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COUNTING PROCESSES AND INTEGRATED CONDITIONAL RATES: A MARTINGALE APPROACH WITH APPLICATION TO DETECTION

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ABSTRACT

COUNTING PROCESSES AND INTEGRATED CONDITIONAL RATES: A MARTINGALE APPROACH WITH APPLICATION TO DETECTION

Martingale theory, as recently developed by Meyer, Kunita, Watanabe and Doléans-Dade, is used to study Counting Processes (CP) and their likelihood functions. Here a CP is a stochastic process having right-continuous sample paths constant except for randomly located positive jumps of size one.

First the problem of modeling and description of CP's is examined. Let (F_t) be an increasing right-continuous family of σ-algebras to which the CP (N_t) is adapted and suppose that the random variable N_t is a.s finite for each t. The Doob-Meyer decomposition for supermartingales associates to the CP (N_t) a unique natural increasing process (A_t) which makes the process $(M_t \triangleq N_t - A_t)$ a local martingale with respect to (F_t). This decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part (M_t) which is not predictable and (A_t) which can be perfectly predicted. The process (A_t) is called the Integrated Conditional Rate (ICR) of (N_t) with respect to (F_t) for the following reason: when (N_t) satisfies some sufficient conditions the ICR takes on the form $(\int_0^t \lambda_s ds)$, where (λ_t) is

a non-negative process called the conditional rate, satisfying $\lambda_t = \lim_{h \to 0} E[h^{-1}(N_{t+h} - N_t) | \mathcal{F}_t]$. Our approach, however, requires only the weak assumption that N_t is a.s finite for each t; there always exists an ICR while in general a conditional rate cannot be defined. Sufficient conditions for the existence of a conditional rate are presented.

Based on the character (e.g., totally inaccessible) of the stopping times defined by its jumps any CP is shown to be uniquely decomposable into the sum of a regular CP and an accessible CP. It is also demonstrated that each class is completely characterized by continuity properties of the ICR. CP's with independent increments are uniquely distinguished by a property of their ICR's: they are deterministic and given by the mean of the CP.

Expressions for probability generating functions and conditional probabilities $\mathcal{P}\{N_t - N_s = n | \mathcal{F}_s\}$ are derived. The technique used (a typical martingale approach) can be specialized to CP's which admit a conditional rate satisfying some kind of conditional independence property and for processes of independent increments. Results in this last case are well known when the mean of the process is continuous, but our derivation extends to the general case.

Likelihood ratios for detecting CP's are computed via an extension of the three-step technique (the Likelihood Ratio Representation Theorem, the Girsanov Theorem and the Innovation Theorem) introduced by Duncan and Kailath in their works on detecting a stochastic signal in white noise.

Suppose (N_t) is under the measure P_o (resp. P_1) a CP with an ICR with respect to the family of σ-algebras F_t^o (resp. F_t^1) of the form $(\int_0^t \lambda_s^o \, dm_s)$ (resp. $(\int_0^t \lambda_s^1 \, dm_s)$) where (λ_t^o) (resp. (λ_t^1)) is a positive process and m_t a continuous deterministic increasing function. Then the likelihood function L_t for the above detection problem and a time of observation $[0,t]$ is shown to be

$$L_t = \left(\prod_{J_n \leq t} \frac{\hat{\lambda}^o_{J_n}}{\hat{\lambda}^1_{J_n}} \right) \exp\left[\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^o) dm_s\right]$$

where $\hat{\lambda}_t^i = E_i[\lambda_t^i | \sigma(N_u, 0 \leq u \leq t)]$, $i = 0,1$ ($E_i(\cdot)$: expectation operator with respect to P_i) and J_n is the time of n^{th} jump of (N_t). Stochastic integral equations which allow us to compute the likelihood function L_t recursively are also derived.

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LIST OF KEY TERMS

LIST OF SYMBOLS AND ABBREVIATIONS

The following list contains symbols and abbreviations frequently used in this dissertation, alphabetically ordered.

Symbol	Meaning	Defined on Page
a.s	Almost surely	
(A_t)	Integrated Conditional Rate	67
A^+	Class of integrable, right-continuous, adapted, increasing processes which are zero at the time origin	45
A	$A^+ - A^+$	45
$\mathcal{B}([a,b])$	Borel sets of the interval [a,b]	9
$\mathbb{C}$	Set of complex numbers	
CP	Counting Process	57
E	Expectation operator with respect to the measure P	
E_i	Expectation operator with respect to the measure P_i	
$E(\cdot \mid F_t)$	Expectation operator conditioned to the σ-algebra F_t	
(F_t)	Increasing family of σ-algebras	11
F_T	σ-algebra associated to the (F_t) stopping time T	14

Symbol	Meaning	Defined on Page
(G_t)	Increasing family of σ-algebras	11
G_T	σ-algebra associated to the (G_t) stopping time T	14
$H(F_t)$	Class of locally bounded predicatble processes	49
I_A	Indicator function of the set A	8
ICR	Integrated Conditional Rate	67
J_n	Time of n^{th} jump of the CP (N_t)	58
LHS	Left Hand Side	
(L_t)	Likelihood ratio	178
(λ_t)	Conditional rate	100
$L^2(X)$	Class of predictable process (H_t) such that $E\int_0^\infty H_s^2 d\langle X\rangle_s < \infty$	40
$L(P,F_t)$	Space of local martingales which are zero at the time origin	46
(M_t)	The local martingale $(N_t - A_t)$	69
$M^2(P,F_t)$	Space of square integrable martingales which are zero at the time origin	38
$M_\ell^2(P,F_t)$	Space of martingales which are locally square integrable and zero at the time origin	38
(N_t)	Counting process	57

Symbol	Meaning	Defined on Page
$\mathcal{N}_t$	σ-algebra $\sigma(N_u, 0 \leq u \leq t)$ generated by the CP (N_t) up to and at time t	57
$\mathbb{N}$	Set of integers	
$\mathcal{P}, \mathcal{P}_0, \mathcal{P}_1$	Probability measures on $(\Omega, \mathcal{F})$	
$\overline{\mathcal{P}}_i^R$	Restriction of the measure $\mathcal{P}_i$ to the σ-algebra $\mathcal{N}_R$	176
$\mathbb{Q}_+$	Set of positive rationals	
RHS	Right Hand Side	
$\mathbb{R}$	Real line	
$\mathbb{R}_+$	Positive real line	
$\mathcal{V}^+$	Class of finite valued, right-continuous, adapted, increasing processes which are zero at the time origin	45
$\mathcal{V}$	$\mathcal{V}^+ - \mathcal{V}^+$	45
(X_t^+)	Positive part of (X_t): $X_t \vee 0$	
(X_t^-)	Negative part of (X_t): $-(X_t \wedge 0)$	
(X_t^c)	Unique local martingale which is the continuous part of the local martingale (X_t)	47
(X_t^d)	Unique local martingale given by $(X_t - X_t^c)$ where (X_t) is a local martingale	47

Symbol	Meaning	Defined on Page
$\langle X \rangle_t$	Natural increasing process associated with the square integrable martingale (X_t)	39
$[X]_t$	Quadratic variation process of the local martingale (X_t)	49
ω	Elementary outcome	7
Ω	Set of elementary outcomes	7
(Ω, F)	Underlying measurable space	7
$\sigma(\cdot)$	σ-algebra generated by	
ϕ	Impossible event, empty set	
$\psi(z,t,s)$	Probability generating function	111
Δf_t	$f_t - f_{t-}$: jump of the function f_t at t	37
$a \wedge b$	Minimum of a and b	
$a \vee b$	Maximum of a and b	

INTRODUCTION

In this thesis we examine the relation between counting processes and martingales and apply the pertinent results to solve the detection problem for a large class of counting processes. By a counting process we mean a process which is a.s zero at the time origin and has a.s right-continuous sample paths which are constant except for positive randomly located jumps of size one. Such a counting process (N_t) can be interpreted as one which counts starting from the time origin the number of points of a point process falling in the interval $(0,t]$. We think of a point process as a sequence of points randomly located on the real line.

To fix ideas in their most simple form suppose now for a moment that (N_t) is a counting process of independent increments and denote its mean EN_t, supposed finite for each t, by m_t. Then it is easy to see by a direct computation that the process

$$M_t = N_t - m_t \tag{I.1}$$

is a martingale. If furthermore (N_t) is of Poisson type with rate λ_t then

$$m_t = \int_0^t \lambda_s ds$$

Recall also that

$$\lambda_t = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \right)$$

The literature (Rubin [R2], Snyder [S1],[S2],[S3], Clark [C1] and lately Brémaud [B1]) reflects interest in the case of a counting process, sometimes called extension of Poisson process, which can be described by a random rate. This random rate, also called intensity function, has the interpretation:

$$\lambda_t = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| \mathcal{F}_t \right) \tag{I.2}$$

where $(\mathcal{F}_t)$ is an increasing family of σ-algebras to which (N_t) is adapted. Usually the σ-algebra $\mathcal{F}_t$ is taken to be the minimal σ-algebra generated by the process (N_t) up to and at time t. We denote this last σ-algebra by $\mathcal{N}_t$. The approach usually taken in the literature ([S1],[S2],[S3], [R2]) to describe such a counting process (N_t) is to assume that the limits

$$\lim_{\Delta t\to 0_+} \frac{1}{\Delta t} [1 - P\{N_{t+\Delta t} - N_t = 0 | \mathcal{N}_t\}]$$

$$= \lim_{\Delta t\to 0_+} \frac{1}{\Delta t} P\{N_{t+\Delta t} = N_t + 1 | \mathcal{N}_t\} \tag{I.3}$$

exist and are equal. Denote this limit by μ_t. This process (μ_t), clearly in the same spirit as the process (λ_t) defined by (I.2), has the following interpretation:

the probability that the counting process (N_t) has a jump in the interval $(t,t+\Delta t]$ given the past is equal to $\mu_t \Delta t + o(\Delta t)$. The technique to obtain results is then to examine what is happening in small cells of size Δt and take limits. But this limiting procedure is not simple (the terms $o(\Delta t)$ are random!) and for validity requires numerous purely technical assumptions on the process (μ_t) (see, for example, in [R2] conditions (2), (3) and (4)). This approach has other drawbacks and difficulties. The existence of counting processes (excluding Poisson processes) for which the above limits (I.3) exist and have all the required properties has never been shown. Also the specification of the process (μ_t) may not define a unique counting process, if indeed any such counting process exits. The problem of existence of counting processes which admit a random rate as defined by (I.2) has been treated only lately by Brémaud [B1] in his dissertation, where a partial answer to this problem is given: the existence of counting processes which possess a <u>bounded</u> random rate with respect to the family of <u>σ-algebras generated by the counting process itself</u> is demonstrated by the use of absolutely continuous changes of measures. We discuss and extend this technique in Section 3.1, while in Section 2.5 sufficient conditions for the existence of a conditional rate are given.

We are interested in the generalization of the above ideas. The basic mathematical tool involved in this is the theory of martingales and related processes. This material may not be familiar to the reader and is reviewed in Chapter 1 in which also the basic notation used throughout this thesis is introduced. In Chapter 2 we consider a counting process (N_t) with the sole assumption that

(i) The random variable N_t is a.s finite for each t. The Doob-Meyer decomposition for supermartingales then implies that any such counting process adapted to a right-continuous increasing family of σ-algebras $(\mathcal{F}_t)$ can be uniquely written as a sum [compare with (I.1) in the case of processes of independent increments]

$$N_t = M_t + A_t \tag{I.4}$$

where the process (M_t) is a local martingale with respect to $(\mathcal{F}_t)$ and (A_t) is a natural increasing process. This process (A_t), when it has a.s absolutely continuous sample paths, can be expressed as

$$A_t = \int_0^t \lambda_s ds \tag{I.5}$$

where furthermore (λ_t) satisfied relation (I.2). This process (λ_t) is then reasonably called the "Conditional Rate" of the process (N_t) with respect to the family $(\mathcal{F}_t)$. For the process (A_t) relation (I.5) suggests then the name "Integrated Conditional Rate" of (N_t) with respect to $(\mathcal{F}_t)$. This terminology will be used even when, as is usually the case, a conditional rate does not exist. Observe that this approach is much more general and goes in the opposite direction of the one taken in previous works ([R2],[S1], [S2],[S3]): we begin with a counting process (N_t) satisfying the very weak assumption (i) and arrive at the notion of integrated conditional rate, instead of defining by (I.3) a conditional rate (subject to numerous assumptions) and then assuming the existence of a hopefully unique counting process corresponding to this conditional rate. Chapter 2 is believed to be the first systematic study of the notion of integrated conditional rate. In Chapter 3 the likelihood ratio for detecting counting processes is computed. This is done using the three-step technique introduced by Kailath [K3] and Duncan [D3] in their works on detection of a stochastic signal in white noise. These three steps are the Likelihood Ratio Representation Theorem ([D3], [B1]), the Girsanov Theorem ([G1],[V1]) and the Innovation Theorem ([K3]). By this method likelihood ratios for a large class of counting processes can be found. Stochastic integral equations which allow us to compute the likelihood ratio continuously in time by recursive techniques are also derived. It is also shown how the Girsanov Theorem can be used to prove the existence of counting processes for which the integrated conditional rate is in a special form.

Results of this third chapter constitute an extension of Brémaud's work [B1]. It should be noted in this connection that Brémaud's proof of the Likelihood Ratio Representation Theorem is erroneous. We will show that the errors cannot be corrected without supplying a missing assumption.

The likelihood ratio formulas presented in Section 3.4 constitute a generalization of the formulas given by Reiffen and Sherman [R4] and Bar-David [B2] in the context of Poisson processes, and Skorokhod [S4] in the context of processes with independent increments.

CHAPTER 1

MATHEMATICAL REVIEW: MARTINGALES AND RELATED PROCESSES

1.0 INTRODUCTION

We assume the reader to be familiar with measure theory in general and as it applies to the study of probabilities and stochastic processes. But he may not be acquainted with concepts such as stopping times, martingales, the Doob-Meyer decomposition and stochastic integrals, concepts which are heavily used in this thesis.

Therefore, the main purpose of this chapter is to introduce and explain the mathematical notions necessary for a good understanding of this study, and to serve as a reference which will hopefully facilitate the reading. At the same time, the terminology used throughout this thesis will be introduced.

The main references for this review chapter are [M1] for Sections 1.1 to 1.6, [M1] and [R3] for Section 1.7, finally [D1] and [M5] for Sections 1.8 and 1.9. Capital letters are systematically used to denote random variables.

1.1 STOCHASTIC PROCESSES

The standard notation $(\Omega,\mathcal{F},P)$ is used to denote a probability space. The set Ω is the set of all possible outcomes of a specified experiment and the sets of the σ-algebra $\mathcal{F}$ are called events. A measurable map from the measurable space $(\Omega,\mathcal{F})$ into a measurable space $(E,\mathcal{E})$, where $\mathcal{E}$

denotes a σ-algebra of subsets of the set E, is called a random variable. The following notation is also standard:

Definition 1.1.1: If A is any set we define the indicator function of the set A to be the function given by:

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Now the definition of stochastic processes: ([M1], Definition 2-IV)

Definition 1.1.2: Let T be an index set. A stochastic process is a system $(\Omega, \mathcal{F}, \mathcal{P}, (X_t, t \in T))$ consisting of (1) a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and (2) a family $(X_t, t \in T)$ of random variables defined on $(\Omega, \mathcal{F})$ with values in a measurable space $(E, \mathcal{E})$.

The measurable space $(E, \mathcal{E})$ is called the state space of the process.

Whenever it causes no ambiguity we will use the simplified notation $(X_t, t \in T)$ or even (X_t).

A stochastic process (X_t) is in particular a mapping from $T \times \Omega$ into E. We denote by $X_t(\omega)$ the image by this mapping of the point (t, ω). The random variable $X_t(\cdot)$ (simplified notation: X_t) is called the state of the process at time t and the mappings $X_\cdot(\omega)$ of T into E are called trajectories or sample paths (or functions) of the process.

Definition 1.1.3: (see [M1], Definition 5-IV) Let $(X_t, t \in T)$ and $(Y_t, t \in T)$ be two stochastic processes defined on the same probability space (Ω, F, P) with values in the same state space (E, E). The process $(Y_t, t \in T)$ is a modification of the process $(X_t, t \in T)$ if $X_t = Y_t$ a.s for each $t \in T$.

If two processes (X_t) and (Y_t) are modifications of each other then they have the same finite dimensional distributions (i.e.: $P\{X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\} = P\{Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n\}$ for every finite system of times $t_1, \ldots, t_n$ and sets $A_1, \ldots, A_n$ of E; in other words (X_t) and (Y_t) are equivalent processes ([M1], Definition 3-IV)). This motivates the fact that, as usually done when dealing with stochastic processes, we will not distinguish between modifications of the same process.

In this thesis, the state space will always be the real line $\mathbb{R}$ equipped with its Borel sets $B(\mathbb{R})$* and the index set T will be the positive real line $\mathbb{R}_+$. From now on we restrict ourselves to this case. We will deal most of the time with stochastic processes having a.s right-continuous trajectories. We call them right-continuous processes for abbreviation. If two processes (X_t) and (Y_t) are two modifications of the same process then we write ·

$$X_t = Y_t \text{ a.s for each } t \in \mathbb{R}_+ \tag{1.1.1}$$

*The notation $B([a,b])$ denotes the Borel sets of an interval $[a,b]$.

By

$$X_t = Y_t \quad \text{for every } t \in \mathbb{R}_+ \text{ a.s} \qquad (1.1.2)$$

we mean

$$P\{X_t = Y_t,\ t \in \mathbb{R}_+\} = 1$$

Condition (1.1.2) obviously implies (1.1.1). The following Remark shows that the converse is true if the processes (X_t) and (Y_t) are both left or right-continuous.

Remark 1.1.4: Suppose (X_t) and (Y_t) are two left or right continuous modifications of the same process. Then clearly the set

$$\Lambda = \{X_t = Y_t,\ t \in \mathbb{Q}_+\}$$

where $\mathbb{Q}_+$ denotes the set of positive rational numbers, is measurable, and

$$P(\Lambda) = 1$$

For $t \notin \mathbb{Q}_+$, let $t_n \in \mathbb{Q}_+$ be a decreasing or increasing (accordingly to the right or left continuity property of the two processes (X_t) and (Y_t)) sequence converging to t. For $\omega \in \Lambda$ we have

$$X_t = \lim_n X_{t_n} = \lim_n Y_{t_n} = Y_t$$

which shows that

$$X_t = Y_t \qquad \text{for every } t \in \mathbb{R}_+ \text{ a.s}$$

1.2 STOPPING TIMES

The basic reference for this section is [M1], Chapter IV and VII.

Let $(\Omega, \mathcal{F})$ be a measurable space and let $(\mathcal{F}_t, t \in \mathbb{R}_+)$ be a family of σ-subalgebras of $\mathcal{F}$ such that one has $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. We say that $(\mathcal{F}_t, t \in \mathbb{R}_+)$ is an increasing family of σ-subalgebras of $\mathcal{F}$ and we often use the simplified notation $(\mathcal{F}_t)$. For each $t \in \mathbb{R}_+$ the σ-subalgebra $\mathcal{F}_t$ is called the σ-algebra of events prior to t. We denote by $\mathcal{F}_\infty$ the σ-algebra generated by the union of the σ-algebras $\mathcal{F}_t$ and we set:

$$\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$$

The family $(\mathcal{F}_t)$ is said to be right-continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for every $t \in \mathbb{R}_+$.

Definition 1.2.1: (see [M1], Definition 31-IV) Let (X_t) be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $(\mathcal{F}_t)$ be an increasing family of σ-subalgebras of $\mathcal{F}$. The process (X_t) is said to be adapted to the family $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{R}_+$.

We often take for the σ-algebra $\mathcal{F}_t$ the σ-algebra $\sigma(X_u, 0 \leq u \leq t)$ generated by the process (X_t) up to time t. If a process (X_t) is adapted to a family $(\mathcal{F}_t)$ we must obviously have the relation $\mathcal{F}_t \supset \sigma(X_u, 0 \leq u \leq t)$.

It is convenient to think of the events of $\mathcal{F}$ as the representation of certain phenomena which can occur in a certain universe. The σ-algebras F_t consist then of the events that occur prior to the instant t. The F_t-measurable random variables are hence those which depend only on the evolution of the universe prior to t. In problems of detection and filtering, the σ-algebra F_t represent, loosely speaking, the information available up to and at time t on which our detection's scheme and estimation are based. The situation where F_t is given by $\sigma(X_u, 0 \leq u \leq t)$ means then that the available information at time t is obtained by observing the process (X_t) up to time t. In the other case where F_t properly contains $\sigma(X_u, 0 \leq u \leq t)$ then we have more information at our disposition than merely that generated by the process (X_t).

We will now introduce the very important notion of stopping times. Suppose an observer watches for the appearance of a specified event and notes the first time $T(\omega)$ it occurs. The event $\{T \leq t\}$ takes place if and only if the event we are watching for is produced at least once before the time t, or at that instant. Therefore the event $\{T \leq t\}$ belongs to the σ-algebra of events prior to t. This motivates the following definition: ([M1], Definition 33-IV)

<u>Definition 1.2.2</u>: Let (Ω, F) be a measurable space and let (F_t) be an increasing family of σ-subalgebras of F. A positive random variable T defined on (Ω, F) is said to

be a stopping time* of the family (F_t) if T satisfies the following property: the event $\{T \leq t\}$ belongs to F_t for every $t \in \mathbb{R}_+$.

Remark 1.2.3:(a) We often allow stopping times to take the value $+\infty$

(b) The notion of stopping time depends on the family (F_t) (but not on a measure).

(c) If the condition $\{T < t\} \in F_t$ for every $t \in \mathbb{R}_+$ is satisfied and if the family (F_t) is right-continuous then T is a stopping time (see [M1], §34-IV).

To get some intuitive feeling for stopping times, here are a couple of examples. In a nuclear reactor, the motion of a particle may be described by a random walk; the first time a particle hits an absorbing barrier, e.g., the shield, is a stopping time. Involved in a betting game you might decide to limit the risks by adopting the following strategy: you will stop playing the first time a given gain or loss is achieved. The time at which this occurs is a stopping time. The two above stopping times are called hitting times (see later Example 1.2.6). Stopping times are a basic tool in the study of Markov processes and martingales. To each stopping time T we can associate in the following

*The name "stopping time" ("optional time," "Markov time" are also used in the literature) comes from the theory of Markov processes. Generally speaking these were times at which decisions were taken or where the process was stopped. A better name, in our opinion, would be "causal time."

way a σ-algebra which can be interpreted as the σ-algebra of events prior to T (see [M1], Definition 35-IV):

Definition 1.2.4: Let T be a stopping time of the family (F_t). We denote by F_T the collection of events $A \in F_T$ such that

$$A \cap \{T \leq t\} \in F_t \text{ for every } t$$

We call F_T the σ-algebra of events prior to T.

It is easily verified that these events do constitute a σ-algebra and that if the stopping time is equal to the constant t, the σ-algebra F_t is recovered.

Theorem 1.2.5: Let S and T be two stopping times such that $S \leq T$ then we have $F_S \subset F_T$.

For the above and other properties of stopping times see [M1], Chapter IV.

In this thesis all stopping times will be of the type presented in the following example ([M1], § 44-IV).

Examples 1.2.6: Let (F_t) be a right continuous family and let (X_t) be a right-continuous stochastic process adapted to the family (F_t). Let B be an open subset of $\mathbb{R}$ and define:

$$D_B = \begin{cases} \inf \{s: X_s \in B\} & \text{if this set is non empty} \\ +\infty & \text{otherwise} \end{cases}$$

We have

$$\{D_B < t\} = \bigcup_{\substack{r \text{ rational} \\ r < t}} \{X_r \in B\}$$

from the right-continuity of the paths. The event on the left thus belongs to F_t and this implies by Remark 1.2.3 (c) that D_B is a stopping time. This stopping time is called the first passage time or hitting time of B.

We now give a classification of stopping times which will be very useful in the rest of this work, in particular to classify point processes.

Definition 1.2.7: (see [M1], Definition 42-VII; [D1])

Let T be a stopping time of the family (F_t).

(a) T is said to be totally inaccessible (with respect to the family (F_t)) if T is not a.s infinite and if for every increasing sequence (S_n) of stopping times majorized by T we have $P\{\lim S_n = T, S_n < T < \infty \text{ for every } n\} = 0$.

(b) The stopping time T is said to be inaccessible (with respect to the family (F_t)) if there exists a totally inaccessible stopping time S such that $P\{T = S < \infty\} > 0$.

(c) A stopping time T is said to be accessible (with respect to the family (F_t)) if it is not inaccessible.

(d) A stopping time T is said to be predictable (with respect to the family (F_t)) if there exists an increasing sequence (S_n) of stopping times which converge a.s to T and such that for every n one has a.s $S_n < T$ on the set $\{T > 0\}$.

It should be strongly emphasized that all these definitions depend on the family $(\mathcal{F}_t)$ choosen. Clearly predictable (resp. totally inaccessible) stopping times are accessible (resp. inaccessible). But in certain circumstances accessible stopping times are predictable. Before elaborating on this result we need the following concepts ([M1], Definitions 39 and 40, VII).

Definition 1.2.8: The family $(\mathcal{F}_t)$ is said to be free of times of discontinuity if for every increasing sequence (S_n) of stopping times

$$\mathcal{F}_{(\lim_n S_n)} = \bigvee_n \mathcal{F}_{S_n}$$

Definition 1.2.9: Let T be a stopping time of a family $(\mathcal{F}_t)$ and let A be an element of $\mathcal{F}_T$. By T_A we denote the stopping time

$$T_A(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in A \\ +\infty & \text{otherwise} \end{cases}$$

The fact that T_A is indeed a stopping time can be easily verified.

Definition 1.2.10: Let T be a stopping time; T is said to be a time of discontinuity for the family $(\mathcal{F}_t)$ if there exists an event $A \in \mathcal{F}_T$ and an increasing sequence (S_n) majorized by T_A such that the event

$$\{\lim_n S_n = T_A\}$$

does not belong to the σ-algebra $\bigvee_n F_{S_n}$.

It can be verified that the two Definitions 1.2.8 and 1.2.10 are compatible (see § 41-VII of [M1]).

Theorem 1.2.11: (Theorem 45-VII of [M1]) Let T be an accessible stopping time of a family (F_t) which is not a time of discontinuity for the family (F_t). Then T is predictable.

We now give some illustrations. In Section 2.4, where the above concepts are applied to counting processes, it is shown in particular that the time of n^{th} occurrence of a Poisson process is a totally inaccessible stopping time with respect to the family of σ-algebras generated by the process itself. Note that any stopping time with respect to a family (F_t) is always predictable with respect to the family (G_t) where $G_t = F_\infty$ for each t. A stopping time which is a constant is obviously predictable with respect to any family of σ-algebras. Define now a family (F_t) by

$$F_t = \{\phi,\Omega\} \quad 0 \leq t < 1$$

$$F_t = \{\phi,\{\omega_1\},\{\omega_2\},\Omega\} \quad 1 \leq t < \infty$$

The stopping time

$$T(\omega) = \begin{cases} 1 & \omega = \omega_1 \\ a & \omega = \omega_2 \end{cases}$$

for any a greater than one is a time of discontinuity for the family (F_t). To see that define the sequence of stopping times $(S_n \triangleq 1 - 1/n)$. Then $S_n < T$ and

$$\{\lim_n S_n = T\} = \{\omega_1\} \notin \bigvee_n F_{S_n} = \{\phi,\Omega\}$$

This stopping time T is accessible but not predictable.

Finally we give a decomposition result for stopping times (Theorem 44-VII of [M1]).

Theorem 1.2.12: Let T be a stopping time. There exists an (essentially unique) partition of the set $\{T < \infty\}$ into two elements of F_T, A and A^1, such that the stopping time T_A is accessible and the stopping time T_{A^1} is totally inaccessible.

1.3 MEASURABLE PROCESSES

Definition 1.3.1: ([M1], Definition 45-IV) Let (Ω,F,P) be a probability space, and let $(F_t, t \in \mathbb{R}_+)$ be an increasing family of σ-subalgebras of F. Let $(X_t, t \in \mathbb{R}_+)$ be a stochastic process. We say that (X_t) is progressively measurable with respect to the family (F_t) if, for each $t \in \mathbb{R}_+$, the mapping $(u,\omega) \to X_u(\omega)$ from $[0,t] \times \Omega$ into $(\mathbb{R},B(\mathbb{R}))$ is measurable with respect to the σ-algebra $B([0,t]) \times F_t$. The process (X_t) is said to be measurable (without reference to a family of σ-algebras) if the mapping $(t,\omega) \to X_t(\omega)$ is measurable over the product σ-algebra $B(\mathbb{R}_+) \times F$.

For a measurable process (X_t) adapted to (F_t) there always exists a modification which is progressively measurable (see [M1] Theorem 46-IV). We will always deal with processes with right-(sometimes left-) continuous trajectories. The following theorem is then what we need ([M1], Theorem 47-IV).

Theorem 1.3.2: Let (X_t) be a right-continuous stochastic process adapted to a family (F_t). The process (X_t) is then progressively measurable with respect to the family (F_t). The same conclusion is true for a process with left-continuous paths.

The following notation will be used constantly ([M1], Definition 48-IV).

Definition 1.3.3: Let (X_t) be a measurable stochastic process defined on (Ω,F,P) and let H be a positive random variable defined on Ω. We denote by X_H the random variable $X(\omega)_{H(\omega)}$.

Usually H is a stopping time and is allowed to take the value $+\infty$ when (X_t) is a process defined on $\mathbb{R}_+ \cup \{\infty\}$.
The following theorem is a basic tool when using stopping times, in particular when studying martingales ([M1], Theorem 49-IV).

Theorem 1.3.4: Let (X_t) be a progressively measurable process with respect to a family (F_t) and let T be a stopping time with respect to (F_t) (possibly infinite). The random variable X_T is then F_T-measurable.

We will often encounter the following situation. Let $(T_t, t \in \mathbb{R}_+)$ be a system of stopping times of a family of σ-algebras $(F_t, t \in \mathbb{R}_+)$ such that the mappings $t \to T_t(\omega)$ are increasing and right-continuous. Let $(X_t, t \in \mathbb{R}_+)$ be a stochastic process measurable with respect to the family $(F_t, t \in \mathbb{R}_+)$. The process $(Y_t = X_{T_t})$ and the family of σ-algebras $(G_t = F_{T_t})$ are respectively called "the transformed from (X_t) by the system (T_t)" and "the family of transformed σ-algebras." We have ([M1], Theorem 57-IV).

<u>Theorem 1.3.5</u>: The process (Y_t) is progressively measurable with respect to the family (G_t).

1.4 CLASS (D) AND (DL) PROCESSES

The following concepts which are generalizations of the notion of uniform integrability will be needed later on when dealing with the Doob-Meyer decomposition of supermartingales.

<u>Definition 1.4.1</u>: ([M1], Definition 17-IV) Let $(X_t, t \in \mathbb{R}_+)$ be a right-continuous stochastic process adapted to a family of σ-algebras $(F_t, t \in \mathbb{R}_+)$. Define τ as the collection of all finite stopping times of the family (F_t) (respectively, τ_a the collection of all stoppings times bounded by a positive constant a). (X_t) is said to belong to the class (D) (respectively belong to the class (D) on the interval [0,a]) if the collection of random variables X_T, $T \in \tau$ (respectively $T \in \tau_a$) is uniformly integrable.

(X_t) is said to belong to the class (DL), (or locally to the class (D)) if (X_t) belongs to the class (D) on every interval $[0,a]$, $(0 \leq a < \infty)$.

Remark 1.4.2(a): A constant time t is a particular case of stopping time. Therefore if a process (X_t) belongs to the class (D), it is a fortiori uniformly integrable. The converse is not true (for a counter example see [J1]).

(b) Every right-continuous and uniformly integrable martingale belongs to the class (D) ([M1], Theorem 19-VI).

(c) If (X_t) is a process such that $|X_t| \leq Y_t$ a.s and if (Y_t) is a process which belongs to the class (D), then it is easy to verify that the process (X_t) also belongs to the class (D).

(d) The notions of class (D) and (DL) arise in the context of the Doob-Meyer decomposition of a supermartingale into the difference of a martingale and an increasing process, but in the continuous parameter case only. While a supermartingale with discrete index set always admits a Doob-Meyer decomposition, such a decomposition exists, in the continuous parameter case, if and only if the supermartingale belongs to the class (DL) (see Section 1.7).

1.5 MARTINGALES

In this section every stochastic process is defined on a fixed probability space (Ω, F, P) and adapted to the same family $(F_t,\ t \in \mathbb{R}_+)$. We suppose that the probability space $(\Omega, \mathcal{F}, P)$ is complete and that the σ-algebra $\mathcal{F}_o$ contains all the P-negligible sets.

Definition 1.5.1: ([M1], Definition 1-V) Let $(\mathcal{F}_t, t \in \mathbb{R}_+)$ be an increasing family of σ-subalgebras of $\mathcal{F}$ and $(X_t, t \in \mathbb{R}_+)$ a real-valued process, adapted to the family $(\mathcal{F}_t)$. The process (X_t) is said to be a martingale (respectively, a supermartingale, a submartingale) with respect to the family $(\mathcal{F}_t)$ if

(a) Each random variable X_t is integrable, and

(b) For every pair s,t of $\mathbb{R}_+$ such that $s \leq t$ we have

$$E(X_t | \mathcal{F}_s) = X_s \text{ a.s (respectively, } \leq X_s, \geq X_s)$$

Remark 1.5.2(a): This definition is not the most general. First the index set $\mathbb{R}_+$ can be in fact any arbitrary set ordered by a relation $\leq$. Secondly, in certain cases the assumption of integrability of X_t can be weakened (see [N1], Section 5 of Chapter IV).

(b) Here again the above definition is very much dependent on the family $(\mathcal{F}_t)$ choosen.

(c) If (X_t) is a supermartingale then the process $(-X_t)$ is a submartingale, and conversely. Thus theorems need only be stated for supermartingales (or submartingales).

We will not give here the basic properties and theorems concerning supermartingales, e.g., fundamental inequalities, optional sampling theorem, convergence theorem, etc. All these results and many others can be found, among other sources, in [M1], Chapter VI.

The simplest example of a martingale is the following ([M1], § 3-V).

Example 1.5.3: Let (F_t, $t \in \mathbb{R}_+$) be an increasing family of σ-subalgebras of F. For each integrable random variable X set

$$X_t = E(X|F_t)$$

The process (X_t) is then a martingale with respect to the family (F_t). By Theorem 19-V of [M1], this martingale is uniformly integrable. Conversly if a martingale (X_t) is uniformly integrable it follows from the supermartingale convergence theorem ([M1], Theorem 6-VI) that this martingale can be written in the above form. More precisely we have: ([M1], Theorem 18-V)

Theorem 1.5.4: Let (F_t) be an increasing family of σ-subalgebras of F. A process (X_t) is a uniformly integrable martingale with respect to the family (F_t) if and only if it can be written in the form

$$X_t = E(X_\infty | F_t)$$

where X_∞ is the limit a.s and in the mean of X_t as t goes to infinity.

Most theorems for supermartingales assume the right-continuity of these supermartingales. The following theorem gives a necessary and sufficient condition for the existence of a right-continuous modification

([M1], Theorem 4-VI)

Theorem 1.5.5: Suppose (X_t) is a supermartingale with respect to a right-continuous increasing family $(\mathcal{F}_t)$. The supermartingale (X_t) then admits a right-continuous modification if and only if the mean function $E\,X_t$ is right-continuous.

Remark 1.5.6: The mean of a martingale being constant it follows immediately that any martingale always admits a right-continuous modification. In accordance with the fact that we do not distinguish between modifications of the same process, we will adopt the following convention: when we speak of a martingale (X_t), we always mean its right-continuous modification. Consequently all martingales which appear in this thesis are right-continuous.

The following very useful result does not appear in our main reference [M1] but in [M3]. Therefore an original proof of this result will be provided, for easy reference, in Appendix A.1.

Lemma 1.5.7: Let $(\mathcal{F}_n, n \in \mathbb{N})$ be an increasing family of σ-subalgebras of $\mathcal{F}$ and $\mathcal{F}_\infty$ be the σ-algebra generated by the union of the $\mathcal{F}_n$. Let $(F_n, n \in \mathbb{N})$ be a sequence of random variables bounded in absolute value by an integrable random variable G and converging a.s to a random variable F. Then $E(F_n|\mathcal{F}_n)$ converges a.s to $E(F|\mathcal{F}_\infty)$.

1.6 POTENTIALS AND THE RIESZ DECOMPOSITION

The hypotheses of the preceding section will be used again in this one.

Definition 1.6.1: ([M1], § 9-VI) Let (X_t) be a right-continuous supermartingale. We say that (X_t) is a potential if the random variables X_t are a.s positive and if $\lim_{t\to\infty} EX_t = 0$.

We have the following theorem ([M1], Theorem 10-VI).

Theorem 1.6.2: (Riesz Decomposition) Let (X_t) be a right-continuous supermartingale with respect to a right-continuous increasing family (F_t). The following two conditions are equivalent:

(a) There exists a submartingale (V_t) such that: $V_t \leq X_t$ a.s for each t.

(b) There exists a martingale (Y_t) and a potential (Z_t) such that $X_t = Y_t + Z_t$ a.s for each $t \in \mathbb{R}_+$.

These two processes are then unique up to modification.

Remark 1.6.3(a): The right-continuity property of the processes involved implies (see Remark 1.1.4)

$$X_t = Y_t + Z_t \text{ for every } t \in \mathbb{R}_+ \text{ a.s}$$

(b) This decomposition is easily obtained when the right-continuous supermartingale is uniformly integrable: by the supermartingale convergence theorem ([M1], Theorem 6-VI), X_t converges a.s and in the mean to a F_∞-measurable random variable X_∞. Define the martingale: $(Y_t = E(X_\infty | F_t))$. Then it is easy to verify that $(Z_t = X_t - E(X_\infty | F_t))$ is a potential. Furthermore we also have, by the convergence theorem again:

$$\lim_{t \to \infty} Z_t = 0$$

(see [M1], § 11-VI).

1.7 DOOB-MEYER DECOMPOSITION

INTRODUCTION

This decomposition of a supermartingale into the difference of a martingale and a continuous increasing process, discovered by Doob in the discrete case and demonstrated by Meyer in the continuous case, will play a very important part in our study of point processes. We will therefore spend some time reviewing the basic concepts and results behind this decomposition. For a complete account of this theory see [M1] and [R3]. To fix some ideas let us first take a look at the discrete case.

Let (Ω, F, P) be a probability space and $(F_n, n \in \mathbb{N})$ be an increasing family of σ-subalgebras of F. Denote by $(X_n, n \in \mathbb{N})$ a supermartingale relative to the family (F_n) and define the random variables Y_n and A_n by induction in the following manner:

$$Y_0 = X_0 \qquad A_0 = 0$$

$$Y_1 = Y_0 + [X_1 - E(X_1 | F_0)] \qquad A_1 = X_0 - E(X_1 | F_0)$$

$$\vdots$$

$$Y_n = Y_{n-1} + [X_n - E(X_n | F_{n-1})], \quad A_n = A_{n-1} + [X_{n-1} - E(X_n | F_{n-1})]$$

The following properties are easily verified:

(a) $X_n = Y_n - A_n$ for every n.

(b) The process (Y_n) is a martingale.

(c) The paths of the process (A_n) are increasing functions of n.

(d) $A_o = 0$ and A_n is F_{n-1}-measurable for every n, and integrable.

Any process (B_n) adapted to the family (F_n), and having sample paths increasing as functions of n and such that $B_o = 0$ will be called an increasing process. The preceding contruction shows that every discrete supermartingale (X_n) is equal to the difference of a martingale and an increasing process. Consider now the uniqueness of such a decomposition. Starting from an increasing process (B_n) and a martingale (Z_n) form the supermartingale $(X_n = Z_n - B_n)$, and construct the processes (Y_n) and (A_n) as above. A simple calculation shows that if B_n is F_{n-1}-measurable then we have: $Y_o = Z_o$ and

$$Y_n = Y_{n-1} + Z_n - Z_{n-1}$$

which implies that $Y_n = Z_n$ and consequently $A_n = B_n$. Conversly if $A_n = B_n$ then B_n is by construction F_{n-1}-measurable. Hence $A_n = B_n$ if and only if B_n is F_{n-1}-measurable. There thus exists only one decomposition of (X_n) by means of an increasing process satisfying property (d).

In the continuous case only supermartingales of class (DL) do have such a decomposition and its uniqueness depends on a property of the increasing process (increasing processes having this property are called natural) which is analogous to, although much more complex than, the discrete case.

We will now define precisely what is meant by a Doob-Meyer decomposition. Every stochastic process in the remainder of this section is defined on a single complete probability space (Ω, F, P) and adapted to an increasing, right-continuous family $(F_t, t \in \mathbb{R}_+)$. We suppose that the σ-algebra F_0 contains all the P-negligible sets. Supermartingales and stopping times are always relative to the above family (F_t). For the following definitions see [M1] Definitions 3 and 5, VII:

Definition 1.7.1: Let $(A_t, t \in \mathbb{R}_+)$ be a real-valued stochastic process, adapted to the family (F_t). We say that (A_t) is an increasing process if

(a) The sample paths of (A_t) are a.s zero for $t = 0$, increasing and right-continuous.

(b) The random variables A_t are integrable.

We say that the increasing process (A_t) is integrable if

$$\sup_t EA_t < \infty$$

Definition 1.7.2: Let (X_t) be a right-continuous supermartingale. We say that (X_t) admits a Doob-Meyer decomposition if there exists a (right continuous) martingale (Y_t) and an increasing process (A_t) such that $X_t = Y_t - A_t$ for every $t \in \mathbb{R}_+$.

Suppose that (A_t) is an integrable increasing process and define the process $(X_t \overset{\Delta}{=} E(A_\infty|F_t) - A_t)$. It is easily verified that (X_t) is a potential of the class (D) and that the above expression is a Doob-Meyer decomposition of (X_t) (note that by our convention $E(A_\infty|F_t)$ is a right-continuous martingale; see Remark 1.5.6). This motivates the following definition ([M1], Definition 6-VII):

Definition 1.7.3: Let (A_t) be an integrable increasing process. The process $(E(A_\infty|F_t) - A_t)$ is called the potential generated by (A_t).

INTEGRATION WITH RESPECT TO AN INCREASING PROCESS

Let (A_t) be an increasing process and (X_t) be a measurable process. Since by Theorem 14, Chapter 11 of [M1] (or Proposition III.1.2 of [N1]) the trajectories of (X_t) are $\mathcal{B}(\mathbb{R}_+)$ measurable we can consider for each $\omega \in \Omega$ the Lebesgue-Stieltjes integral on $\mathbb{R}_+$, if it exists:

$$\int_0^\infty X_t(\omega)\,dA_t(\omega)$$

From Fubini's theorem this integral is an F-measurable function of ω. Now if (X_t) is progressively measurable with respect to the family (F_t) the process (Y_t) defined by:

$$Y_t = \int_0^t X_s dA_s$$

(where the point t is included in the interval of integration*) is, if it exists, F_t-measurable for every $t \in \mathbb{R}_+$

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*The notation $\int_a^b$ is used for $\int_{(a,b]}$.

and has right-continuous paths. It is hence progressively measurable (see Theorem 1.3.2). Then if T is a stopping-time the random variable

$$Y_T = \int_0^T X_s dA_s$$

is F_T-measurable (Theorem 1.3.4).

UNIQUENESS

As said in the introduction the uniqueness of the Doob-Meyer decomposition depends on a property of the increasing process, which we now define as ([M1], Definition 18-VII).

<u>Definition 1.7.4</u>: Let (A_t) be an increasing process. We say that (A_t) is a natural increasing process if

$$E\int_0^t Y_s dA_s = E\int_0^t Y_{s^-} dA_s$$

for every $t \in \mathbb{R}_+$ and every positive, bounded, right-continuous martingale (Y_t).

<u>Remark 1.7.5(a)</u>: The martingale property of a process is very much dependent on the family (F_t) choosen and therefore so is the above definition. To be more precise we should speak of an increasing process as being natural with respect to a given family (F_t). We will see later on that the same process can be natural with respect to one family but not with respect to another one.

(b) If the process (A_t) is integrable then the condition is equivalent to ([M1], Theorem 19-VII)

$$E\int_0^\infty Y_s dA_s = E\int_0^\infty Y_{s-} dA_s$$

(c) Deterministic increasing processes are natural with respect to any family $(\mathcal{F}_t)$: From the Fubini's theorem it follows that

$$E\int_0^t Y_s dA_s = \int_0^t (EY_s) dA_s$$

and

$$E\int_0^t Y_{s-} dA_s = \int_0^t (EY_{s-}) dA_s$$

Now for a martingale $EY_s = EY_{s-}$ (See [M1], Theorem 4-VI) and the result follows.

(d) Continuous increasing processes are obviously natural. We can now state the uniqueness Theorem ([M1], Theorem 21-VII).

<u>Theorem 1.7.6</u>: (Uniqueness) Let (X_t) be a right-continuous supermartingale. There exists at most one natural increasing process (A_t) such that the process $(X_t + A_t)$ is a martingale.

We reexamine now for an increasing process the property of being natural. The next theorem gives another characterization of this property. But first ([M1], Definition 48-VII and Theorem 49-VII):

<u>Definition 1.7.7</u>: Let (A_t) be an increasing process and T be a stopping time. We say that (A_t) charges T if

$$P\{A_T = A_{T-}\} > 0$$

<u>Theorem 1.7.8</u>: Let (A_t) be an integrable increasing process. Then (A_t) is natural if and only if the following two properties are satisfied:

(a) For every sequence of stopping times (S_n) which increases to a stopping time S, the random variable A_S is measurable with respect to the σ-algebra $\bigvee_n \mathcal{F}_{S_n}$.

(b) (A_t) charges no totally inaccessible stopping times.

Recall that in the discrete parameter case (see the Introduction to this section) property (d) for an increasing process (A_n is $\mathcal{F}_{n-1}$ measurable) is the condition under which the Doob-Meyer decomposition is unique. Condition (a) in the above theorem is clearly the analogue in the continuous parameter case, of property (d). But condition (b) above has no equivalent in the discrete case. From Definition 1.7.4 the property of being natural has clearly something to do with the existence of martingales which would jump at the same times as the increasing process. Hence in view of Theorems 46 and 47, Chapter VII, of [M1], condition (b) is not unexpected.

The next result tells us that stopped natural increasing processes are still natural:

Theorem 1.7.9: Let (A_t) be a natural increasing process and T be a stopping time. Then the increasing process $(A_{t \wedge T})$ is natural with respect to the two families (F_t) and $(F_{t \wedge T})$.

This theorem appears in [M1] (Theorem 19-VII, (3)); but there, it is not clear with respect to which family (F_t) or $(F_{t \wedge T})$ the stopped process $(A_{t \wedge T})$ is natural. This is why we provide a proof of this result in Appendix A.2.

The uniqueness theorem and the above result immediately give us:

Lemma 1.7.10: Suppose (X_t) is a right-continuous supermartingale with a unique Doob-Meyer decomposition with respect to a family (F_t) given by:

$$X_t = Y_t - A_t$$

where (Y_t) is a martingale and (A_t) a natural increasing process. Let T be a stopping time. Then the unique Doob-Meyer decomposition with respect to the family $(F_{t \wedge T})$ of the supermartingale $(X_{t \wedge T})$ (with respect to $(F_{t \wedge T})$) is given by:

$$X_{t \wedge T} = Y_{t \wedge T} - A_{t \wedge T}$$

EXISTENCE

We have seen (Definition 1.7.3) how potentials of the class (D) can be generated by increasing integrable processes. The next existence theorem states the converse result ([M1], Theorem 29-VII)

Theorem 1.7.11: Let (X_t) be a right-continuous potential of class (D). There then exists an integrable natural, increasing process (A_t) which generates (X_t), and this process is unique.

Since the natural increasing process (A_t) that generates a potential (X_t) is uniquely determined by (X_t), the continuity property of the process (A_t) follows from a property of (X_t) ([M1], Definition 33-VII)

Definition 1.7.12: Let (X_t) be a right-continuous supermartingale of the class (DL). We say that the supermartingale (X_t) is regular if, for every increasing sequence (T_n) of stopping times which converges to a bounded stopping time T,

$$\lim_n EX_{T_n} = EX_T$$

For example every right-continuous martingale is regular. A supermartingale which has with some positive probability a jump at a fixed time t cannot be regular.

Theorem 1.7.13: ([M1], Theorem 37-VII) Let (X_t) be a right-continuous potential of the class D, and let (A_t) be the natural, integrable, increasing process which generates (X_t). The process (A_t) is continuous if and only if the potential (X_t) is regular.

We get now the existence theorem for supermartingales of the class (DL) from the above results for potentials of the class (D), by using the Riesz decomposition (see Theorem 1.6.2). A limiting argument is also involved here to get the extension from the class (D) to the class (DL) (see [M1], Theorem 31-VII)

Theorem 1.7.14(a): A right-continuous supermartingale (X_t) has a Doob-Meyer decomposition

$$X_t = Y_t - A_t$$

where (Y_t) denotes a right-continuous martingale and (A_t) an increasing process, if and only if (X_t) belongs to the class (DL). There then exists such a decomposition for which the process (A_t) is natural, and this decomposition is unique.

(b) The natural increasing process (A_t) is continuous if and only if the supermartingale (X_t) is regular.

The following simple remark, a direct consequence of the uniqueness theorem, is often used later on:

Remark 1.7.15: Let (X_t) be a right-continuous supermartingale of the class (D) and denote its Riesz decomposition (see Theorem 1.6.2) by:

$$X_t = P_t + Y_t \tag{1.7.1}$$

where (P_t) denotes a potential and (Y_t) a right-continuous martingale. By Remark 1.6.3(b) the martingale (Y_t)

is uniformly integrable (this implies by Remark 1.4.2(b) that it belongs to the class (D)) and is given by:

$$Y_t = E(X_\infty | F_t) \qquad (1.7.2)$$

The potential (P_t), which is the difference (see (1.7.1)) of two processes of class (D), also belongs to this class and by the above Theorem 1.7.11 there exists a natural integrable increasing process, say (A_t), which generates (P_t). That is (see Definition 1.7.3):

$$P_t = E(A_\infty | F_t) - A_t \qquad (1.7.3)$$

Introducing the two relations (1.7.2) and (1.7.3) in (1.7.1) we get:

$$X_t = E(A_\infty + X_\infty | F_t) - A_t \qquad (1.7.4)$$

The first term in the RHS of Eq. (1.7.4) is a right-continuous martingale and the second term, (A_t), is by definition a natural, increasing, integrable process. Relation (1.7.4) is therefore the unique Doob-Meyer decomposition of (X_t).

The above is only a summary of results concerning the Doob-Meyer decomposition. For other facts (e.g., the natural increasing process (A_t) can be obtained as the weak limit of a sequence of absolutely continuous natural increasing processes) we refer the reader to the original source which is [M1], Chapter VII, or [R3].

1.8 SQUARE INTEGRABLE MARTINGALES

INTRODUCTION

Ito integrals are now well known. Doob also defined stochastic integrals, in particular with respect to processes of independent increments. The generalization of these concepts to stochastic integration with respect to local martingales was rendered possible by the Doob-Meyer decomposition. Doléans-Dade and Meyer on the one hand and Kunita and Watanabe on the other did the pioneering work in this area. But their results are not similar and this has created some confusion. By giving here the following summary of results (Sections 1.8 and 1.9) we hope to introduce as well as clarify some of the definitions and results concerning this relatively new and still developing subject. The basic references for this résumé are [D1] and [M5] (see also [K2]).

The basic assumptions of the preceding section (just above Definition 1.7.1) are used again in this section and the next one. If f_t is a right-continuous function with left-hand limits we denote the jump of f_t at time t by $\Delta f_t \triangleq f_t - f_{t-}$.

Square integrable martingales play an important part in the theory of stochastic integration and also later on in this thesis:

Definition 1.8.1(a): We say that a right-continuous martingale (X_t) is square integrable if we have

$$\sup_t EX_t^2 < \infty$$

We denote by $M^2(P,F_t)$ (or simply M^2 when it does not create confusion) the space of all square integrable martingales (X_t), with respect to the measure P and family (F_t), such that $X_o = 0$.

The subspace of M^2 consisting of the continuous martingales is denoted by M_c^2.

We equip M^2 with a scalar product $((X_t),(Y_t)) = E(X_\infty Y_\infty)$ for (X_t) and (Y_t) belonging to M^2.

(b) $M_\ell^2(P,F_t)$ denotes the space of (P,F_t) martingales (X_t) such that $X_o = 0$ and $EX_t^2 < \infty$ for each t.

Remark 1.8.2(a): By Theorem 22-II of [M1], a square integrable martingale (X_t) is uniformly integrable and hence (Theorem 1.5.4) can be expressed as $X_t = E(X_\infty | F_t)$ where $EX_\infty^2 < \infty$.

(b) If $(X_t) \in M_\ell^2$ then $(X_{t\wedge a}) \in M^2$ for any constant a. This implies that all the following results stated for martingales in M^2 can be extended to the case of martingales in M_ℓ^2.

Theorem 1.8.3: ([D1], Theorem 1) M^2 is a Hilbert space and the subspace M_c^2 is closed in M^2.

NATURAL INCREASING PROCESSES ASSOCIATED WITH SQUARE INTEGRABLE MARTINGALES AND STOCHASTIC INTEGRALS

The following result is a consquence of the Doob-Meyer decomposition ([M1], § 23-VIII; [D1], Theorem 2).

Theorem 1.8.4: Let $(X_t) \in M^2$. There exists a unique natural increasing process denoted $(\langle X\rangle_t)$ such that the process $(X_t^2 - \langle X\rangle_t)$ is a martingale.

We will say that $(\langle X\rangle_t)$ is the increasing process associated with the martingale (X_t).

For two stopping times S and T such that $S \leq T$ we have the basic relation:

$$E(X_T^2 - X_S^2 | F_S) = E[(X_T - X_S)^2 | F_S] = E(\langle X\rangle_T - \langle X\rangle_S | F_S)$$

More generally if (X_t) and (Y_t) are two square integrable martingales we set (see [D1]):

Definition 1.8.5: $\langle X,Y\rangle_t = \frac{1}{2}(\langle X+Y\rangle_t - \langle X\rangle_t - \langle Y\rangle_t)$

Remark 1.8.6: It is easy to see that

(a) $\langle X\rangle_t = \langle X,X\rangle_t$

(b) The process $(\langle X,Y\rangle_t)$ is the difference of two natural increasing processes.

(c) The process $(X_tY_t - \langle X,Y\rangle_t)$ is a martingale.

(d) The process $(\langle X,Y\rangle_t)$ is the unique process satisfying properties (b) and (c) above.

We will now define stochastic integrals with respect to square integrable martingales. Recall that if f_t is a right-continuous function of bounded variation on the real line then $f_b - f_a$ is the integral of df over (a,b], whose indicator function is left-continuous. It is therefore natural to start with the integration of left-continuous simple processes:

<u>Definition 1.8.7</u>: (see [M5]) A process (H_t) defined as follows is called a simple process of the family $(\mathcal{F}_t)$. Take some finite subdivision $0 = t_o < t_1 < \ldots < t_n$ of the positive real line $\mathbb{R}_+$ and suppose that:

(a) H_o is $\mathcal{F}_o$ measurable and bounded

(b) on $(t_{i-1},t_i]$, $i = 1,\ldots,n$, $H_t = H_i$ where H_i is $\mathcal{F}_{t_{i-1}}$ measurable and bounded

(c) after t_n, $H_t = 0$.

These simple processes give rise to the class of predictable processes ([M5], Definition 1):

<u>Definition 1.8.8</u>: The process (H_t) is predictable (with respect to the family $(\mathcal{F}_t)$) if the function $(t,\omega) \to H_t(\omega)$ is measurable over the σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by all simple processes of the family $(\mathcal{F}_t)$.

<u>Definition 1.8.9</u>: ([M5], Definition 2) Let $(X_t) \in M^2$. We say that the process (H_t) belongs to the space $L^2(X)$ if (H_t) is predictable and

$$E\int_0^\infty H_s^2 \, d\langle X\rangle_s < \infty$$

We equip the space $L^2(X)$ with the semi norm

$$\|(H_t)\| = \left(E\int_0^\infty H_s^2 \, d\langle X\rangle_s\right)^{1/2}, \quad (H_t) \in L^2(X)$$

Let $(X_t) \in M^2$ and (H_t) be a simple process. We define the stochastic integral by:

$$\int_0^\infty H_s dX_s = H_o X_o + H_1(X_{t_1} - X_{t_o}) + \cdots + H_n(X_{t_n} - X_{t_{n-1}})$$

and we denote this process by $((H\cdot X)_t)$. We observe:

(a) $((H\cdot X)_t) \in M^2$

(b) $E(\int_0^\infty H_s dX_s)^2 = E\int_0^\infty H_s^2 d\langle X\rangle_s$

This last property (b) defines a norm preserving operator from the space of simple processes (which is a dense subset of $L^2(X)$) into the space M^2. By applying the procedure of functional completion we obtain the definition of stochastic integrals with respect to square integrable martingales ([D1], Theorem 3; [M5], Theorem 1):

Theorem 1.8.10(a): The mapping $(H_t) \to ((H\cdot X)_t)$ from simple processes to martingales can be uniquely extended as a norm preserving operator from $L^2(X)$ to M^2

(b) This stochastic integral is uniquely characterized by the following property: Let $(H_t) \in L^2(X)$. For any $(Y_t) \in M^2$ we have

$$<(H\cdot X),Y>_t = \int_0^t H_s\,d<X,Y>_s$$

(c) For almost all ω, we have

$$\Delta(H\cdot X)_t = H_t.\Delta X_t$$

DECOMPOSITION OF THE SPACE M^2

Definition 1.8.11: ([M5], Definition 4) A stable subspace S of M^2 is a closed subspace of M^2 such that $(X_t) \in S$ and $(H_t) \in L^2(X)$ imply $(H\cdot X)_t \in S$.

Remark 1.8.12: The stable subspace generated by $(X_t) \in M^2$ is given by $\{(H\cdot X)_t;(H_t) \in L^2(X)\}$

Definition 1.8.13: ([M1], Definition 26-VIII; [M5], Definition 5) Two martingales (X_t) and (Y_t) belonging to M^2 are said to be orthogonal if $<X,Y>_t = 0$.

Remark 1.8.14: The above definition is equivalent to saying that the process (X_tY_t) is a martingale. If $<X,Y>_t = 0$ then by Remark 1.8.6(c) (X_tY_t) is a martingale. Conversly if (X_tY_t) is a martingale then $<X,Y>_t$ is also a martingale; by Remark 1.8.6(b) and Lemma 1.9.4 we must have $<X,Y>_t \equiv 0$.

If S is a stable subspace of M^2, $S^\perp$ denote the subspace of all square integrable martingales which are orthogonal to S. Now similarly to the projection theorem in Hilbert space theory we have ([M5], Theorem 2):

<u>Theorem 1.8.15</u>: Let S be a stable subspace of M^2. Every $(X_t) \in M^2$ can be decomposed uniquely into $(Y_t + Z_t)$ with $(Y_t) \in S$ and $(Z_t) \in S^{\perp}$

As an application we have ([D1], Theorem 4; [M5]):

<u>Theorem 1.8.16</u>: Let $(X_t) \in M^2$. Then there exists a unique decomposition of (X_t) into a sum of two square integrable martingales (X_t^c) and (X_t^d) where $(X_t^c) \in M_c^2$ and $(X_t^d) \in M_c^{2\perp}$.

<u>Remark 1.8.17</u>: The martingale (X_t^d) is not simply the process having constant sample paths except for jumps which are the same as those of (X_t). Such a process would not necessarily be a martingale. Now (see the remark, on p. 90, following Proposition 3 of [D1]; [M5]) if $(X_t) \in M^2$ and has a.s sample paths of bounded variation on every finite interval then $(X_t^c) \equiv 0$, i.e., $(X_t) \in M_c^{2\perp}$. In fact $M_c^{2\perp}$ is the closure in M^2 of such martingales of bounded variation. If $(X_t) \in M^2$, Meyer calls the process (X_t^d) the compensated sum of the jumps of (X_t).

QUADRATIC VARIATION PROCESSES AND STOCHASTIC INTEGRALS

The above decomposition (Theorem 1.8.16) allows us to associate to any square integrable martingale another increasing process (this one not natural), ([D1], p. 87; [M5])

<u>Definition 1.8.18(a)</u>: Let $(X_t) \in M^2$. We call quadratic variation process associated to (X_t) the following

increasing process:

$$[X]_t \triangleq \langle X^c\rangle_t + \sum_{s\leq t} (\Delta X_s)^2$$

where $\langle X^c\rangle_t$ is the natural increasing process associated with the martingale (X_t^c) (see Theorem 1.8.4).

(b) If (X_t) and (Y_t) are two elements of M^2 we set:

$$[X,Y]_t = \frac{1}{2}([X+Y]_t - [X]_t - [Y]_t)$$

We have ([D1], Theorem 5):

<u>Theorem 1.8.19</u>: The process $(X_t^2 - [X]_t)$ is a martingale.

<u>Remark 1.8.20</u>: Recall that $(X_t^2 - \langle X\rangle_t)$ is a martingale. Hence the process $([X]_t - \langle X\rangle_t)$ is also a martingale.

Let $(X_t) \in M^2$. The fact that the process $(X_t^2 - \langle X\rangle_t)$ is a martingale allowed us to construct a norm preserving operator from the space $L^2(X)$ to M^2 and to define stochastic integrals. Similarly we can define a stochastic integral and construct a norm preserving operator starting this time from the martingale property of the process $(X_t^2 - [X]_t)$. It turns out (Theorem 6 of [D1]) that the class of integrable stochastic processes and the stochastic integral associated with the process $([X]_t)$ are the same as those associated with the process $(\langle X\rangle_t)$. As before we also have ([D1], Theorem 6; [M5], Theorem 4):

Theorem 1.8.21: Let (X_t) and (Y_t) belong to M^2 and (H_t) to $L^2(X)$. Then

(a) $E\int_0^\infty |H_s||d[X,Y]_s| < \infty$

(b) The stochastic integral $((H\cdot X)_t)$ is uniquely characterized by the property:

$$[(H\cdot X),Y]_t = \int_0^t H_s d[X,Y]_s$$

for every $(Y_t) \in M^2$.

The interest of the process $([X]_t)$ is that it allows an extension of stochastic integrals to local martingales while the process $(\langle X\rangle_t)$ does not. This is the subject of the next section.

1.9 GENERALIZATIONS OF MARTINGALES

LOCAL MARTINGALES AND STOCHASTIC INTEGRALS

Definition 1.9.1(a): V^+ is the class of all finite valued, right-continuous adapted increasing processes (A_t) such that $A_0 = 0$

(b) $V = V^+ - V^+$. V is in fact the space of all right-continuous, adapted processes having sample paths of bounded variation on every finite interval, and which are zero at the time origin.

(c) A^+ is the subspace of V^+ consisting of integrable increasing processes and $A = A^+ - A^+$.

Definition 1.9.2(a): ([M6], Definition 4) A right-continuous adapted process (X_t) is a local martingale if there exists a sequence of stopping times (T_n) increasing a.s to ∞ such that for every n the process $(X_{t\wedge T_n})$ on $\{T_n > 0\}$ is a uniformly integrable martingale.

(b) If the stopped process $(X_{t\wedge T_n})$ is a square integrable martingale then we say that (X_t) is a square integrable local martingale.

(c) We denote by $L(P,F_t)$ (or simply L) the space of all local martingales (X_t) such that $X_o = 0$.

(d) We say that a sequence of stopping times (T_n) reduces the local martingale (X_t) if the stopped process $(X_{t\wedge T_n})$ is a uniformly integrable martingale.

Remark 1.9.3(a): The restriction to uniformly integrable martingales, in the above definition, is not important: if (T_n) is a sequence of stopping times such that $(X_{t\wedge T_n})$ is a martingale then the sequence of stopping times $(R_n \triangleq T_n \wedge n)$ makes the process $(X_{t\wedge R_n})$ a uniformly integrable martingale.

(b) In the above definitions of spaces, a subscript c indicates the subclass of continuous processes (e.g., L_c denotes the space of continuous local martingales).

The following result will be most useful later on (compare with Remark 1.8.17):

Lemma 1.9.4: Let $(X_t) \in L_c \cap V$. Then

$$P\{X_t \equiv 0, \ t \in \mathbb{R}_+\} = 1$$

Remark 1.9.5: The proof of this result when (X_t) is a martingale is given in [F1], Lemma 3.2.1. The extension to local martingales is trivial. If furthermore the martingale (X_t) belongs to A the above result follows from the uniqueness of the Doob-Meyer decomposition. In this case (X_t) can be expressed as a difference $(A_t - B_t)$ where both (A_t) and (B_t) belong to A_c^+. The process $(Y_t \stackrel{\Delta}{=} X_t - A_t)$ is then a supermartingale of the class (D) and thus admits a unique Doob-Meyer decomposition. But $(X_t - A_t)$ and $(0 - B_t)$ are precisely both such a unique decomposition ((A_t) and (B_t) are both natural because they are continuous). So we must have $(X_t = 0)$ and $(A_t = B_t)$.

The extension of Theorem 1.8.16 to local martingales is given by (see Theorem 7 of [D1]; [M6], Theorem 1):

Theorem 1.9.6: Let $(X_t) \in L$. (X_t) can be written in a unique way as $X_t = X_t^c + X_t^d$ where $(X_t^c) \in L_c$ and (X_t^d) is such that for every $(Y_t) \in L_c$ the process $(X_t^d Y_t) \in L$.

By definition of a square integrable local martingale (X_t) (Definition 1.9.2(b)) there exists a sequence of stopping times (T_n) increasing a.s to ∞ such that the stopped process $(X_t^n \stackrel{\Delta}{=} X_{t \wedge T_n})$ is a square integrable martingale for each n. Now if $(X_t) \in L_c$, but is not necessarily square integrable, we can construct such a sequence of stopping times as follows. Let

$$T_n = \begin{cases} \inf\{t: \ |X_t| \geq n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

Because (X_t) is continuous, the stopped process (X_t^n) is bounded by n thus square integrable. Furthermore because martingales have sample paths which are a.s bounded on every compact interval (see Theorem 3-VI of [M1]) the above sequence (T_n) increases to ∞. Hence in both of these cases the process $(<X^n>_t)$ makes sense and by the uniqueness of this process we can uniquely define an increasing natural process $(<X>_t)$ such that $(X_t - <X>_t) \varepsilon L$. Now if $(X_t) \varepsilon L$ but is not continuous or square integrable, the above is no longer possible because it is not always true that there exists a sequence of stopping times (T_n) which will make the stopped process $(X_{t\wedge T_n})$ a square integrable martingale. Hence it is possible to extend the definition of the process $(<X>_t)$ to continuous or square integrable local martingales only; but this, in turn, allow us to extend the definition of the process $([X]_t)$ (see Definition 1.8.18) to local martingales (see [D1], p. 98, [M6]). Stochastic integrals with respect to local martingales can then be defined as in Theorem 1.8.21.

Definition 1.9.7(a): Let $(X_t) \in L$. By $[X]_t$ we denote the quadratic variation process:

$$[X]_t = \langle X^c \rangle_t + \sum_{s \leq t} (\Delta X_s)^2$$

(b) If (X_t) and (Y_t) both belong to L, we set:

$$[X,Y]_t = \frac{1}{2}([X+Y]_t - [X]_t - [Y]_t)$$

Remark 1.9.8(a): The fact that $[X]_t$ is finite follows from [D1], Theorem 7 (see also [A1]).

(b) The process $(X_t Y_t - [X,Y]_t) \in L$ (see [D1], p. 106).

We now give the results on stochastic integrals with respect to local martingales ([D1], Section 4, [M6]).

Definition 1.9.9: ([D1], p. 98) $H(F_t)$ denotes the class of all locally bounded predictable (with respect to the family (F_t)) processes (H_t), locally bounded meaning that there exists a sequence of stopping times (T_n) increasing to ∞ and a sequence of positive numbers (M_n) such that

$$|H_{T_n \wedge t} \cdot I_{\{T_n > 0\}}| \leq M_n < \infty \text{ a.s}$$

Remark 1.9.10(a): (see [D1], remark on p. 100) Let (H_t) be a right-continuous process with left-hand finite limits. Then $(H_{t^-}) \in H$.

(b) By Theorem 3-VI of [M1] every right-continuous supermartingale (X_t) has sample paths with finite left-hand limits. Hence the process $(X_{t^-}) \in H$. This result extends to local-martingales (or semimartingales, defined later on).

Theorem 1.9.11: ([D1], Proposition 5; [M6], Theorem 2) Given $(X_t) \in L$ and $(H_t) \in H$ there is one and only one process $((H \cdot X)_t)$ such that

$$[(H \cdot X), Y]_t = \int_0^t H_s \, d[X,Y]_s$$

for every $(Y_t) \in L$. The stochastic integral $((H \cdot X)_t)$ belongs to L.

The following very important lemma makes the connection between stochastic integrals and Stieljes integrals when they both exist ([D1], Proposition 3; [M6], Lemma 2):

Lemma 1.9.12: If $(X_t) \in L \cap V$ and $(H_t) \in H$ the integral of (H_t) with respect to (X_t) is the same in its stochastic and Stieltjes definition.

It might be appropriate now to compare the definitions of Doléans-Dade and Meyer on the one hand and the approach of Kunita and Watanabe on the other. There are not the same. First of all when Kunita and Watanabe speak of a local martingale (X_t) they mean a square integrable local martingale. This allows them to deal only with the natural increasing process $(\langle X \rangle_t)$. The class of integrable processes is also different. Instead of the class of locally bounded predictable processes they use the class $\bar{K}_{rc}$ which is the closure of the class K_{rc} with respect to the seminorm

$$(E\int_0^t K_s^2 d\langle X\rangle_s)^{1/2}$$

for $(K_t) \in K_{rc}$ and where

$K_{rc} = \{(K_t)$: bounded right-continuous adapted processes having left-hand limits$\}$

Kunita and Watanabe do not need the notion of predictability because they assume that the family $(\mathcal{F}_t)$ is free of times of discontinuity. In [D1], Doléans-Dade and Meyer do not assume this last condition. But Meyer does in [M7] and that allows him to integrate a larger class of processes.

SEMIMARTINGALES AND THE CHANGE OF VARIABLES FORMULA

<u>Definition 1.9.13(a)</u>: A semimartingale is a process (X_t) which can be written as a sum:

$$X_t = X_o + L_t + A_t$$

where X_o is $\mathcal{F}_o$-measurable, $(L_t) \in \mathcal{L}$ and $(A_t) \in \mathcal{V}$.

(b) A process (X_t) with values in $\mathbb{R}^n$ is a semimartingale if all its components are real semimartingales.

Examples of semimartingales are sub- and supermartingales, and processes of independent increments.

The above decomposition is not unique. The only intrinsic elements are (1) X_o and (2) (L_t^c) (see [D1], Section 5). The natural increasing process $(\langle L^c\rangle_t)$ is hence uniquely determined by (X_t). We set:

$$X_t^c \triangleq L_t^c \text{ and } \langle X^c\rangle_t \triangleq \langle L^c\rangle_t$$

The stochastic integral $((H.X)_t)$, where $(H_t) \in \mathcal{H}$ and (X_t) is a semimartingale with a decomposition $X_t = X_o + L_t + A_t$ is defined by:

$$(H\cdot X)_t = H_o\cdot X_o + (H\cdot L)_t + (H\cdot A)_t$$

where $((H\cdot L)_t)$ is a stochastic integral and $((H\cdot A)_t)$ is the usual Stieltjes integral $\int_0^t H_s dA_s$. The next theorem, a generalization of the Ito differentiation formula was first obtained, for locally square integrable martingales, by Kunita and Watanabe (see [K1], Theorems 2.2 and 5.1) and finally for semimartingales by Doléans-Dade and Meyer (see [D1], Theorem 8; for the most general version of this theorem (for martingales taking values in a Hilbert space), see [K2], Theorem 3):

Theorem 1.9.14: (Change of variables formula) Let (X_t) be a vector $(\mathbb{R}^n)$ valued semimartingale (we denote by X_t^i the i^{th} component of X_t) and F a twice continuously differentiable function of $\mathbb{R}^n$ into $\mathbb{C}$. Denote by D^i the derivation operator with respect to the i^{th} coordinate. We then have for each finite t:

$$F(X_t) = F(X_o) + \int_0^t \sum_{i=1}^n D^i F(X_{s^-}) dX_s^i$$

$$+ \frac{1}{2} \int_0^t \sum_{\substack{i=1 \\ j=1}}^n D^i D^j F(X_{s^-}) d\langle X^{ic}, X^{jc} \rangle_s$$

$$+ \sum_{s \leq t} [\Delta F(X_s) - \sum_{i=1}^n D^i F(X_{s^-}) \Delta X_s^i]$$

where the sum $\sum_{s \leq t} [\ldots]$ in the RHS converges a.s for each finite t. In particular the process $(F(X_t))$ is a semimartingale.

This formula gives rise to a lot of applications (see [D1], p. 106; [M6] Theorem 4 on integration by parts, etc.). We give only one of them, very important, which is the subject of a paper of Doléans-Dade [D2].

EXPONENTIAL FORMULA

<u>Theorem 1.9.15</u>: Let (X_t) be a semimartingale such that $X_o = 0$.

(a) There exists one and only one semimartingale (Z_t) satisfying the stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_{s^-} dX_s$$

(b) The solution (Z_t) is given by:

$$Z_t = \exp(X_t - \frac{1}{2}\langle X^c\rangle_t) \prod_{s\leq t} (1 + \Delta X_s)e^{-\Delta X_s}$$

where the product in the RHS converges a.s for each t.

This theorem itself generates numerous applications, in particular on multiplicative decomposition of martingales (see [D2]).

CHAPTER 2

COUNTING PROCESSES AND INTEGRATED CONDITIONAL RATES

2.0 INTRODUCTION

In this chapter we use the Doob-Meyer decomposition to uniquely decompose any counting process (N_t) for which the random variable N_t is a.s finite for each t into a sum of a square integrable local martingale and a natural increasing process. This last process is then called the Integrated Condition Rate as explained in the INTRODUCTION. After defining counting processes and establishing some notation in Section 2.1, we define and study the notion of integrated conditional rate in Section 2.3 (Section 2.2 is concerned with a preliminary result). In Section 2.4 three classes of counting processes are defined: regular, accessible and predictable counting processes, these latter constituting a subclass of accessible counting processes. We show that any counting process can be uniquely decomposed into the sum of two counting processes which are respectively regular and accessible. Regular counting processes have, loosely speaking, totally unexpected times of jump. Poisson processes are of this type. On the contrary, the times of jump of an accessible counting process can be predicted with some chance of success. A counting process which jumps with some positive probability at given fixed times is an example of this kind of processes. Properties of integrated conditional rate of counting processes belonging to these three classes are derived and examples are presented. In Section 2.5 we give sufficient conditions for the existence of a conditional rate. Counting processes of independent increments play an important part in solving the detection problem. These processes are precisely those which have a deterministic integrated conditional rate and this is the topic of Section 2.6.

Finally in Section 2.7 we obtain, using the change of variables formula originally due to Ito [I2] and extended by Doléans-Dade and Meyer [D1], some results related to probability generating functions.

2.1 BASIC DEFINITIONS AND ASSUMPTIONS

The notation introducted in the previous chapter is used consistently is this one. As before, the state space and the index set of all stochastic processes (see Definition 1.1.2) are respectively given by the real line $\mathbb{R}$ and its positive part $\mathbb{R}_+$. By a continuous (right-continuous, etc.) process we mean a process with continuous (right-continuous, etc.) sample paths. We do not distinguish between modifications of the same process (see Section 1.1); this allows us in particular to consider only right-continuous martingales (c.f. Remark 1.5.6). If (X_t) and (Y_t) are two right- (or left-) continuous processes which are modifications of each other (i.e., $X_t = Y_t$ a.s for each t) then we have $X_t = Y_t$ a.s for every t (see Remark 1.1.4). Recall that the notion of martingale is relative to a probability measure P and an increasing family of σ-algebras (F_t), while stopping times depend only on the family (F_t) (see Sections 1.2 and 1.5). We emphasize this by speaking of a (P,F_t) martingale (or simply (F_t) martingale, when only one probability measure is involved) and of a (F_t) stopping time. Every stochastic process in this chapter is defined on a single probability space (Ω,F,P).

Definition 2.1.1: A counting process (N_t) (hereafter abbreviated CP) is a stochastic process having sample paths which are zero at the time origin, right-continuous step functions with positive jumps of size one.

As seen in the INTRODUCTION CP's are naturally associated to point processes. If (N_t) is a CP associated to a point process then observing (N_t) to time t tells us of the points occurring up to and including t. Note that this would not be the case had we choosen the sample paths of (N_t) to be left-continuous (Rubin in [R2] makes this unnatural left-continuity assumption). With every CP (N_t) we associate an increasing family (F_t) of σ-subalgebras of F to which (N_t) is adapted (see Definition 1.2.1). Loosely speaking the σ-algebra F_t represents the information to which we have access at time t (see Section 1.2). In particular we will denote by $N_t \triangleq \sigma(N_u,\ 0 \leq u \leq t)$ the minimal σ-algebra generated by the CP (N_t) up to and at time t. Numerous results we will be using depend on the right-continuity of the family (F_t) (e.g., optional sampling theorem, existence of right-continuous modifications for supermartingales, etc.). If the family (F_t) to which the CP (N_t) is adapted does not have this property then we will consider instead, following Meyer [M8], the family (F_{t^+}) (see Section 1.2). This family (F_{t^+}) is by construction right-continuous; the σ-algebra F_t is contained in F_{t^+} so that the CP (N_t) is adapted to this family. Thus we will always assume in the remainder of this thesis that <u>the family (F_t) is in fact right-continuous</u>. We also suppose that the probability space (Ω, F, P) is complete and that the σ-algebra F_o contains all the P-negligible sets.

The points in time at which a CP (N_t) jumps are basic to this study:

<u>Definition 2.1.2</u>: The stopping time:

$$J_n = \begin{cases} \inf \{t: \ N_t \geq n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

is called the time of n^{th} jump of the CP (N_t).

The fact that J_n is a stopping time with respect to any family (F_t) to which the CP (N_t) is adapted can be easily verified: the set $\{J_n \leq t\} = \{N_t \geq n\}$ belongs to F_t for every t (see also Example 1.2.6). If (N_t) is a CP bounded by m then for $n > m$ J_n is equal to infinity and is not properly speaking a time of jump of (N_t). But the above definition has the practical advantage that, when the random variable N_t is a.s finite for each t, the sequence (J_n)

increases to infinity. For example a given property may be shown to hold on the interval $[0,J_n]$ for each n (on this interval the CP (N_t) has the nice behavior of being bounded by n). Then the fact that the sequence (J_n) increases to infinity shows that this property holds for all t. Recall also that the definition of local martingales involves a sequence of stopping times increasing to infinity.

2.2 A PRELIMINARY RESULT

The following result is basic to the establishment of the Likelihood Ratio Representation Theorem (Theorem 3.3.1) given in the next chapter. We state it here because it is also used in this chapter, although not in its full generality. This lemma is basically a generalization to supermartingales of a result on energy for potentials. The known result is the following:

Lemma 2.2.1: Let (P_t) be a potential of class (D) and denote by (A_t) the unique natural integrable increasing process which generates (P_t) (see Definition 1.7.3 and Theorem 1.7.11). Then we have the following chain of inequalities:

$$EA_\infty^2 \leq 4E(\sup_t P_t)^2 \leq 16EA_\infty^2$$

For a proof of this Lemma see Chapter VII, Section 6 of [M1].

If (X_t) is a supermartingale we denote its positive and negative part respectively by $(X_t^+ \triangleq X_t \vee 0)$ and

($X_t^- \triangleq -(X_t \wedge 0)$). If (X_t) is of class (D) (or even just uniformly integrable) then by the martingale convergence theorem ([M1], Theorem 6-VI) there exists an integrable random variable X_∞ such that a.s and in the mean $X_\infty = \lim_t X_t$. Define $X_\infty^+ \triangleq X_\infty \vee 0$ and $X_\infty^- \triangleq -(X_\infty \wedge 0)$. Then because the two functions $(\cdot \vee 0)$ and $(\cdot \wedge 0)$ are continuous we also have by Theorem 4.6 of [R1]

$$\lim_{t\to\infty} X_t^+ = \lim_{t\to\infty}(X_t \vee 0) = (\lim_t X_t \vee 0) = (X_\infty \vee 0) \triangleq X_\infty^+$$

Similarly

$$\lim_{t\to\infty} X_t^- = X_\infty^-$$

Now the result

<u>Lemma 2.2.2</u>: Let (X_t) be a supermartingale of the class (D) with respect to a family (F_t). Denote its unique Doob-Meyer decomposition by

$$X_t = Y_t - A_t \qquad (2.2.1)$$

where (Y_t) is a uniformly integrable martingale and (A_t) an integrable natural increasing process. Then:

(a) $EA_\infty^2 \leq 8[E(\sup_t X_t^+)^2 + E(X_\infty^-)^2]$

(b) $E(\sup_t X_t)^2 \leq 8[EA_\infty^2 + E(X_\infty^+)^2]$

(c) The three following statements are equivalent:

(1) $E(\sup_t X_t^+)^2 < \infty$ and $E(X_\infty^-)^2 < \infty$

(2) $E(\sup_t X_t)^2 < \infty$ and $EX_\infty^2 < \infty$

(3) $\sup_t EY_t^2 < \infty$ i.e., (Y_t) is a square integrable martingale and $EA_\infty^2 < \infty$

<u>Proof</u>: (a) (X_t) being of the class (D), it has the unique Riesz decomposition (see Remark 1.6.3(b))

$$X_t = P_t + E(X_\infty | F_t) \tag{2.2.2}$$

where (P_t) is a potential of the class (D) and $X_\infty = \lim_{t\to\infty} X_t$ a.s and in the mean. Denote by (B_t) the unique natural integrable increasing process which generates (P_t) (see Theorem 1.7.11). By Remark 1.7.15, the relation

$$X_t = E(B_\infty + X_\infty | F_t) - B_t \tag{2.2.3}$$

is also a unique Doob-Meyer decomposition of (X_t). Hence we have (see (2.2.1))

$$A_t = B_t \tag{2.2.4}$$

$$Y_t = E(A_\infty + X_\infty | F_t) \tag{2.2.5}$$

Now by Theorem 23-VII of [M1]

$$EA_\infty^2 = E\int_0^\infty (P_t + P_{t^-})dA_t \tag{2.2.6}$$

Using (2.2.2) we get (see also [M1], Theorem 4-VI)

$$EA_\infty^2 = E\{ \int_0^\infty (X_t + X_{t^-}) dA_t - \int_0^\infty [E(X_\infty | \mathcal{F}_t) + E(X_\infty | \mathcal{F}_{t^-})] dA_t \}$$

Hence

$$EA_\infty^2 \leq E\sup_t (X_t^+ + X_{t^-}^+) A_\infty + E\int_0^\infty [E(X_\infty^- | \mathcal{F}_t) + E(X_\infty^- | \mathcal{F}_{t^-})] dA_t \quad (2.2.7)$$

Now $\sup_t X_t^+ = \sup_t X_{t^-}^+$; the process (A_t) is natural so that (see [M1], Theorem 20-VII) the last term in the RHS of (2.2.7) is equal to $2E\int_0^\infty E(X_\infty^- | \mathcal{F}_t) dA_t$ and by Theorem 16-VII of [M1]:

$$E\int_0^\infty E(X_\infty^- | \mathcal{F}_t) dA_t = E(X_\infty^- A_\infty)$$

So from (2.2.7) we have

$$EA_\infty^2 \leq 2E[(\sup_t X_t^+ + X_\infty^-) A_\infty]$$

and by the Schwarz inequality

$$(EA_\infty^2)^2 \leq 4E(\sup_t X_t^+ + X_\infty^-)^2 EA_\infty^2$$

Then we finally obtain

$$EA_\infty^2 \leq 4E(\sup_t X_t^+ + X_\infty^-)^2 \leq 8[E(\sup_t X_t^+)^2 + E(X_\infty^-)^2]$$

(b) By (2.2.2)

$$X_t = P_t + E(X_\infty | \mathcal{F}_t) \leq P_t + E(X_\infty^+ | \mathcal{F}_t)$$

So

$$E(\sup_t X_t)^2 \le 2\{E(\sup_t P_t)^2 + E[\sup_t E(X_\infty^+|F_t)]^2\} \qquad (2.2.8)$$

By Lemma 2.2.1

$$E(\sup_t P_t)^2 \le 4EA_\infty^2 \qquad (2.2.9)$$

Now $(E(X_\infty^+|F_t))$ is a positive martingale. Hence by Remark 2-VI of [M1]:

$$E[\sup_t E(X_\infty^+|F_t)]^2 \le 4\sup_t E[E(X_\infty^+|F_t)]^2$$

Furthermore by the Jensen inequality

$$[E(X_\infty^+|F_t)]^2 \le E[(X_\infty^+)^2|F_t]$$

so that

$$E[\sup_t E(X_\infty^+|F_t)]^2 \le 4\sup_t E\{E[(X_\infty^+)^2|F_t]\} = 4E(X_\infty^+)^2 \qquad (2.2.10)$$

Using the two above inequalities (2.2.9) and (2.2.10) in (2.2.8) we get the desired

$$E(\sup_t X_t)^2 \le 8[EA_\infty^2 + E(X_\infty^+)^2]$$

(c) First we show that (1) ⇔ (2)

(1) ⇒ (2) If, given a sample path of (X_t), there exists a time t_o such that $X_{t_o} \ge 0$ then $\sup_t X_t = \sup_t X_t^+$ and $\inf_t X_t^- = 0$. If not, then $X_t \le 0$ all t, $\sup_t X_t^+ = 0$ and $\sup_t X_t = -\inf_t X_t^-$. Hence we have the relations

$$\sup_t X_t = \sup_t X_t^+ - \inf_t X_t^- \qquad (2.2.11)$$

$$(\sup_t X_t)^2 = (\sup_t X_t^+)^2 + (\inf_t X_{t^-})^2 \qquad (2.2.12)$$

Also $0 \leq (\inf_t X_t^-)^2 \leq (X_\infty^-)^2$ and this implies from (2.2.12)

$$E(\sup_t X_t)^2 \leq E(\sup_t X_t^+)^2 + E(X_\infty^-)^2$$

and the RHS of this relation is finite by assumption. Clearly $0 \leq X_\infty^+ \leq \sup_t X_t^+$. Hence $E(X_\infty^+)^2$ is also finite by assumption, i.e., EX_∞^2 is finite. (1) $\Leftarrow$ (2) By (2.2.12)

$$(\sup_t X_t^+)^2 = (\sup_t X_t)^2 - (\inf_t X_t^-)^2 \leq (\sup_t X_t)^2$$

and obviously $E(X_\infty^-)^2$ is finite by assumption. Now we show that (2) $\Leftrightarrow$ (3)

(2) $\Rightarrow$ (3) We have (see Eq. (2.2.5))

$$Y_t = E(A_\infty + X_\infty | F_t)$$

By (a) and the implication (2) $\Rightarrow$ (1) then

$$EA_\infty^2 < \infty \qquad (2.2.13)$$

By the Jensen inequality

$$Y_t^2 = [E(A_\infty + X_\infty | F_t)]^2 \leq E[(A_\infty + X_\infty)^2 | F_t]$$

Thus

$$\sup_t EY_t^2 \le E(A_\infty + X_\infty)^2 \le 2EA_\infty^2 + 2EX_\infty^2$$

and the RHS of this relation is finite by (2.2.13) and by assumption.

(3) ⇒ (2) If (Y_t) is a square integrable martingale then in particular $EY_\infty^2 < \infty$ so that (see (2.2.1))
$EX_\infty^2 \le 2(EY_\infty^2 + EA_\infty^2) < \infty$ and by (b) $E(\sup_t X_t)^2 < \infty$.

□

2.3 INTEGRATED CONDITIONAL RATE

DOOB-MEYER DECOMPOSITION FOR COUNTING PROCESSES

As a direct application to CP's of the Doob-Meyer decomposition of supermartingales into the sum of a martingale and an increasing process we have (see Section 1.7; [I1]).

Theorem 2.3.1: (Doob-Meyer Decomposition for CP's) Let (N_t) be a CP adapted to an increasing family $(\mathcal{F}_t)$.

(a) If for each $t \in \mathbb{R}_+$, N_t is a.s finite then there exists a unique natural increasing process (A_t) such that the process $(M_t \triangleq N_t - A_t)$ is a square integrable $(P,\mathcal{F}_t)$ local martingale. The unique decomposition $(N_t = M_t + A_t)$ is called the Doob-Meyer decomposition for the CP (N_t) with respect to the family $(\mathcal{F}_t)$.

(b) If furthermore EN_t is finite for each t then the process $(M_t = N_t - A_t)$ is a $(P,\mathcal{F}_t)$ martingale.

Proof: (a) Let J_n be the time of n^{th} jump of the CP (N_t) and define $(N_t^n \triangleq N_{t\wedge J_n})$ and $(F_t^n \triangleq F_{t\wedge J_n})$. By assumption N_t is a.s finite for each t. Hence the sequence of stopping times (J_n) increases a.s to infinity. Also by construction the stopped process (N_t^n) is bounded by n. For $t \geq s$ we obviously have

$$E(-N_t^n | F_s) \leq -N_s^n$$

Thus $(-N_t^n)$ is a bounded (F_t^n) supermartingale and by the Doob-Meyer decomposition we can obtain (see Theorem 1.7.14 and Lemma 2.2.2) the unique decomposition:

$$N_t^n = M_t^n + A_t^n \qquad (2.3.1)$$

where (M_t^n) is a square integrable (F_t^n) martingale and (A_t^n) a natural integrable increasing process. Now for $n \leq m$ the unique Doob-Meyer decomposition of (N_t^n) with respect to (F_t^n) is also given by (see Lemma 1.7.10)

$$N_t^n = M_{t\wedge J_n}^m + A_{t\wedge J_n}^m \qquad (2.3.2)$$

Therefore comparing (2.3.1) and (2.3.2) we get

$$M_{t\wedge J_n}^m = M_t^n$$

$$A_{t\wedge J_n}^m = A_t^n$$

Hence we can uniquely define for all t an increasing natural process (A_t) and a square integrable local martingale (M_t) by

$$A_t \stackrel{\Delta}{=} A_t^n \quad \text{for} \quad t \leq J_n$$

$$M_t \stackrel{\Delta}{=} M_t^n \quad \text{for} \quad t \leq J_n$$

and we clearly have

$$N_t = M_t + A_t \quad \text{for every } t$$

This proves part (a).

(b) If EN_t is finite for each t then the process $(-N_t)$ is a right-continuous negative supermartingale. By Theorem 19-VI of [M1], this supermartingale belongs to the class (DL). Then result (b) follows directly from the Doob-Meyer decomposition (Theorem 1.7.14). □

INTEGRATED CONDITIONAL RATE: DEFINITION

For every CP (N_t) with N_t a.s finite for each t and adapted to a family (F_t), the uniqueness of the Doob-Meyer decomposition for this CP (N_t) allows us to propose:

Definition 2.3.2: We will call Integrated Conditional Rate (hereafter abbreviated ICR) with respect to the family (F_t) the unique natural increasing process which appears in the Doob-Meyer decomposition of (N_t) with respect to the family (F_t).

The terminology "Integrated Conditional Rate" is motivated by the following (see also the INTRODUCTION): when the ICR (A_t) of a CP (N_t) with respect to a family (F_t) is absolutely continuous (sufficient conditions for that are given in Theorem 2.5.1) it can be expressed as

$$A_t = \int_0^t \lambda_s ds \tag{2.3.3}$$

Furthermore

$$\lambda_t = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| \mathcal{F}_t\right) \tag{2.3.4}$$

so that the process (λ_t) is called the Conditional Rate with respect to the family $(\mathcal{F}_t)$. Expression (2.3.3) is then a justification for the terminology introduced in Definition 2.3.2, even though a conditional rate does not generally exist. The existence of CP's which admit a bounded conditional rate with respect to the family of σ-algebras generated by the CP itself is shown in Section 3.1. Also if (N_t) is a Poisson process (see [P1] Chapter 4) then the notion of conditional rate with respect to the family of σ-algebras generated by the process itself reduces to the usual notion of rate.

If the random variable N_t is not a.s finite for each t then the sequence (J_n) of times of jump of (N_t) does not converge a.s to infinity. Define $J \triangleq \lim_n J_n$. By Theorem 42-IV of [M1], J is a stopping time. For $t \geq J$, $N_t = \infty$ and the best we can do in this case is to consider what is happening on the stochastic interval $[0,J)$ only. If now a local martingale (X_t) is redefined as being a process such that there exists a sequence of stopping times (R_n) increasing to J a.s (instead of ∞) which makes the stopped process

$(X_{t\wedge R_n})$ a uniformly integrable martingale then as above we can associate uniquely to the CP (N_t) an ICR on the stochastic interval $[0,J)$.

From now on, when speaking of CP (N_t) we always assume that the random variable (N_t) is a.s finite for each t since this is clearly the weakest condition under which we can define an ICR on the entire positive real line. Note that this assumption is very weak as it is violated only if the times of jump of the CP (N_t) considered converge with some positive probability to a finite time, or, in other words, that the point process associated with the CP (N_t) contains with some positive probability a point of accumulation, an unlikely situation in practice. Hence if (A_t) is the ICR of a CP (N_t) with respect to a family $(\mathcal{F}_t)$ the process $(N_t - A_t)$, that we will systematically denote by (M_t), is in the general case a square integrable $(\mathcal{F}_t)$ local martingale, and a $(\mathcal{F}_t)$ martingale when the mean EN_t is finite for each t. We will see later on (Corollary 2.4.12) that this Doob-Meyer decomposition $(N_t = M_t + A_t)$ is intuitively a decomposition into the part of the CP (N_t) which is not predictable or expected (this is (M_t)) and the one which can be perfectly predicted or contains no "surprises" (this is the ICR (A_t)). We refer to that as the separating property of the Doob-Meyer decomposition for CP's.

EXAMPLES AND FIRST PROPERTIES

Let (N_t) be a CP and denote by $(\mathcal{N}_t)$ the family of σ-algebras generated by (N_t). Let J_n be the time of n^{th} jump. Clearly for each n the stopped process $(N_{t\wedge J_n})$ is a submartingale with respect to any family $(\mathcal{F}_t)$ such that $\mathcal{F}_t \supset \mathcal{N}_t$. Hence we can define an ICR with respect to any such family. By definition an ICR is a natural process. This last property is dependent on the family $(\mathcal{F}_t)$ choosen (see Remark 1.7.5(a)). So we expect the ICR of the CP (N_t) to be dependent on the family $(\mathcal{F}_t)$ considered. That this is actually the case is demonstrated in Example 2.3.5. For emphasis we therefore speak of a "$(\mathcal{F}_t)$ ICR." This Example 2.3.5 is constructed with the help of the two next Propositions which also constitute our first examples of ICR's. Let (N_t) be a CP and (A_t) its $(\mathcal{F}_t)$ ICR. The first example is an extreme case in the sense that the family $(\mathcal{F}_t)$ considered is given for each t by $\mathcal{F}_t = \mathcal{N}_\infty$; hence at each time t, if we think of the available information as being given by the family $(\mathcal{F}_t = \mathcal{N}_\infty)$, everything is known about the process (N_t). In other words the CP (N_t) contains no surprises with respect to the family $(\mathcal{F}_t = \mathcal{N}_\infty)$. Thus in the light of the separating property of the Doob-Meyer decomposition the following result was to be expected:

<u>Proposition 2.3.3</u>: The $(\mathcal{F}_t)$ ICR (A_t) of a CP (N_t) where for each t $\mathcal{F}_t = \mathcal{N}_\infty$ is given by $A_t = N_t$.

Proof: Let (T_n) be a sequence of stopping times reducing the local martingale $(M_t = N_t - A_t)$, i.e., for each n the process $(M_t^n \overset{\Delta}{=} M_{t \wedge T_n})$ is a uniformly integrable $(F_t^n \overset{\Delta}{=} F_{t \wedge T_n})$ martingale which can be expressed as $E(M_\infty^n | F_t^n)$ by Theorem 1.5.4. Now:

$$N_\infty \overset{\Delta}{=} F_o = F_{o \wedge T_n} = F_o^n \subset F_t^n = F_{t \wedge T_n} \subset F_t \overset{\Delta}{=} N_\infty$$

i.e., for each t and n, $F_t^n = N_\infty = F_o^n$. Hence we can write (recall $M_o = N_o - A_o = 0$):

$$M_t^n = E(M_\infty^n | F_t^n) = E(M_\infty^n | F_o^n) = M_o^n = M_o = 0$$

which clearly implies $M_t = 0$ for each t and hence the result.

Another proof consists in showing directly that the increasing process (N_t) is natural with respect to the family $(F_t = N_\infty)$. Now for every sequence of stopping times (S_n) increasing to a stopping time S the random variable N_S is clearly $(\bigvee_n F_{S_n} = N_\infty)$ measurable. Also every $(F_t = N_\infty)$ stopping time R is predictable (the sequence of stopping times (R - 1/n) increases to R). Therefore totally inaccessible $(F_t = N_\infty)$ stopping times simply do not exist. Hence the process (N_t) charges no totally inaccessible stopping times. The two conditions (a) and (b) of Theorem 1.7.8 are satisfied and this shows that (N_t) is a natural increasing process with respect to the family $(F_t = N_\infty)$. Hence $(N_t = 0 + N_t)$ is the unique Doob-Meyer decomposition of (N_t), i.e., (0) is a uniformly $(F_t = N_\infty)$ martingale and $(A_t \equiv N_t)$.

□

The last part of the proof shows that the times of jump of the CP (N_t) are predictable. We will show later on (Corollary 2.4.11) that a CP (N_t) has predictable times of jump with respect to a family (F_t) if and only if its (F_t) ICR is given by (N_t) itself. The next example of ICR is about processes of independent increments:

Proposition 2.3.4: Let (N_t) be a CP of independent increments with a finite mean m_t for each t. Then the (N_t) ICR (A_t) is given by

$$A_t = m_t$$

Proof: For $t \geq s$ we have

$$\begin{aligned} E(N_t - m_t | N_s) &= E(N_t - N_s | N_s) + N_s - m_t \\ &= m_t - m_s + N_s - m_t \\ &= N_s - m_s \end{aligned}$$

i.e., the process $(N_t - m_t)$ is a (N_t) martingale. Furthermore the increasing process m_t is natural because it is deterministic (see Remark 1.7.5(c)). Now (N_t) has the Doob-Meyer decomposition

$$N_t = (N_t - m_t) + m_t$$

and the uniqueness requires m_t to be the (N_t) ICR. □

We will reexamine CP's of independent increments later on (Section 2.6) and prove in particular a converse result to the above proposition: namely that if a CP (N_t) has a deterministic $(\mathcal{N}_t)$ ICR then it is a process of independent increments.

Example 2.3.5: The two above results show that for a CP (N_t) of independent increments with finite mean m_t, the $(\mathcal{N}_t)$ ICR is given by m_t and the $(\mathcal{F}_t = \mathcal{N}_\infty)$ ICR by N_t. This example illustrates clearly the dependence of ICR's on the family of conditioning σ-algebras.

Given a CP (N_t) and its ICR's with respect to two distinct families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ such that $\mathcal{F}_t \supset \mathcal{G}_t \supset \mathcal{N}_t$, it is natural to ask how these two ICR's are related. This is what we examine now. Assume that the CP (N_t) has a finite mean. We will see that even in this case there is no simple useful answer to this problem. Denote respectively by $(A_t^{\mathcal{F}})$ and $(A_t^{\mathcal{G}})$ the ICR of (N_t) with respect to the families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$. We know that the processes

$$(M_t^{\mathcal{F}} \triangleq N_t - A_t^{\mathcal{F}}) \tag{2.3.5}$$

and

$$(M_t^{\mathcal{G}} \triangleq N_t - A_t^{\mathcal{G}}) \tag{2.3.6}$$

are respectively $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ martingales. But it is easy to show (see Appendix A.3) that the process

$$(X_t \triangleq N_t - C_t) \tag{2.3.7}$$

where

$$(C_t \triangleq E(A_t^{\mathcal{F}} | \mathcal{G}_t))$$

is a $(\mathcal{G}_t)$ martingale. The process (C_t) is not necessarily increasing and more over may not be natural with respect to the family $(\mathcal{G}_t)$. This last point is shown in the following example: let (N_t) be a CP of independent increments with finite mean m_t. If we choose $\mathcal{G}_t = \mathcal{N}_t$ and $\mathcal{F}_t = \mathcal{N}_\infty$ then we have seen (Propositions 2.3.3 and 2.3.4) that

$$A_t^{\mathcal{G}} = m_t \text{ and } A_t^{\mathcal{F}} = N_t$$

But

$$C_t \overset{\Delta}{=} E(A_t^{\mathcal{F}} | \mathcal{G}_t) = E(N_t | \mathcal{N}_t) = N_t \neq A_t^{\mathcal{G}}$$

so that by the uniqueness Theorem 1.7.6 (C_t) cannot be a natural process. The above shows that the relation

$$A_t^{\mathcal{G}} = C_t = E(A_t^{\mathcal{F}} | \mathcal{G}_t)$$

which seems very plausible at first glance does not hold in general. What is true is that the process (C_t) is a $(\mathcal{G}_t)$ submartingale: for $t \geq s$.

$$E(C_t | \mathcal{G}_s) = E[E(A_t^{\mathcal{F}} | \mathcal{G}_t) | \mathcal{G}_s] = E(A_t^{\mathcal{F}} | \mathcal{G}_s) \geq E(A_s^{\mathcal{F}} | \mathcal{G}_s) = C_s$$

Suppose (C_t) is in fact a right-continuous version of $E(A_t^{\mathcal{F}} | \mathcal{G}_t)$ (the mean $EC_t = EA_t^{\mathcal{F}}$ is right continuous so that such a right-continuous version exists by Theorem 1.5.5). By Theorem 19-VI of [M1] this positive submartingale belongs to the class (DL) and we denote its unique Doob-Meyer decomposition by

$$C_t = Y_t + B_t \tag{2.3.8}$$

where (Y_t) is a $(\mathcal{G}_t)$ martingale and (B_t) a natural (with respect to $(\mathcal{G}_t)$) increasing process. Introducing (2.3.8) in (2.3.7) we get

$$N_t = (X_t + Y_t) + B_t$$

which is, as (2.3.6), the unique ((B_t) is natural) Doob-Meyer decomposition of (N_t) with respect to $\mathcal{G}_t$. Hence the relation between $(A_t^{\mathcal{G}})$ and $(A_t^{\mathcal{F}})$ is

$$A_t^{\mathcal{G}} = B_t = E(A_t^{\mathcal{F}}|\mathcal{G}_t) - Y_t$$

It is also clear that if $(A_t^{\mathcal{F}})$ is in fact adapted to the family $(\mathcal{G}_t)$ then

$$A_t^{\mathcal{F}} = A_t^{\mathcal{G}}$$

In conclusion there is no simple way to related the two ICR's $(A_t^{\mathcal{G}})$ and $(A_t^{\mathcal{F}})$ in the general case. But when conditional rates with respect to the two families $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ exist then these two conditional rates are simply related (see Proposition 2.5.2).

We finish this section by two simple propositions. The first one shows the intuitive result that a.s no jump occurs in an interval on which the ICR is a.s constant as a function of time.

<u>Proposition 2.3.6</u>: Suppose (N_t) is a CP adapted to a family $(\mathcal{F}_t)$ which has an ICR with respect to this family that is a.s constant as a function of time on the stochastic interval [T,S] (T and S are stopping times, finite

or not such that $T < S$ a.s). Then (N_t) is a.s constant as a function of time for $t \in [T,S]$.

Proof: Let (R_n) be a squence of stopping times reducing the local martingale $(M_t \triangleq N_t - A_t)$ where (A_t) is the ICR of (N_t). The stopped process $(M_t^n \triangleq M_{t\wedge R_n})$ is a uniformly integrable martingale for each n. Define similarly $(N_t^n \triangleq N_{t\wedge R_n})$ and $(A_t^n \triangleq A_{t\wedge R_n})$. The process (A_t^n) is clearly also constant as a function of time for $t \in [T,S]$. The process (M_t^n) having zero mean we can write:

$$E(N_S^n - N_T^n) = E(M_S^n - M_T^n) - E(A_S^n - A_T^n) = 0 - 0 = 0$$

But

$$N_S^n - N_T^n \geq 0 \quad \text{a.s}$$

Thus

$$N_S^n = N_T^n \quad \text{a.s}$$

Hence

$$N_S = \lim_n N_S^n = \lim_n N_T^n = N_T \quad \text{a.s} \qquad \square$$

Proposition 2.3.7: Let (A_t) be the (F_t) ICR of a CP (N_t). Then $EN_t < \infty$ if and only if $EA_t < \infty$ and $EN_t = EA_t$.

Proof: $(\Rightarrow)$ If $EN_t < \infty$ then by Theorem 2.3.1(b) the process $(M_t \triangleq N_t - A_t)$ is a zero mean (F_t) martingale so that $EA_t = EN_t < \infty$.

$(\Leftarrow)$ Let J_n be the time of n^{th} jump of (N_t). Then (see the proof of Theorem 2.3.1) the process $(M_{t\wedge J_n} \triangleq N_{t\wedge J_n} - A_{t\wedge J_n})$

is a zero mean martingale. Hence $EN_{t\wedge J_n} = EA_{t\wedge J_n}$. Furthermore $A_{t\wedge J_n}$ increases to A_t as n goes to ∞ (similarly for $N_{t\wedge J_n}$) so that by the monotone convergence theorem

$$EN_t = \lim_n EN_{t\wedge J_n} = \lim_n EA_{t\wedge J_n} = EA_t < \infty$$

□

2.4 REGULAR AND ACCESSIBLE COUNTING PROCESSES

DEFINITION AND DECOMPOSITION

Let (N_t) be a CP adapted to a family (F_t). Denote by J_n the time of n^{th} jump. It is natural to classify CP's in terms of the properties of their stopping times J_n.

Definition 2.4.1: A CP (N_t) is called respectively regular, accessible or predictable with respect to the family (F_t) in accordance with the total inaccessibility, accessibility or predictability of its times of jump J_n with respect to this same family (see Definition 1.2.7).

While a process can be none of these, the next theorem will show that any CP (N_t) can be decomposed uniquely into the sum of a regular CP and an accessible CP. Here again these definitions are dependent on the particular family (F_t) choosen. We will see later on (below Theorem 2.4.7) that a CP can be regular with respect to one family and predictable with respect to another.

The term regular was previously used (Definition 1.7.12) to characterize a supermartingale (or submartingale) (X_t) such that for any sequence of stopping times

(S_n) increasing to a bounded stopping time S we have

$$\lim_n EX_{S_n} = EX_S$$

We show by the next Proposition that our terminology is consistent: regular CP's as in Definition 2.4.1 are also regular in the above sense (Definition 1.7.12), and conversely. On the contrary Rubin [R2] uses the term "regular CP" in a different sense. It denotes (if anything) a CP with a random rate which must possess numerous technical properties.

Proposition 2.4.2: Let (N_t) be a CP. Then the three following statements are equivalent:

(a) The CP (N_t) is regular in the sense of Definition 2.4.1

(b) For any stopping time S such that $EN_S < \infty$ the process $(N_{t\wedge S})$ is a regular submartingale in the sense of Definition 1.7.12.

(c) $\lim_n EN_{R_n} = EN_R$ for any sequence of stopping times increasing a.s to R and such that $EN_R < \infty$.

Proof: Let S be a stopping time such that $EN_S < \infty$ and (T_n) any sequence of stopping times increasing to T a.s. If the relation

$$\lim_n N_{T_n\wedge S} = N_{T\wedge S} \quad \text{a.s} \qquad (2.4.1)$$

is verified then by the monotone convergence theorem we have

$$\lim_n EN_{T_n \wedge S} = EN_{T \wedge S} \qquad (2.4.2)$$

Conversely if relation (2.4.2) holds we have $E(N_{T\wedge S} - \lim_n N_{T_n \wedge S}) = 0$ by the monotone convergence theorem. As the random variable $N_{T\wedge S} - \lim_n N_{T_n\wedge S}$ is positive, relation (2.4.1) must be verified. Hence conditions (2.4.1) and (2.4.2) are equivalent. We show now that (a) is equivalent to (b). If (a) is verified then the times of jump of the submartingale $(N_{t\wedge S})$ are totally inaccessible (the time of n^{th} jump of $(N_{t\wedge S})$ is equal to J_n on the set $\{J_n < S\}$ and to ∞ otherwise). Therefore relation (2.4.1) is verified and, being equivalent to (2.4.2), (b) follows (see Definition 1.7.12). Conversely if (b) is true, relation (2.4.2) is satisfied. Then (2.4.1) holds which implies that the times of jump of $(N_{t\wedge S})$ are totally inaccessible (otherwise we reach a contradiction). By taking $S = J_n$, the time of n^{th} jump of (N_t), we get that J_n is a totally inaccessible stopping time. This is true for each n so that (a) follows. We show now that (b) is equivalent to (c). If (b) is true and (R_n) is any sequence of stopping times increasing a.s to R and such that $EN_R < \infty$ then $(N_{t\wedge R})$ is a regular submartingale. In particular (see Definition 1.7.12) $\lim_n EN_{R_n\wedge R} = EN_{R\wedge R}$ so that (c) follows. Conversely if (c) is true and (T_n) is any sequence of stopping times increasing a.s to T then

$$\lim_n EN_{T_n\wedge R} = EN_{T\wedge R}$$

which shows that $(N_{t\wedge R})$ is a regular submartingale. □

Now the announced decomposition result:

Theorem 2.4.3: Let (N_t) be a CP adapted to a family (F_t). Then there exists two CP's, (N_t^r) and (N_t^a) which are respectively regular and accessible with respect to the above family and such that

$$N_t = N_t^r + N_t^a \quad \text{for every } t$$

This decomposition is unique.

Remark 2.4.4: The (F_t) ICR of (N_t) is given by

$$A_t = A_t^r + A_t^a$$

where (A_t^r) and (A_t^a) are respectively the (F_t) ICR's of (N_t^r) and (N_t^a).

Proof: As usual, denote by J_n the time of n^{th} jump. By J_n^A we mean the stopping time (Definition 1.2.9)

$$J_n^A = \begin{cases} J_n & \text{if } \omega \in A \\ \infty & \text{otherwise} \end{cases}$$

for $A \in F_{J_n}$. By Theorem 44-VII of [M1] there exists for each n an essentially unique partition of the set $\{J_n < \infty\}$ into two sets of F_{J_n}, A and R, such that the stopping times J_n^A and J_n^R are respectively accessible and totally inaccessible. The two CP's

$$N_t^a \triangleq \sum_n I_{\{t \geq J_n^A\}}$$

$$N_t^r \triangleq \sum_n I_{\{t \geq J_n^R\}}$$

clearly satisfy the conditions of the theorem. The uniqueness of this decomposition follows from the essentially uniqueness of the partition of each set $\{J_n < \infty\}$. □

Example 2.4.5: Take $\Omega = [0,1]$ and let J_1 be a random variable uniformly distributed on Ω. Define the random variables $J_{n+1} = J_1 + n$, for $n \geq 1$. Let (N_t) be the CP having J_n as time of n^{th} jump, i.e.,

$$N_t = \sum_n I_{\{t \geq J_n\}}$$

For each $m,n \geq 1$, it is trivial to check that the random variable $T^n_m \triangleq J_n + 1 - 1/m$ is a stopping time with respect to any family $(\mathcal{F}_t)$ such that $\mathcal{F}_t \supset \mathcal{N}_t$. But for each $n \geq 1$ the sequence (T^n_m) converges to J_{n+1} as m goes to ∞. Hence the times of jump J_n for $n \geq 2$ are predictable. The time of jump J_1, being uniformly distributed on Ω, is totally inaccessible by Corollary A.4.2. Thus the decomposition $N_t = N^r_t + N^a_t$ is given by

$$N^r_t = I_{\{t \geq J_1\}} \text{ and } N^a_t = \sum_{n \geq 2} I_{\{t \geq J_n\}}$$

In this very simple example the CP (N^a_t) is in fact predictable. This is not always the case. If we assume in the above example that jumps may be skipped independently of each other with a positive probability then (N^r_t) and (N^a_t) are still given as above but the CP (N^a_t) is no longer predictable (see Example 2.4.13).

For clarity we outline now some of the results we are going to investigate. First regular, then accessible CP's are studied in detail. In particular we will see that a CP is regular with respect to a family (F_t) if and only if its (F_t) ICR is continuous (Theorem 2.4.7); when the family (F_t) is free of times of discontinuity (Definition 1.2.8) then accessible CP's are predictable (Proposition 2.4.9). Predictable CP's are uniquely characterized by the fact that their ICR is given by the CP itself (Corollary 2.4.11). In other words predictable CP's are natural processes. Combining these facts with the above decomposition for CP's (Theorem 2.4.3) gives, when the family (F_t) is free of times of discontinuity, the separating property of the unique Doob-Meyer decomposition for CP's. Namely if $(N_t = M_t + A_t)$ is the unique Doob-Meyer decomposition of (N_t) then the local martingale (M_t) contains only jumps of size one which take place at totally inaccessible stopping times while the (F_t) ICR (A_t) also has jumps of size one but at predictable stopping times (Corollary 2.4.12). In other words (M_t) represents the part of (N_t) which is unexpected and the ICR (A_t) the one which can be perfectly predicted. The case where the family (F_t) does contain times of discontinuity is more complex. Most of these results are obtained by studying the different terms in the equation $\Delta N_T = \Delta M_T + \Delta A_T$ in relation to the appropriate property of the stopping time T (Theorem 2.4.10).

REGULAR COUNTING PROCESSES

Let (N_t) be a regular CP with respect to a family (F_t). By definition the times of jump J_n of (N_t) are totally inaccessible. This has the immediate consequence that the probability a jump occurs at time t is zero. Because if not there exists a constant a and an m such that $P\{J_m = a\} > 0$. The sequence of stopping times $(R_n \triangleq J_m \wedge (a-1/n))$ is such that $P\{\lim_n R_n = J_m < \infty, R_n < J_m \ \forall\ n\} = P\{J_m = a\} > 0$ which shows that the time of jump J_m is not totally inaccessible, a contradiction. Also if T is a (F_t) stopping time we cannot make with a positive probability a prediction of any time of jump after T, the prediction being based on the information available up to and at time T. More precisely we have:

Proposition 2.4.6: Let (N_t) be a regular CP with respect to a family (F_t) and T a (F_t) stopping time. Assume W is a strictly positive F_T measurable random variable. Then for each n

$$P\{T + W = J_n\} = 0$$

where J_n is the time of n^{th} jump of (N_t).

Proof: By contradiction. Assume that for $n = n_o$ there exists $W = W_o$, a strictly positive (F_t) measurable random variable, with

$$P\{T + W_o = J_{n_o}\} = p > 0$$

By Theorem 38-IV of [M1] the random variable

$$T_e = T + (1 - 1/e)W_o$$

is a (F_t) stopping time for each e. This sequence (T_e) is increasing and

$$P\{\lim_e T_e = J_{n_o}\} = p > 0$$

i.e., the time of n_o^{th} jump is not totally inaccessible, a contradiction. □

The next theorem is a direct consequence of Proposition 2.4.2 and a result on the Doob-Meyer decomposition of regular supermartingales (Theorem 1.7.14).

Theorem 2.4.7: Let (N_t) be a CP adapted to a family (F_t). Then the (F_t) ICR (A_t) of (N_t) is continuous if and only if the CP (N_t) is regular with respect to this family.

Proof: Let J_n be the n^{th} time jump and define $(N_t^n \triangleq N_{t\wedge J_n})$, $(A_t^n \triangleq A_{t\wedge J_n})$, $(F_t^n \triangleq F_{t\wedge J_n})$. Note that by the uniqueness of the Doob-Meyer decomposition (A_t^n) is the (F_t^n) ICR of (N_t^n) (see Lemma 1.7.10). By Theorem 1.7.14(b) the process (A_t^n) is continuous for each n if and only if the CP (N_t) is regular. The result follows then by taking the limit as the sequence (J_n) increases to ∞. One uses here the fact that on any interval $[0,t_o]$, $A_t = A_t^n$ for sufficiently large n (depending on ω). □

Examples of regular CP's with respect to the family (N_t) are, by Proposition 2.3.4 and the above Theorem, any CP's of independent increments with continuous mean, in particular Poisson processes. Note that these processes of independent increments with continuous mean are not regular but predictable if we take the family $(F_t = N_\infty)$ (see Proposition 2.3.3).

For a regular CP (N_t) with ICR (A_t) we have just proved that all the jumps are contained in the local martingale $(M_t = N_t - A_t)$. But these jumps completely determined the CP (N_t). This suggests that there is a direct relation between (M_t) and the ICR (A_t). This point is made clear in the following theorem. Recall (see Sction 1.9, below Theorem 1.9.6; [K1]) that if (X_t) is a square integrable local martingale then $(\langle X\rangle_t)$ denotes the unique natural increasing process which makes the process $(X_t^2 - \langle X\rangle_t)$ a local martingale. Also if $(N_t = M_t + A_t)$ is the unique Doob-Meyer decomposition of (N_t) then by Theorem 2.3.1 (M_t) is a square integrable local martingale.

Theorem 2.4.8: Let (N_t) be a regular CP with respect to a family (F_t). Denote by (A_t) its (F_t) ICR and by (M_t) the square integrable local martingale $(N_t - A_t)$. We have (a) $A_t = \langle M\rangle_t$

(b) If EN_t is finite then so is EM_t^2

Property (b) will be used later on to prove a result on martingale representation, a result essential in solving the detection problem for CP's.

Proof: (a) We have $N_t = M_t + A_t$ (2.4.3)

Let $M_t = M_t^c + M_t^d$ where $(M_t^c) \in L_c$ is the unique decomposition of Theorem 1.9.6. Now relation (2.4.3) shows that $(M_t) \in L \cap V$. Consequently (this is an easy extension of Remark 1.8.17) $(M_t^c \equiv 0)$, $(\langle M^c \rangle_t \equiv 0)$ and the quadratic variation process (Definition 1.9.7) is given by

$$[M]_t = \sum_{s \leq t} (\Delta M_s)^2 \qquad (2.4.4)$$

But (N_t) is a regular CP and by Theorem 2.4.7 its ICR (A_t) is continuous so that $\Delta M_s = \Delta N_s$. Now N_s is either 0 or 1 hence $(\Delta M_s)^2 = (\Delta N_s)^2 = \Delta N_s$ which implies by (2.4.4)

$$[M]_t = N_t \qquad (2.4.5)$$

The two processes $(M_t^2 - \langle M \rangle_t)$ and $(M_t^2 - [M]_t)$ are local martingales (see Section 1.9, below Theorem 1.9.6 and Remark 1.9.8(b)). Thus so is their difference $(X_t \triangleq [M]_t - \langle M \rangle_t)$ and by (2.4.5) we get

$$N_t = X_t + \langle M \rangle_t \qquad (2.4.6)$$

where $(X_t) \in L$. The increasing process $(\langle M \rangle_t)$ is natural by definition so that relation (2.4.6), as (2.4.3), is the unique Doob-Meyer decomposition of the CP (N_t). Then we must have

$$A_t = \langle M \rangle_t \text{ and } X_t = M_t$$

(b) We have seen above that the process $(M_t^2 - [M]_t) \in L$ or by (2.4.5) $(M_t^2 - N_t) \in L$. Let (T_n) be a sequence of stopping times reducing this local martingale i.e., the process $(M^2_{t\wedge T_n} - N_{t\wedge T_n})$ is a uniformly integrable martingale. In particular

$$E(M^2_{t\wedge T_n} - N_{t\wedge T_n}) = E(M^2_o - N_o) = 0$$

Hence

$$EM^2_{t\wedge T_n} = EN_{t\wedge T_n} \qquad (2.4.7)$$

Since $M_{t\wedge T_n}$ converges to M_t, Fatou's lemma implies

$$EM^2_t \leq \liminf_n EM^2_{t\wedge T_n}$$

and by (2.4.7) and the monotone convergence theorem ($N_{t\wedge T_n}$ increases to N_t) we get

$$EM^2_t \leq \liminf_n EM^2_{t\wedge T_n} = \lim_n EN_{t\wedge T_n} = EN_t \qquad \square$$

ACCESSIBLE COUNTING PROCESSES

Theorem 2.4.7, which says that the ICR of a CP is continuous if and only if this CP is regular, implies that the ICR (A_t) of an accessible CP (N_t) is discontinuous. We could conjecture that the times of jump of the ICR (A_t) are the same as those of the accessible CP (N_t). As we will see this would be true, and we would have in fact $(A_t \equiv N_t)$ but for the possible presence of times of discontinuity for the family (F_t) considered (see Definitions 1.2.8 and 1.2.10). Recall that an accessible (F_t) stopping time which is not a time of discontinuity for the family (F_t) is (F_t) predictable (see Theorem 1.2.11). This immediately gives us:

Proposition 2.4.9: An accessible CP (N_t) with respect to a family (F_t) which is free of times of discontinuity is predictable.

Let (N_t) be any CP with ICR (A_t). We examine now the jump ΔA_T in relation to the property of the stopping time T. We already know that for a regular CP $\Delta A_T = 0$ for any stopping time T (Theorem 2.4.7). The next result will lead to a unique characterization of predictable CP's (Corollary 2.4.11) and the separating property of the unique Doob-Meyer decomposition for CP's (Corollary 2.4.12).

Theorem 2.4.10: Suppose (N_t) is any CP adapted to a family (F_t). Denote by (A_t) its (F_t) ICR.

(a) If T is (F_t) predictable then

$$\Delta A_T = E(\Delta N_T | \bigvee_n F_{T_n})$$

where (T_n) is any sequence of stopping times increasing to T. In particular $0 \leq \Delta A_T \leq 1$, and $\Delta A_T = 1$ (or 0) a.s if and only if $\Delta N_T = 1$ (or 0) a.s.

(b) If T is (F_t) accessible but not a time of discontinuity for (F_t) then

$$\Delta A_T = \Delta N_T$$

(c) If T is (F_t) totally inaccessible then $\Delta A_T = 0$.

(d) Let J_n be the time of n^{th} jump of (N_t). Then

$$\Delta A_{J_n} = \Delta N_{J_n} = 1$$

if and only if J_n is a predictable (F_t) stopping time. In particular $\Delta A_{J_n} = 1$ if J_n is accessible but not a time of discontinuity for the family (F_t).

<u>Proof</u>: (a) (see [M1], § 51-VII) Let J_n be the time of n^{th} jump of (N_t), and define $(N_t^n \triangleq N_{t\wedge J_n})$, $(A_t^n \triangleq A_{t\wedge J_n})$. We know (Theorem 2.3.1) that the process $(M_t^n \triangleq N_t^n - A_t^n)$ is a square integrable $(F_{t\wedge J_n})$ martingale. By Remark 1.8.2(a) this martingale is uniformly integrable and by Lemma A.2.1 it is also a (F_t) martingale. Thus for $i \geq m$ and any set $H \in F_{T_m}$ where (T_m) is a sequence of stopping times increasing to T we have

$$\int_H (M_T^n - M_{T_i}^n)dP = \int_H E(M_T^n - M_{T_i}^n | F_{T_m})dP = 0$$

so that using the relation $(M_t^n = N_t^n - A_t^n)$ one gets

$$\int_H (A_T^n - A_{T_i}^n)dP = \int_H (N_T^n - N_{T_i}^n)dP$$

Letting i increase to ∞ one obtains, by the monotone convergence theorem

$$\int_H \Delta A_T^n dP = \int_H \Delta N_T^n dP \quad \forall\ H \in F_{T_m}$$

This implies

$$E(\Delta A_T^n | F_{T_m}) = E(\Delta N_T^n | F_{T_m}) \quad a.s$$

and taking the limit with respect to m, by Lemma 1.5.7

$$E(\Delta A_T^n | \bigvee_m F_{T_m}) = E(\Delta N_T^n | \bigvee_m F_{T_m})$$

The process (A_t^n) is natural with respect to the family (F_t) (Theorem 1.7.9). Then by Theorem 1.7.8', the random variable ΔA_T^n is $(\bigvee_m F_{T_m})$ measurable. Thus the above relation gives

$$\Delta A_T^n = E(\Delta N_T^n | \bigvee_m F_{T_m})$$

and by the bounded convergence theorem we get the desired result letting n go to ∞.

(b) By Theorem 1.2.11, T is predictable so that part (a) is applicable. Furthermore $F_T = \bigvee_m F_{T_m}$ (T is not a time of discontinuity of (F_t)). Hence

$$\Delta A_T = E(\Delta N_T | \bigvee_m F_{T_m}) = E(\Delta N_T | F_T) = \Delta N_T$$

Part (c) is just a restatement of condition (b) of Theorem 1.7.8 and is given here for completeness.

(d) (⇐) J_n is predictable and $\Delta N_{J_n} = 1$ so that by part (a) $\Delta A_{J_n} = 1$

(⇒) Assume $\Delta A_{J_n} = 1$. Let $C_t^n = \Delta A_{J_n} I_{\{t \geq J_n\}} = I_{\{t \geq J_n\}}$

The process (C_t^n) is natural because it satisfies the necessary and sufficient conditions (a) and (b) of Theorem 1.7.8 (if not then the natural process (A_t) would not satisfy these two conditions, a contradiction). By Theorem 52-VII of [M1] then J_n is a predictable stopping time. □

<u>Corollary 2.4.11</u>: A CP (N_t) with ICR (A_t) with respect to a family $(\mathcal{F}_t)$ is predictable with respect to this family if and only if $(A_t = N_t)$.

<u>Proof</u>: (⇒) (N_t) is predictable so by (d) of Theorem 2.4.10

$$\Delta A_{J_n} = 1 \quad \text{for each } n$$

where J_n is the time of n^{th} jump of (N_t). This implies

$$A_t \geq N_t$$

In particular for each n

$$M_{t \wedge J_n} \triangleq N_{t \wedge J_n} - A_{t \wedge J_n} \leq 0 \tag{2.4.8}$$

But $(M_{t \wedge J_n})$ is a zero mean martingale so that (2.4.8) implies

$$M_{t \wedge J_n} = 0 \quad \text{a.s}$$

or after taking the limit

$$N_t = A_t \quad \text{a.s}$$

(⇐) If $(N_t = A_t)$ then $\Delta A_{J_n} = 1$ for each n and by (d) of Theorem 2.4.10 J_n is a predictable stopping time for each n, i.e., (N_t) is predictable. □

<u>Corollary 2.4.12</u>: Let (N_t) be a CP with (F_t) ICR (A_t) and define $(M_t \triangleq N_t - A_t)$. Then if the family (F_t) is free of times of discontinuity

(a) The local martingale (M_t) has only jumps of size one taking place at (F_t) totally inaccessible stopping times.

(b) The (F_t) ICR (A_t) has only jumps of size one at (F_t) predictable stopping times.

<u>Proof</u>: Let

$$N_t \triangleq N_t^r + N_t^a \qquad (2.4.9)$$

denote the unique decomposition of Theorem 2.4.3 where (N_t^r) is a regular CP and (N_t^a) an accessible CP. Let respectively (A_t^r) and (A_t^a) be the (F_t) ICR of (N_t^r) and (N_t^a). By Theorem 2.4.7, (A_t^r) is continuous so that the local martingale

$$M_t^r \triangleq N_t^r - A_t^r \qquad (2.4.10)$$

has only jumps of size one taking place at totally inaccessible stopping times (namely the times of jump of (N_t^r)).
By assumption the family (F_t) is free of times of discontinuity so that by Proposition 2.4.9 (N_t^a) is a predictable CP and by Corollary 2.4.11

$$A_t^a = N_t^a \qquad (2.4.11)$$

Introducing (2.4.10) in (2.4.9) one gets

$$N_t = M_t^r + (A_t^r + N_t^a)$$

which is a unique Doob-Meyer decomposition of (N_t) as, by (2.4.11), $(A_t^r + N_t^a = A_t^r + A_t^a)$ is a natural increasing process. But $(N_t = M_t + A_t)$ is also such a unique decomposition so that one must have

$$M_t = M_t^r$$
$$A_t = A_t^r + N_t^a$$

and the result follows. □

Let $(N_t = M_t + A_t)$ denote the unique Doob-Meyer decomposition of the CP (N_t) with respect to the family (F_t). When this family (F_t) is free of times of discontinuity the above Corollary 2.4.12 completely describes the discontinuities of the local martingale (M_t) and of the (F_t) ICR (A_t): either (M_t) or (A_t) (but not both) have a discontinuity which is of size one and can only take place at a time of jump of (N_t). When the family (F_t) does have times of discontinuity the situation is a little more complex. Suppose T is a time of discontinuity for (F_t). Then (see Definitions 1.2.9 and 1.2.10) there exists an event $A \in F_T$ and a sequence of stopping times (S_n) increasing to S such that $S \leq T_A$ a.s and

$$\{S = T_A\} \notin \bigvee_n F_{S_n}$$

This has the following consequence for a uniformly integrable

$(\mathcal{F}_t)$ martingale $(X_t = E(X_\infty|\mathcal{F}_t))$: we have (see Theorem 13-VI of [M1] and Lemma 1.5.7)

$$\Delta X_S \triangleq X_S - \lim_n X_{S_n} = E(X_\infty|\mathcal{F}_S) - E(X_\infty|\bigvee_n \mathcal{F}_{S_n})$$

Since the set $\{S = T_A\}$ belongs to $\mathcal{F}_S$ (Theorem 41-IV of [M1]) but not to $\bigvee_n \mathcal{F}_{S_n}$ it may be that ΔX_S is different from zero with some positive probability on the set $\{S = T_A\}$. Hence it is not surprising for a uniformly integrable martingale to have a jump at a time of discontinuity of $(\mathcal{F}_t)$ and similarly for local martingales. In terms of the unique Doob-Meyer decomposition $(N_t = M_t + A_t)$ this suggests that two situations, which do not occur when $(\mathcal{F}_t)$ is free of times of discontinuity, may now take place. They will be illustrated in Example 2.4.13.

(a) Let T be a stopping time which is a time of discontinuity for $(\mathcal{F}_t)$ and such that $\Delta N_T = 0$ a.s. As explained above it is likely that ΔM_T will be different from zero so that both ΔM_T and $\Delta A_T = -\Delta M_T$ are different from zero with some positive probability although $\Delta N_T = 0$ a.s. By Corollary 2.4.12 this does not happens when T is not a time of discontinuity for $(\mathcal{F}_t)$. Let T_A and $T_{A'}$ be respectively the accessible and totally inaccessible part of T (see Theorem 1.2.12). By Theorem 2.4.10(c), $\Delta A_{T_{A'}} = 0$ a.s so that $\Delta M_{T_{A'}} = 0$ a.s. If now T_A is predictable, by Theorem 2.4.10(a), $\Delta A_{T_A} = 0$ a.s and $\Delta M_{T_A} = 0$ a.s. Hence ΔA_T and ΔM_T may be both different from zero only if T_A is not predictable (T_A is of course a time of discontinuity for $(\mathcal{F}_t)$, as T has this property).

(b) Let J_n be the time of n^{th} jump of (N_t) and assume it is a time of discontinuity for $(\mathcal{F}_t)$. Suppose also that J_n is an accessible stopping time (if not we can decompose it into its totally inaccessible and accessible parts (Theorem 1.2.12); on the totally inaccessible part we already know that (Theorem 2.4.7) $\Delta A_{J_n} = 0$ so that $\Delta M_{J_n} = 1$). As before it is likely that ΔM_{J_n} will be different from zero with some positive probability. Now by Theorem 2.4.7 ΔM_{J_n} cannot be one a.s so that both (A_t) and (M_t) have a discontinuity at J_n. Recall that this cannot happen when J_n is not a time of discontinuity for $(\mathcal{F}_t)$ as, by Theorem 2.4.1(b), $\Delta A_{J_n} = 1$ a.s so that $\Delta M_{J_n} = 0$ a.s in this case. That both situations (a) and (b) actually take place is illustrated in the next example.

<u>Example 2.4.13</u>: Take $\Omega = \{\omega_1, \omega_2\}$ with $P\{\omega_1\} = p$ where $0 < p < 1$. Define the following CP (N_t):

$$N_t(\omega_1) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$$

$$N_t(\omega_2) = 0 \qquad t \geq 0$$

The family $(\mathcal{N}_t)$ is then given by

$$\mathcal{N}_t = \begin{cases} \{\phi, \Omega\} & \text{for } t < 1 \\ \{\phi, \{\omega_1\}, \{\omega_2\}, \Omega\} & \text{for } t \geq 1 \end{cases}$$

and the unique time of jump of (N_t) by

$$J(\omega) = \begin{cases} 1 & \omega = \omega_1 \\ \infty & \omega = \omega_2 \end{cases}$$

This stopping time J is obviously accessible. Observe also that J is a time of discontinuity for the family $(\mathcal{N}_t)$: (see Definition 1.2.10) $(S_n \triangleq 1 - 1/n)$ is an increasing sequence of stopping times, $S_n < J$ for each n and the set

$$\{\omega: \lim_n S_n = J\} = \{\omega_1\} \notin \bigvee_n \mathcal{N}_{S_n} = \{\phi, \Omega\}$$

Denote a (not necessarily unique) Doob-Meyer decomposition of (N_t) by

$$N_t = M_t + A_t \tag{2.4.12}$$

where (M_t) is a uniformly integrable martingale ((N_t) is bounded) and (A_t) an increasing (not necessarily natural) process. By Theorem 1.5.4 the martingale (M_t) can be expressed as

$$M_t = E(M_\infty | \mathcal{F}_t)$$

where M_∞ is a random variable measurable with respect to $\mathcal{N}_\infty = \{\phi, \{\omega_1\}, \{\omega_2\}, \Omega\}$. Furthermore we know that $M_0 = 0$ so that $EM_\infty = 0$. From that it is easy to see that any martingale (M_t) is given by

$$M_t(\omega) = \begin{cases} 0 & \forall\, \omega\ ,\ t < 1 \\ \begin{cases} a & \omega = \omega_1\ ,\ t \geq 1 \\ b & \omega = \omega_2\ ,\ t \geq 1 \end{cases} \end{cases}$$

where a and b are two constants such that

$$\frac{a}{b} = -\left(\frac{1-p}{p}\right) \tag{2.4.13}$$

Then by (2.4.12) we must have

$$A_t(\omega) = \begin{cases} 0 & \forall\, \omega\ ,\ t < 1 \\ \begin{cases} 1-a & \omega = \omega_1\ ,\ t \geq 1 \\ -b & \omega = \omega_2\ ,\ t \geq 1 \end{cases} \end{cases}$$

By the uniqueness theorem only one set of values a and b makes the increasing process (A_t) natural. These values are $a = 1 - p$ and $b = -p$ (this choice obviously satisfies (2.4.13)) as in this case

$$A_t = pI_{[1,\infty)}(t)$$

is a deterministic hence natural process (see Remark 1.7.5(c)). Hence the ICR of (N_t) with respect to the family $(\mathcal{N}_t)$ is $pI_{[1,\infty)}(t)$ and the martingale $(M_t = N_t - pI_{[1,\infty)}(t))$ is given by (see Figure 2.4.14):

$$M_t = \begin{cases} 0 & \forall\ \omega,\ t < 1 \\ \begin{cases} 1-p & \omega = \omega_1,\ t \geq 1 \\ -p & \omega = \omega_2,\ t \geq 1 \end{cases} \end{cases}$$

Therefore both the ICR $pI_{[1,\infty)}(t)$ and the above martingale have a discontinuity at the time of jump J of (N_t). This illustrates case (b). As stated above, this is a consequence of the fact that the accessible stopping time J is not predictable and is also a time of discontinuity for (N_t). Also if we define the stopping time T

$$T = \begin{cases} \infty & \omega = \omega_1 \\ 1 & \omega = \omega_2 \end{cases}$$

then

$$\Delta A_T = \begin{cases} 0 & \omega = \omega_1 \\ p & \omega = \omega_2 \end{cases}$$

even though $\Delta N_T = 0$ for any ω. This illustrates case (a). It is easy to check that T is a time of discontinuity for (N_t) which is accessible but not predictable.

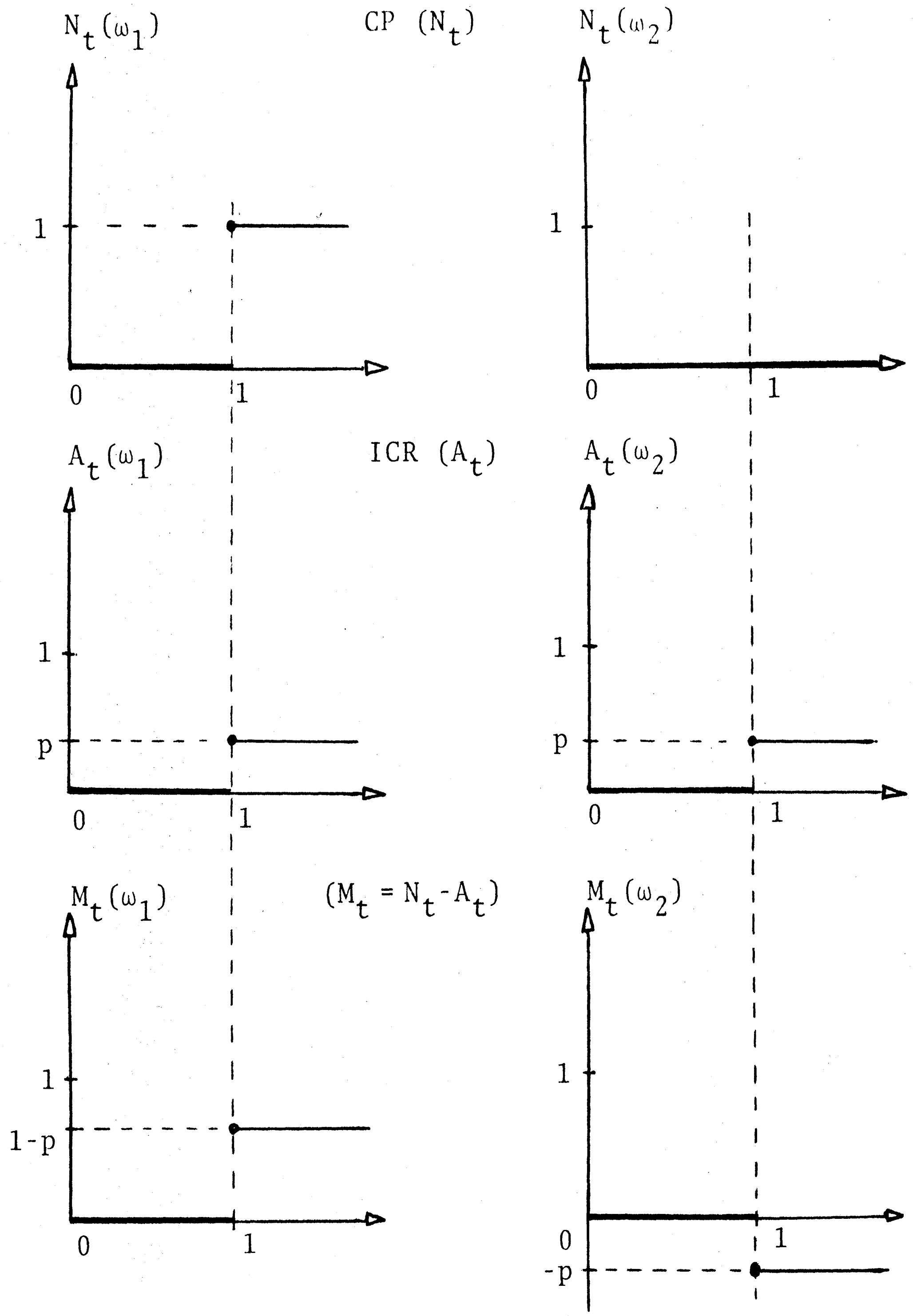

N_t(ω_1)
CP (N_t)
N_t(ω_2)
1
0
1
A_t(ω_1)
ICR (A_t)
A_t(ω_2)
1
p
0
1
M_t(ω_1)
(M_t = N_t - A_t)
M_t(ω_2)
1
1-p
0
1
-p

Figure 2.4.14

2.5 CONDITIONAL RATE

In the previous section we have seen that we can decompose uniquely any CP (N_t) adapted to a family $(\mathcal{F}_t)$ into a sum of two CP's which are respectively regular and accessible with respect to this family $(\mathcal{F}_t)$ (Theorem 2.4.3). Regular CP's relatively to a family $(\mathcal{F}_t)$ are precisely those which have a continuous $(\mathcal{F}_t)$ ICR (Theorem 2.4.7). But a continuous ICR may not have absolutely continuous sample paths. For example consider a CP of independent increments with a continuous, but not absolutely continuous mean.

In the next theorem we give sufficient conditions under which the ICR (A_t) of a CP (N_t) with respect to a family $(\mathcal{F}_t)$ is absolutely continuous or in other words when does a random process (λ_t) adapted to $(\mathcal{F}_t)$ exist such that we can express the ICR (A_t) as

$$A_t = \int_0^t \lambda_s ds \tag{2.5.1}$$

Under these conditions, we also have

$$\lambda_t = \lim_{h \to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| \mathcal{F}_t \right) \tag{2.5.2}$$

and because of this relation we call the process (λ_t) the "conditional rate" of the CP (N_t) with respect to the family $(\mathcal{F}_t)$. Expression (2.5.1) is then a justification for the terminology "Integrated Conditional Rate" (ICR) introduced in Section 2.3, terminology used even though a conditional rate does not generally exist. Note also that if the (F_t) ICR of a CP (N_t) with finite mean is given by $(\int_0^t \lambda_s ds)$ where (λ_t) is a right-continuous bounded process adapted to the family (F_t) then, the process $(N_t - \int_0^t \lambda_s ds)$ being a (F_t) martingale (Theorem 2.3.1), we have

$$E\left(\frac{N_{t+h} - N_t}{h} \middle| F_t \right) = E\left(\frac{1}{h} \int_t^{t+h} \lambda_s ds \middle| F_t \right)$$

and by the dominated convergence theorem ((λ_t) is bounded)

$$\lim_{h \to 0} E\left(\frac{1}{h} \int_t^{t+h} \lambda_s ds \middle| F_t \right) = E\left(\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \lambda_s ds \middle| F_t \right)$$

$$= E(\lambda_t | F_t) = \lambda_t$$

so that relation (2.5.2) is also verified in this case.

Although there is a lot of emphasis in the literature ([C1],[B1],[R2],[S1],[S2],[S3]) on CP's which admit conditional rates, the problem of existence of these CP's has been partially treated, as explained in the INTRODUCTION, only lately by Brémaud in his dissertation ([B1]), using a technique of absolutely continuous change of measure. We will examine a generalization of this technique but only in the next chapter on detection, these two problems being related. This question of existence of CP's with conditional rates is difficult and may in fact not be of great importance: for example (and this will be demonstrated in the next chapter) the solution to the detection problem does not require the existence of conditional rates for the CP's involved (but they must be regular). We now give sufficient conditions under which a CP with finite mean does possess a conditional rate. Note that these conditions are a kind of conditional version of the conditions which uniquely define a Poisson counting process (see Chapter 4 of [P1]).

Theorem 2.5.1: If for a CP (N_t) with finite mean and adapted to a family (F_t)

(i) for each t the following limit exists a.s

$$\lim_{h\to 0} \frac{1}{h} Q_m(t,h,\omega) \triangleq \lambda_m(t,\omega) \quad m = 1,2,\ldots$$

where $Q_m(t,h,\omega) \triangleq P\{N_{t+h} - N_t \geq m | F_t\}$

(ii) for almost all ω there exists $h_o(\omega) > 0$ such that the series $\sum_m \frac{1}{h} Q_m(t,h,\omega)$ converges uniformly for $h \in (0,h_o(\omega)]$ and is bounded by a function $a(t,\omega)$ such that $\int_0^t a(s,\omega)ds < \infty$ for each t. Then

(a) The series $\sum_m \lambda_m$ is convergent. Define the process $(\lambda_t \triangleq \sum_m \lambda_m)$. We have the relation:

$$\lambda_t = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h} \middle| F_t \right) \quad \text{a.s for every } t$$

(b) The (F_t) ICR of (N_t) is given by

$$A_t = \int_0^t \lambda_s ds$$

Proof: By (i) and (ii)

$$\lim_{h\to 0} \frac{1}{h} \sum_m Q_m(t,h,\omega) = \sum_m \lim_{h\to 0} \frac{1}{h} Q_m(t,h,\omega) = \sum_m \lambda_m(t,\omega) \overset{\Delta}{=} \lambda_t(\omega) \tag{2.5.3}$$

where the first equality follows by the uniform convergence on $(0,h_o(\omega)]$ (see Theorem 7.11 of [R1]). Assumption (ii) also implies for almost all ω and $h \leq h_o(\omega)$

$$\sum_m Q_m(t,h,\omega) \leq a(t,\omega)h_o(\omega) < \infty$$

and this is enough to justify the equality

$$\sum_m m(Q_m - Q_{m+1}) = \sum_m Q_m$$

But

$$Q_m - Q_{m+1} = P\{N_{t+h} - N_t = m|F_t\}$$

So that the above relation gives for $h \leq h_o(\omega)$

$$E(N_{t+h} - N_t|F_t) = \sum_m Q_m(t,h,\omega) \tag{2.5.4}$$

and by (2.5.3)

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \sum_m Q_m(t,h,\omega) = \lim_{h\to 0} E\left(\frac{N_{t+h} - N_t}{h}\middle| F_t\right) \tag{2.5.5}$$

(b) The CP (N_t) is right-continuous; by Theorem 1.5.5 there exists a right-continuous modification for the submartingale $(E(N_{t+h}|F_t))$ (see Definition 27-VII of [M1]) and we denote by $(p_h\lambda_t)$ a right-continuous modification of the process $(E(\frac{N_{t+h} - N_t}{h}|F_t))$. We have seen above that

$$\lim_{h\to 0} p_h\lambda_t = \lambda_t \quad \text{a.s}$$

By (ii) and (2.5.4)

$$0 \le p_h\lambda_t = \sum_m \frac{1}{h} Q_m(t,h,\omega) \le a(t,\omega)$$

for $h \le h_o(\omega)$. Hence the integral

$$\int_0^t p_h\lambda_s ds$$

is well defined for almost all ω and by the dominated convergence theorem

$$\lim_{h\to 0} \int_0^t p_h\lambda_s ds = \int_0^t \lambda_s ds \quad \text{a.s} \tag{2.5.6}$$

Denote by (A_t) the (F_t) ICR of (N_t) and define as usual the martingale $(M_t \triangleq N_t - A_t)$. Let c be any positive constant and define

$$P_t^c \triangleq E(A_c|F_t) - A_{t\wedge c} \tag{2.5.7}$$

It is easy to check that (P_t^c) is a potential (see Definition 1.6.1) and by Theorem 29-VII of [M1] we know that for each t

$$\int_0^t \frac{1}{h} E(P_s^c - P_{s+h}^c | F_s) ds \xrightarrow[h \to 0]{\sigma(L_1, L_\infty)} A_{t \wedge c} \qquad (2.5.8)$$

Now $(A_{t \wedge c} = N_{t \wedge c} - M_{t \wedge c})$ so that by (2.5.7)

$$P_t^c = [E(A_c | F_t) + M_{t \wedge c}] - N_{t \wedge c}$$

where $(M_{t \wedge c})$ is not only a $(F_{t \wedge c})$ but also a (F_t) martingale. Hence for $s \leq t$ and if we choose $c > t + h$

$$E(P_s^c - P_{s+h}^c | F_s) = E(N_{s+h} - N_s | F_s)$$

Thus on the one hand by (2.5.8)

$$\int_0^t \frac{1}{h} E(N_{s+h} - N_s | F_s) ds \xrightarrow[h \to 0]{\sigma(L_1, L_\infty)} A_t$$

and on the other by (2.5.6)

$$\int_0^t \frac{1}{h} E(N_{s+h} - N_s | F_s) ds \xrightarrow[h \to 0]{a.s} \int_0^t \lambda_s ds$$

so that we must have a.s for each t

$$A_t = \int_0^t \lambda_s ds \qquad (2.5.9)$$

By the right-continuity of the processes involved relation (2.5.9) is valid for every t a.s (see Remark 1.1.4) and result (b) follows. □

The next result shows that the two conditional rates of a same CP (N_t) but with respect to two families (F_t) and (G_t) such that $F_t \supset G_t \supset N_t$ are related by a simple expression.

Proposition 2.5.2: Let (N_t) be a CP with finite mean. Denote its conditional rate with respect to the family (F_t) by (λ_t). Let (G_t) be another family such that $N_t \subset G_t \subset F_t$. Then the conditional rate $(\hat{\lambda}_t)$ of (N_t) with respect to (G_t) exists and is given by

$$\hat{\lambda}_t = E(\lambda_t | G_t)$$

Note that this result makes good intuitive sense, the conditional rate $(\hat{\lambda}_t)$ being the best mean square estimate of the conditional rate (λ_t).

Proof: Part of this proof is a consequence of the innovation theorem which is given in the next chapter, i.e., the process $(N_t - \int_0^t \hat{\lambda}_s ds)$ is a (G_t) martingale. Now the process $(\int_0^t \hat{\lambda}_s ds)$ is increasing, continuous hence natural (see Remark 1.7.5(d)) and consequently is the (G_t) ICR of (N_t) by the uniqueness of the Doob-Meyer decomposition. □

2.6 COUNTING PROCESSES OF INDEPENDENT INCREMENTS

Let (N_t) be a CP of independent increments with finite mean m_t for each t. We have already seen (Proposition 2.3.4) that the (N_t) ICR of (N_t) is given by m_t. Hence this (N_t) ICR is deterministic. Using a technique of proof due to Kunita and Watanabe [K1] and a result of Doléans-Dade [D2] we now show that the converse is true. CP's of independent increments will play an important part in solving the detection problem.

Theorem 2.6.1: Let (N_t) be a CP with finite mean m_t for each t. Denote its (N_t) ICR by (A_t). Then

(a) (N_t) is a CP of independent increments if and only if the ICR (A_t) is deterministic.

(b) If the ICR (A_t) is deterministic then

$$A_t = m_t$$

(c) A CP of independent increments is regular with respect to the family (N_t) if and only if its means is continuous.

(d) The characteristic function of a CP with independent increments is given by: $E\, e^{iu(N_t - N_s)} =$

$$\exp\{(e^{iu} - 1)(m_t - m_s)\}$$

$$\cdot \prod_{s<v\leq t} [\{1 + (e^{iu}-1)\Delta m_v\}\exp\{(1-e^{iu})\Delta m_v\}] \tag{2.6.1}$$

Proof: The "only if" part of (a) is simply a restatement of Proposition 2.3.4. Now assume that the (N_t) ICR (A_t) is deterministic. The CP (N_t) is a right-continuous step process with ΔN_t being either zero or one so that for $t \geq s$

$$e^{iuN_t} - e^{iuN_s} = \sum_{s<v\leq t} \Delta e^{iuN_v} = \sum_{s<v\leq t} (e^{iuN_v} - e^{iuN_{v-}})$$

$$= \sum_{s<v\leq t} (e^{iu\Delta N_v} - 1)e^{iuN_{v-}}$$

$$= \sum_{s<v\leq t} (e^{iu} - 1)e^{iuN_{v-}} \Delta N_v$$

$$= (e^{iu} - 1)\int_s^t e^{iuN_{v-}} dN_v$$

where $\sum$ is the sum over the discontinuities of (N_t) in $(s,t]$. Using the expression $(N_t \overset{\Delta}{=} M_t + A_t)$ in the above gives

$$e^{iuN_t} = e^{iuN_s} + (e^{iu} - 1)\left[\int_s^t e^{iuN_{v-}} dM_v + \int_s^t e^{iuN_{v-}} dA_v\right] \tag{2.6.2}$$

The process $(M_t \overset{\Delta}{=} N_t - A_t) \in A$ is a martingale by Theorem 2.3.1, $|e^{iuN_{t-}}| \leq 1$, so that by Proposition 2 of [D1] the process $(\int_0^t e^{iuN_{v-}} dM_v)$ is a martingale. In particular

$$E\left(\int_s^t e^{iuN_{v-}}\, dM_v \,|\, N_s\right) = 0$$

so that by (2.6.2) (multiplying both sides by e^{-iuN_s})

$$E[e^{iu(N_t-N_s)} \,|\, N_s] = 1+(e^{iu}-1)E\left[\int_s^t e^{iu(N_{v-}-N_s)} dA_v \,|\, N_s\right] \quad (2.6.3)$$

We examine now the last term in the RHS. For any set $H \varepsilon N_s$ one can write by Fubini's Theorem and the definition of conditional expectations (note that we use here the fact that A_t is a deterministic function):

$$\int_H \int_s^t e^{iu(N_{v-}-N_s)} dA_v dP = \int_s^t \int_H e^{iu(N_{v-}-N_s)} dP dA_v$$

$$= \int_s^t \int_H E[e^{iu(N_{v-}-N_s)} \,|\, N_s] dP dA_v$$

$$= \int_H \int_s^t E[e^{iu(N_{v-}-N_s)} \,|\, N_s] dA_v dP$$

(2.6.4)

We also have

$$\int_H \int_s^t e^{iu(N_{v-}-N_s)} dA_v dP = \int_H E\left[\int_s^t e^{iu(N_{v-}-N_s)} dA_v \,|\, N_s\right] dP \quad (2.6.5)$$

so that for any $H \in \mathcal{N}_s$ by (2.6.4) and (2.6.5)

$$\int_H E\Big[\int_s^t e^{iu(N_{v^-}-N_s)} dA_v \,|\, \mathcal{N}_s\Big] dP = \int_H \int_s^t E\big[e^{iu(N_{v^-}-N_s)} \,|\, \mathcal{N}_s\big] dA_v \, dP$$

which implies a.s

$$E\Big[\int_s^t e^{iu(N_{v^-}-N_s)} dA_v \,|\, \mathcal{N}_s\Big] = \int_s^t E\big[e^{iu(N_{v^-}-N_s)} \,|\, \mathcal{N}_s\big] dA_v \qquad (2.6.6)$$

Introducing the above relation (2.6.6) in (2.6.3) one gets

$$E\big[e^{iu(N_t-N_s)} \,|\, \mathcal{N}_s\big] = 1+(e^{iu}-1)\int_s^t E\big[e^{iu(N_{v^-}-N_s)} \,|\, \mathcal{N}_s\big] dA_v \qquad (2.6.7)$$

Now $((e^{iu}-1)A_t)$ is a semimartingale which belongs to $\mathcal{A}$. Then by Theorem 1.9.15 the unique solution of (2.6.7) is a semimartingale given by

$$E\big[e^{iu(N_t-N_s)} \,|\, \mathcal{N}_s\big]$$

$$= \exp[(e^{iu}-1)(A_t-A_s)] \prod_{s<v\leq t} \{[1+(e^{iu}-1)\Delta A_v]$$

$$\cdot \exp[(1-e^{iu})\Delta A_v]\} \qquad (2.6.8)$$

The RHS of (2.6.8) is a deterministic function and it follows that (N_t) is a process of independent increments. This shows part (a). We have (Proposition 2.3.7) $EA_t = m_t$ and part (b) follows trivially. Part (c) results from (a) and Theorem 2.4.7 and (d) is a restatement of (2.6.8) where we have used the fact that $A_t = m_t$. □

If we define a non-homogeneous Poisson process (N_t) as being a CP of independent increments with a characteristic function Ee^{iuN_t} given by $\exp\{(e^{iu}-1)\int_0^t \lambda_s ds\}$ where λ_t is a nonnegative function called the rate, then we have:

Corollary 2.6.2: A CP (N_t) of independent increments with finite mean m_t for each t is a non-homogeneous Poisson process if and only if the mean m_t is absolutely continuous. The rate λ_t is then given by the Radon-Nikodym derivative $\frac{dm_t}{dt}$.

Proof: By Theorem 2.6.1(d) it is easy to see that

$$Ee^{iuN_t} = \exp\{(e^{iu}-1)\int_0^t \lambda_s ds\}$$

if and only if

$$m_t = \int_0^t \lambda_s ds \qquad \square$$

2.7 PROBABILITY GENERATING FUNCTION

PRELIMINARIES

Let (N_t) be a CP with finite mean for each t and adapted to a family (F_t). Denote its (F_t) ICR by (A_t). Recall that the process $(M_t \overset{\Delta}{=} N_t - A_t)$ is a (F_t) martingale (Theorem 2.3.1). The conditional probability generating function $\psi(z,t,s)$ is defined for $t \geq s$ by:

$$\psi(z,t,s) \overset{\Delta}{=} E[z^{(N_t-N_s)}|F_s] = \sum_n z^n P\{N_t-N_s = n|F_s\} \qquad (2.7.1)$$

where z is a complex number with $|z| \leq 1$ and we have

$$P\{N_t - N_s = n | F_s\} = \frac{1}{n!} \frac{d^{(n)}}{dz^n} \psi(z,t,s) \Big|_{z=0} \qquad (2.7.2)$$

We can compute the probability generating function proceeding exactly as in the proof of Theorem 2.6.1 (let $z = e^u$) and find (note that (N_t) is not necessarily of independent increments):

$$\psi(z,t,s) = 1+(z-1)E\left[\int_s^t z^{(N_{v^-}-N_s)} dA_v | F_s\right] \qquad (2.7.3)$$

The above formula can be generalized to the case where the jumps of the process (N_t) are of random size. This formula would then contain, in place of the term $(z-1)$, a random quantity which is a function of the random size jumps ΔN_t and z. This additional randomness makes this formula harder to manipulate and practically useless.

APPLICATION TO COUNTING PROCESSES OF INDEPENDENT INCREMENTS

Suppose now that (N_t) is a process of independent increments with finite mean m_t and take $F_t = N_t$. In this case the (N_t) ICR (A_t) is given by m_t (Theorem 2.6.1). We compute now the probability generating function $\psi(z,t,s)$ and the probability $P\{N_t - N_s = n\}$. The method used to derive these formulas is appealing as it does not require the mean m_t to be continuous (in this case the formulas are well known). First, one gets the probability generating function by letting $z = e^u$ in Theorem 2.6.1(d)

$$\psi(z,t,s) = \exp\{(z-1)(m_t - m_s)\} \cdot \prod_{s<v\leq t} (1+(z-1)\Delta m_v)\exp\{(1-z)\Delta m_v\} \tag{2.7.4}$$

For a process of independent increments we clearly have

$$P\{N_t - N_s = n | N_s\} = P\{N_t - N_s = n\} \tag{2.7.5}$$

If the mean m_t is continuous we then immediately get using (2.7.2)

$$P\{N_t - N_s = n\} = \frac{1}{n!}(m_t - m_s)^n \exp[-(m_t - m_s)] \tag{2.7.6}$$

This relation motivates the following

<u>Definition 2.7.1</u>: A CP (N_t) of independent increments with a continuous finite mean is called a Generalized Poisson process.

If furthermore (N_t) is a non-homogeneous Poisson process (see the definition above Corollary 2.6.2) with rate λ_t then we get the well-known formula

$$P\{N_t - N_s = n\} = \frac{1}{n!}\left(\int_s^t \lambda_v dv\right)^n \exp\left(-\int_s^t \lambda_v dv\right) \tag{2.7.7}$$

However the interesting case is when the mean m_t is discontinuous. We know that an increasing function finite for each t has at most a countable number of jumps in any finite interval. Denote by (t_i, i = 1,2...) the times of jump of m_t in the interval (s,t] and define

$$\Delta_s^t = \sum_{s<v\leq t} \Delta m_v = \sum_i \Delta m_{t_i} \tag{2.7.8}$$

and

$$\delta_s^t = m_t - m_s - \Delta_s^t \tag{2.7.9}$$

Formula (2.7.4) can be rewritten with the above relations in the form

$$\psi(z,t,s) = \exp\{-\delta_s^t\}\exp\{z\delta_s^t\} \prod_i [1+(z-1)\Delta m_{t_i}] \tag{2.7.10}$$

We exmine now the infinite product

$$\prod_i [1 + (z-1)\Delta m_{t_i}] \tag{2.7.11}$$

Observe that: (a) for each n the partial product

$$f_n(z) = \prod_{i=1}^{n} [1 + (z-1)\Delta m_{t_i}] \tag{2.7.12}$$

is analytic on the complex plane and (b) the series

$$\sum_i |(z - 1)\Delta m_{t_i}|$$

is uniformly convergent in the region $|z| \leq 1$. This last point follows from the Weierstrass test:

$$|(z - 1)\Delta m_{t_i}| \leq 2\Delta m_{t_i}$$

and because the mean m_t is finite for each t the series

$$\Lambda_s^t = \sum_i \Delta m_{t_i} < m_t < \infty$$

is convergent. Conditions (a) and (b) above imply that the infinite product (2.7.11) converges uniformly to a function f(z) which is analytic in the region $|z| < 1$ (see [H1], Corollary to Theorem 8.6.3; or [D3], Theorem 5.4.8). Hence we can get a Taylor series expansion for $f(z) = \prod_i [1 + (z-1)\Delta m_{t_i}]$ in the region $|z| \leq 1$.

$$f(z) = \sum_e a_e z^e = \prod_i [1 + (z-1)\Delta m_{t_i}] \qquad (2.7.13)$$

We compute now

$$\frac{d^{(n)}}{dz^n}\left(\exp\{z\delta_s^t\} \sum_e a_e z^e\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{d^{(n-k)}}{dz^{n-k}} (\exp\{z\delta_s^t\}) \frac{d^{(k)}}{dz^k}\left(\sum_e a_e z^e\right) \qquad (2.7.14)$$

Now

$$\frac{d^{(n-k)}}{dz^{n-k}} (\exp\{z\delta_s^t\}) = (\delta_s^t)^{n-k} \exp\{z\delta_s^t\} \qquad (2.7.15)$$

The power series $(\sum_e a_e z^e)$ is uniformly convergent and can be differentiated term by term in the region $|z| < 1$ so that

$$\frac{d^{(k)}}{dz^k}\left(\sum_e a_e z^e\right) = \sum_e a_e \frac{d^{(k)}}{dz^k} z^e = \sum_{e \geq k} a_e \frac{e!}{(e-k)!} z^{e-k}$$

(2.7.16)

Introducing the two above relations (2.7.15) and (2.7.16) in (2.7.14) we get

$$\frac{d^{(n)}}{dz^n} (\exp\{z\delta_s^t\} \sum_e a_e z^e) = \frac{d^{(n)}}{dz^n} (\exp\{z\delta_s^t\} \prod_i [1+(z-1)\Delta m_{t_i}])$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\delta_s^t)^{n-k} \exp\{z\delta_s^t\} (\sum_{e\geq k} a_e \frac{e!}{(e-k)!} z^{e-k}) \quad (2.7.17)$$

Evaluating (2.7.17) at z = 0, we have (see (2.7.10))

$$\frac{d^{(n)}}{dz^n} \psi(z,t,s) \Big|_{z=0} = \exp\{-\delta_s^t\} \sum_{k=0}^{n} \binom{n}{k} (\delta_s^t)^{n-k} a_k k! \quad (2.7.18)$$

and finally (see (2.7.2))

$$P\{N_t - N_s = n\} = \exp\{-\delta_s^t\} \sum_{k=0}^{n} \frac{a_k}{(n-k)!} (\delta_s^t)^{n-k} \quad (2.7.19)$$

Now if the mean m_t has only a finite number of jumps $j \geq 1$ in the interval (s,t] then the coeffiencts a_k are such that

$$\sum_{e=1}^{j} a_e z^e = \prod_{e=1}^{j} [1 + (z-1)\Delta m_{t_e}]$$

and can be computed by

$$a_o = \sum_{i=1}^{j} (1 - \Delta m_{t_i}) \quad (2.7.20)$$

For $0 < k < j$

$$a_k = \frac{1}{k!} \sum_{\substack{\text{all permutations} \\ \{e_q,\ q=1,\dots,j\} \\ \text{of} \\ \{1,\dots\dots\dots,j\}}} \left(\prod_{q=1}^{k} \Delta m_{t_{e_q}}\right)\left[\prod_{q=k+1}^{j} (1-\Delta m_{t_{e_q}})\right] \tag{2.7.21}$$

$$a_j = \prod_{i=1}^{j} \Delta m_{t_i} \tag{2.7.22}$$

and finally for $k > j$, $a_k = 0$.

If $j = 0$ (continuous case) then $\prod_i [1 + (z-1)\Delta m_{t_i}] = 1$ so that $a_o = 1$ and $a_k = 0$, $k \geq 1$ and result (2.7.19) reduces to (2.7.6) ($\delta_s^t = m_t - m_s$ in this case).

We summarize the above results in

<u>Theorem 2.7.2</u>: Let (N_t) be a CP of independent increments with finite mean $m_t \stackrel{\Delta}{=} EN_t$ for each t. Let $s \leq t$ and denote by t_i the (at most countable) times of jump of m_t on the interval (s,t]. Define

$$\delta_s^t \stackrel{\Delta}{=} m_t - m_s - \sum_{s<v\leq t} \Delta m_v$$

(a) If the number of jumps of m_t in (s,t] is infinite then the infinite product

$$\prod_i [1 + (z-1)\Delta m_{t_i}]$$

is uniformly convergent in the region $|z| \leq 1$ to an analytic function and we denote by a_k the coefficients of the Taylor's expansion of the above infinite product.

(b) If the number j of jumps of m_t in (s,t] is finite then the coefficients a_k can be computed by formulas (2.7.20) to (2.7.22). If m_t is continuous (j = 0) then $a_o = 1$, $a_k = 0$, $k \geq 1$ and $\delta_s^t = m_t - m_s$.

(c) The probability generating function $\psi(z,t,s)$ is given by

$$\psi(z,t,s) = \exp\{(z-1)\delta_s^t\} \prod_i [1 + (z-1)\Delta m_{t_i}]$$

$$= \exp\{(z-1)\delta_s^t\} \sum_i a_i z^i$$

(d) We have

$$P\{N_t - N_s = n\} = \exp\{-\delta_s^t\} \sum_{k=0}^{n} \frac{a_k}{(n-k)!} (\delta_s^t)^{n-k}$$

APPLICATION TO COUNTING PROCESSES WITH A CONDITIONAL RATE

Assume now that (N_t) is a CP with finite mean for each t and for which a conditional rate (λ_t) with respect to a family (F_t) exists and satisfies the condition

$$E(z^{N_{v-}} \lambda_v | F_s) = E(z^{N_{v-}} | F_s) E(\lambda_v | F_s) \tag{2.7.23}$$

for all $v \geq s$. This condition will be discussed later on. From (2.7.3) we get

$$\psi(z,t,s) = 1 + (z-1) E\left(\int_s^t z^{(N_{v-} - N_s)} \lambda_v dv | F_s \right) \tag{2.7.24}$$

Now the CP has a finite mean so that (Proposition 2.3.7) $E\int_0^t \lambda_s ds$ is finite which implies ($|z| \leq 1$)

$$E\int_s^t |z^{(N_{v^-}-N_s)}|\lambda_v dv < \infty$$

Then by Fubini's Theorem (applied as in the proof of Theorem 2.6.1) one gets

$$E[\int_s^t z^{(N_{v^-}-N_s)}\lambda_v dv|F_s] = \int_s^t E[z^{(N_{v^-}-N_s)}\lambda_v|F_s]dv$$

Hence by the above relation, (2.7.23) and (2.7.24) one has

$$\psi(z,t,s) = 1 + (z-1)\int_s^t \psi(z,v^-,s)\hat{\lambda}_v^s dv \qquad (2.7.25)$$

where $\hat{\lambda}_v^s \triangleq E(\lambda_v|F_s)$ i.e., $(\hat{\lambda}_v^s)$ is the best prediction of (λ_v) based on the past information up to and at time s. As before, this equation has a unique solution which is a semimartingale (Theorem 1.9.15):

$$\psi(z,t,s) = \exp\{(z-1)\int_s^t \hat{\lambda}_v^s dv\} \qquad (2.7.26)$$

and

$$P\{N_t - N_s = n|F_s\} = \frac{1}{n!}(\int_s^t \hat{\lambda}_v^s dv)^n \exp\{-\int_s^t \hat{\lambda}_v^s dv\} \qquad (2.7.27)$$

It is interesting to note that both these formulas are identical to the formulas for Poisson processes where we have substituted for the rate λ_v the best estimate

All this is very appealing but is true only if condition (2.7.23) is satisfied. This condition which can be rewritten as

$$E[z^{(N_{v^-}-N_s)}(\lambda_v - \lambda_s)|F_s] = E[z^{(N_{v^-}-N_s)}|F_s]E[(\lambda_v-\lambda_s|F_s] \tag{2.7.28}$$

is difficult to interpret. But if we take s = 0 and assume $F_0 = \{\phi,\Omega\}$ (this is in particular the case for N_0) then the above condition (2.7.23) becomes

$$E(z^{N_{v^-}}\lambda_v) = E(z^{N_{v^-}})E(\lambda_v) \quad v \geq 0 \tag{2.7.29}$$

and is satisfied if for each t the two random variables N_{t^-} and λ_t are independent. This seems a reasonable assumption if we think that the value of N_{t^-} does not influence the rate at time t. Then under this condition (2.7.29) relation (2.7.27) gives

$$P\{N_t = n\} = \frac{1}{n!}(\int_0^t (E\lambda_v)dv)^n \exp\{-\int_0^t (E\lambda_v)dv\} \tag{2.7.30}$$

In conclusion what we have done is to use the formula of change of variables (Theorem 1.9.14) to get an expression (Eq. (2.7.3)) for the conditional probability generating function. This expression contains an integral which can under certain circumstances be manipulated so as to give an integral equation for the conditional probability generating function. Furthermore this integral equation has, by Theorem 1.9.15, a unique solution which is a semimartingale. This method is particularly suitable for CP's of independent increments and can also be applied to CP's with a random rate which satisfies (2.7.23). But in general we cannot obtain an integral equation for the conditional probability generating function.

CHAPTER 3

DETECTION

3.0 INTRODUCTION

In this chapter we examine the detection problem for CP's by the likelihood ratio technique. This approach is well known and will not be motivated here. Recall that the (P,F_t) ICR (A_t) of a CP (N_t) is the unique natural increasing process which makes the process $(M_t \triangleq N_t - A_t)$ a square integrable (P, F_t) local martingale (Theorem 2.3.1 and Definition 2.3.2) and that this ICR is continuous if and only if the CP is regular with respect to the family (F_t) (Theorem 2.4.7).

We first obtain the likelihood ratio in the case where one of the CP's is of independent increments with continuous mean m_t while the other has an (F_t) ICR of the form $(\int_0^t \lambda_s dm_s)$ where $(\lambda_t) \in H(F_t)$ is a positive process. Note that both these CP's are regular. Then taking advantage of the chain rule for likelihood ratios we extend this result to the more general case where both ICR's are of the form $(\int_0^t \lambda_s^i dm_s)$ for i = 0,1. Stochastic integral equations which allow us to compute the likelihood ratio continuously in time are also derived. The results described above are given in the last Section 3.4. The method used to obtain the likelihood ratio consists in the three steps procedure introduced by Duncan ([D3],[D4]) and Kailath [K3]:

Step 1 gives a general description of the likelihood ratio dP_0/dP where P is a measure under which a given CP is of independent increments and P_0 a measure absolutely continuous with respect to P.

Step 2 is a Girsanov type theorem and

Step 3 is the Innovation Theorem.

In very general terms the Girsanov, respectively the Innovation Theorem, tells us how to transform local martingales of interest into new processes which are also local martingales when a change of measure, respectively a change of family of σ-algebras, is made. These two theorems are presented in Section 3.1. It is also shown there that the Girsanov Theorem can be used to prove, under suitable assumptions, the existence of CP's for which the (F_t) ICR is of the form $(\int_0^t \lambda_s dA_s)$ where $(\lambda_t) \in H(F_t)$ is a nonnegative process and (A_t) is itself the (F_t) ICR of a CP. Let now (N_t) be a regular CP of independent increments with mean m_t. Recall that the process $(M_t \overset{\Delta}{=} N_t - m_t)$ belongs to $M_\ell^2(N_t)$ (Theorem 2.4.8) where N_t is the minimal σ-algebra generated by (N_t) up to and at time t. In Section 3.2 we show that any martingale in $M_\ell^2(N_t)$ can be represented as a stochastic integral with respect to (M_t). This result is basic to Section 3.3 where the likelihood ratio representation theorem is demonstrated (this is Step 1).

In this chapter we deal with a measurable space (Ω,F) on which the probability measures P, P_0 and P_1 are defined. We denote respectively by $E(\cdot)$, $E_0(\cdot)$ and $E_1(\cdot)$ the expectation operator with respect to the measures P, P_0 and P_1. The standard notation $P_0 << P$ means that the measure P_0 is absolutely continuous with respect to P_1 while $P_0 \sim P$ indicates that the two measures are equivalent (i.e., $P_0 << P$ and $P << P_0$). Every stochastic process is defined on the measurable space (Ω,F) equipped with a given probability measure P, P_0 or P_1. The general assumptions of the previous chapter (see Section 2.1) are used again in this one.

3.1 TWO BASIC THEOREMS IN DETECTION

ABSOLUTELY CONTINUOUS CHANGE OF MEASURE: THE GIRSANOV THEOREM

The Girsanov theorem is a basic step in finding likelihood ratios. The version we present here, in the context of CP's, will also enable us to create from a regular CP with (F_t) ICR (A_t) other CP's for which the (F_t) ICR's are of the form $(\int_0^t \lambda_s dA_s)$ where $(\lambda_t) \in H(F_t)$ (c.f. Definition 1.9.9) is a nonnegative process. The original version of the Girsanov Theorem dates back to 1960 [G1] and was concerned with Brownian motion. The extension of this result to the case of local martingales became possible with the new calculus being developed for these latter processes. We give now the version of Girsanov Theorem which is appropriate for CP's.

<u>Theorem 3.1.1</u>: Under the measure P let (N_t) be a regular CP with respect to a family (F_t). Denote its (P,F_t) ICR by (A_t). Suppose $(\lambda_t) \in H(F_t)$ is a nonnegative process and define

$$L_t = \left(\prod_{J_n \leq t} \lambda_{J_n} \right) \exp\left(\int_0^t (1-\lambda_s) dA_s \right) \qquad (3.1.1)$$

where J_n denotes the time of n^{th} jump of (N_t). (By convention the product $(\prod_{J_n \leq t} \lambda_{J_n})$, when empty (e.g., for $t = 0$), is set equal to one.)

(a) The process (L_t) is a (P,F_t) local martingale which is the unique solution of the stochastic integral equation

$$L_t = 1 + \int_0^t L_{s-}(\lambda_s - 1) dM_s \qquad (3.1.2)$$

where $(M_t \triangleq N_t - A_t) \in L$.

(b) If (L_t) is a uniformly integrable (P,F_t) martingale define a new measure P_0 on (Ω,F) by

$$P_0(A) = \int_A L_\infty dP \qquad A \in F$$

where L_∞ is the limit a.s and in the mean of L_t as t goes to ∞. Then the process $(N_t - \int_0^t \lambda_s dA_s)$ is a (P_0,F_t) local martingale, i.e., the process $(\int_0^t \lambda_s dA_s)$ is the (P_0,F_t) ICR of (N_t).

To prove the above theorem we will need:

Lemma 3.1.2: Let P_o be a measure absolutely continuous with respect to P and define the uniformly integrable (P,F_t) martingale (L_t) by

$$L_t = E\left(\frac{dP_o}{dP} \middle| F_t \right)$$

Then the process (X_t), adapted to (F_t), is a (P_o,F_t) local martingale if and only if the process (L_tX_t) is a (P,F_t) local martingale.

Proof of the Lemma: $(\Leftarrow)$ Assume (L_tX_t) is a (P,F_t) local martingale and let (T_n) be a sequence of stopping times reducing (X_tL_t), i.e., $(L^n_tX^n_t)$ is a uniformly integrable (P,F_t) martingale for each n where $(L^n_t \triangleq L_{t\wedge T_n})$ and $(X^n_t \triangleq X_{t\wedge T_n})$. For $s \leq t$, if $(F^n_t \triangleq F_{t\wedge T_n})$

$$E(X^n_tL^n_t - X^n_sL^n_s | F^n_s) = 0 = E[X^n_tE(\frac{dP_o}{dP}|F^n_t) - X^n_sE(\frac{dP_o}{dP}|F^n_s)|F^n_s]$$

$$= E[E(X^n_t \frac{dP_o}{dP}|F^n_t) - E(X^n_s \frac{dP_o}{dP}|F^n_s)|F^n_s]$$

i.e.,

$$0 = E[(X^n_t - X^n_s)\frac{dP_o}{dP}|F^n_s]$$

Hence, by definition of conditional expectations, $\forall A \in F^n_s$

$$0 = \int_A E[(X_t^n - X_s^n)\frac{dP_o}{dP} | F_s^n] dP = \int_A (X_t^n - X_s^n)\frac{dP_o}{dP} dP = \int_A (X_t^n - X_s^n) dP_o$$

which implies

$$E_o(X_t^n - X_s^n | F_s^n) = 0$$

so that (X_t) is a (P_o, F_t) local martingale. The argument can be easily reversed. □

Proof of Theorem 3.1.1: (a) This is a direct consequence of Theorem 1.9.15. The unique solution of (3.1.2), a local martingale because the process $(\int_0^t (\lambda_s - 1) dM_s)$ is one (see Theorem 1.9.11), is given by

$$L_t = \prod_{s \leq t} [1 + (\lambda_s - 1)\Delta M_s] \exp[(1 - \lambda_s)\Delta M_s] \exp(\int_0^t (\lambda_s - 1) dM_s)$$

The CP (N_t) is regular by assumption so that by Theorem 2.4.7 the ICR (A_t) is continuous. Thus $\Delta M_t = \Delta N_t = 0$ or 1 and the product term $\prod_{s \leq t} (\cdot)$ is equal to

$$\prod_{s \leq t} [1 + (\lambda_s - 1)\Delta N_s] \exp[(1 - \lambda_s)\Delta N_s] = \prod_{J_n \leq t} \lambda_{J_n} \exp(1 - \lambda_{J_n})$$

$$= (\prod_{J_n \leq t} \lambda_{J_n}) \exp[\sum_{J_n \leq t} (1 - \lambda_{J_n})] = (\prod_{J_n \leq t} \lambda_{J_n}) \exp(\int_0^t (1 - \lambda_s) dN_s)$$

so that upon introducing the relation $M_t = N_t - A_t$ in the term $\exp(\int_0^t (\lambda_s - 1) dM_s)$ we get the desired result (3.1.1).

(b) Because of Lemma 3.1.2 we only have to show that the process $(L_t Y_t)$ is a (P, F_t) local martingale where

$$Y_t \triangleq N_t - \int_0^t \lambda_s dA_s \qquad (3.1.3)$$

Define

$$F_t \triangleq \prod_{J_n \le t} \lambda_{J_n} \qquad (3.1.4)$$

and

$$X_t \triangleq \exp\left(\int_0^t (1 - \lambda_s) dA_s\right) \qquad (3.1.5)$$

i.e.,

$$L_t = F_t X_t$$

and apply the formula of change of variables to the product $(F_t X_t Y_t)$. We get

$$\begin{aligned} L_t Y_t = F_t X_t Y_t &= \int_0^t X_s Y_{s-} dF_s + \int_0^t F_{s-} Y_{s-} dX_s \\ &+ \int_0^t F_{s-} X_s dY_s \\ &+ \sum_{s \le t} [X_s \Delta(F_s Y_s) - X_s Y_{s-} \Delta F_s \\ &- F_{s-} X_s \Delta Y_s] \end{aligned} \qquad (3.1.7)$$

We have

$$\int_0^t X_s Y_{s-} dF_s = \sum_{s \le t} X_s Y_{s-} \Delta F_s \tag{3.1.8}$$

because (F_t) is a step process. By (3.1.5)

$$dX_s = -X_s(\lambda_s - 1)dA_s$$

so that

$$\int_0^t F_{s-} Y_{s-} dX_s = -\int_0^t L_{s-} Y_{s-} (\lambda_s - 1) dA_s \tag{3.1.9}$$

Also $\Delta Y_s = \Delta N_s$ and $Y_s - N_s = -\int_0^t \lambda_s dA_s$ (see (3.1.3)) hence

$$\int_0^t F_{s-} X_s dY_s - \sum_{s \le t} F_{s-} X_s \Delta Y_s = \int_0^t F_{s-} X_s d(Y_s - N_s) = -\int_0^t L_{s-} \lambda_s dA_s \tag{3.1.10}$$

Using the relations $Y_s = Y_{s-} + \Delta N_s$, $F_s \Delta N_s = F_{s-} \lambda_s \Delta N_s$ so that $\Delta F_s = F_{s-}(\lambda_s - 1)\Delta N_s$ we obtain

$$\sum_{s \le t} X_s \Delta(F_s Y_s) = X_s(F_s Y_{s-} + F_s \Delta N_s - F_{s-} Y_{s-})$$

$$= X_s Y_{s-} \Delta F_s + X_s F_s \Delta N_s$$

$$= X_s Y_{s-} F_{s-} (\lambda_s - 1) \Delta N_s + X_s F_{s-} \lambda_s \Delta N_s$$

$$\sum_{s\leq t} X_s \Delta(F_s Y_s) = \int_0^t L_{s-}Y_{s-}(\lambda_s - 1)dN_s + \int_0^t L_{s-}\lambda_s dN_s \tag{3.1.11}$$

Introducing all the above relations (3.1.8)-(3.1.11) into (3.1.7) we finally get

$$L_t Y_t = -\int_0^t L_{s-}Y_{s-}(\lambda_s - 1)dA_s - \int_0^t L_{s-}\lambda_s dA_s$$

$$+ \int_0^t L_{s-}Y_{s-}(\lambda_s - 1)dN_s + \int_0^t L_{s-}\lambda_s dN_s$$

$$L_t Y_t = \int_0^t L_{s-}Y_{s-}(\lambda_s - 1)dM_s + \int_0^t L_{s-}\lambda_s dM_s \tag{3.1.12}$$

The process (M_t) is $(\mathcal{P},\mathcal{F}_t)$ local martingale and the following processes belong to $\mathcal{H}(\mathcal{F}_t)$: (L_{t-}) (by (a) (L_t) is a local martingale and see Remark 1.9.10(b)), (λ_t) (by assumption) and (Y_{t-}) (easy to check). Therefore the above relation (3.1.12) shows that $(L_t Y_t)$ is a $(\mathcal{P},\mathcal{F}_t)$ local martingale (see Theorem 1.9.11). □

We remark the following. The generalization of the original Girsanov Theorem [G1] was obtained by Brémaud [B1], Gualtierotti [G2] and us (and maybe others we are unaware of). By now the most general version has been obtained by Van Schuppen and Wong [V1] (1973) who took good advantage of the work of Doléans-Dade [D2]. But their result ([V1], Theorem 3.2, see in particular their comments on p. 10) is incorrect in the sense that they only assume that the process (L_t) is a positive martingale with $EL_t = 1$. But this is not enough. Although we know by the supermartingale convergence theorem that L_t converges a.s as t goes to ∞ (denote the limit by L_∞) it is not true, unless the martingale (L_t) is uniformly integrable, that the required condition $EL_\infty = 1$ (i.e., $P_0(\Omega) = 1$) is satisfied (by Fatou's lemma we have $EL_\infty \leq 1$). There is no reason to believe that positive martingales are necessarily uniformly integrable. In Appendix A.5 we exhibit a discrete positive martingale (X_n), $EX_n = 1$ which is not uniformly integrable: $E\lim_n X_n = 0$. Brémaud makes the same error in his dissertation (Theorem 2-1-i of [B1]).

Let (N_t) be a regular CP with a (P,F_t) ICR given by (A_t). When the process (L_t) defined by (3.1.1) is a uniformly integrable martingale the above theorem shows the existence of CP's which have (P_0,F_t) ICR's of the form $(\int_0^t \lambda_s dA_s)$ where $(\lambda_t) \in H(F_t)$ is a nonnegative process. Unfortunately this uniform integrability requirement for (L_t) seems difficult to meet. By the above theorem

$$L_t = 1 + \int_0^t L_{s-}(\lambda_s - 1)dM_s$$

and by Proposition 2 of [D1] a sufficient condition for

(L_t) to be a martingale (not necessarily uniformly integrable) is

$$E\int_0^t L_{s-}|\lambda_s-1||dM_s| < \infty \tag{3.1.13}$$

i.e.,

$$\left.\begin{array}{l} E\int_0^t L_{s-}|\lambda_s-1|dA_s < \infty \\ \text{and} \\ E\int_0^t L_{s-}|\lambda_s-1|dN_s < \infty \end{array}\right\} \tag{3.1.14}$$

Suppose now that (λ_t) satisfies

$$\lambda_t \leq c \quad \text{a.s} \tag{3.1.15}$$

where c is a constant such that $c \geq 1$. By (3.1.1) then

$$L_t \leq c^{N_t}\exp(A_t) \tag{3.1.16}$$

so that

$$E\int_0^t L_{s-}|\lambda_s-1|dA_s \leq cEc^{N_t}\int_0^t \exp(A_s)dA_s = cEc^{N_t}[\exp(A_t)-1] \tag{3.1.17}$$

and

$$E\int_0^t L_{s-}|\lambda_s-1|dN_s \leq cEN_tc^{N_t}\exp(A_t) \tag{3.1.18}$$

Hence if relation (3.1.15) is satisfied and if

$$E(N_t + 1)c^{N_t} \exp(A_t) < \infty \qquad (3.1.19)$$

then condition (3.1.13) is satisfied and (L_t) is a martingale. This martingale is not necessarily uniformly integrable and consequently we have to consider finite intervals $[0,a]$ instead of $\mathbb{R}_+$. In practice this is the usual case as we deal with finite observation times. For any a, we define a new measure P_0 on (Ω,F) by $dP_0/dP = L_a$ and by the above theorem the CP $(N_{t\wedge a})$ under this measure P_0 has the process $(\int_0^{t\wedge a} \lambda_s dA_s)$ for $(P_0, F_{t\wedge a})$ ICR. The condition (3.1.19) is in particular satisfied if (a) the constant c is equal to one and (b) the (P,F_t) ICR (A_t) is bounded a.s by a deterministic function K_t (this last condition implies by Proposition 2.3.7 that $EN_t = EA_t \leq K_t < \infty$). All these conditions are pretty strong. There is an important case where condition (3.1.19) is satisfied: for generalized Poisson processes (i.e., regular CP's of independent increments with finite mean, see Definition 2.7.1). In this situation the (P,N_t) ICR of (N_t) is given by m_t and (see (2.7.7))

$$P\{N_t = n\} = \exp(-m_t)\frac{1}{n!}(m_t)^n$$

so that condition (3.1.19) becomes

$$E(N_t+1)c^{N_t} = \exp(-m_t)\sum_n (n+1)c^n \frac{1}{n!}(m_t)^n < \infty$$

Furthermore for any increasing deterministic function m_t with $m_o = 0$ there exists a measure P such that the CP (N_t) is of independent increments with mean m_t. Thus we can summarize this last result as

Corollary 3.1.3: Let m_t be a deterministic increasing continuous function with $m_o = 0$. Then for any positive constant a there exists a measure P_o and a CP $(N_{t \wedge a})$ for which the $(P_o, N_{t \wedge a})$ ICR is of the form $(\int_0^{t \wedge a} \lambda_s dm_s)$ where $(\lambda_t) \in H(N_t)$ is a nonnegative uniformly bounded process (i.e., $\lambda_t \leq c$, c is a constant).

This shows in particular the existence of uniformly bounded conditional rates with respect to the family (N_t) (take $m_t = t$).

INNOVATION THEOREM

In the Girsanov Theorem we make a change of measure while keeping the family of σ-algebras (F_t) the same. The Innovation Theorem is concerned with exactly the reverse problem: only one measure P but two families of σ-algebras (F_t) and (G_t), $F_t \supset G_t$, are considered; how (P, F_t) martingales of interest (e.g., of the type $(N_t - \int_0^t \lambda_s dm_s)$ should be modified to become (P, G_t) martingales is the question answered by this theorem. This result is of much simpler nature than the Girsanov Theorem.

Theorem 3.1.4: Let (X_t) and (Y_t) be two processes respectively adapted to the families of σ-algebras (G_t)

and $(\mathcal{F}_t)$, where $\mathcal{G}_t \subset \mathcal{F}_t$. Suppose $(X_t - \int_0^t Y_s dm_s)$ is a $(P,\mathcal{F}_t)$ martingale, where $m_t \in \mathcal{V}$ is a deterministic function with $m_o = 0$. Then if

$$E\int_0^t |Y_s||dm_s| < \infty$$

the process $(X_t - \int_0^t \hat{Y}_s dm_s)$ is a $(P,\mathcal{G}_t)$ martingale, where $\hat{Y}_t = E(Y_t|\mathcal{G}_t)$.

Remark 3.1.5: Denote by Λ the union of all intervals of $\mathbb{R}_+$ on which the function m_t is constant. Note that the process (Y_t) may be infinite for $t \in \Lambda$ and that the process $(\hat{Y}_t)$ is then not well defined. The value of the integral $(\int_0^t Y_s dm_s)$ is not affected if one changes the values of (Y_t) for $t \in \Lambda$ so that, to avoid the above problem, we adopt the following convention: for $t \in \Lambda$ we set $(Y_t \equiv 1)$.

Proof: $E(X_t - X_s|\mathcal{G}_s)$

$$= E[E(X_t - X_s|\mathcal{F}_s)|\mathcal{G}_s] \qquad (\mathcal{F}_s \supset \mathcal{G}_s)$$

$$= E[E(\int_s^t Y_u dm_u|\mathcal{F}_s)|\mathcal{G}_s]$$

$$= E(\int_s^t Y_u dm_u|\mathcal{G}_s)$$

$$= \int_s^t E(Y_u|\mathcal{G}_s)dm_u \qquad \text{(Fubini's Theorem)}$$

$$= \int_s^t E[E(Y_u|\mathcal{G}_u)|\mathcal{G}_s]dm_u \qquad (\mathcal{G}_u \supset \mathcal{G}_s)$$

$$= E[\int_s^t E(Y_u|\mathcal{G}_u)dm_u|\mathcal{G}_s] \qquad \text{(Fubini's Theorem)}$$

$$= E(\int_s^t \hat{Y}_u dm_u | \mathcal{G}_s) \qquad \square$$

The above theorem is in fact only a trivial modification of Theorem 1.1 of [B1] which deals with the function t and the family $\sigma(X_u, 0 \leq u \leq t)$ instead of m_t and $(\mathcal{G}_t)$ and contains the unnecessary assumption that $(X_t - \int_0^t \lambda_s dm_s)$ is square integrable. We give this theorem here for completeness and because of the shortness of the proof. Observe that if $(\int_0^t \lambda_s dm_s)$ is the $(\mathcal{P},\mathcal{F}_t)$ ICR of a CP (N_t) with finite mean then the above result shows that $(\int_0^t \hat{\lambda}_s dm_s)$, where $\hat{\lambda}_s = E(\lambda_s | \mathcal{G}_s)$, is the $(\mathcal{P},\mathcal{G}_t)$ ICR of (N_t). Now by Lemma A.3.1 we also know that the process $(N_t - E(\int_0^t \lambda_s dm_s | \mathcal{G}_t))$ is a martingale. These two processes, $(\int_0^t \hat{\lambda}_s dm_s)$ and $(E(\int_0^t \lambda_s dm_s | \mathcal{G}_t))$, are different because the first one is continuous, thus natural, which is not necessarily true for the second.

3.2 MARTINGALE REPRESENTATION

Let (N_t) with the measure $\mathcal{P}$ carried on $(\Omega,\mathcal{F})$ be a Poisson process with rate one. Brémaud ([B1], Lemma 3) has shown by applying results of Kunita and Watanabe ([K1], Theorem 4.2) on additive functionals of a Hunt process that any martingale $(X_t) \in \mathcal{M}_\ell^2(\mathcal{P},\mathcal{N}_t)$ can be represented as a stochastic integral with respect to $(M_t = N_t - t)$. An analogous result for Brownian motions is well known (see [W1]). Recall also that if (N_t) is a regular CP with finite mean and ICR (A_t) then by Theorem 2.4.8 the martingale $(M_t \overset{\Delta}{=} N_t - A_t) \in M_\ell^2$. In this section we extend the above result to the case where the CP (N_t) is not necessarily of Poisson type as above but more generally is a regular CP of independent increments with finite mean m_t. This is done by making a nonrandom change of time which is motivated by the following lemma. This is essentially Theorem 12-VII of [M1].

Lemma 3.2.1: Let a_t be a positive right-continuous increasing extended real function defined on $\mathbb{R}_+$. Define

$$t_\infty = \begin{cases} \inf\{t: \ a_t = a_s, \ \forall\ s \geq t\} \\ \infty \text{ if the above set is empty} \end{cases}$$

i.e., t_∞ is the first time from which the function a_t remains constant. For all $t \in \mathbb{R}_+$ let

$$c_t = \begin{cases} \inf\{s: \ a_s > t\} \\ \infty \text{ if the above set is empty} \end{cases}$$

Then (a) For $t \geq a_{t_\infty}$, $c_t = \infty$; otherwise the function c_t is finite, right-continuous and increasing

(b)

$$a_s = \begin{cases} \inf\{t: \ c_t > s\} \\ \infty \text{ if the above set is empty} \end{cases}$$

(c) If the function a_t is finite and continuous, the function c_t is strictly increasing and

$$a_{c_t} = t$$

(d) Suppose $a_0 = 0$ and let f_t be a Borel function on $\mathbb{R}_+$ such that either $\int_0^\infty f_s^+ da_s$ or $\int_0^\infty f_s^- da_s$ is finite. Then

$$\int_0^\infty f_s da_s = \int_0^{a_\infty} f_{c_s} ds \qquad (3.2.1)$$

<u>Proof</u>: (a) For $t \geq a_{t_\infty}$ the set $\{s: a_s > t\}$ is empty and $c_t = \infty$ by definition. For $t < a_{t_\infty}$, this set is non-empty, decreasing as t increases. Hence the function c_t is finite and increasing. Suppose that c_t is not right-continuous at a point $t < a_{t_\infty}$. Then for any positive ε there exists a number h such that

$$c_t < h < c_{t+\varepsilon}$$

This implies, by the definition of c_t and the increasing property of a_t that

$$t < a_h$$

and, $\forall\ \varepsilon > 0$, $a_h \leq t + \varepsilon$ i.e., $a_h \leq t$ and we have reached a contradiction. Hence the function c_t is right-continuous.

(b) By definition

$$c_{a_t} = \inf\{s: a_s > a_t\} \geq t$$

Thus

$$c_{a_{s+\varepsilon}} \geq s + \varepsilon > s \quad \text{for any } \varepsilon > 0$$

Also

$$c_u > s \text{ implies } u \geq \inf\{t: c_t > s\}$$

so that

$$a_{s+\varepsilon} \geq \inf\{t: c_t > s\} \quad \text{for any } \varepsilon > 0$$

and by the right-continuity of a_t we thus have

$$a_s \geq \inf\{t: c_t > s\} \tag{3.2.2}$$

Let t be such that $c_t > s$; then by definition of a_t, $a_s < t$ so that

$$a_s \leq \inf\{t: c_t > s\} \tag{3.2.3}$$

Relations (3.2.2) and (3.2.3) prove part (b).

(c) Assume c_t is not strictly increasing; then there exists $t_o < t_1$ such that for any $t \in [t_o, t_1)$, $c_t = c_o =$ constant. By relation (b), $a_{c_o} = \inf\{t: c_t > c_o\} \geq t_1$ and for any $\varepsilon > 0$, $a_{c_o - \varepsilon} = \inf\{t: c_t > c_o - \varepsilon\} \leq t_o$, i.e., the function a_t is not continuous at c_o, a contradiction. Thus the function c_t is strictly increasing. From that and (b) we get

$$a_s = \inf\{t: c_t \geq s\}$$

so we can write

$$a_{c_s} = \inf\{t: \ c_t \geq c_s\} = s$$

(d) Let f_t be the indicator function of the interval $[0,s]$, i.e., $f_t = I_{[0,s]}(t)$. We first show that relation (3.2.1) is verified for this type of function. We have

$$\int_0^\infty f_t da_t = \int_0^\infty I_{[0,s]}(t) da_t = a_s$$

Now

$$f_{c_t} = I_{[0,s]}(c_t) = I_{\{u: c_u \leq s\}}(t)$$

Thus

$$\int_0^{a_\infty} f_{c_t} dt = \int_0^{a_\infty} I_{\{u: c_u \leq s\}}(t) dt$$

$$= \text{"length of the interval } \{u: c_u \leq s\}\text{"}$$

$$= \inf\{t: \ c_t > s\}$$

$$= a_s$$

where the last equality follows by (b). Hence relation (3.2.1) is verified in the case where f_t is an indicator function. This implies (see the end of the proof of

Theorem 12-VII of [M1]) that this relation (3.2.1) is verified for all bounded Borel functions. The fact that any positive Borel function is the limit of an increasing sequence of bounded Borel functions and the monotone convergence theorem show that relation (3.2.1) is true for any positive Borel functions. If f_t is now any Borel function we apply (3.2.1) to f_t^+ and f_t^- to get the desired result as the sum $\int_0^\infty f_t^+ da_t - \int_0^\infty f_t^- da_t$ is well defined by assumption. □

We can now prove the desired result on martingale representation. Under the measure P let (N_t) be a regular CP of independent increments with finite mean m_t. Recall that by Theorem 2.4.8 the martingale $(M_t = N_t - m_t) \varepsilon$ $\mathcal{M}_\ell^2(P,\mathcal{N}_t)$ and $\langle M\rangle_t = m_t$.

Theorem 3.2.2: Let (N_t) be the CP described just above. Then any martingale $(X_t) \varepsilon \mathcal{M}_\ell^2(P,\mathcal{N}_t)$ has the form

$$X_t = \int_0^t F_s dM_s \qquad (3.2.4)$$

where $(M_t = N_t - m_t)$ and (F_t) is a process belonging to $\mathcal{H}(\mathcal{N}_t)$ and such that

$$E\int_0^t F_s^2 dm_s < \infty \quad \text{for each } t \ \varepsilon\ \mathbb{R}_+ \qquad (3.2.5)$$

Proof: Define t_∞ and c_t as in Lemma 3.2.1 where we now use the continuous mean function m_t in place of a_t (see

Figure 3.2.3). For each t, c_t being a constant is trivially a stopping time. Define $N_t^* = N_{c_t}$, $M_t^* = M_{c_t}$, $X_t^* = X_{c_t}$ and $\mathcal{N}_t^* = \mathcal{N}_{c_t}$. We obviously have

$$\mathcal{N}_t^* = \sigma(N_u,\ 0 \le u \le c_t) \quad = \quad \sigma(N_{c_v},\ 0 \le c_v \le c_t) \quad = \quad \sigma(N_v^*,\ 0 \le v \le t) \tag{3.2.6}$$

where the last equality follows because c_t is strictly increasing by Lemma 3.2.1(c). Symmetrically for each t m_t is a stopping time and

$$\mathcal{N}_{m_t}^* \quad = \quad \mathcal{N}_t \tag{3.2.7}$$

Consider now the two following cases

<u>Case 1</u>: t_∞ is infinite (m_{t_∞} may be finite or infinite) and let $T^* = [0, m_{t_\infty})$.

<u>Case 2</u>: t_∞ is finite (m_{t_∞} is then also finite) and let $T^* = [0, m_{t_\infty}]$.

We show now that in these two cases (M_t^*) and (X_t^*) are $(\mathcal{N}_t^*)$ martingales with $E(M_t^*)^2 < \infty$ and $E(X_t^*)^2 < \infty$ for $t \in T^*$.

Case 1: By Lemma 3.2.1(a), c_t is finite for $t \in T^*$ and by the Optional Sampling Theorem (X_t^*) and (M_t^*) are $(\mathcal{N}_t^*)$ martingales. Clearly $E(M_t^*)^2$ and $E(X_t^*)^2$ are finite for $t \in T^*$.

Case 2: By Proposition 2.3.6 the CP (N_t) is a.s constant as a function of time for $t \geq t_\infty$; this implies $\mathcal{N}_t = \mathcal{N}_{t_\infty}$ for $t \geq t_\infty$ so that the two martingales (M_t) and (X_t) can respectively be expressed as $(M_t = E(M_{t_\infty}|\mathcal{N}_t))$ and $(X_t = E(X_{t_\infty}|\mathcal{N}_t))$; by Theorem 1.5.4 they are uniformly integrable and by the Optional Sampling Theorem we have for $t \in \mathbb{R}_+$ (although $c_t = \infty$ for $t \geq m_{t_\infty}$) that $(M_t^* = E(M_{t_\infty}|\mathcal{N}_t^*))$ and $(X_t^* = E(X_{t_\infty}|\mathcal{N}_t^*))$ belong to $\mathcal{M}_\ell^2(\mathcal{N}_t^*)$. For $t \geq m_{t_\infty}$, $c_t = \infty$, $M_t^* = M_\infty = M_{t_\infty}$ and $X_t^* = X_\infty = X_{t_\infty}$. Hence we only have to consider the index set T*.

Now by Lemma 3.2.1(c) one gets

$$N_t^* = M_t^* + m_{c_t} = M_t^* + t \qquad (3.2.8)$$

i.e., by Corollary 2.6.2 (N_t^*) is a Poisson process with rate one. We have seen just above that (X_t^*) is a $(\mathcal{N}_t^*)$ martingale with $E(X_t^*)^2 < \infty$ for $t \in T^*$. Furthermore (see (2.3.6)) $\mathcal{N}_t^* = \sigma(N_u^*, 0 \leq u \leq t)$ so that by Lemma 3 of [B1] there exists a process $(F_t^*) \in \mathcal{H}(\mathcal{N}_t^*)$ such that for $t \in T^*$.

$$E\int_0^t (F_s^*)^2 ds < \infty \qquad (3.2.9)$$

and

$$X_t^* = \int_0^t F_s^* dM_s^* = \int_0^t F_s^* dN_s^* - \int_0^t F_s^* ds \qquad (3.2.10)$$

Define

$$F_t = F^*_{m_t}$$

By (3.2.7) the process (F_t) is adapted to $(\mathcal{N}_t)$ and in fact $(F_t) \varepsilon \mathcal{H}(\mathcal{N}_t)$ since m_t is a continuous function and $(F^*_t) \varepsilon \mathcal{H}(\mathcal{N}^*_t)$.

By Lemma 3.2.1(c) and (d) we can write

$$\int_0^{c_t} F_s dm_s = \int_0^{c_t} F^*_{m_s} dm_s = \int_0^{\infty} I_{\{s \leq c_t\}}(s) \ F^*_{m_s} dm_s$$

$$= \int_0^{m_\infty} I_{\{s \leq c_t\}}(c_s) F^*_{m_{c_s}} ds = \int_0^{m_\infty \wedge t} F^*_s ds$$

For $t \varepsilon T^*$, $t \leq m_\infty$ so

$$\int_0^{c_t} F_s dm_s = \int_0^t F^*_s ds \qquad (3.2.11)$$

Similarly we get

$$\int_0^{c_t} F^2_s dm_s = \int_0^t (F^*_s)^2 ds$$

Hence by (3.2.9)

$$E\int_0^{c_t} F^2_s dm_s < \infty \qquad (3.2.12)$$

Also

$$\int_0^{c_t} F_s dN_s = \sum_{0\le s\le c_t} F^*_{m_s} \Delta N_s = \sum_{0\le u\le t} F^*_u \Delta N_{c_u}$$

$$= \int_0^t F^*_u dN^*_u \qquad (3.2.13)$$

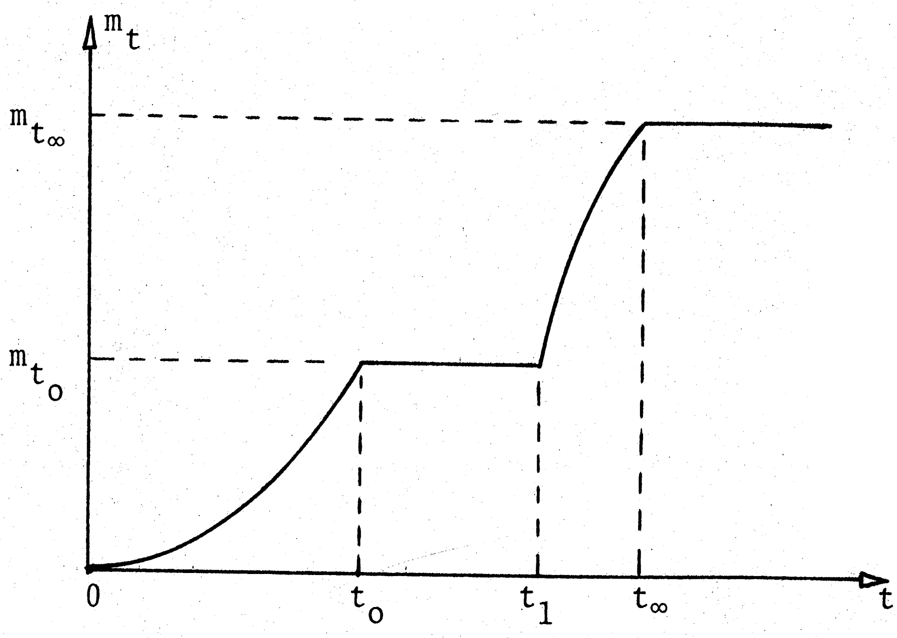
where we have used the change of variables $s = c_u$. Therefore by (3.2.10), (3.2.11) and (3.2.13) we have obtained

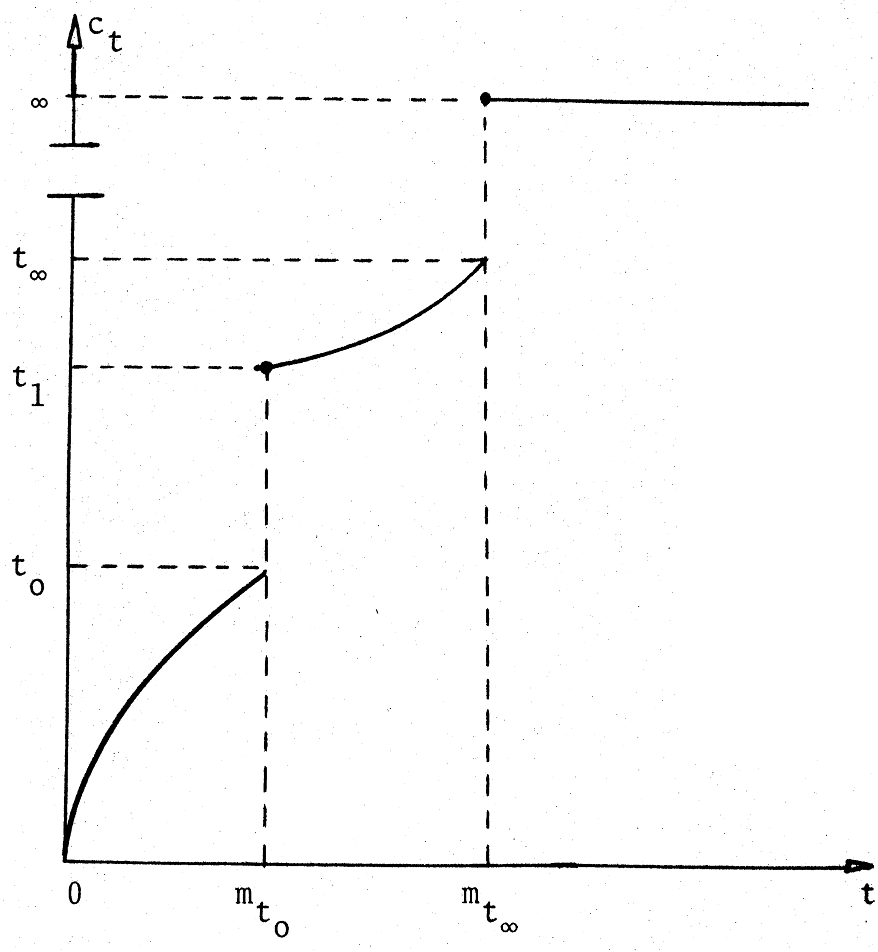
$$X^*_t = X_{c_t} = \int_0^{c_t} F_s dN_s - \int_0^{c_t} F_s dm_s = \int_0^{c_t} F_s dM_s \quad (3.2.14)$$

which shows together with (3.2.12) the desired result for all t in the range of c_t. If t is not in the range of c_t then $t \in [t_o, t_1)$ where t_o and $t_1 = \sup\{t: m_t = m_{t_o}\}$ delimit a flat of m_t. We include here case 2 where $t_o = t_\infty$ and $t_1 = \infty$. Note that t_1 belongs to the range of c_t (by right-continuity of c_t, see Lemma 3.2.1(a)); hence by (3.2.14) (if $t_1 = \infty$, $X_{t_1} = X_{t_\infty}$ and $M_{t_1} = M_{t_\infty}$ are well defined, see case 2)

$$X_{t_1} = \int_0^{t_1} F_s dM_s \qquad (3.2.15)$$

Let $t \in [t_o, t_1)$. By Proposition 2.3.6, $N_t = N_{t_1}$ a.s so that $\mathcal{N}_t = \mathcal{N}_{t_1}$ and therefore





$T^* = [0, m_{t_\infty}]$

Figure 3.2.3

$$X_t = E(X_{t_1}|\mathcal{N}_t) = E(X_{t_1}|\mathcal{N}_{t_1}) = X_{t_1} \qquad (3.2.16)$$

Similarly

$$M_t = M_{t_1} \qquad (3.2.17)$$

Also

$$F_t = F^*_{m_t} = F^*_{m_{t_1}} = F^*_{m_{t_o}} = F_{t_1} = F_{t_o}$$

Hence we also have in this case

$$\int_0^t F_s dM_s = \int_0^{t_1} F_s dM_s = X_{t_1} = X_t$$

where the first equality follows by (3.2.17), the second by (3.2.15) and the last one by (3.2.16). □

We remark the following. If (N_t) is now any regular CP with $(\mathcal{N}_t)$ ICR (A_t) then we can define a stochastic change of time by

$$c_t = \begin{cases} \inf\{s: A_s > t\} \\ \infty \text{ if the above set is empty} \end{cases}$$

and using the notation of the theorem we also have that $(N^*_t = M^*_t + t)$ is a Poisson process with rate one. But now $\mathcal{N}^*_t$ is not necessarily given by $\sigma(N^*_u, 0\le u\le t)$ (we have $\mathcal{N}^*_t \supset \sigma(N^*_u, 0\le u\le t)$). The $(\mathcal{N}^*_t)$ martingale (X^*_t) is not necessarily $(\sigma(N_u^*, 0 \leq u \leq t))$ measurable so that we cannot in this case apply Brémaud's result to express the martingale (X_t^*) as a stochastic integral with respect to (M_t^*), i.e., express (X_t) as a stochastic integral with respect to (M_t). Later on we will need the apparently more general result

Corollary 3.2.4: Let (N_t) be a regular CP of independent increments with finite mean m_t. Suppose T is a $(\mathcal{N}_t)$ stopping time. Then if $(X_t) \in \mathcal{M}_\ell^2(\mathcal{N}_{t \wedge T})$ there exists a process $(F_t) \in \mathcal{H}(\mathcal{N}_t)$ such that

$$E\int_0^t F_s^2 dm_s < \infty \qquad \text{for each } t \in \mathbb{R}_+$$

$$F_t = F_t I_{\{t \leq T\}}$$

and

$$X_t = \int_0^t F_s dM_s$$

where $(M_t = N_t - m_t)$.

Proof: By assumption $(X_t) \in \mathcal{M}_\ell^2(\mathcal{N}_{t \wedge T})$ and by Lemma A.2.1 we also have $(X_t) \in \mathcal{M}_\ell^2(\mathcal{N}_t)$. Thus by Theorem 3.2.2 there exists a process $(F_t) \in \mathcal{H}(\mathcal{N}_t)$ such that $E\int_0^t F_s^2 dm_s < \infty$ and

$$X_t = \int_0^t F_s dM_s$$

For $t \geq T$, $X_t = X_T$ so that one can take $F_t = F_t \cdot I_{\{t \leq T\}}$ □

As a consequence of Theorem 3.2.2 we also have

<u>Corollary 3.2.5</u>: Let (N_t) be a regular CP of independent increments with finite mean m_t. Then the family $(\mathcal{N}_t)$ is free of times of discontinuity.

<u>Proof</u>: By assumption the CP (N_t) is regular so that the times of jump of (N_t) are totally inaccessible (see Definition 2.4.1); the mean m_t is continuous by Theorem 2.4.7 so that the times of jump of the martingale $(M_t \triangleq N_t - m_t)$ are also totally inaccessible. Let (S_n) be any increasing sequence of stopping times. We have to show that $\bigvee_n \mathcal{N}_{S_n} = \mathcal{N}_{\lim_n S_n}$. Let A be any set belonging to $\mathcal{N}_{\lim_n S_n}$ and define the bounded martingale

$$X_t = E(I_A | \mathcal{N}_t)$$

By Theorem 3.2.2 there exists a process $(F_t) \in H(\mathcal{N}_t)$ so that

$$X_t = \int_0^t F_s dM_s$$

The above relation implies that the times of jump of (X_t) are totally inaccessible because as we have seen above those of (M_t) are. Hence

$$\lim_n X_{S_n} = X_{\lim_n S_n} \tag{3.2.19}$$

Now by choice of A

$$X_{\lim_n S_n} = E(I_A | N_{\lim_n S_n}) = I_A$$

Also

$$X_{S_n} = E(I_A | N_{S_n})$$

and by Lemma 1.5.7

$$\lim_n X_{S_n} = E(I_A | \bigvee_n N_{S_n})$$

so that (3.2.19) is equivalent to

$$E(I_A | \bigvee_n N_{S_n}) = I_A$$

which implies that for any $A \varepsilon N_{\lim_n S_n}$ then $A \varepsilon \bigvee_n N_{S_n}$ or $\bigvee_n N_{S_n} \supset N_{\lim_n S_n}$.

But we have, because S_n is an increasing sequence of stopping times $\bigvee_n N_{S_n} \subset N_{\lim_n S_n}$ so that the desired result $\bigvee_n N_{S_n} = N_{\lim_n S_n}$ has been obtained. □

3.3 LIKELIHOOD RATIO REPRESENTATION

MAIN RESULT

With respect to the measure P^* carried on (Ω, F) let (N_t) be a regular (for (N_t)) CP of independent increments with finite mean m_t. Denote by P the restriction of the measure P^* to the σ-algebra N_∞ and suppose P_o is another measure defined on (Ω, N_∞) which is absolutely continuous with respect to P. It is then meaningful to define the process

$$L_t = E\left(\frac{dP_o}{dP} \middle| N_t\right)$$

which is a uniformly integrable martingale and where dP_o/dP is the limit a.s and in the mean of L_t as t goes to ∞ (see Theorem 1.5.4). Observe that L_t is the likelihood ratio for the interval of observation $[0,t]$ (see Section 3.4: [D6], Chapter VIII). The following theorem gives a description of this martingale.

Theorem 3.3.1: Let (N_t) and (L_t) be the two processes defined above. Assume that T is a stopping time with

Property (H): There exists an increasing sequence of stopping times (T_n) such that $E(\ln^- L_{T_n})^2 < \infty$ for each n and $T = \lim_n T_n$ a.s.

Then there exists a positive process $(F_\cdot) \in H(N_t)$ such that

$$\int_0^{t \wedge \iota} F_s dm_s$$

and

$$L_{t\wedge T} = \left(\prod_{J_n \leq t\wedge T} F_{J_n}\right) \exp\left(m_{t\wedge T} - \int_0^{t\wedge T} F_s dm_s\right) \qquad (3.3.1)$$

where J_n is the time of n^{th} jump of (N_t) and the product term $\left(\prod_{J_n \leq t\wedge T} F_{J_n}\right)$ when empty (i.e., for $t\wedge T < J_1$) is set equal to one by convention.

Remark 3.3.2: Let T, T* be two stopping times with property (H) and (F_t), (F_t^*) their corresponding positive processes. The continuous parts of $(L_{t\wedge T})$ and $(L_{t\wedge T^*})$ are a.s equal on the set $\{t \leq T\wedge T^*\}$. Hence, from (3.3.1), we have

$$\int_0^{t\wedge T\wedge T^*} F_s dm_s = \int_0^{t\wedge T\wedge T^*} F_s^* dm_s \qquad \text{a.s}$$

i.e., $F_t = F_t^*$ a.s on the set $\{t \leq T\wedge T^*\}$. Note that the values of both these integrals are not affected by a change of the values of (F_t) and (F_t^*) on the intervals of constancy of m_t. By our convention (Remark 3.1.5), $F_t = F_t^* = 1$ on these intervals.

The stopping time T, which is the first time after which the likelihood ratio (L_t) can behave badly, may take the value $+\infty$. In fact it is desirable for T to be as large as possible. In the next section where we solve the detection problem we will identify the process (F_t) when the measure P_0 is generated by another CP. Brémaud

(Theorem 5.2.i of [B1]) states the above result in the case where the CP (N_t) is a Poisson process with rate one, but without assuming property (H). His proof is clearly wrong and we will see later on by an example that at least an assumption of the type $E(\ln^- L_{T_n}) < \infty$ (using the notation of the above theorem) is necessary to prove the result by the technique used. We delay a more extensive discussion and explanation of the above theorem and property (H) as this can best be achieved by first providing a proof of this result. This proof, which requires two additional lemmas is long and for clarity we outline it now:

<u>Step 1</u>: We show that there exists an increasing sequence of stopping times (S_n) converging to T such that (a) the process $(Z_t^n \triangleq \ln(L_{t\wedge S_n}))$ is a <u>regular supermartingale of class (D)</u> and (b) the martingale (Y_t^n) in the Doob-Meyer decomposition of (Z_t^n) belongs to $M^2(P,N_{t\wedge S_n})$. This part will be demonstrated with the help of Lemma 3.3.3 given below.

<u>Step 2</u>: By Corollary 3.2.4 we then express (Y_t^n) as a stochastic integral with respect to the martingale $(M_t = N_t - m_t)$ and get

$$L_{t\wedge S_n} = \exp\left(\int_0^t G_s I_{\{s\leq S_n\}} dN_s - B_t^n\right) \tag{3.3.2}$$

where $(B_t^n) \in V_c$ and $(G_t) \in H(N_t)$. Continuity of (B_t^n) follows from the regularity of both the CP (N_t) and the supermartingale (Z_t^n) and is necessary for the next step.

Step 3: By Lemma 3.3.4 below, the process (B_t^n) is of the form

$$B_t^n = \int_0^t [\exp(G_s I_{\{s \le S_n\}}) - 1] dm_s$$

Introducing the above relation in (3.3.2) and showing we can take the limit we obtain

$$L_{t \wedge T} = \exp\left(\int_0^{t \wedge T} G_s dN_s + m_{t \wedge T} - \int_0^{t \wedge T} \exp(G_s) dm_s\right)$$

The final result follows then by letting $F_t = \exp(G_t)$.

Lemma 3.3.3: Let $(\mathcal{F}_t)$ be an increasing family of σ-algebras and X_∞ be a nonnegative integrable random variable measurable with respect to $\mathcal{F}_\infty$. Define the uniformly integrable martingale $(X_t \triangleq E(X_\infty | \mathcal{F}_t))$ and the process $(Z_t \triangleq \ln X_t)$. Then

(a) The process $((Z_t^+)^2)$ is of class (D).

(b) The two following statements are equivalent

(1) (Z_t) is a supermartingale of class (D)

(2) $E(\ln^- X_\infty) < \infty$

(c) If $E(\ln^- X_\infty)^2 < \infty$ then (Z_t) is a supermartingale of class (D). Furthermore the process (Z_t^2) is also of class (D). In particular for any stopping time T, $EZ_T^2 < \infty$.

Proof: (a) Recall the relation

$$0 \leq \ln^2 x \leq 4x \qquad \text{for } x \geq 1$$

Now $Z_t^+ = \ln(X_t \vee 1)$. Hence

$$(Z_t^+)^2 = \ln^2(X_t \vee 1) \leq 4(X_t \vee 1) \qquad (3.3.3)$$

On the set $\{X_t \leq 1\}$, $Z_t^+ = 0$ and $X_t \geq 0$ so that (3.3.3) implies

$$0 \leq (Z_t^+)^2 \leq 4X_t$$

This shows that the process $((Z_t^+)^2)$ is of class (D) because the martingale (X_t) is uniformly integrable and hence of class (D) (see Remark 1.4.2(b) and (c)).

(b) (1) ⇒ (2) If (Z_t) is a supermartingale of class (D) then so is $(-Z_t^-)$ (see Theorem 6-V of [M1]; Proposition IV.5.1 of [N1]). Hence (Z_t^-) is a submartingale of class (D), a fortiori uniformly integrable (see Remark 1.4.2(a)). So by the supermartingale convergence theorem Z_t^- converges a.s and in the mean to an integrable random variable, say Z_∞^-, as t goes to ∞. Now it follows from the continuity and monotonicity of the logarithm that

$$Z_\infty^- = \ln^- X_\infty \qquad (3.3.4)$$

(even on the set $\{X_\infty = 0\}$); hence

$$E\ln^- X_\infty = EZ_\infty^- < \infty$$

(2) ⇒ (1) We have

$$Z_t^- = -\ln(X_t \wedge 1) \tag{3.3.5}$$

By proposition IV.5.1 of [N1], the process $(X_t \wedge 1)$ is a supermartingale (this is in fact a direct consequence of the Jensen inequality). The martingale (X_t) is uniformly integrable, hence so is the supermartingale $(X_t \wedge 1)$ so that

$$X_t \wedge 1 \geq E(X_\infty \wedge 1 | F_t) \tag{3.3.6}$$

By assumption and the Jensen inequality we get

$$-Z_t^- = \ln(X_t \wedge 1) \geq \ln[E(X_\infty \wedge 1 | F_t)] \geq E[\ln(X_\infty \wedge 1) | F_t]$$

i.e.,

$$0 \leq Z_t^- \leq E(\ln^- X_\infty | F_t) \tag{3.3.7}$$

Now the RHS of (3.3.7) is uniformly integrable (Theorem 1.5.4) and hence of class (D) (Remark 1.4.2(b)) so that the process (Z_t^-) is also of class (D) (Remark 1.4.2(c)) and by Theorem 6-V of [M1] (Z_t^-) is then a positive submartingle of class (D). Now the relation

$$Z_t^+ \leq (Z_t^+)^2 + \frac{1}{4}$$

together with part (a) and Remark 1.4.2(c) show that (Z_t^+) is of class (D) and consequently so is the process $(Z_t = Z_t^+ - Z_t^-)$. Then by Theorem 6-V of [M1] the process $(Z_t \stackrel{\Delta}{=} \ln X_t)$ is a supermartingale of class (D).

(c) The relation $E(\ln^{-} X_{\infty})^2 < \infty$ obviously implies $E\ln^{-} X_{\infty} < \infty$. We have just shown above that this last condition implies that (Z_t^-) is a submartingale of class (D), hence uniformly integrable (Remark 1.4.2(a)). We can then write (Theorem 13-VI of [M1])

$$Z_t^- \leq E(Z_{\infty}^- | F_t) \tag{3.3.8}$$

Now by the Jensen inequality, relations (3.3.4) and (3.3.8) we have

$$0 \leq (Z_t^-)^2 \leq [E(Z_{\infty}^- | F_t)]^2 \leq E[(Z_{\infty}^-)^2 | F_t] = E[(\ln^{-} X_{\infty})^2 | F_t]$$

where the RHS exists by assumption and is a martingale of class (D) (Theorem 1.5.4 and Remark 1.4.2(b)). Thus the above inequality shows that the process $((Z_t^-)^2)$ is of class (D) (Remark 1.4.2(c)). Then since $(Z_t^2 = (Z_t^+)^2 + (Z_t^-)^2)$, (Z_t^2) is of class (D) by virtue of part (a). □

The second lemma is essentially already contained in Lemma 6.1 of [K1]. Our result is more general in the sense we do not require local martingales to be square integrable nor do we assume the underlying probability space to be generated by a Hunt process. On the other hand it is more restrictive as we only consider processes which belong to V.

Lemma 3.3.4: Let (N_t) be a CP regular with respect to a family (F_t) and denote its (F_t) ICR by (A_t). Suppose that the process (B_t) belongs to $\mathcal{V}_c$ and that (F_t) is a predictable (with respect to $(\mathcal{F}_t)$) process. If the process (X_t) defined by

$$X_t \overset{\Delta}{=} \exp\left(\int_0^t F_s dN_s - B_t\right) \tag{3.3.9}$$

is a local martingale then

$$B_t = \int_0^t [\exp(F_s) - 1] dA_s$$

Proof: Let

$$X_t = \exp Z_t \tag{3.3.10}$$

where

$$Z_t \overset{\Delta}{=} \int_0^t F_s dN_s - B_t \tag{3.3.11}$$

and apply the change of variables formula (Theorem 1.9.14) to the RHS of (3.3.10). We get

$$X_t = 1 + \int_0^t X_{s^-} dZ_s + \sum_{0<s\leq t} [\exp(Z_s) - \exp(Z_{s^-}) - \exp(Z_{s^-})\Delta Z_s] \tag{3.3.12}$$

By relation (3.3.11):

$$\Delta Z_s = F_s \Delta N_s$$

Thus:

$$\sum_{0<s\leq t} [\exp(Z_s) - \exp(Z_{s^-}) - \exp(Z_{s^-})\Delta Z_s]$$

$$= \sum_{0<s\leq t} \{\exp(Z_{s-})[\exp(\Delta Z_s) - 1 - \Delta Z_s]\}$$

$$= \sum_{0<s\leq t} \{X_{s-}[\exp(F_s) - 1 - F_s]\Delta N_s\}$$

$$= \int_0^t X_{s-}[\exp(F_s) - 1]dN_s - \int_0^t X_{s-}F_s dN_s \qquad (3.3.13)$$

Now:

$$\int_0^t X_{s-}dZ_s = \int_0^t X_{s-}F_s dN_s - \int_0^t X_{s-}dB_s \qquad (3.3.14)$$

Introducing (3.3.13) and (3.3.14) into (3.3.12) we obtain

$$X_t - 1 = \int_0^t X_{s-}[\exp(F_s) - 1]dN_s - \int_0^t X_{s-}dB_s \qquad (3.3.15)$$

or taking into account the relation

$$N_t = M_t + A_t$$

$$X_t - 1 + W_t - Y_t = \int_0^t X_{s-}\exp(F_s)dA_s - \int_0^t X_{s-}d(A_s+B_s) \qquad (3.3.16)$$

where

$$W_t \triangleq \int_0^t X_{s-}dM_s$$

and

$$Y_t \triangleq \int_0^t X_{s-} \exp(F_s) dM_s$$

The process $(X_{t-}) \in H$ (see Remark 1.9.10(b)) so that the process (W_t) is a local martingale (Theorem 1.9.11). As we will show later on, the process (Y_t) is also a local martingale. Thus (3.3.16) implies that the process

$$\left(\int_0^t X_{s-} \exp(F_s) dA_s - \int_0^t X_{s-} d(A_s + B_s)\right) \in L_c \cap V$$

and by Lemma 1.9.4 we must have then:

$$B_t = \int_0^t [\exp(F_s) - 1] dA_s$$

since the process (X_t) is different from zero (see (3.3.9)), which is the desired result. To complete the proof we now show that the process (Y_t) is indeed a local martingale. Let

$$(F_s^n = F_s^+ \wedge n) \qquad (3.3.17)$$

so that

$$X_t^n \triangleq X_t \exp\left[\int_0^t (F_s^n - F_s^+) dN_s\right] \qquad (3.3.18)$$

increases to X_t by the monotone convergence theorem. Introducing the expression (3.3.9) of X_t into the above relation we get

$$X_t^n = \exp\left[\int_0^t (F_s^n - F_s^-)dN_s - B_t\right] \qquad (3.3.19)$$

Using the change of variables formula as above we find

$$X_t^n - 1 = V_t^n + \int_0^t X_{s-}^n \exp(F_s^n - F_s^-)dA_s - \int_0^t X_{s-}^n d(A_s+B_s) \qquad (3.3.20)$$

where

$$V_t^n \triangleq \int_0^t X_{s-}^n [\exp(F_s^n - F_s^-) - 1]dM_s$$

By Remark 1.9.10(b) the process $(X_{t-}) \varepsilon H$ and so does the process $(\exp(F_t^n - F_t^-) - 1)$, by construction. Hence the process (V_t^n) is a local martingale. Let (U_m) be a sequence of stopping times which makes the process (X_{t-}) locally bounded (see Definition 1.9.9). The process $(\exp(F_t^n - F_t^-)-1)$ is bounded by n and $0 \leq X_{t-}^n \leq X_{t-}$ so that, for each n, (U_m) is a sequence of stopping times reducing the local martingale (V_t^n). Let (R_m) be a sequence of stopping times reducing the local martingales (M_t), (X_t) and (W_t) and (S_m) the sequence of stopping times defined by

$$S_m = \begin{cases} \inf\{t: \left|\int_0^t X_{s-}d(A_s+B_s)\right| \geq m\} \\ \infty \text{ if the above set is empty} \end{cases}$$

The sequence (S_m) increases a.s to $+\infty$ because the process $(\int_0^t X_{s-}d(A_s+B_s)) \varepsilon V$ so that the sequence of stopping times given by $(T_m = U_m \wedge R_m \wedge S_m)$ also increases a.s to $+\infty$. Note also that, because of the continuity of the processes (A_t) and (B_t),

$$E\left|\int_0^{t\wedge T_m} X_{s-}\,d(A_s+B_s)\right| \leq m$$

Hence (see (3.3.20) and note, for each n, $EV^n_{t\wedge T_m} = EV^n_o = 0$)

$$EX^n_{t\wedge T_m} - 1 = E\int_0^{t\wedge T_m} X^n_{s-}\exp(F^n_s - F^-_s)\,dA_s - E\int_0^{t\wedge T_m} X^n_{s-}\,d(A_s+B_s)$$

and by the monotone convergence theorem (recall: $\lim_n (F^n_s - F^-_s) = F^+_s - F^-_s = F_s$)

$$E\int_0^{t\wedge T_m} X_{s-}\exp(F_s)\,dA_s$$

$$= EX_{t\wedge T_m} - 1 + E\int_0^{t\wedge T_m} X_{s-}\,d(A_s+B_s) < \infty \qquad (3.3.21)$$

where the RHS is finite by construction of the sequence (T_m). Then (3.3.16) shows that

$$EY_{t\wedge T_m} \triangleq E\int_0^{t\wedge T_m} X_{s-}\exp(F_s)\,dM_s < \infty \qquad (3.3.22)$$

and consequently (recall $M_t = N_t - A_t$)

$$E\int_0^{t\wedge T_m} X_{s-}\exp(F_s)|dM_s| = EY_{t\wedge T_m} + 2E\int_0^{t\wedge T_m} X_{s-}\exp(F_s)dA_s < \infty$$

where the last inequality follows by (3.3.21) and (3.3.22). This and Proposition 2 of [D1] (and the remark following it) which states that if $(M_t) \in A$ is a martingale and if (C_t) is a predictable process such that $E\int_0^t |C_s||dM_s| < \infty$ then $(\int_0^t C_s dM_s)$ is a martingale, finally shows that $(Y_t = \int_0^t X_{s-}\exp(F_s)dM_s)$ is a local martingale. □

Now we go back to the Proof of Theorem 3.3.1:

Step 1: By definition $L_t = E(L_\infty | F_t)$ where $L_\infty = dP_0/dP$ Define the process $Z_t \triangleq \ln L_t$. Recall that (T_n) is an increasing sequence of stopping times such that $E(\ln^- L_{T_n})^2 < \infty$ for each n, and that $T = \lim_n T_n$. Let (S_n) be any sequence of stopping times with $S_n \leq T_n$ a.s. Then we also have

$$E(\ln^- L_{S_n})^2 < \infty \text{ each } n \qquad (3.3.23)$$

This follows directly by applying Lemma 3.3.3(c) to the stopped process $(Z_{t\wedge T_n} \triangleq \ln L_{t\wedge T_n})$. Define now

$$R_n = \begin{cases} \inf\{t: \ L_t > e^n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

(R_n) is a sequence of stopping times increasing to ∞

because (L_t) is a right-continuous martingale and by Theorem 3-VI of [M1] the sample paths of such a martingale are a.s bounded on every compact interval. Let

$$S_n = T_n \wedge R_n$$

Clearly (S_n) is a sequence of stopping times increasing to T and $S_n \leq T_n$ each n. Define then $L^n_t \triangleq L_{t\wedge S_n}$, $Z^n_t \triangleq Z_{t\wedge S_n}$ and $N^n_t \triangleq N_{t\wedge S_n}$. By relation (3.3.23) we have

$E(\ln^- L^n_\infty)^2 < \infty$. The optional sampling theorem and Lemma 3.3.3(c) imply that (Z^n_t) is a (N^n_t) supermartingale of class (D) and for any stopping time Q

$$E(Z^n_Q)^2 < \infty \qquad (3.3.24)$$

We prove now that the supermartingale (Z^n_t) is regular. Let (V_m) be any increasing sequence of stopping times converging to a bounded stopping time V. We have (Theorem 13-VI of [M1])

$$L^n_{V_m} = E(L^n_V | N^n_{V_m})$$

By Lemma 1.5.7

$$\lim_m L^n_{V_m} = E(L^n_V | \bigvee_m N^n_{V_m}) \qquad (3.3.25)$$

By Corollary 3.2.5 the family (N_t) is free of times of discontinuity. Then obviously the family (N^n_t) has the same property so that

$$\bigvee_m \mathcal{N}^n_{V_m} = \mathcal{N}^n_V$$

and therefore (3.3.25) implies

$$\lim_m L^n_{V_m} = E(L^n_V | \mathcal{N}^n_V) = L^n_V$$

Consequently (by continuity of the logarithm)

$$\lim_m Z^n_{V_m} = Z^n_V$$

and because (Z^n_t) is of class (D) we can interchange limit and expectation operations and finally get

$$\lim_m EZ^n_{V_m} = EZ^n_V$$

which precisely means that (Z^n_t) is a regular supermartingale (Definition 1.7.12). Denote now the unique Doob-Meyer decomposition of (Z^n_t) by

$$Z^n_t = Y^n_t - A^n_t \tag{3.3.26}$$

where (Y^n_t) is a uniformly integrable martingale and (A^n_t) a continuous natural integrable increasing process (see Theorem 1.7.14(a) and (b)). We want to show now that $(Y^n_t) \in M^2(P, \mathcal{N}^n_t)$. By construction of the sequence (R_n) and the fact that $S_n \leq R_n$ we have

$$Z^n_t \leq n \vee Z^n_{S_n}$$

which clearly implies

$$0 \leq {}^{+}Z_t^n \leq n + ({}^{+}Z_{S_n}^n) \tag{3.3.27}$$

Hence

$$[\sup_t ({}^{+}Z_t^n)]^2 \leq 2[n^2 + ({}^{+}Z_{S_n}^n)^2]$$

and by (3.3.24)

$$E[\sup_t ({}^{+}Z_t^n)]^2 < \infty \tag{3.3.28}$$

Then relations (3.3.24) and (3.3.28) allow us to apply Lemma 2.2.2(c) to the supermartingale (Z_t^n) and get the desired result $(Y_t^n) \in \mathcal{M}^2(P,N_t^n)$. In résumé we have obtained an increasing sequence of stopping times (S_n) converging to T such that $(Z_t^n \triangleq \ln(L_{t\wedge S_n}))$ is a regular supermartingale of class (D) and the martingale (Y_t^n) in the Doob-Meyer decomposition of (Z_t^n) belongs to $\mathcal{M}^2(P,N_t^n)$.

Step 2: By Corollary 3.2.4 there exists a process (G_t^n) $\in \mathcal{H}(N_t^n)$ such that $E\int_0^t (G_s)^2 dm_s < \infty$ for each t and

$$Y_t^n = \int_0^t G_s^n I_{\{s\leq S_n\}} dM_s \tag{3.3.29}$$

By Lemma 1.7.10 $(Y_{t\wedge S_n}^m - A_{t\wedge S_n}^m)$, for $m \geq n$, is also as (3.3.26) a unique Doob-Meyer decomposition of (Z_t^n) so that we must have

$$Y_t^n = Y_{t\wedge S_n}^m$$

Hence we can define a process (G_t) by

$$G_t = G_t^n \quad \text{for } t \leq S_n$$

so that

$$Y_t^n = \int_0^t G_s I_{\{s \leq S_n\}} dM_s \qquad \forall\, n \tag{3.3.30}$$

Using the relation $M_t = N_t - m_t$ we get

$$Y_t^n = \int_0^t G_s I_{\{s \leq S_n\}} dN_s - \int_0^t G_s I_{\{s \leq S_n\}} dm_s$$

and we can rewrite (Z_t^n) as (see (3.3.26))

$$L_t^n = \exp(Z_t^n) = \exp\left(\int_0^t G_s I_{\{s \leq S_n\}} dN_s - B_t^n\right) \tag{3.3.31}$$

where

$$B_t^n \triangleq \int_0^t G_s I_{\{s \leq S_n\}} dm_s + A_t^n \tag{3.3.32}$$

Now by Corollary 3.2.4 $E\int_0^t G_s^2 I_{\{s \leq S_n\}} dm_s < \infty$ for each t.

Both the processes (m_t) and (A_t^n) are continuous because respectively the CP (N_t) and the supermartingale (Z_t^n) are regular. Hence $(B_t^n) \in \mathcal{V}_c$ so that

Step 3: By Lemma 3.3.4 we must have

$$B_t^n = \int_0^t [\exp(G_s I_{\{s \leq S_n\}}) - 1] dm_s$$

or

$$B_t^n = \int_0^t [\exp(G_s) - 1] I_{\{s \leq S_n\}} dm_s \qquad (3.3.33)$$

and $((B_t^n) \in \mathcal{V}_c)$

$$\int_0^t \exp(G_s) I_{\{s \leq S_n\}} dm_s < \infty \qquad (3.3.34)$$

Introducing (3.3.33) in (3.3.31) we get

$$L_t^n \stackrel{\Delta}{=} L_{t \wedge S_n} = \exp\{\int_0^t G_s I_{\{s \leq S_n\}} dN_s + \int_0^t I_{\{s \leq S_n\}} dm_s$$

$$- \int_0^t \exp(G_s) I_{\{s \leq S_n\}} dm_s\} \qquad (3.3.35)$$

Now we take the limit:

LHS of (3.3.35): we can write

$$L_{t \wedge S_n} = E(L_T | \mathcal{N}_{t \wedge S_n})$$

and by Lemma 1.5.7

$$\lim_n L_{t \wedge S_n} = E(L_T | \bigvee_n \mathcal{N}_{t \wedge S_n})$$

The family $(\mathcal{N}_t)$ is free of times of discontinuity (Corollary 3.2.5), i.e., $\bigvee_n \mathcal{N}_{t \wedge S_n} = \mathcal{N}_{t \wedge T}$, therefore

$$\lim_n L_{t \wedge S_n} = E(L_T | \mathcal{N}_{t \wedge T}) = L_{t \wedge T} \qquad (3.3.36)$$

RHS of (3.3.35):

$$\lim_n \int_0^t G_s I_{\{s\le S_n\}} dN_s = \lim_n \int_0^t G_s^+ I_{\{s\le S_n\}} dN_s$$

$$- \lim_n \int_0^t G_s^- I_{\{s\le S_n\}} dN_s$$

$$= \int_0^t G_s^+ I_{\{s\le T\}} dN_s - \int_0^t G_s^- I_{\{s\le T\}} dN_s$$

where the last equality follows by the monotone convergence theorem. Hence

$$\lim_n \int_0^t G_s I_{\{s\le S_n\}} dN_s = \int_0^{t\wedge T} G_s dN_s \qquad (3.3.37)$$

Similarly we get

$$\lim_n \int_0^t I_{\{s\le S_n\}} dm_s = \int_0^{t\wedge T} dm_s = m_{t\wedge T} \qquad (3.3.38)$$

and

$$\lim_n \int_0^t \exp(G_s) I_{\{s\le S_n\}} dm_s = \int_0^{t\wedge T} \exp(G_s) dm_s \qquad (3.3.39)$$

Hence we finally get

$$L_{t\wedge T} = \exp\{ \int_0^{t\wedge T} G_s dN_s + m_{t\wedge T} - \int_0^{t\wedge T} \exp(G_s) dm_s \}$$

or

$$L_{t\wedge T} = \left(\prod_{J_n \leq t\wedge T} F_{J_n}\right) \exp\{m_{t\wedge T} - \int_0^{t\wedge T} F_s dm_s\}$$

where we have defined $F_s = \exp(G_s)$ and relation (3.3.34) implies

$$\int_0^{t\wedge T} F_s dm_s < \infty \qquad \square$$

DISCUSSION OF ASSUMPTIONS

We use the notation introduced in Theorem 3.3.1 and its proof: $L_t = E(dP_0/dP|N_t)$ and $Z_t = \ln L_t$. In this theorem we assume the existence of a stopping time T with property (H): There exists an increasing sequence of stopping times (T_n) such that $E(\ln^- L_{T_n})^2 < \infty$ for each n and $T = \lim_n T_n$ a.s. Consider now the weaker condition (H'): There exists an increasing sequence of stopping times (T_n) such that $E\ln^- L_{T_n} < \infty$ for each n and $T = \lim_n T_n$ a.s.

By Lemma 3.3.3(b), (H') is a necessary and sufficient condition for the stopped process $(Z_{t\wedge T_n})$ to be a supermartingale of class (D) for each n. Thus <u>(H') is the weakest condition which allows us to carry on the first part (part (a)) of Step 1</u>. We give now a concrete example where this assumption (H') is not satisfied.

Example 3.3.5: Let (N_t) be a Poisson process with constant rate λ. Denote its time of first jump by J_1. We have $P\{N_t = k\} = e^{-\lambda t}(\lambda t)^k(k!)^{-1}$ and the set equality $\{J_1 > t\} = \{N_t = 0\}$ so that the probability distribution and the probability density of J_1 are respectively given by

$$F_{J_1}(t) = P\{J_1 \le t\} = 1 - P\{J_1 > t\} = 1 - P\{N_t = 0\} = 1 - e^{-\lambda t}$$

$$f_{J_1}(t) = \lambda e^{-\lambda t}$$

Define now

$$L_t = E\left(\frac{dP_o}{dP} \middle| N_t\right) \tag{3.3.40}$$

where

$$\frac{dP_o}{dP} = \alpha \exp(-\lambda/J_1) \tag{3.3.41}$$

and α is a normalizing constant making $E\,\frac{dP_o}{dP} = 1$. We want to show that in this case the random variable $\ln^- L_R$ is not integrable for any choice of stopping time R. To do that it is enough to consider stopping times $R \le J_1$ since by construction dP_o/dP is a (N_{J_1}) measurable random variable and hence $L_R = L_{J_1}$ for $R > J_1$. Now by Proposition A.4.1 any stopping time $R \le J_1$ is of the form $R = J_1 \wedge a$ where a is any positive constant. We have

$$L_R = L_{J_1 \wedge a} = \begin{cases} \frac{dP_o}{dP} & \text{on the set } A = \{J_1 \leq a\} \\ \leq \alpha & \text{otherwise} \end{cases}$$

Hence

$$\ln L_R \leq I_A \ln \left(\frac{dP_o}{dP} \right) + I_{A^c} \ln \alpha$$

so that

$$E \ln L_R \leq EI_A \ln\left(\frac{dP_o}{dP}\right) + \ln \alpha P\{A^c\} \qquad (3.3.42)$$

Now by (3.3.41)

$$\ln \frac{dP_o}{dP} = \ln \alpha - \lambda/J_1$$

and by introducing the above relation in (3.3.42) we get:

$$E \ln L_R \leq \ln \alpha - EI_A \lambda/J_1 \qquad (3.3.43)$$

Since $EI_A\lambda/J_1 = \lambda^2 \int_0^a \frac{1}{t} e^{-\lambda t} dt = +\infty$, (3.3.43) shows that $E \ln L_R = -\infty$, which is the desired result.

Brémaud's Theorem 5.2.i [B1], which consists of Theorem 3.3.1 in the case where the CP (N_t) is a Poisson process with rate one, is stated without assumption (H'). The above shows that this is a mistake. In fact the error in his proof is easy to pinpoint. Define the sequence of stopping times

$$V_n = \begin{cases} \inf\{t: \ L_t < 1/n \text{ or } L_t > n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

Now Brémaud's proof requires the process $(\ln L_{t \wedge V_n})$ to be a bounded supermartingale. But this is not necessarily so as (V_n) may be a time of jump of (L_t) and hence $(L_{t \wedge V_n})$ may not be bounded (see above example). At this point Brémaud follows too closely Duncan's proof of Theorem 3 [D3]. There the problem of detecting a stochastic signal in white noise is examined; in this situation the martingale (L_t) is continuous so that the process $(\ln L_{t \wedge V_n})$ is now a bounded supermartingale as required by the rest of the proof. It may be worth noting that when (L_t) is continuous our assumption (H) is automatically satisfied by taking

$$T_n = \begin{cases} \inf\{t: \ L_t < 1/n\} \\ \infty \text{ if the above set is empty} \end{cases}$$

so that we could recover Duncan's result (Theorem 3, [D3]) by the technique of proof of Theorem 3.3.1.

We now examine assumption (H), which is stronger than (H'). We have just seen above that (H') is the weakest assumption which implies that the process $(Z_{t \wedge T_n})$ is a supermartingale of class (D). Now (step 2 of the proof) we would like to express, using Corollary 3.2.4, the martingale in the Doob-Meyer decomposition of $(Z_{t\wedge T_n})$ as a stochastic integral with respect to the martingale $(M_t \triangleq N_t - m_t)$. Hence this martingale should be square integrable, which is not necessarily the case as Example 3.3.6 below will show. This is why we need assumption (H) as it implies (see the proof of Theorem 3.3.1) the existence of a sequence of stopping times (S_n) increasing to T and such that the martingale (Y_t^n) in the Doob-Meyer decomposition of $(Z_{t\wedge S_n})$ is square integrable. Now the converse is also true: if there exists a sequence of stopping times (S_n) increasing to T and such that the martingale (Y_t^n) in the Doob-Meyer decomposition of $(Z_{t\wedge S_n})$ is square integrable then using Lemma 2.2.2(c) it is easy to show that assumption (H) is satisfied. Hence property (H) is the weakest assumption under which step 2 of the proof can be undertaken.

Example 3.3.6: This is a case where (H'), but not (H), is satisfied. The CP (N_t) is as in the previous Example 3.3.5. We let now

$$\frac{dP_o}{dP} \triangleq \alpha \exp(-\lambda/\sqrt{J_1})$$

where α is again a normalizing constant which makes $E\, dP_o/dP = 1$. As usual $L_t \triangleq E(d\, P_o/dP|N_t)$ and $Z_t \triangleq \ln L_t$. It is easy to show that in this case the random variable $\ln^- L_\infty$ is integrable and hence by Lemma 3.3.3(b) the process (Z_t) is a supermartingale of class (D). But for any positive stopping time R the random variable $(\ln^- L_R)^2$ is not integrable, i.e., assumption (H) is not satisfied.

In conclusion if we could extend Theorem 3.2.2 (on martingale representation), and consequently Corollary 3.2.4, to non square integrable martingales then Theorem 3.3.1 could be proved for the wider class of stopping times T satisfying the weaker assumption (H'). We have tried to generalize the above Theorem 3.3.1 doing the following. Let $L_\infty^n = L_\infty + 1/n$ and define $L_t^n = E(L_\infty^n | \mathcal{N}_t)$. Clearly $L_t^n = L_t + 1/n$, i.e., L_t^n converges uniformly to L_t in (t,ω). Now we can apply Theorem 3.3.1 to (L_t^n) as assumption (H) is obviously satisfied and obtain an expression of the form (3.3.1) for (L_t^n). The problem here is in taking the limit: we have not been able to show that this expression of (L_t^n) converges to the desired result.

3.4 DETECTION FORMULAS

INTRODUCTION

The problem of detection by the likelihood ratio technique is now considered. Let $(\Omega,\mathcal{F})$ be a measure space on which two measures P_0 and P_1 are defined. Suppose that (N_t) is a CP defined on $(\Omega,\mathcal{F})$ and denote as usual by $\mathcal{N}_t$ the minimal σ-algebra generated by (N_t) up to and at time t. The notation $E_i(\cdot)$ for i = 0,1 is intended for the expectation operator with respect to the measure P_i.

<u>Definition 3.4.1</u>: For a $(\mathcal{N}_t)$ stopping time R (possibly infinite) denote by $\overline{\mathcal{P}}_i^R$ for i = 0,1 the restriction of the measure $\mathcal{P}_i$ to the σ-algebra $\mathcal{N}_R$.

We have the inclusion $\mathcal{N}_R \subset \mathcal{F}$ so that if $\mathcal{P}_o << \mathcal{P}_1$ then $\overline{\mathcal{P}}_o^R << \overline{\mathcal{P}}_1^R$ and the Radon-Nikodym derivative $d\overline{\mathcal{P}}_o^R/d\overline{\mathcal{P}}_1^R$ is well defined. We examine now the meaning of this Radon-Nikodym derivative. In the case where the stopping time R is equal to a constant a then $\mathcal{N}_R = \mathcal{N}_a = \sigma(N_u,\ 0 \leq u \leq a)$ so that $d\overline{\mathcal{P}}_o^a/d\overline{\mathcal{P}}_1^a$ is the likelihood ratio for testing the two hypotheses H_i for i = 0,1: $\mathcal{P}_i$ is the probability measure on $(\Omega,\mathcal{F})$, by observations on the CP (N_t) for $t \leq a$. The detection scheme then consists in selecting H_o or H_1 according as $d\overline{\mathcal{P}}_o^a/d\overline{\mathcal{P}}_1^a$ is above or below a given threshold. Now in the case where R is a stopping time which is not a constant we know that $\mathcal{N}_R \supset \sigma(N_{u \wedge R},\ 0 \leq u)$ (this follows from the fact that $N_{u \wedge R}$ is $(\mathcal{N}_R)$ measurable by Theorem 1.3.4) but the reverse inclusion is not necessarily true. For this reason $d\overline{\mathcal{P}}_o^R/d\overline{\mathcal{P}}_1^R$ is not the likelihood ratio for our detection problem when the time of observation is the stochastic interval [0,R], as one could have conjectured. But one can interpret $d\overline{\mathcal{P}}_o^R/d\overline{\mathcal{P}}_1^R$ as a likelihood ratio if we assume that the information accessible to the observer is $\mathcal{N}_R$ and not simply $\sigma(N_{u \wedge R},\ 0 \leq u)$.

Let now $L_\infty \triangleq d\overline{\mathcal{P}}_o^\infty/d\overline{\mathcal{P}}_1^\infty$, i.e., L_∞ is the likelihood ratio when the time of observation of the CP (N_t) is the positive real line $\mathbb{R}_+$, and define

$$L_t = E_1(L_\infty | N_t) \tag{3.4.1}$$

Then it is easy to see

Lemma 3.4.2:

$$\frac{d\bar{P}_o^R}{d\bar{P}_1^R} = L_R$$

Proof: $N_\infty \supset N_R$ so that $\forall\ A \in N_R$ we can write

$$\int_A \frac{d\bar{P}_o^R}{d\bar{P}_1^R}\, dP_1 = \int_A \frac{d\bar{P}_o^R}{d\bar{P}_1^R}\, d\bar{P}_1^R = \bar{P}_o^R(A) = \bar{P}_o^\infty(A)$$

and by definition of conditional expectations

$$\int_A E_1\left(\frac{d\bar{P}_o^\infty}{d\bar{P}_1^\infty}\Big| N_R\right) dP_1 = \int_A \frac{d\bar{P}_o^\infty}{d\bar{P}_1^\infty}\, dP_1 = \bar{P}_o^\infty(A)\ ;$$

hence we have the equality

$$\int_A \frac{d\bar{P}_o^R}{d\bar{P}_1^R}\, dP_1 = \int_A E_1\left(\frac{d\bar{P}_o^\infty}{d\bar{P}_1^\infty}\Big| N_R\right) dP_1 \qquad \forall\ A \in N_R$$

which implies

$$\frac{d\bar{P}_o^R}{d\bar{P}_1^R} = E_1(L_\infty | N_R) = L_R$$

As a result of the likelihood ratio representation and the Girsanov and Innovation Theorems, an expression for the martingale (L_t) can be found under suitable circumstances.

LIKELIHOOD RATIO: FIRST RESULT

With the measure P_1 carried on (Ω, F) we suppose that (N_t) is a regular CP of independent increments with mean $E_1 N_t = m_t$. Under the measure P_o assume that (N_t) is a CP which has an (F_t) ICR of the form $(\int_0^t \lambda_s dm_s)$ where (F_t) is a family of σ-algebras such that $F_t \supset N_t$ and $(\lambda_t) \in H(F_t)$ is a nonnegative process.

Theorem 3.4.3: Let (N_t) be the CP described above. Assume

(a) $P_o << P_1$. Define the uniformly integrable martingale (see (3.4.1))

$$L_t = E_1\left(\frac{d\overline{P}_o^\infty}{d\overline{P}_1^\infty} \middle| N_t\right)$$

(b) The stopping time T is such that there exists an increasing sequence of stopping times (T_n) for which $T = \lim_n T_n$ a.s and $E(\ln^- L_{T_n})^2 < \infty$ for each n.

(c) $E_o \int_0^t \lambda_s dm_s < \infty$

Then

$$\frac{d\overline{P}_o^{t\wedge T}}{d\overline{P}_1^{t\wedge T}} = L_{t\wedge T} = \left(\prod_{J_n \leq t\wedge T} \hat{\lambda}_{J_n}\right) \exp\left(m_{t\wedge T} - \int_0^{t\wedge T} \hat{\lambda}_s dm_s\right) \quad (3.4.2)$$

where $\hat{\lambda}_t \triangleq E_0(\lambda_t|N_t)$ and J_n is the time of n^{th} jump of (N_t). By convention the product $\Pi(\cdot) = 1$ for $J_1 > t \wedge T$.

We adopt here the same convention as in Remark 3.1.5, i.e., we set $(\lambda_t = 1)$ on the intervals of constancy of m_t. Condition (c) then insures that the process $(\hat{\lambda}_t)$ is well defined. Recall that the meaning of $d\overline{P}_0^R/d\overline{P}_1^R$ has been given in the Introduction to this section. In particular if $t \leq T$ so that $t \wedge T = t$ then (3.4.2) is the likelihood ratio for our detection problem when the time of observation of (N_t) is the interval $[0,t]$.

Proof: By assumption (c) and Proposition 2.3.7 we have $E_0 N_t < \infty$ so that by Theorem 2.3.1(b) the process $(N_t - \int_0^t \lambda_s dm_s)$ is a (P_0, F_t) martingale and by the Innovation Theorem 3.1.4 $(N_t - \int_0^t \hat{\lambda}_s dm_s)$ is a (P_0, N_t) martingale, where $\hat{\lambda}_t = E_0(\lambda_t|N_t)$. Then by the Optional Sampling Theorem the process

$$(X_{t\wedge T} \triangleq N_{t\wedge T} - \int_0^{t\wedge T} \hat{\lambda}_s dm_s) \qquad (3.4.3)$$

is a $(P_0, N_{t\wedge T})$ martingale. Now by (a) $\overline{P}_0^\infty << \overline{P}_1^\infty$ because $N_\infty \subset F$. Thus the martingale $L_t = E_1(d\overline{P}_0^\infty/d\overline{P}_1^\infty|N_t)$ is well defined. Then by assumption (b) and the likelihood ratio representation Theorem 3.3.1 there exists a positive process $(F_t) \in H(N_t)$ such that

$$\int_0^{t\wedge T} F_s dm_s < \infty \qquad (3.4.4)$$

and

$$L_{t\wedge T} = \left(\prod_{J_n \le t\wedge T} F_{J_n}\right) \exp(m_{t\wedge T} - \int_0^{t\wedge T} F_s dm_s) \qquad (3.4.5)$$

By Girsanov Theorem 3.1.1(b) (applied to $(N_{t\wedge T})$) the process

$$(Y_{t\wedge T} \triangleq N_{t\wedge T} - \int_0^{t\wedge T} F_s dm_s) \qquad (3.4.6)$$

is a $(P_o, N_{t\wedge T})$ local martingale. Subtracting (3.4.3) from (3.4.6) one gets that

$$(Y_{t\wedge T} - X_{t\wedge T} = \int_0^{t\wedge T} (\hat{\lambda}_s - F_s) dm_s) \qquad (3.4.7)$$

is a $(P_o, N_{t\wedge T})$ local martingale. By assumption (c) and Fubini's Theorem

$$E_o \int_0^{t\wedge T} \hat{\lambda}_s dm_s \le E_o \int_0^t \hat{\lambda}_s dm_s = \int_0^t E_o \lambda_s dm_s = E_o \int_0^t \lambda_s dm_s < \infty \qquad (3.4.8)$$

Furthermore m_t is continuous (the CP (N_t) is assumed regular, see Theorem 2.4.7) so that by (3.4.4) and (3.4.8) the RHS of (3.4.7) belongs to V_c. Hence $(Y_{t\wedge T} - X_{t\wedge T}) \in L \cap V_c$ and by Lemma 1.9.4, $(Y_{t\wedge T} - X_{t\wedge T})$ is identically zero a.s which implies $\hat{\lambda}_s = F_s$. The result then follows by introducing this relation in (3.4.5). Finally by Lemma 3.4.2, $L_{t\wedge T} = d\overline{P}_o^{t\wedge T}/d\overline{P}_1^{t\wedge T}$ □

GENERALIZATION

We now take advantage of the chain rule for Radon-Nikodym derivatives to extend the previous result. For $i = 0,1$, with the measure P_i carried on (Ω,F) suppose that the CP (N_t) has the process $(\int_0^t \lambda_s^i dm_s)$ for (F_t^i) ICR, where (F_t^i) is a family of σ-algebras with $F_t^i \supset N_t$, $(\lambda_t^i) \in H(F_t^i)$ is a positive process and m_t is an increasing deterministic function with $m_o = 0$.

By Theorem 2.6.1 there exists a measure P which makes (N_t) a CP of independent increments with mean $EN_t = m_t$.

Theorem 3.4.4: For $i = 0,1$ let (N_t) be, under the measure P_i, the CP described above. Assume

(a) $P_o << P$ and $P \sim P_1$ and define for $i = 0,1$ the (P,N_t) martingale

$$L_t^i = E\left(\frac{d\bar{P}_i^\infty}{d\bar{P}^\infty}\Big| N_t\right)$$

(b) For $i = 0,1$ the stopping time T^i is such that there exists an increasing sequence of stopping times (T_n^i) for which $T^i = \lim_n T_n^i$ a.s and $E(\ln^- L^i_{T_n^i})^2$ for each n. Let $T = T^1 \wedge T^o$

(c) For $i = 0,1$ $E_i \int_0^t \lambda_s^i dm_s < \infty$

Then

$$\frac{d\bar{P}_o^{t\wedge T}}{d\bar{P}_1^{t\wedge T}} = \left(\prod_{J_n\leq t\wedge T} \frac{\hat{\lambda}^o_{J_n}}{\hat{\lambda}^1_{J_n}}\right)\exp\left(\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^o)dm_s\right) \qquad (3.4.9)$$

where $\hat{\lambda}^i_t \overset{\Delta}{=} E_i(\lambda^i_t | N_t)$ for i = 0,1 and J_n is the time of n^{th} jump of (N_t). By convention the product $\Pi(\cdot) = 1$ for $J_1 > t \wedge T$.

Remark that this indeed generalizes the preceeding result with $\lambda^1_t \equiv 1$.

Proof: By the previous Theorem 3.4.3 we get for i = 0,1

$$\frac{d\overline{P}_i^{t\wedge T}}{d\overline{P}^{t\wedge T}} = \left(\prod_{J_n \leq t\wedge T} \hat{\lambda}^i_{J_n} \right) \exp(m_{t\wedge T} - \int_0^{t\wedge T} \hat{\lambda}^i_s dm_s) \qquad (3.4.10)$$

where $\hat{\lambda}^i_t \overset{\Delta}{=} E_i(\lambda^i_t | N_t)$. Now $P \sim P_1$ so that $\overline{P}^{t\wedge T} \sim \overline{P}_1^{t\wedge T}$. Indeed

$$\frac{d\overline{P}^{t\wedge T}}{d\overline{P}_1^{t\wedge T}} = \left(\frac{d\overline{P}_1^{t\wedge T}}{d\overline{P}^{t\wedge T}} \right)^{-1}$$

since $N_{t\wedge T} \subset F$; hence

$$\frac{d\overline{P}_o^{t\wedge T}}{d\overline{P}_1^{t\wedge T}} = \left(\frac{d\overline{P}_o^{t\wedge T}}{d\overline{P}^{t\wedge T}} \right) \left(\frac{d\overline{P}_1^{t\wedge T}}{d\overline{P}^{t\wedge T}} \right)^{-1}$$

and the result follows by a simple computation from (3.4.10). □

INTEGRAL EQUATIONS FOR LIKELIHOOD RATIOS

We show here that the likelihood ratio of our detection problem can be obtained as the unique solution of a stochastic integral equation.

Theorem 3.4.5: The likelihood ratio $d\bar{P}_o^{t\wedge T}/d\bar{P}_1^{t\wedge T}$ of Theorem 3.4.4 is the unique solution of the following stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_{s-} dX_{s\wedge T} \qquad (3.4.11)$$

where

$$X_t = \int_0^t \left(\frac{\hat{\lambda}_s^o}{\hat{\lambda}_s^1} - 1\right) dN_s + \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^o) dm_s \qquad (3.4.12)$$

Proof: By assumption (λ_t^i), $i = 0,1$, is positive (see above Theorem 3.4.4) and is a.s finite for all t (by condition (c) of Theorem 3.4.4). Furthermore the process (N_t) has a finite number of jumps in any finite interval so that the process $(\int_0^{t\wedge T} ((\hat{\lambda}_s^o/\hat{\lambda}_s^1) - 1) dN_s) \in V$. The process $(\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^o) dm_s)$ also belongs to this class V by assumption (c) of Theorem 3.4.4. Therefore $(X_{t\wedge T}) \in V$ is a semimartingale with $\langle X^c\rangle_{t\wedge T} = 0$ (see Remark 1.8.17). Then by Theorem 1.9.15 the unique solution of (3.4.11) is given by

$$Z_t = \exp(X_{t\wedge T}) \prod_{s\le t} (1 + \Delta X_{s\wedge T}) \exp(-\Delta X_{s\wedge T}) \qquad (3.4.13)$$

Now $\Delta X_{s\wedge T} = ((\hat{\lambda}_s^o/\hat{\lambda}_s^1) - 1)\Delta N_{s\wedge T}$ and

$$\prod_{s\leq t}(\cdot) = \prod_{s\leq t} [1 + \left(\frac{\hat{\lambda}_s^o}{\hat{\lambda}_s^1} - 1\right)\Delta N_{s\wedge T}] \exp[\sum_{s\leq t\wedge T} - \left(\frac{\hat{\lambda}_s^o}{\hat{\lambda}_s^1} - 1\right)\Delta N_{s\wedge T}]$$

$$= \prod_{J_n\leq t\wedge T} \left(\frac{\hat{\lambda}^o_{J_n}}{\hat{\lambda}^1_{J_n}}\right) \exp\left(-\int_0^{t\wedge T} \left(\frac{\hat{\lambda}_s^o}{\hat{\lambda}_s^1} - 1\right) dN_s\right)$$

Substituting the above relation and expression (3.4.12) in (3.4.13) gives the desired result (compare with (3.4.9))

$$Z_t = \prod_{J_n\leq t\wedge T} \left(\frac{\hat{\lambda}^o_{J_n}}{\hat{\lambda}^1_{J_n}}\right) \exp\left(\int_0^{t\wedge T} (\hat{\lambda}_s^1 - \hat{\lambda}_s^o) dm_s\right) = \frac{d\overline{P}_o^{t\wedge T}}{d\overline{P}_1^{t\wedge T}} \qquad \square$$

Observe that if under the measure P_1 the CP (N_t) is a process of independent increments with mean m_t then $P \equiv P_1$, $\lambda_t^1 = 1$ and Eq. (3.4.12) becomes

$$X_t = \int_0^t (\hat{\lambda}_s^o - 1) d(N_s - m_s) \qquad (3.4.14)$$

The process $(M_t \triangleq N_t - m_t)$ is a (P, N_t) martingale. Hence (3.4.14) shows that the process $(X_{t\wedge T})$ is a local martingale. This in turn implies by (3.4.11) that the process (Z_t) is a local martingale. This is consistent with what we have seen in the previous section (Theorem 3.4.3) since in this case $Z_t = L_{t\wedge T} = E_1(d\overline{P}_o^\infty/d\overline{P}_1^\infty | N_{t\wedge T})$, i.e. (Z_t) is in fact a uniformly integrable martingale. In the more general case the likelihood ratio $(Z_t = d\overline{P}_o^{t\wedge T}/d\overline{P}_1^{t\wedge T})$ is not necessarily a local martingale.

In application Eqs. (3.4.11) and (3.4.12) give a way of implementing the computation of the likelihood ratio continuously in time. They represent recursive equations if one also obtains the best estimates $(\hat{\lambda}_t^i)$ in a recursive way. The block diagram of this implementation is given in Figure 3.4.6.

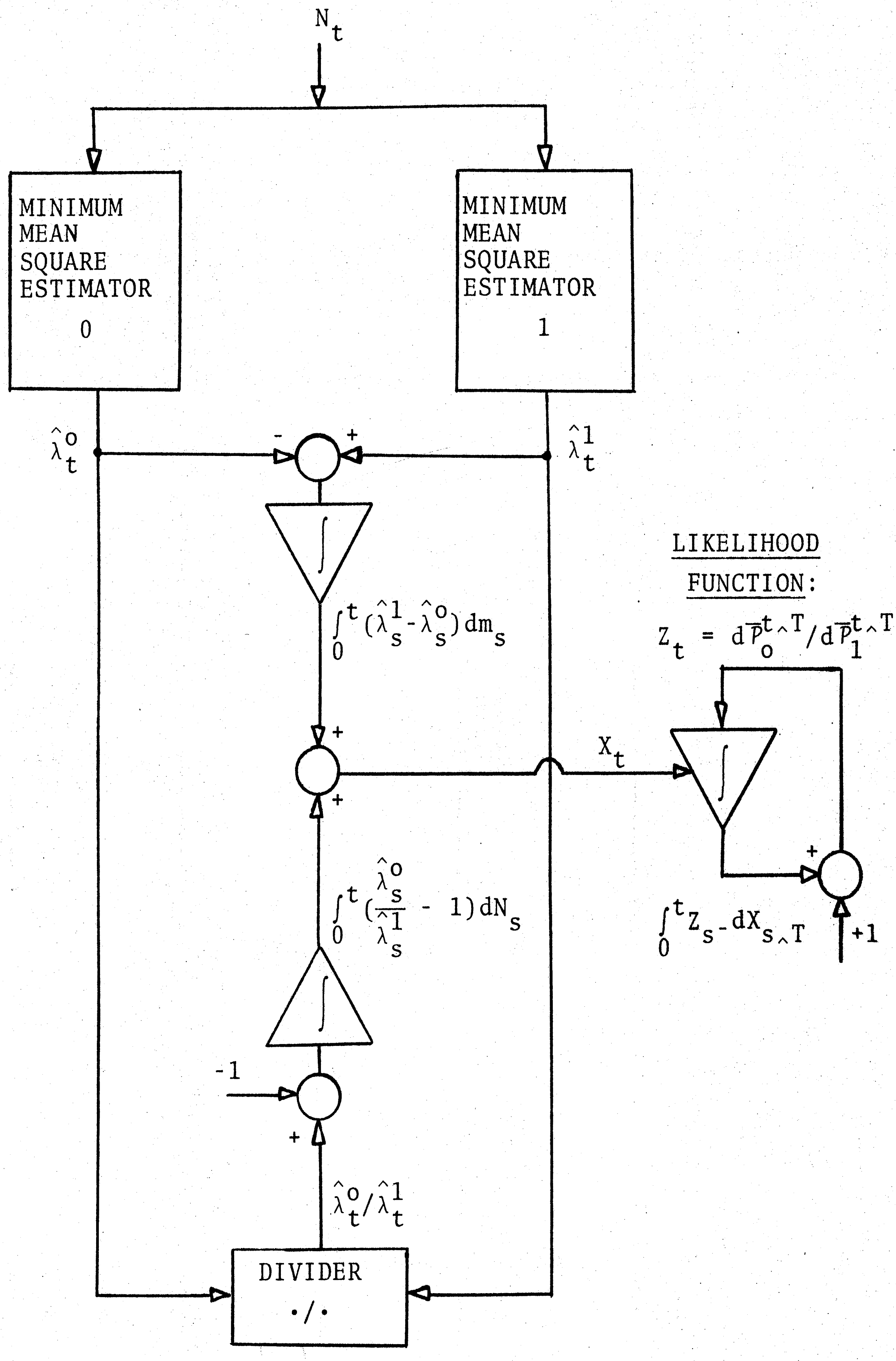

N_t
MINIMUM MEAN SQUARE ESTIMATOR 0
MINIMUM MEAN SQUARE ESTIMATOR 1
$\hat\lambda^o_t$
$\hat\lambda^1_t$
$\int_0^t (\hat\lambda^1_s - \hat\lambda^o_s) dm_s$
LIKELIHOOD FUNCTION:
$Z_t = d\bar{P}_o^{t\wedge T} / d\bar{P}_1^{t\wedge T}$
X_t
$\int_0^t Z_{s-} dX_{s\wedge T}$
+1
$\int_0^t (\frac{\hat\lambda^o_s}{\hat\lambda^1_s} - 1) dN_s$
-1
$\hat\lambda^o_t / \hat\lambda^1_t$
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Figure 3.4.6

CONCLUSION

We mention here--for future research--some of the problems which have not been solved in this dissertation.

In Chapter 2 (Chapter 1 is a mathematical review) Counting Processes (CP) and their Integrated Conditional Rates (ICR) were examined. It was shown that, given a CP (N_t) adapted to a right-continuous increasing family of σ-algebras $(\mathcal{F}_t)$ and for which--sole assumption--the random variable N_t is a.s finite for each t, there always exists a $(\mathcal{F}_t)$ ICR and this ICR is unique (Theorem 2.3.1 and Definition 2.3.2). Now given a natural increasing process (A_t) with respect to a family $(\mathcal{F}_t)$, there does not always exist a CP (N_t) adapted to $(\mathcal{F}_t)$ and for which (A_t) is the $(\mathcal{F}_t)$ ICR (e.g., take $(A_t = 2I_{[0,1)}(t))$, see Corollary 2.4.11). Hence the following problem:

(1) Find necessary and sufficient conditions for a natural increasing process (A_t) to be the ICR of a CP (N_t). Then the question arises:

(2) If (A_t) is a natural increasing process which is the ICR of a CP (N_t), when is this CP unique?

If the process (A_t) is continuous then the two above problems can be reformulated as (see Theorem 2.4.8):

(1') Find necessary and sufficient conditions for (A_t) to be the natural increasing process associated with a square integrable local martingale (M_t) (i.e., $A_t = \langle M\rangle_t$)

such that the process $(M_t + A_t)$ is a CP.

(2') Do there exist two distinct square integrable local martingales, (M_t) and (M_t^*), to which is associated the same natural increasing process $A_t = \langle M\rangle_t = \langle M^*\rangle_t$ and such that both $(M_t + A_t)$ and $(M_t^* + A_t)$ are CP's?

In Section 2.5 we give sufficient conditions for the existence of conditional rates. It is obviously desirable to find:

(3) Necessary conditions for the existence of conditional rates.

In Chapter 3 the Girsanov Theorem was used to prove the existence of CP's with (F_t) ICR's of the form $(\int_0^t \lambda_s dA_s)$ where the processes (λ_t) and (A_t) $((\lambda_t) \in H(F_t)$ is a nonnegative process and (A_t) is itself the ICR of a given CP) were such that the process (L_t) (see Theorem 3.1.1) was a martingale (which had to be uniformly integrable if one wanted to consider the positive real line instead of finite time intervals). Some strong conditions on (λ_t) and (A_t) which insured that (L_t) was a martingale were provided (Chapter 3, p. 133). Weaker conditions might be obtained. Hence:

(4) Find necessary and sufficient conditions on the processes (λ_t) and (A_t) for which the process (L_t) (see Theorem 3.1.1) is a martingale, or a uniformly integrable martingale.

The detection problem for a large class of CP's was solved. The generality of the method was limited mainly by the scope of the Martingale Representation Theorem (a basic step in proving the Likelihood Ratio Representation Theorem) which was demonstrated in the context of CP's of independent increments. Hence

(5) Find a larger class of CP's for which the Martingale Representation Theorem is still valid.

Finally, the likelihood ratio was obtained as a function of the best estimates $(\hat{\lambda}_t^i)$, i = 0,1 (Theorems 3.4.4 and 3.4.5). Thus to get a complete solution to the detection problem

(6) Recursive equations to compute $(\hat{\lambda}_t^i)$ should be obtained.

APPENDIX

APPENDIX A.1

The following lemma does not appear in our main reference [M1] but in [M3]. For ease of reference we provide here an original proof, due to Prof. F. J. Beutler, of this result.

Lemma 1.5.7: Let $(\mathcal{F}_n, n \in \mathbb{N})$ be an increasing family of σ-subalgebras of $\mathcal{F}$ and $\mathcal{F}_\infty$ be the σ-algebra generated by the union of the $\mathcal{F}_n$. Let $(F_n, n \in \mathbb{N})$ be a sequence of random variables bounded in absolute value by an integrable random variable G and converging a.s to a random variable F. Then $E(F_n|\mathcal{F}_n)$ converges a.s to $E(F|\mathcal{F}_\infty)$.

Proof of Lemma 1.5.7: Let $U_m = \inf_{n \geq m} F_n$ and $V_m = \sup_{n \geq m} F_n$. The sequences (U_m) and (V_m) are, respectively, increasing and decreasing, and both converge to F. We also have for $n \geq m$:

$$U_m \leq F_n \leq V_m$$

which implies

$$E(U_m|\mathcal{F}_n) \leq E(F_n|\mathcal{F}_n) \leq E(V_m|\mathcal{F}_n)$$

Fix m and let n tend to infinity. By the supermartingale convergence theorem (Theorem 6-VI of [M1]), the uniformly integrable martingale $E(U_m|\mathcal{F}_n)$ tends to $E(U_m|\mathcal{F}_\infty)$. Similarly for $E(V_m|\mathcal{F}_n)$. Thus we get the following chain of inequalities:

$$E(U_m|\mathcal{F}_\infty) \le \liminf_n E(F_n|\mathcal{F}_n) \le \limsup_n E(F_n|\mathcal{F}_n) \le E(V_m|\mathcal{F}_\infty)$$

Letting m go to infinity we have by the monotone convergence theorem:

$$E(F|\mathcal{F}_\infty) \le \liminf_n E(F_n|\mathcal{F}_n) \le \limsup_n E(F_n|\mathcal{F}_n) \le E(F|\mathcal{F}_\infty)$$

which implies the result. □

APPENDIX A.2

The proof of Theorem 1.7.9 will be clearer if we show first the following simple result:

Lemma A.2.1: Let $(\mathcal{F}_t)$ be a right-continuous increasing family of σ-algebras and T be a $(\mathcal{F}_t)$ stopping time. Suppose (X_t) is a $(\mathcal{F}_{t\wedge T})$ local martingale (resp. martingale). Then (X_t) is also a $(\mathcal{F}_t)$ local martingale (resp. martingale).

Proof: First we show that the lemma is true when (X_t) is a uniformly integrable $(\mathcal{F}_{t\wedge T})$ martingale. By Theorem 1.5.4 (X_t) can be expressed in this case as

$$X_t = E(X_T|\mathcal{F}_{t\wedge T})$$

Define now the uniformly integrable $(\mathcal{F}_t)$ martingale

$$Y_t = E(X_T|\mathcal{F}_t)$$

It suffices to show $Y_t = X_t$. In the first place

$$Y_{t\wedge T} = E(X_T | F_{t\wedge T}) = X_t$$

so $Y_t = X_t$ on the set $\{t < T\}$. Now $T \vee t$ is a stopping time (see Theorem 36-IV of [M1]) such that $T \vee t \geq T$. Hence

$$Y_{T\vee t} = E(X_T | F_{T\vee t}) = X_T$$

$$X_{T\vee t} = E(X_T | F_{(T\vee t)\wedge T}) = X_T$$

This shows $X_t = Y_t$ on the set $\{t \geq T\}$ and the lemma is verified for uniformly integrable martingale. Finally if (X_t) is a local $(F_{t\wedge T})$ martingale, let (T_n) be a sequence of stopping times reducing (X_t), i.e., $(X_{t\wedge T_n})$ is a uniformly integrable $(F_{t\wedge T_n\wedge T})$ martingale. Then the above shows that $(X_{t\wedge T_n})$ is also a uniformly integrable (F_t) martingale, i.e., (X_t) is a (F_t) local martingale. □

Note that the above result is a kind of converse to the Optional Sampling Theorem since this latter implies that if (Y_t) is a (F_t) martingale then $(Y_{t\wedge T})$ is a $(F_{t\wedge T})$ martingale.

We now go back to Theorem 1.7.9. This theorem appears in [M1] (Theorem 19-VII) but there is a gap in the proof which is supplemented by the above lemma.

<u>Theorem 1.7.9</u>: Let (A_t) be a natural increasing process and T be a stopping time. Then the increasing process $(A_{t\wedge T})$ is natural with respect to the two families

(F_t) and $(F_{t\wedge T})$.

Proof: If (Y_t) is a bounded positive (F_t) martingale then by the optional sampling theorem $(\overline{Y}_t \triangleq Y_{t\wedge T})$ is a $(F_{t\wedge T})$ martingale. By Lemma A.2.1 $(\overline{Y}_t)$ is also a (F_t) martingale so that by definition of a natural process (Definition 1.7.4)

$$E\int_0^t \overline{Y}_s dA_s = E\int_0^t \overline{Y}_{s-} dA_s$$

and proceeding as in [M1] (Theorem 19-VII) we get

$$E\int_0^{t\wedge T} Y_s dA_s = E\int_0^{t\wedge T} Y_{s-} dA_s$$

The LHS of the above expression can be rewritten as

$$E\int_0^{t\wedge T} Y_s dA_s = E\int_0^t Y_s dA_{s\wedge T}$$

and likewise for the RHS. Hence $(A_{t\wedge T})$ is natural with respect to (F_t). Finally since

$$E\int_0^t Y_s dA_{s\wedge T} = E\int_0^t Y_{s-} dA_{s\wedge T}$$

for any bounded (F_t) martingale, it holds in particular for bounded $(F_{t\wedge T})$ martingales, since the latter are also (F_t) martingales by Lemma A.2.1. This shows that $(A_{t\wedge T})$ is natural with respect to $(F_{t\wedge T})$ □

APPENDIX A.3

Lemma A.3.1: Let (N_t) be a CP with finite mean and $(\mathcal{N}_t)$ the family of σ-algebras generated by (N_t). Suppose $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ are two families of σ-algebras such that $\mathcal{F}_t \supset \mathcal{G}_t \supset \mathcal{N}_t$. Denote by (A_t) the $(\mathcal{F}_t)$ ICR of (N_t) and define the process

$$C_t \triangleq E(A_t | \mathcal{G}_t) \qquad \text{(A.3.1)}$$

Then the process $(N_t - C_t)$ is a $(\mathcal{G}_t)$ martingale.

Proof: We can write for $t \geq s$

$$\begin{aligned}
E(N_t - C_t | \mathcal{G}_s) &= E[N_t - E(A_t | \mathcal{G}_t) | \mathcal{G}_s] && \text{(by (A.3.1))} \\
&= E(N_t - A_t | \mathcal{G}_s) && (\mathcal{G}_t \supset \mathcal{G}_s) \\
&= E[E(N_t - A_t | \mathcal{F}_s) | \mathcal{G}_s] && (\mathcal{F}_s \supset \mathcal{G}_s) \\
&= E(N_s - A_s | \mathcal{G}_s) && \text{((}N_t\text{-}A_t\text{) is a (}\mathcal{F}_t\text{) martingale by Theorem 2.3.1)} \\
&= N_s - C_s && \text{(}N_s\text{ is (}\mathcal{G}_s\text{) measurable and by (A.3.1))}
\end{aligned}$$

which shows the result. □

APPENDIX A.4

Proposition A.4.1: Let (N_t) be a CP and J_1 its time of first jump. Let R be any $(\mathcal{N}_t)$ stopping time such that $R \leq J_1$ a.s. Then R is of the form

$$R = J_1 \wedge c \quad a.s$$

where c is some nonnegative constant.

<u>Corollary A.4.2</u>: Let J_1 be as in the above Proposition. Then J_1 is totally inaccessible with respect to the family (N_t) if and only if

$$P\{J_1 = a\} = 0$$

for any nonnegative constant a.

<u>Proof of Proposition A.4.1</u>: The σ-algebra N_0 is given by $\{\phi,\Omega\}$ so that either R = 0 a.s or R ≠ 0 a.s. In the case where R = 0 a.s the proposition is trivially verified so suppose that R ≠ 0 a.s and define the positive number $c = \sup\{e: P\{R > e\} > 0\}$. For any $0 < b < c$, the set $\{R > b\}$, which belongs to the σ-algebra N_b (see Theorem 41-IV of [M1]), is a set of positive measures and up to sets of measure zero $\{R > b\} \subset \{J_1 > b\}$ since $R \leq J_1$ a.s. Now the set $\{J_1 > b\} = \{N_b = 0\}$ is an atom (for a definition see [H2]) of N_b so that one must have modulo sets of measure zero $\{R > b\} = \{J_1 > b\}$ for any $0 < b < c$. This implies $R = J_1$ a.s on the set $\{R < c\}$, and proves the result in the case where c is infinite. When c is finite $P\{R > d\} = 0$ for any $d > c$ so that R = c a.s on the set $\{R \geq c\}$.

□

<u>Proof of Corollary A.4.2</u>: ($\Rightarrow$) By contradiction: if there exists an a such that $P\{J_1 = a\} = p > 0$ then let $(S_n = (a-1/n)\wedge J_1)$. We have $P\{\lim_n S_n = J_1,\ S_n < J_1\} \geq p > 0$ which shows that J_1 is not totally inaccessible, a contradiction.

($\Leftarrow$) By Proposition A.4.1 any increasing sequence of stopping times inferior or equal to J_1 a.s is of the form $S_n = J_1 \wedge a_n$ a.s where a_n is any increasing sequence of numbers. Let $a \triangleq \lim_n a_n$. Then

$$P\{\lim_n S_n = J_1,\ S_n < J_1\} \leq P\{J_1 = a\} = 0$$

that is: J_1 is totally inaccessible. □

<u>APPENDIX A.5</u>

We give here an example of a positive discrete martingale which is not uniformly integrable. Let $\Omega = [0,1)$ and P be the Lebesgue measure on $[0,1)$. Let $F_o = \{\phi,\Omega\}$ and F_n be the σ-algebra generated by the disjoint sets A^n_m, $m = 0,\ldots,3^n-1$ where

$$A^n_m = [m.3^{-n},(m+1).3^{-n})$$

Define

$$X_o = 1$$

and

$$X_n = 3^n I_{\{A^n_{(3^n-1)/2}\}}$$

We have

$$\int_{A_m^n} E(X_{n+1}|F_n)dP = \int_{A_m^n} X_{n+1}dP = \begin{cases} 3^{n+1}P\{A^{n+1}_{(3^{n+1}-1)/2}\} = 1 \\ \qquad \text{for } m = (3^n-1)/2 \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$\int_{A_m^n} X_n dP = \begin{cases} 1 & \text{if } m = (3^n-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

which implies $E(X_{n+1}|F_n) = X_n$, i.e., (X_n) is a positive martingale with $EX_n = 1$. By the supermartingale convergence theorem the limit $X_\infty = \lim_n X_n$ exists a.s. In this case we clearly have $X_\infty = 0$ a.s so that $X_n \neq E(X_\infty|F_n) = 0$ which shows, by Theorem 1.5.4, that (X_n) is not uniformly integrable.

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MESH HALFTONE WEDGES

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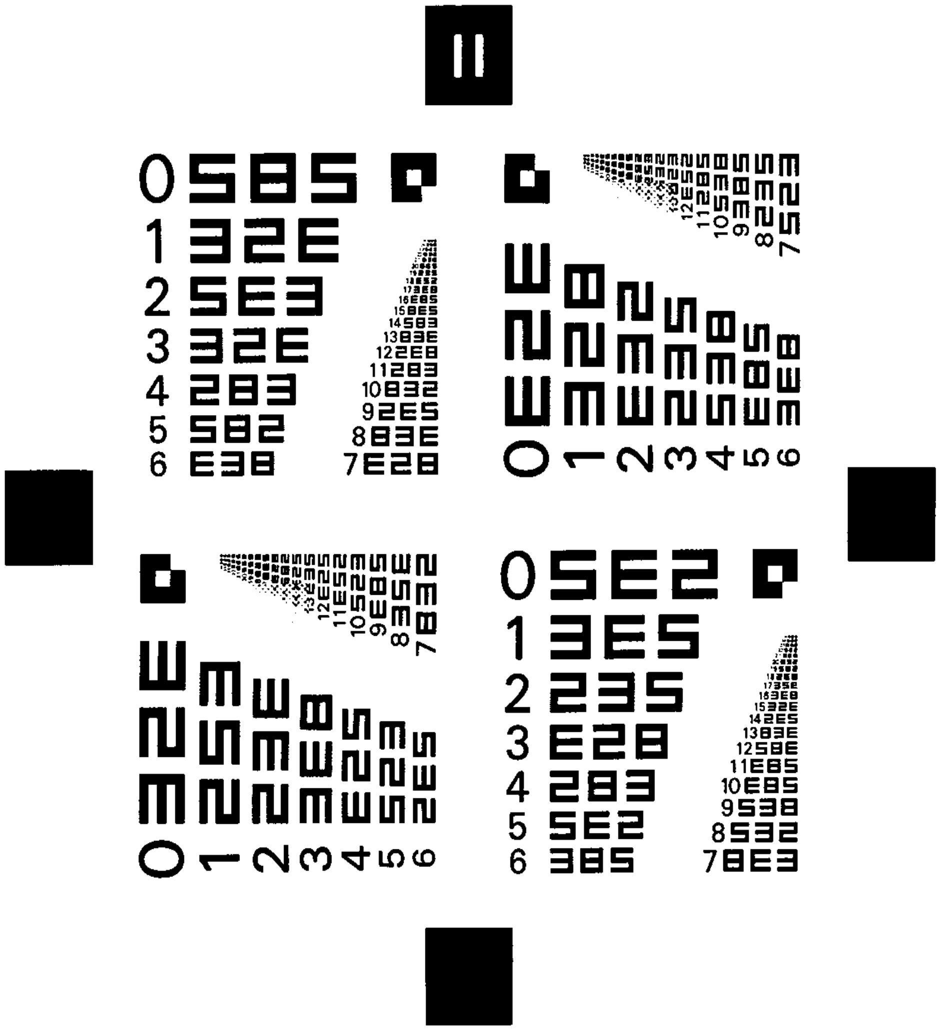

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