max\left(2,\frac{\alpha^2}{\lambda}\,E(0,u)\right),
\end{aligned}$$

where we have used the estimate $3\alpha^2-a\alpha\leqq\alpha^2$. Substituting the above estimate for $\mathscr{I}(0)$ into the inequality for $E(t,u)$ yields (10).

If u is a solution with finite energy, say $u=\pi_1(S(t)\,h)$, we choose $h_n=(\phi_n,\psi_n)\in(C_0^\infty(\Omega))^2$ with $(\phi_n,\psi_n)\to h$ in H, and let $u_n=\pi_1\,S(t)\,h_n$. Then $u_n\in C^\infty(\mathbb{R}\times\bar{\Omega})$ and $E(t,u_n)\to E(t,u)$ by virtue of the continuity of S. Applying inequality (10) to u_n and passing to the limit $n\to\infty$ proves the theorem.

Theorem 1 has strong implications for the spectrum of $S(t)$. On H, let $|\ |$ be the continuous seminorm defined by

$$|(\phi,\psi)|^2=\int_\Omega |\phi|^2\,dx.$$

This seminorm is compact in the following sense.

Definition. If Y is a Banach space and $p\colon Y\to\mathbb{R}$ is a continuous seminorm, then p is *compact* if every bounded sequence $\{y_n\}$ in Y has a subsequence $\{y_{n_k}\}$ such that $p(y_{n_k}-y_{n_l})\to 0$ as $k,l\to\infty$.

Notice that if $\{(\phi_n,\psi_n)\}$ is bounded in H, then $\{\phi_n\}$ is bounded in $\mathring{H}_1(\Omega)$. Hence by Rellich's theorem there is a subsequence $\{\phi_{n_k}\}$ convergent in $L_2(\Omega)$.

This proves the compactness of $|\ |$. As a second observation note that if $A\colon Y \to Y$ is a continuous linear map and $p\colon Y \to \mathbb{R}$ is a compact seminorm, then $p \circ A$ is a compact seminorm. For any $t>0$, let $q\colon H \to \mathbb{R}$ be the seminorm defined by $q(h)=|S(t)h|$. The above remarks show that q is compact. The basic estimate (10) shows that

$$\|S(t)h\|_H \leqq c_1\, e^{-ta_{\min}/2}\, \|h\|_H + \sqrt{c_2}\, q(h). \tag{12}$$

Though this is not as strong as the estimate $\|S(t)\| \leqq \text{const}\, e^{-ta_{\min}/2}$, we will show that it is also not much weaker (see Theorem 4).

First we study the spectrum $\sigma(S(t))$ of $S(t)$. Recall that an isolated point λ of $\sigma(S)$ is an eigenvalue of finite multiplicity if $[z-S]^{-1}$ has a pole at $z=\lambda$. The (geometric) multiplicity of λ as an eigenvalue is the rank of the projection

$$(2\pi i)^{-1} \oint_{\Gamma} (z-S)^{-1}\, dz,$$

where Γ is a circle enclosing λ but no other part of $\sigma(S)$.

Theorem 2. *Let D be the disc $\{z \mid |z| \leqq c_1\, e^{-ta_{\min}/2}\}$. Then $\sigma(S(t)) \cap \mathbb{C} \setminus D$ is a discrete subset of $\mathbb{C} \setminus D$ consisting only of eigenvalues of finite multiplicity.*

Theorem 2 follows by applying the next lemma, which is a standard Fredholm type result (proof omitted).

Lemma. *Suppose H is a Hilbert space, $S\colon H \to H$ is a bounded linear transformation, and q a compact seminorm on H. If*

$$\|Sh\|_H \leqq C\|h\|_H + C'\, q(h), \qquad h \in H,$$

then $\sigma(S) \cap \{|z| > C\}$ consists of isolated eigenvalues with finite multiplicity.

If G is the generator of the group $S(t)$, then $\sigma(G) \subset \{\operatorname{Re} z \leqq 0\}$. The following more detailed result also holds.

Theorem 3. *We have $\sigma(G) \subset \{\operatorname{Re} z < 0\}$. Moreover for any $\varepsilon > 0$ the set $\sigma(G) \cap \{\operatorname{Re} z \geqq \varepsilon + a_{\min}/2\}$ consists of a finite number of isolated eigenvalues of finite multiplicity.*

Proof. The second assertion follows from Theorem 2 and the basic spectral theory of semigroups, for example, Theorems 16.7.1 and 16.7.2 of [3]. It remains to show that there are no purely imaginary eigenvalues. If $Gh = i\mu h$ with $\mu \in \mathbb{R}$, and $h = (\phi, \psi) \in H$, then

$$0 = \operatorname{Re}(Gh, h) = -2 \int_{\Omega} a\, |\psi|^2\, dx.$$

Since $a > 0$ we see that $\psi = 0$. The equation $Gh = i\mu h$ implies $\psi = i\mu\phi$. Hence if $\mu \neq 0$ it follows that $h = 0$. If $\mu = 0$ we have $Gh = 0$, so $\Delta\phi = a\psi$. Now $\psi = 0$, whence $\phi \in \mathring{H}_1(\Omega)$ and $\Delta\phi = 0$. Thus $\phi = 0$ and again $h = 0$. This completes the proof.

Fix $\varepsilon > 0$ and let Γ be a smooth simple closed curve enclosing the eigenvalues of G with real parts greater than or equal to $\varepsilon - a_{\min}/2$, but enclosing no other part of $\sigma(G)$. Let P be the corresponding spectral projection, that is,

$$P=\frac{1}{2\pi i}\oint_{\Gamma}(z-G)^{-1}\,dz.$$

By Theorem 3, P has finite rank. The general spectral theory of closed operators shows that the decomposition $H=PH\oplus(I-P)H$ reduces G and S in the following sense:

(i) $S(t)$ maps PH and $(I-P)H$ into themselves.
(ii) $PH\subset D(G)$, and the generator of the group $S(t)|_{PH}$ is $G|_{PH}$. The generator of $S(t)|_{(I-P)H}$ is $G|_{D(G)\cap(I-P)H}$.
(iii) $\sigma(S(t)|_{PH})\subset\{|z|\geqq e^{t(\varepsilon-a_{\min}/2)}\}$

and

$$\sigma(S(t)|_{(I-P)H})\subset\{|z|\leqq e^{t(\varepsilon'-a_{\min}/2)}\}\ \text{ for some } \varepsilon'\in[0,\varepsilon).$$

From this we derive the following basic decay theorems.

Theorem 4. *For $\varepsilon>0$, let P be the spectral projection of G corresponding to the eigenvalues of G with real part $\geqq -a_{\min}/2+\varepsilon$. Then $H=PH\oplus(I-P)H$, PH is finite dimensional, both PH and $(I-P)H$ are invariant under $S(t)$, and there is an $\varepsilon'<\varepsilon$ such that*

$$\|S(t)|_{(I-P)H}\|\leqq C\,e^{t(\varepsilon'-a_{\min}/2)},\qquad t\geqq 0. \tag{14}$$

Proof. All that needs to be proved is the decay estimate. Let $\tilde{S}=S(1)|_{(I-P)H}$. Then by (iii), $\sigma(\tilde{S})\subset\{|z|\leqq\rho\}$ for some $\rho<e^{-a_{\min}/2+\varepsilon}$. The spectral radius formula now implies

$$\rho=\lim_{n\to\infty}\|\tilde{S}^n\|^{1/n}=\lim\|S(n)|_{(I-P)H}\|^{1/n}.$$

It follows that

$$\|S(n)|_{(I-P)H}\|\leqq\text{const.}\ e^{(\varepsilon'-a_{\min}/2)n}$$

for $n=0,1,2,\ldots$. Since S is a contraction semigroup for $t\geqq 0$, we have

$$\|S(t)|_{(I-P)H}\|\leqq\|S([t])|_{(I-P)H}\|\leqq\text{const.}\ e^{(\varepsilon'-a_{\min}/2)[t]}$$

where [ ] is the greatest integer function. This inequality implies the desired estimate.

The above theorem renders precise the heuristic description of decay described in the introduction.

Corollary 5. *If $r=\sup\{\operatorname{Re} z\,|\,z\in\sigma(G)\}$ and $r>-a_{\min}/2$, then there is an integer m and a real number c such that for all $t\geqq 0$ we have*

$$\|S(t)\|\leqq c(1+t^m)e^{rt}. \tag{15}$$

Proof. Choose $\varepsilon>0$ so that $\varepsilon-a_{\min}/2<r$, and then write $H=PH\oplus(I-P)H$ as in Theorem 4. The estimate (14) suffices for the consideration of $S|_{(I-P)H}$. How-

ever, PH is finite dimensional and $\sigma(G|_{PH}) \subset \{\operatorname{Re} z \leqq r\}$, so for $t \geqq 0$

$$\|S(t)|_{PH}\| \leqq \text{const.}\,(1+t^m)e^{rt}$$

with $m+1 \leqq \dim PH$.

Remark. It is a simple matter to derive decay estimates for the higher derivatives of u. For example, if $h \in D(G)$ and $u = \pi_1(S(t)h)$, then $u_t = \pi_1(S(t)Gh)$ is also a finite energy solution of (1), (2). Hence $E(t, u_t)$ decays exponentially, and in particular $\|u_t\|_{L_2}$ and $\|u_{tt}\|_{L_2}$ decay like $e^{-(r-\varepsilon)t}$. From equation (1) it follows that $\|\Delta u\|_{L_2}$ decays at the same rate. Thus $\|u\|_{H_2(\Omega)} = O(e^{-(r-\varepsilon)t})$, since $\Delta\colon H_2(\Omega) \cap \mathring{H}_1(\Omega) \to L_2(\Omega)$ is an isomorphism. In the same way, if $h \in D(G^k)$ then $\|D^{\beta}_{t,x} u\|_{L_2} = O(e^{-(r-\varepsilon)t})$ for $|\beta| \leqq k+1$.

The occurence of eigenvalues of G with real part greater than $-a_{\min}/2$ is connected with overdamping. If $a_{\max}$ is sufficiently small, one might expect that these eigenvalues do not occur and that $\|S(t)\| = O(e^{t(\varepsilon - a_{\min}/2)})$ as $t \to \infty$ for any $\varepsilon > 0$.

Theorem 6. *Let λ be the smallest eigenvalue of $-\Delta$ with Dirichlet boundary conditions. If*

$$a_{\min}(2a_{\max} - a_{\min}) < 4\lambda$$

then $\sigma(G) \subset \{\operatorname{Re} z \leqq -a_{\min}/2\}$.

Remark. If a is constant the above inequality becomes $4\lambda > a^2$. Since $\lambda = -\lambda_1$ in the notation of (6), this is precisely the condition on a needed to rule out overdamping.

Proof. Suppose there is a value μ with $\operatorname{Re}\mu > -a_{\min}/2$ and an $h \in H \setminus \{0\}$ such that $Gh = i\mu h$. If $h = (\phi, \psi)$ then $u = e^{\mu t}\phi$ satisfies (1), (2). We now apply the method of proof used in Theorem 1. Let $\alpha = a_{\min}/2$ and $v = e^{\alpha t}u$. Then

$$\begin{aligned} e^{2\alpha t}\|\nabla u\|^2_{L_2} &\leqq E(t, v) \\ &\leqq \mathscr{I}(t) + (\alpha a_{\max} - \alpha^2)\|v(t)\|^2_{L_2} \\ &\leqq \mathscr{I}(0) + (\alpha a_{\max} - \alpha^2)e^{2\alpha t}\|u(t)\|^2_{L_2}. \end{aligned}$$

Since $h \neq 0$ and $Gh = i\mu h$, it follows that $\phi \neq 0$. Hence $\|u(t)\|_{L_2} \neq 0$. Then as $t \to \infty$

$$\frac{\|\nabla\phi\|^2_{L_2}}{\|\phi\|^2_{L_2}} = \frac{\|\nabla u\|^2_{L_2}}{\|u\|^2} = \alpha a_{\max} - \alpha^2 + o(1).$$

Letting $t \to \infty$ we have

$$\lambda \leqq \frac{\|\nabla\phi\|^2_{L_2}}{\|\phi\|^2_{L_2}} \leqq \alpha a_{\max} - \alpha^2,$$

that is, $4\lambda \leqq a_{\min}(2a_{\max} - a_{\min})$.

Remark. If one applies estimate (10) to the solution u, this gives the weaker estimate $2\lambda \leqq a_{\max}a_{\min}$. In particular, when a is constant this does not give a sharp result. The reason is that in (10) the need to estimate $\|u_t\|^2_{L_2}$ introduces some leeway into the estimate for ∇u.

Based on Theorem 6 and the case $a=$constant, we make the following conjecture:

If λ_N is the Nth largest (counting multiplicity) eigenvalue of $-\Delta$ with Dirichlet boundary conditions, and if $a_{\max}^2<4\lambda_N$, then there are at most N eigenvalues (counting multiplicity) of G with real part larger than $-a_{\min}/2$.

We have been concerned with upper bounds for $S(t)$, and the theme has been that "essentially" $\|S(t)\| \leqq \text{const.}\, e^{-t a_{\min}/2}$. There is a corresponding lower bound which, somewhat surprisingly, is true without qualification.

Theorem 7. *If u is a finite energy solution of* (1), (2), *then there is a positive constant c such that*

$$E(t,u) \geqq c\, e^{-a_{\max} t} E(0,u), \qquad t \geqq 0.$$

Proof. Let $\alpha=a_{\max}/2$ and $v=e^{\alpha t}u$. Then for v we have the differential equation

$$v_{tt}-\Delta v+(\alpha^2+a\alpha)v=(a_{\max}-a)v_t.$$

Since $a_{\max}-a\geqq 0$ the functional

$$J(t)=\int_\Omega \left(|v_t(t)|^2+|\nabla v(t)|^2+(\alpha^2+a\alpha)|v(t)|^2\right)dx$$

is increasing in t. Notice also that $\alpha^2+a\alpha\geqq 0$, so the norms $J^{1/2}$ and $E(t,v)^{1/2}$ are equivalent. Thus, for positive constants independent of t, u, v, we have

$$\begin{aligned} e^{-a_{\max} t} E(t,u) &\geqq \text{const. } E(t,v) \\ &\geqq \text{const. } J(t) \geqq \text{const. } J(0) \\ &\geqq \text{const. } E(0,v) \geqq \text{const. } E(0,u). \end{aligned}$$

In terms of S this theorem asserts that

$$\|S(t)h\|_H \geqq c\, e^{-t a_{\max}/2}\|h\|_H$$

for all $t\geqq 0$ and $h\in H$. Since $S(-t)=S(t)^{-1}$ this shows that $\|S(-t)\|\leqq c\, e^{t a_{\max}/2}$ for all $t\geqq 0$.

Corollary 8. $\sigma(G)\subset\{z \mid \operatorname{Re} z\geqq -a_{\max}/2\}$.

Proof. For $\operatorname{Re} z<-a_{\max}/2$ we have

$$(G-zI)^{-1}=\int_{-\infty}^{0} e^{-zt}S(t)\,dt,$$

an absolutely convergent integral central in the theory of semigroups (see [3]).

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