The Fundamental Solution for the Axially Symmetric Wave Equation

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I. Introduction

We propose to discuss here the relation between the standard free space functions of Green for the axially symmetric Helmholtz equation in three dimensions and two others which have appeared explicitly or implicitly in recent years [2, 6]. We have already stated the advantage of such alternate forms for axially symmetric potential theory [3] and first summarize the formulations which lead to them.

The Helmholtz equation in three dimensions

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+k^2\phi=0$$

has the fundamental solution $e^{ik\omega}/4\pi\omega$ where

$$\omega=\{(x-\zeta)^2+(y-\eta)^2+(z-\xi)^2\}^{1/2}.$$

This solution satisfies a radiation condition for k real; that is, it is asymptotic to an outward going spherical wave when $\sqrt{x^2+y^2+z^2}\to\infty$. If we now write this equation in cylindrical coordinates about the x axis by putting $y=g\cos\theta$ and $z=g\sin\theta$, $0\leqq\theta\leqq 2\pi$, we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial g^2}+\frac{1}{g}\frac{\partial\phi}{\partial g}+\frac{1}{g^2}\frac{\partial^2\phi}{\partial\theta^2}+k^2\phi=0.$$

Each Fourier coefficient of the fundamental solution satisfies the equation

$$\frac{\partial^2\phi_n}{\partial x^2}+\frac{\partial^2\phi_n}{\partial g^2}+\frac{1}{g}\frac{\partial\phi_n}{\partial g}-\frac{n^2}{g^2}\phi_n+k^2\phi_n=0$$

where $\phi_n(x,g)$ is either

$$\frac{1}{4\pi}\int_0^{2\pi}\cos n\theta\frac{e^{ik\omega}}{\omega}d\theta$$

or

$$\frac{1}{4\pi}\int_0^{2\pi}\sin n\theta \frac{e^{ik\omega}}{\omega}\,d\theta.$$

Upon introducing $\eta = h\cos\theta'$ and $\xi = h\sin\theta'$, the cosine term becomes

$$\frac{1}{4\pi}\int_0^{2\pi}\cos n\theta \frac{e^{ik\sqrt{(x-\zeta)^2+h^2+g^2-2hg\cos(\theta-\theta')}}}{\sqrt{(x-\zeta)^2+h^2+g^2-2hg\cos(\theta-\theta')}}\,d\theta.$$

This in turn may be simplified by writing

$$\cos n\theta = \cos n(\theta-\theta')\cos n\theta' - \sin n(\theta-\theta')\sin n\theta'$$

and observing that since n is an integer we may use symmetry to simplify the integral to

$$\frac{\cos n\theta'}{4\pi}\int_0^{2\pi}\cos n\psi \frac{e^{ik\sqrt{(x-\zeta)^2+h^2+g^2-2hg\cos\psi}}}{\sqrt{(x-\zeta)^2+h^2+g^2-2hg\cos\psi}}\,d\psi.$$

The sine term differs from the cosine term by the replacement of $\cos n\theta'$ by $\sin n\theta'$.

The substitution $\phi_n(x,g) = (hg)^n A_n(x,g)$* enables us to write the equation for ϕ_n as

$$\frac{\partial^2 A_n}{\partial x^2} + \frac{\partial^2 A_n}{\partial g^2} + \frac{(2n+1)}{g}\frac{\partial A_n}{\partial g} + k^2 A_n = 0$$

At the same time, an identity due to Jacobi** states that if $s = \cos\psi$, then

$$\frac{\cos(n \arccos s)}{\sqrt{1-s^2}} = \frac{(-)^n}{1.3\ldots(2n-1)}\frac{d^n(1-s^2)^{n-\frac{1}{2}}}{ds^n}, \qquad n \geqq 0.$$

Then with the substitution $s = \cos\psi$, we can write

$$A_n(x,g) = \frac{1}{4\pi(hg)^n}\int_0^{2\pi}\cos n\psi \frac{e^{ik\sqrt{(x-\zeta)^2+h^2+g^2-2hg\cos\psi}}}{\sqrt{(x-\zeta)^2+h^2+g^2=2gh\cos\psi}}\,d\psi,$$

and this may be transformed by n integration by parts to give us a first alternate form, for the Fourier coefficient ϕ_n, namely

$$A_n(x,g) = \frac{ik^{2n+1}}{2^{n+3}\Gamma(n+\frac{1}{2})}\int_0^{\pi}\sin^{2n}\psi\, H^{(1)}_{n+\frac{1}{2}}(k\omega)/(k\omega)^{n+\frac{1}{2}}\,d\psi,$$

* We have introduced the factor hg rather than g in order to simplify some subsequent expressions which will be derived.

** C.G.J. Jacobi, J. für Math. 15 (1836), 3. This and related results are discussed in "Aufgaben und Lehrsätze aus der Analysis" by G. Polyá and G. Szegö, Springer-Verlag, (1964), pp. 75 and 76.

where $\theta-\theta'$ has been replaced by ψ in the definition of ω. $H^{(1)}_{n+\frac{1}{2}}(k\omega)$ is a Hankel function of the first kind, of order $n+\frac{1}{2}$.

The factor $\sin^{2n}\psi$ suggests that it might be possible to reinterpret this first alternate form in a space of $2n+3$ dimensions by the following device. As we shall see in the next paragraph, there is a transformation from a Euclidean space of $(2n+3)$ dimensions to one which preserves one Euclidean coordinate and transforms the remaining ones to those of the "spherical type" – that is, to those which depend on a radial vector which is a function of the conventional magnitude of a vector and $2n+1$ angles which will be defined in the next paragraph (incidentally, we note that this radial vector is perpendicular to the one Euclidean coordinate and therefore this new coordinate system may be viewed as a "$2n+3$ dimensional cylindrical coordinate system"). The Jacobian of this transformation contains a factor which depends on a specific angle in the transformation, this factor being precisely the sine term which we have in our alternate form for ϕ_n. We shall therefore examine the Helmholtz equation in a space of $m=2n+3$ dimensions in terms of these new coordinates and seek angularly independent fundamental solutions. Once we have invoked the radiation condition and normalized its source strength, it will be possible to show that this fundamental solution is the first alternate form. *This procedure therefore relates the nth Fourier coefficient of the three dimensional fundamental solution with the angularly independent fundamental solution in a space of $2n+3$ dimensions.*

In an m dimensional Euclidean space we have the Helmholtz equation

$$\sum_{i=1}^{m}\frac{\partial^2\phi}{\partial x_i^2}+k^2\phi=0 \tag{1.1}$$

where k is the same constant which we used in the three dimensional case. We transform the Euclidean coordinates x_i, $i=1,\dots,m$, to "cylindrical coordinates" by the transformation

$$\begin{aligned}
x_1&=r\cos\theta_1\\
x_2&=r\sin\theta_1\cos\theta_2\\
&\;\dots\dots\dots\dots\dots\\
x_{m-2}&=r\sin\theta_1\dots\cos\theta_{m-2}\\
x_{m-1}&=r\sin\theta_1\dots\sin\theta_{m-2}\\
x_m&=x_m
\end{aligned}$$

where

$$r=\left\{\sum_{i=1}^{m-1}x_i^2\right\}^{1/2};\quad 0\leqq\theta_i\leqq\pi,\ i=1\dots m-3;\ 0\leqq\theta_{m-2}\leqq 2\pi;$$

and $m\geqq 3$.* Solutions of equation (1.1) which are independent of the angles θ_i will satisfy the partial differential equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{m-2}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial x_m^2}+k^2\phi=0 \tag{1.2}$$

* This transformation is not in the same form which was used in [3]. There are also some misprints in [3].

Clearly if $m=2n+3$, $r=g$, and $x_m=x$, the functions ϕ and A_n will satisfy the same differential equation. We shall refer to equation (1.2) as the axially symmetric wave equation in m dimensions.

It is now possible to give an obvious form of a free space Green's function (not unique) by recalling that the "spherically symmetric" form of equation (1.1) is

$$\frac{d^2\phi}{d\rho^2}+\frac{m-1}{\rho}\frac{d\phi}{d\rho}+k^2\phi=0 \tag{1.3}$$

where $\rho=\left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$. This ordinary differential equation has two linearly independent solutions

$$J_{\frac{m-2}{2}}(k\rho)/(k\rho)^{\frac{m-2}{2}} \quad \text{and} \quad Y_{\frac{m-2}{2}}(k\rho)/(k\rho)^{\frac{m-2}{2}} \tag{1.4}$$

where $J_{\frac{m-2}{2}}$ and $Y_{\frac{m-2}{2}}$ are the customary symbols for the Bessel function of the first and second kind respectively. For $m\geqq 3$, the first solution is regular in the neighborhood of $\rho=0$ and the second solution is $O(\rho^{2-m})$ in this neighborhood. Now since either of the solutions (1.4) satisfy equation (1.1), so do

$$J_{\frac{m-2}{2}}(k\rho_1)/(k\rho_1)^{\frac{m-2}{2}} \quad \text{and} \quad Y_{\frac{m-2}{2}}(k\rho_1)/(k\rho_1)^{\frac{m-2}{2}} \tag{1.4a}$$

where $\rho_1=\left(\sum_{i=1}^{m}(x_i-\xi_i)^2\right)^{1/2}$. Observe that while the functions in (1.4) are spherically symmetric about the origin, those in (1.4a) are not if a single $\xi_i\neq 0$. It is from these latter functions that we form the axially symmetric ones. First we fix the source point at $(\xi_1, 0, \ldots, \xi_m)$, so that ρ_1 then becomes

$$[(x_m-\xi_m)^2+r^2+\xi_1^2-2r\xi_1\cos\theta_1]^{1/2}$$

where θ_1 is the angle between the $m-1$ dimensional vectors $\bar{r}=(x_1,\ldots,x_{m-1})$ and $\bar{\xi}=(\xi_1,\ldots,0)$*. Note also that θ_1 is the first angle of our coordinate transformation.

How then do we extract axially symmetric fundamental solutions from the functions (1.4a)? We recall that (1.2) was written under the assumption that ϕ was independent of the angles θ_i. Had we retained the θ derivatives, this equation would have the form [7]

$$\frac{\partial^2\phi}{\partial r^2}+\frac{m-2}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial x_m^2}+\Lambda\phi+k^2\phi=0 \tag{1.2a}$$

where Λ is an operator defined by

$$\Lambda\phi=\frac{1}{t}\sum_{i=1}^{m-2}\frac{\partial}{\partial\theta_i}\frac{t}{t_i}\frac{\partial\phi}{\partial\theta_i}$$

* The choice of θ_1 in reference [3] is inadequate.

and is self-adjoint on the unit hypersphere, center at the origin, in a space of $m-1$ dimensions. Here $t_1=1$, $t_2=\sin^2\theta_1, \ldots, t_{m-2}=\sin^2\theta_1 \ldots \sin^2\theta_{m-3}$ and

$$t=\sin^{m-3}\theta_1 \sin^{m-4}\theta_2 \ldots \sin\theta_{m-3}.$$

The partial differential equation

$$\Delta Z+q(q+m-2)Z=0, \qquad q=0,\ldots,$$

which arises from the separation of variables of (1.2a) and invokes regularity conditions which gives the separation constant its special form, has solutions

$$Z_q(\theta_1,\ldots,\theta_{m-2}), \qquad q=0,1,\ldots$$

which form a complete and orthogonal set over the $m-1$ dimensional unit sphere. There is one Z_q, namely that corresponding to $q=0$ which is constant, independent of the angles θ_i. The set of functions Z_q is called the set of hyperspherical functions.* For $m=3$ these are the sets $\{\cos z\theta_1\}$ and $\{\sin z\theta_1\}$, $z=0,1,\ldots$ while for $m=4$ they are the regular spherical harmonic. In this sense, we are using a generalization of some elementary orthogonal functions.

Now it is known that since the functions (1.4a) are analytic in θ_1 if $x_m \neq \xi_m$ or $r\neq\xi_1$, we may expand them in a series of hyperspherical functions [7]. That is,

$$\frac{AY_{\frac{m-2}{2}}(k\rho_1)+BJ_{\frac{m-2}{2}}(k\rho_1)}{(k\rho_1)^{\frac{m-2}{2}}}=\sum_{p=0}^{\infty} A_p(x_m,\xi_m,r\ \xi_1)Z_p,$$

where Z_p are the hyperspherical functions which depend in general on $\theta_1,\ldots,\theta_{m-2}$: in particular Z_0 is constant. But since the Z_p's are orthogonal on the $m-2$ dimensional unit hypersphere,

$$\int_{\partial\Omega_{m-1}} Z_p Z_q d\Omega=0, \qquad q\neq p$$

where

$$d\Omega=\sin^{m-3}\theta_1 \ldots \sin\theta_{m-3}\, d\theta_1 \ldots d\theta_{m-2}.$$

The expression for $d\Omega$ is valid for $m\geqq 3$ and does not contain any negative exponents. The coefficients of Z_0 in the expansion is the axially symmetric solution, that is, the solution independent of the angles $\theta_1,\ldots,\theta_{m-2}$ and has the form

$$\int_{\partial\Omega_{m-1}} \frac{AY_{\frac{m-2}{2}}(k\rho_1)+BJ_{\frac{m-2}{2}}(k\rho_1)}{(k\rho_1)^{\frac{m-2}{2}}}\,d\Omega \tag{1.5}$$

* See [7] for a brief but usable account of these functions. A more complete account appears in "Fonctions Hypergéométrique et Hypersphériques", P. APPELL and J. KAMPÉ de Feriet (1926).

The normalizing constant of the Z_0 has been absorbed into the constants A and B. These constants will be determined by two specific requirements in the next paragraph.

For subsequent convenience, we shall normalize (1.5) as we do in classical potential theory by requiring that the "flow" from the point $(\xi_1, 0, \ldots 0, \xi_m)$ over a sphere be unity. That is,

$$\operatorname*{Lim}_{\rho_1 \to 0} \int_{\partial\Omega} A \frac{\partial}{\partial \rho_1} \left\{ Y_{\frac{m-2}{2}}(k\rho_1)/(k\rho_1)^{\frac{m-2}{2}} \right\} d\Omega = -1$$

and hence

$$A = -2^{-(m+2)/2}(k^2/\pi)^{(m-2)/2}.$$

The constant B remains arbitrary under this condition. We can determine it, however, by demanding that (1.5) be an "outward going spherical wave", that is, satisfy a radiation condition for $\operatorname{Im} k \geqq 0$, $\operatorname{Re} k > 0$. Since

$$J_\nu(k\rho_1) = \left(\frac{2}{\pi k \rho_1}\right)^{1/2} \cos\left(k\rho_1 - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(k\rho_1)^{-3/2}$$

and

$$Y_\nu(k\rho_1) = \left(\frac{2}{\pi k \rho_1}\right)^{1/2} \sin\left(k\rho_1 - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(k\rho_1)^{-3/2}$$

as $k\rho_1 \to \infty$, we may choose $B = -iA$ and the desired exponential behavior is obtained. (1.5) then becomes

$$\text{(1.5a)} \qquad \frac{ik^{m-2}}{2^{\frac{m}{2}+1}\pi^{\frac{m}{2}-1}} \int_{\partial\Omega_{m-1}} H^{(1)}_{\frac{m-2}{2}}(k\rho_1)/(k\rho_1)^{\frac{m-2}{2}} d\Omega$$

and this in turn can be reduced to

$$\text{(1.5b)} \qquad \frac{ik^{m-2}}{2^{\frac{m}{2}}\Gamma\left(\frac{m-2}{2}\right)} \int_0^\pi \sin^{m-3}\theta_1 H^{(1)}_{\frac{m-2}{2}}(k\rho_1)/(k\rho_1)^{\frac{m-2}{2}} d\theta,$$

which is again $A_n(x, g)$.

We shall now replace x_m, ξ_m, g, ξ_1 and θ_1 by x, a, r, b and θ respectively. Since (1.5b) is a solution of (1.2) provided m is merely greater than 2 (to insure the convergence of the integral in (1.5b) when $x \neq a$ and $r \neq b$), its relation to the coordinate transformation may be omitted. This provides us, in the terminology of A. Weinstein, a fundamental solution of equation (1.2) in a space of $m > 2$ dimensions where m is no longer an integer.

Henceforth we write $(m-2)/2 = \alpha > 0$ and have for (1.5b)

$$\text{(1.5c)} \qquad \frac{ik^{2\alpha}}{2^{\alpha+1}\Gamma(\alpha)} \int_0^\pi \sin^{2\alpha-1} H^{(1)}_\alpha(k\rho_1)/(k\rho_1)^\alpha d\theta.$$

We mention two useful properties of (1.5c). If $R=\sqrt{x^2+r^2}$ and $R\to\infty$, (1.5c) is asymptotic to

$$\frac{k^{\alpha-1/2}\,2^{\alpha-3/2}}{R^{\alpha+1/2}\sqrt{\pi}}\,e^{i(k\pi-\alpha\pi/2-\pi/4)}.$$

A second property deals with the behavior of (1.5c) in the neighborhood of the point $x=a$, $r=b$. Here we see from the properties of the Hankel function that (1.5c) is of the order

$$-\ln\sqrt{(x-a)^2+(r-b)^2}/2\pi\, r^{2\alpha}.$$

The logarithmic factor indicates that the Green's function in space of "$2\alpha+2$" dimensions enjoys an important property of the classical two dimensional one.

We shall next derive two forms which are alternatives to (1.5c). That is, we shall show that we may write (1.5c) as

$$\text{(1.5d)}\qquad \frac{ikr^{1-\alpha}}{4}\int_0^\pi \sin^\alpha\theta J_{\alpha-1}(kr\sin\theta)\,\frac{H_\alpha^{(1)}\big[k\sqrt{(x-a+ir\cos\theta)^2+b^2}\big]\,d\theta}{[(x-a+ir\cos\theta)^2+b^2]^{\alpha/2}}$$

$b>r$ or

$$\text{(1.5e)}\qquad \frac{ikb^{1-\alpha}}{4}\int_0^\pi \sin^\alpha\theta J_{\alpha-1}(kb\sin\theta)\,\frac{H_\alpha^{(1)}\big[k\sqrt{(x-a+ib\cos\theta)^2+r^2}\big]\,d\theta}{[(x-a+ib\cos\theta)^2+r^2]^{\alpha/2}}$$

$r>b$, where we have now replaced x_m by x and ξ_m by a. These in turn, may be cast into the complex form used by VEKUA [7] in the regular case ($m=2$). They may be described as representations which are analogous to the one Riemann found in the study of the corresponding hyperbolic partial differential equation, although in this case the equation is singular. As a byproduct of this development, we will derive the case $m=2$ via the correspondence principle. Finally, we remark that there is something to be gained in keeping the Green's function in evidence, since it does point to some mathematical restrictions which we shall discuss in Section V [3].

II. The Derivation of the Singular Parts of the Identities (1.5d) and (1.5e)

In order to derive the identities (1.5d) and (1.5e), we shall make use of two identities which were derived in [3]. There we observed that for $x\neq a$, $r\neq b$, the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial r^2}+\frac{2\alpha}{r}\frac{\partial\phi}{\partial r}=0\qquad(\alpha>0)$$

had a fundamental solution with the following alternative, but equivalent, forms:

(2.1)
$$\frac{1}{2\pi}\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{[(x-a)^2+r^2+b^2-2rb\cos\theta]^\alpha}$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\sin^{2\alpha-1}\theta d\theta}{[(x-a+ir\cos\theta)^2+b^2]^\alpha}$$

(2.1b)
$$\frac{1}{2\pi}\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{[(x-a-ib\cos\theta)^2+r^2]^\alpha}.$$

It is from these that we can build the fundamental solution of equation (1.2) for the case α is not an even positive integer. If α is an even positive integer, there are terms with logarithmic character which appear in the evaluation of (1.2), (1.2a) and (1.2b). In order to avoid the complications which such terms will present to us, we have made the assumption that α is not positive and even and show later that we can obtain the fundamental solution of (1.2) for $\alpha=2n$, $n=1,\ldots$ as a limiting case. The case $\alpha=0$ can be handled by means of the correspondence principle [6].

We recall that (see [5])

$$H_\alpha^{(1)}(k\rho_1)=\frac{iJ_\alpha(k\rho_1)\,e^{-i\alpha\pi}-iJ_{-\alpha}(k\rho_1)}{\sin\alpha\pi},$$

so that the singular part of the fundamental solution of (1.5d) is

(2.2)
$$\frac{k^\alpha}{2^{\alpha+1}\,\Gamma(\alpha)\sin\alpha\pi}\int_0^\pi \sin^{2\alpha-1}\theta\,\frac{J_{-\alpha}(k\rho_1)}{\rho_1^\alpha}\,d\theta.$$

Now

$$J_{-\alpha}(k\rho_1)=\sum_{\sigma=0}^\infty \frac{(-)^\sigma}{\sigma!\,\Gamma(\sigma+1-\alpha)}\left(\frac{k\rho_1}{2}\right)^{2\sigma-\alpha}$$

and hence (2.2) becomes

(2.3)
$$\frac{1}{2\Gamma(\alpha)\sin\alpha\pi}\sum_{\sigma=0}^\infty \frac{(-)^\sigma}{\sigma!\,\Gamma(\sigma+1-\alpha)}\left(\frac{k}{2}\right)^{2\sigma}\int_0^\pi \rho_1^{2\sigma-2\alpha}\sin^{2\alpha-1}\theta\,d\theta.$$

We shall now show that (2.3) can be written as an integral of the product of two Bessel functions, one of which is of the order $\alpha-1$ and argument $kr\sin\theta$, while the other is of the order $-\alpha$ and argument $k[(x-a+ir\cos\theta)^2+b^2]^{1/2}$. There is a second integral of the same form with r and b interchanged.

We shall first examine the integral for $\sigma=0$ in (2.3) and observe that it may be replaced by (2.1a) and (2.1b). We shall only work with (2.1a) since (2.1b) is found from (2.1a) by interchanging r and b.[Since replacing (2.1) by (2.1a) does not involve any change in the quantity $x-a$, we shall omit the quantity a in this part of our work].

We next examine that second term

$$\int_0^\pi \rho_1^{2-2\alpha}\sin^{2\alpha-1}\theta\,d\theta$$

and observe that this may be obtained from the term $\sigma=0$ by an integration over the variable x with the additional weight factor x. To make use of this idea, we begin by considering the identity

$$\begin{aligned}\int_0^x t\,dt\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{[t^2+r^2+b^2-2rb\cos\theta]^\alpha}\\ =\int_0^x t\,dt\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{[(t+ir\cos\theta)^2+b^2]^\alpha}.\end{aligned}\tag{2.4}$$

We may write the right side in the form

$$\begin{aligned}&\int_0^\pi \sin^{2\alpha-1}\theta\,d\theta\int_0^x \frac{(t+ir\cos\theta-ir\cos\theta)\,dt}{[(t+ir\cos\theta)^2+b^2]^\alpha}\\ &=\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{2(1-\alpha)}\{[x+ir\cos\theta)^2+b^2]^{1-\alpha}-(b^2-r^2\cos^2\theta)^{1-\alpha}\}\\ &\quad -ir\int_0^x dt\int_0^\pi \frac{\sin^{2\alpha-1}\theta\cos\theta\,d\theta}{[(t+ir\cos\theta)^2+b^2]^\alpha}.\end{aligned}$$

Now since $\alpha>0$, we may integrate the inner integral by parts to show that the right side of (2.4) is, for $b>r$

$$\begin{aligned}&\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{2(1-\alpha)}\{[(x+ir\cos\theta)^2+b^2]^{1-\alpha}-(b^2-r^2\cos^2\theta)^{1-\alpha}\}\\ &\quad+\int_0^\pi \frac{r^2\sin^{2\alpha+1}\theta\,d\theta}{2\alpha}\{[(x+ir\cos\theta)^2+b^2]^{-\alpha}-(b^2-r^2\cos^2\theta)^{-\alpha}\}.\end{aligned}$$

But by direct calculation the left integral of (2.4) also equals

$$\int_0^\pi \frac{\sin^{2\alpha-1}\theta\,d\theta}{2(1-\alpha)}\{(x^2+r^2+b^2-2rb\cos\theta)^{1-\alpha}-(r^2+b^2-2rb\cos\theta)^{1-\alpha}\},$$

the first part of this being of course essentially the desired integral.

Now we shall show that the terms independent of x in the last two displays cancel; that is we shall derive the identity,

$$\begin{aligned}&\int_0^\pi \frac{\sin^{2\alpha-1}\theta}{2(1-\alpha)}(r^2+b^2-2rb\cos\theta)^{1-\alpha}\,d\theta\\ &=\int_0^\pi \frac{\sin^{2\alpha-1}\theta}{2(1-\alpha)}(b^2-r^2\cos^2\theta)^{1-\alpha}\,d\theta\\ &\quad+\int_0^\pi r^2\frac{\sin^{2\alpha+1}\theta}{2\alpha}(b^2-r^2\cos^2\theta)^{-\alpha}\,d\theta,\qquad r^2/b^2<1.\end{aligned}\tag{2.5}$$

To this end, we note that if we put $\lambda=r^2/b^2$, the right side of (2.5) becomes

$$b^{2-2\alpha}\int_0^\pi \sin^{2\alpha-1}\theta\left[\frac{(1-\lambda\cos^2\theta)^{1-\alpha}}{1-\alpha}+\frac{\lambda(1-\lambda\cos^2\theta)^{-\alpha}}{\alpha}\sin^2\theta\right]d\theta.$$

The substitution $\cos^2\theta = t$ reduces this last integral* to

$$\frac{b^{2-2\alpha}\Gamma(1/2)\Gamma(\alpha)}{\Gamma(\alpha+1/2)}\left[\frac{F(\alpha-1, 1/2, \alpha+1/2, \lambda)}{1-\alpha}+\frac{\lambda F(\alpha, 1/2, \alpha+3/2, \lambda)}{\alpha+1/2}\right]$$

where F is the hypergeometric function. This in turn simplifies to

$$\frac{b^{2-2\alpha}\Gamma(1/2)\Gamma(\alpha)}{(1-\alpha)\Gamma(\alpha+1/2)}F(\alpha-1, -1/2, \alpha+1/2, \lambda), \quad \lambda<1. \tag{2.6}$$

If we now examine the integral

$$\int_0^\pi \frac{\sin^{2\alpha-1}\theta(r^2+b^2-2rb\cos\theta)^{1-\alpha}}{2(1-\alpha)}\,d\theta$$

we find that it may be transformed into

$$\frac{(r+b)^{2-2\alpha}2^{2\alpha-1}}{(1-\alpha)}\int_0^1[t(1-t)]^{\alpha-1}\left[1-\frac{4rbt}{(r+b)^2}\right]^{1-\alpha}dt \tag{2.7}$$

upon using the substitution $t=\cos^2\frac{\theta}{2}$. The quadratic transformation of GOURSAT [1] transforms (2.7) into (2.6) and we have therefore derived (2.5). This being done, our earlier calculation then yields the result

$$\begin{aligned}\int_0^\pi &\frac{\sin^{2\alpha-1}\theta}{2(1-\alpha)}(x^2+r^2+b^2-2rb\cos\theta)^{1-\alpha}\,d\theta\\ &=\int_0^\pi\frac{\sin^{2\alpha-1}\theta}{2(1-\alpha)}[(x+ir\cos\theta)^2+b^2]^{1-\alpha}\,d\theta\\ &\quad+\int_0^\pi\frac{r^2\sin^{2\alpha+1}\theta}{2\alpha}[(x+ir\cos\theta)^2+b^2]^{-\alpha}\,d\theta, \qquad r/b<1.\end{aligned} \tag{2.8}$$

This is our required expression for the second term ($\sigma=1$) in (2.3).

The derivation of the identity (2.8) indicates how we may derive the following identity, which provides us with the conversion for a general term. Let $\rho_1=(x^2+r^2+b^2-2rb\cos\theta)^{1/2}$ and $\rho_2=[(x+ir\cos\theta)^2+b^2]^{1/2}$. We claim that

$$\begin{aligned}\int_0^\pi&\frac{\rho_1^{2n-2\alpha}\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(n+1-\alpha)}\\ &=\Gamma(\alpha)\,n!\int_0^\pi\sum_{j=0}^n\frac{r^{2j}\rho_2^{2n-2j-2\alpha}\sin^{2\alpha+2j-1}\theta\,d\theta}{(n-j)!\,j!\,\Gamma(\alpha+j)\,\Gamma(n+1-j-\alpha)}, \qquad \lambda<1.\end{aligned} \tag{2.9}$$

We have of course already proved (2.9) for $n=0$ and 1, and an induction argument will now show that it is true for any positive integer.

We proceed as we did in the case $n=1$ by multiplying (2.9) by x and integrating with respect to x from 0 to x. We will find that n is replaced by $n+1$ in (2.9) and there are terms from the lower limit of integration just as in the case

* L.V. AHLFORS *Complex Analysis*, McGraw-Hill, (1966), p. 309.

$n=1$. These latter terms are independent of x and we will demonstrate that the one on the left side is equal to the sum on the right side. This will imply, as in the case $n=1$, that the term dependent on x on the left side is equal to the sum of those dependent on x on the right side.

We therefore have to show that

$$\text{(2.10)}\qquad \int_0^\pi \frac{(r^2+b^2-2rb\cos\theta)^{n-\alpha}\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(n+1-\alpha)} = \Gamma(\alpha)\,n!\sum_{j=0}^{n}\int_0^\pi \frac{r^{2j}(b^2-r^2\cos^2\theta)^{n-j-\alpha}\sin^{2\alpha+2j-1}\theta\,d\theta}{(n-j)!\,j!\,\Gamma(\alpha+j)\,\Gamma(n+1-j-\alpha)}, \qquad \lambda<1.$$

The left side of (2.10) may be converted into a term whose dependence on λ is given by the quadratic transformation of Goursat; we get in this way

$$\frac{b^{2n-2}\,\Gamma(\alpha)\,\Gamma(1/2)}{\Gamma(1+n-\alpha)\,\Gamma(\alpha+1/2)}\,F(\alpha-n,1/2-n,\alpha+1/2,\lambda), \qquad \lambda<1.$$

Now we shall examine the right side of (2.10). With the substitutions used for the case $n=1$, we get

$$\Gamma(\alpha)\,n!\,b^{2n-2\alpha}\sum_{j=0}^{n}\int_0^1 \frac{\lambda^j(1-\lambda t)^{n-j-\alpha}(1-t)^{\alpha+j-1}\,t^{-1/2}\,dt}{(n-j)!\,j!\,\Gamma(\alpha+j)\,\Gamma(n+1-j-\alpha)},$$

$$\Gamma(1/2)\,\Gamma(\alpha)\,n!\,b^{2n-2\alpha}\sum_{j=0}^{n}\frac{\lambda^j F(\alpha+j-n,1/2,\alpha+j+1/2,\lambda)}{(n-j)!\,j!\,\Gamma(n+1-j-\alpha)\,\Gamma(\alpha+j+1/2)},$$

where F is the hypergeometric function.* Upon using the power series expansion of the hypergeometric function about $\lambda=0$, we get

$$\frac{b^{2n-2\alpha}\,\Gamma(\alpha)\,n!\,\sin\pi\alpha}{\pi}\sum_{j=0}^{n}\sum_{\sigma=0}^{\infty}\frac{\Gamma(\alpha+j+\sigma-n)\,\Gamma(1/2+\sigma)(-)^{j+n}\lambda^{j+\sigma}}{\Gamma(\alpha+1/2+j+\sigma)(n-j)!\,j!\,\sigma!}.$$

The double sum can be written as

$$\sum_{j=0}^{n}\sum_{\sigma=j}^{\infty}\frac{\Gamma(\alpha-n+\sigma)\,\Gamma(1/2+\sigma-j)(-)^{j+n}\lambda^{\sigma}}{\Gamma(\alpha+\sigma+1/2)(n-j)!\,j!\,(\sigma-j)!}$$

or

$$\text{(2.11)}\qquad \begin{aligned}&\sum_{j=0}^{n}\sum_{\sigma=j}^{n}\frac{\Gamma(\alpha-n+\sigma)\,\Gamma(1/2+\sigma-j)(-)^{j+n}\lambda^{\sigma}}{\Gamma(\alpha+\sigma+1/2)(n-j)!\,(\sigma-j)!\,j!}\\&\quad+\sum_{j=0}^{n}\sum_{\sigma=n+1}^{\infty}\frac{\Gamma(\alpha-n+\sigma)\,\Gamma(1/2+\sigma-j)(-)^{j+n}\lambda^{\sigma}}{\Gamma(\alpha+\sigma+1/2)(n-j)!\,j!\,(\sigma-j)!}\\&=\sum_{\sigma=0}^{n}\frac{\lambda^\sigma\,\Gamma(\alpha-n+\sigma)}{\Gamma(\alpha+\sigma+1/2)}\sum_{j=0}^{\sigma}\frac{\Gamma(1/2+\sigma-j)(-)^{j+n}}{(n-j)!\,j!\,(\sigma-j)!}\\&\quad+\sum_{\sigma=n+1}^{\infty}\frac{\lambda^\sigma\,\Gamma(\alpha-n+\sigma)}{\Gamma(\alpha+\sigma+1/2)}\sum_{j=0}^{n}\frac{\Gamma(1/2+\sigma-j)(-)^{j+n}}{(n-j)!\,j!\,(\sigma-j)!}.\end{aligned}$$

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* AHLFORS, loc. cit.

We shall now show that the finite sums in (2.11) can be evaluated and therefore verify the identity in (2.10). The second sum

$$\sum_{j=0}^{n} \frac{\Gamma(1/2+\sigma-j)(-)^j}{(n-j)!\,j!\,(\sigma-j)!} = g(\sigma, n) \qquad \sigma \geqq n+1$$

can be evaluated by the following elementary method. We multiply this sum by $\sigma - n + 1/2$ and obtain

$$\sum_{j=0}^{n} \frac{\Gamma(3/2+\sigma-j)(-)^j}{(n-j)!\,j!\,(\sigma-j)!} - \sum_{j=0}^{n-1} \frac{\Gamma(1/2+\sigma-j)(-)^j}{(n-j-1)!\,j!\,(\sigma-j)!} = (\sigma-n+1/2)\,g(\sigma, n).$$

The upper index in the second sum is now $n-1$ because the term $j=n$ vanishes. Upon shifting the index in the second sum and combining it with the first term we get

$$\sum_{j=0}^{n} \frac{\Gamma(\sigma-j+3/2)(-)^j}{(n-j)!\,j!\,(\sigma+1-j)} = \frac{(\sigma-n+1/2)\,g(\sigma, n)}{\sigma+1} = g(\sigma+1, n)$$

From this we can derive immediately

$$g(\sigma, n) = \frac{\Gamma(1/2-n+\sigma)}{\Gamma(1/2-n)\,\sigma!}\,g(0, n)$$

and since

$$g(0, n) = \Gamma(1/2)/n!$$

we have finally

$$g(\sigma, n) = \frac{\Gamma(1/2-n+\sigma)\,\Gamma(1/2)}{\Gamma(1/2-n)\,\sigma!\,n!}.$$

For the first sum

$$\sum_{j=0}^{\sigma} \frac{\Gamma(1/2+\sigma-j)(-)^j}{(n-j)!\,j!\,(\sigma-j)!} \qquad \sigma \leqq n$$

we note that it can be written as

$$\sum_{j=0}^{n} \frac{\Gamma(1/2+\sigma-j)(-)^j}{(n-j)!\,j!\,(\sigma-j)!}$$

since all terms from $j=\sigma+1$ to $j=n$ vanish and therefore this sum is still $g(\sigma, n)$. It follows therefore that if (2.11) is written in the notation of the hypergeometric function, we get

$$\frac{\pi}{\sin \pi\alpha}\,\frac{\Gamma(1/2)}{\Gamma(\alpha+1/2)\,\Gamma(1-\alpha+n)\,n!}\,F(\alpha-n, 1/2-n, \alpha+1/2, \lambda).$$

Upon inserting the factor $\dfrac{b^{2n-2\alpha}\,\Gamma(\alpha)\,n!\,\sin \pi\alpha}{\pi}$ preceding the double sum, we see that the right and left sides agree for $r/b<1$.

With the derivation of (2.10) we have proved (2.9) and we can now express (2.3) as

$$\frac{1}{2\sin\pi\alpha}\int_0^\pi d\theta\sum_{\sigma=0}^{\infty}(-)^\sigma\left(\frac{k}{2}\right)^{2\sigma}\sum_{j=0}^{\sigma}\frac{r^{2j}\rho_2^{2\sigma-2j-2\alpha}\sin^{2\alpha+2j-1}\theta}{(\sigma-j)!\,j!\,\Gamma(\alpha+j)!\,\Gamma(\sigma+1-\alpha-j)}.$$

But this may be rewritten as

$$\frac{1}{2\sin\pi\alpha}\int_0^\pi d\theta\sum_{j=0}^{\infty}\sum_{\sigma=j}^{\infty}\frac{(-)^\sigma\left(\frac{k}{2}\right)^{2\sigma}r^{2j}\rho_2^{2\sigma-2j-2\alpha}\sin^{2\alpha+2j-1}\theta}{(\sigma-j)!\,j!\,\Gamma(\alpha+j)\,\Gamma(\sigma+1-\alpha-j)}$$

from which we recognize, after putting $\sigma-j=\sigma'$ that this may be rewritten as

$$\frac{1}{2\sin\pi\alpha}\int_0^\pi d\theta\sum_{j=0}^{\infty}\frac{(-)^j\left(\frac{k}{2}\right)^{2j}r^{2j}\sin^{2\alpha+2j-1}\theta}{j!\,\Gamma(\alpha+j)}\sum_{\sigma'=0}^{\infty}\frac{\left(\frac{k}{2}\right)^{2\sigma'}(-)^{\sigma'}\rho_2^{2\sigma'-2\alpha}}{\sigma'!\,\Gamma(\sigma'+1-\alpha)}.$$

The infinite series may be identified as Bessel functions of order $\alpha-1$ and $-\alpha$ respectively and we finally have

$$\text{(2.12)}\qquad \frac{k}{4\sin\alpha\pi}\int_0^\pi J_{\alpha-1}(kr\sin\theta)\,J_{-\alpha}(k\rho_2)\,r^{1-\alpha}\rho_2^{-\alpha}\sin^\alpha\theta\,d\theta$$
$$=\frac{k^\alpha}{2^{\alpha+1}\sin\alpha\pi}\int_0^\pi\sin^{2\alpha-1}\theta\,J_{-\alpha}(k\rho_1)\,\rho_1^{-\alpha}\,d\theta$$

$\alpha>0$. Since we have derived this subject to the restriction $r<b$, we also have the identity

$$\text{(2.12a)}\qquad \frac{k}{4\sin\alpha\pi}\int_0^\pi J_{\alpha-1}(kr\sin\theta)\,\frac{J_{-\alpha}(k\sqrt{b^2-r^2\cos^2\theta})\,r^{1-\alpha}\sin^\alpha\theta\,d\theta}{(b^2-r^2\cos^2\theta)^{\alpha/2}}$$
$$=\frac{k^\alpha}{2^{\alpha+1}\sin\alpha\pi}\int_0^\pi\sin^{2\alpha-1}\theta\,\frac{J_{-\alpha}(k\sqrt{r^2+b^2-2rb\cos\theta})\,d\theta}{(r^2+b^2-2rb\cos\theta)^{\alpha/2}}.$$

We also note that if we start with (2.1b) and replace ρ_2 by $\rho_3=[x+ib\cos\theta)^2+r^2]^{1/2}$, an identity similar to (2.12) is derived with r and b interchanged. For $r>b$, (2.12a) merely involves the interchange of r and b.

III. The Derivation of the Regular Part of the Identities (1.5d) and (1.5e)

The regular part of (1.5c) is

$$\frac{-k^\alpha e^{-i\alpha\pi}}{2^{\alpha+1}\Gamma(\alpha)\sin\alpha\pi}\int_0^\pi\sin^{2\alpha-1}\theta\,J_\alpha(k\rho_1)\,\rho_1^{-\alpha}\,d\theta$$

which, in view of the power series expansion of the Bessel function, can be written as

$$(3.1)\qquad \frac{-k^{2\alpha}e^{-i\alpha\pi}}{2^{2\alpha+1}\Gamma(\alpha)\sin\alpha\pi}\sum_{\sigma=0}^{\infty}\frac{(-)^{\sigma}}{\sigma!\,\Gamma(\sigma+1+\alpha)}\left(\frac{k}{2}\right)^{2\sigma}\int_0^{\pi}\rho_1^{2\sigma}\sin^{2\alpha-1}\theta\,d\theta.$$

For the time being we shall omit the numerical constants before the summation sign. We note that $\rho_1^{2\sigma}$ is a polynomial in x, r, b and $\cos\theta$ and furthermore, it is simple to verify that

$$(3.2)\qquad \int_0^{\pi}\frac{\rho_1^2\sin^{2\alpha-1}\theta\,d\theta}{\alpha+1}=\int_0^{\pi}\frac{\rho_2^2\sin^{2\alpha-1}\theta\,d\theta}{\alpha+1}+\frac{r^2}{\alpha}\int_0^{\pi}\sin^{2\alpha+1}\theta\,d\theta.$$

On the surface it appears that (3.2) may be used to derive alternate forms of the integrals in (3.1). Unfortunately, the situation is somewhat different from that in Section II, although much of the machinery carries over. If we multiply equation by x and integrate from 0 to x, we get after some simplification

$$(3.3)\qquad \begin{aligned}\frac{1}{4}\int_0^{\pi}\frac{\rho_1^4\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+2)}&=\frac{1}{4}\int_0^{\pi}\frac{\rho_2^2\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\\&+\frac{(\alpha+2)\,r^2}{2}\int_0^{\pi}\frac{\rho_2^2\sin^{2\alpha+1}\theta\,d\theta}{\Gamma(\alpha+1)\,\Gamma(\alpha+2)}+\frac{1}{4}\int_0^{\pi}\frac{(r^2+b^2-2rb\cos\theta)^2\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\\&-\frac{1}{4}\int_0^{\pi}\frac{(b^2-r^2\cos^2\theta)^2\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+2)}-\frac{r^2(\alpha+2)}{2}\int_0^{\pi}\frac{(b^2-r^2\cos^2\theta)\sin^{2\alpha+1}\theta\,d\theta}{\Gamma(\alpha+1)\,\Gamma(\alpha+2)}\end{aligned}$$

Now the last three terms on the right side of this equation simplify to

$$\frac{(\alpha+2)\,r^4}{4\Gamma(\alpha+1)\,\Gamma(\alpha+2)}\int_0^{\pi}\sin^{2\alpha+3}\theta\,d\theta$$

so that equation (3.3) becomes

$$(3.3a)\qquad \begin{aligned}\frac{1}{4}\int_0^{\pi}\frac{\rho_1^4\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+3)}&=\frac{1}{4}\int_0^{\pi}\frac{\rho_2^4\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\,\Gamma(\alpha+3)}+\frac{r^2}{2}\int_0^{\pi}\frac{\rho_2^2\sin^{2\alpha+1}\theta\,d\theta}{\Gamma(\alpha+1)\,\Gamma(\alpha+2)}\\&+\frac{r^4}{4\Gamma(\alpha+1)\,\Gamma(\alpha+2)}\int_0^{\pi}\sin^{2\alpha+3}\theta\,d\theta.\end{aligned}$$

This situation contrasts the results in Section II where the terms from the lower limit of integration combined to vanish.

Having been guided by this special case, we can now derive the following identity which we will need in our subsequent integrations. We claim that

$$(3.4)\qquad \begin{aligned}&\frac{1}{n!}\int_0^{\pi}\frac{(r^2+b^2-2rb\cos\theta)^n\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\Gamma(\alpha+n+1)}\\&\qquad=\sum_{j=0}^{n}r^{2j}\int_0^{\pi}\frac{(b^2-r^2\cos^2\theta)^{n-j}\sin^{2\alpha+2j-1}\theta\,d\theta}{j!\,(n-j)!\,\Gamma(\alpha+j)\,\Gamma(\alpha+1+n-j)}.\end{aligned}$$

In order to demonstrate this relation, we simply expand $(b^2 - r^2 \cos^2 \theta)^{n-j}$ and evaluate the integrals. After some minor simplification we will obtain the left side of (3.4). Indeed we have from the right side of (3.4) that

$$\sum_{j=0}^{n} r^{2j} \int_0^{\pi} \sum_{\sigma=0}^{n-j} \frac{(-)^\sigma b^{2n-2j-2\sigma} r^{2\sigma} \cos^{2\sigma}\theta \sin^{2\alpha+2j-1}\theta \, d\theta}{\sigma!\, j!\, (n-j-\sigma)!\, \Gamma(\alpha+j)\, \Gamma(\alpha+1+n-j)}$$

$$= \sum_{j=0}^{n} \sum_{\sigma=0}^{n-j} \frac{r^{2j+2\sigma} (-)^\sigma b^{2n-2j-2\sigma} \Gamma(\sigma+1/2)}{\sigma!\, j!\, (n-j-\sigma)!\, \Gamma(\alpha+1+n-j)\, \Gamma(\alpha+j+\sigma+1/2)}$$

and upon replacing $j+\sigma$ by γ we get

$$\sum_{j=0}^{n} \sum_{\gamma=j}^{n} \frac{r^{2\gamma} (-)^{\gamma+j} b^{2n-2\gamma} \Gamma(\gamma-j+1/2)}{j!\, (\gamma-j)!\, (n-\gamma)!\, \Gamma(\alpha+1+n-j)\, \Gamma(\alpha+\gamma+1/2)}$$

or finally, upon interchanging the order of summation,

$$\sum_{\gamma=0}^{n} \sum_{j=0}^{\gamma} \frac{r^{2\gamma} (-)^{\gamma+j} b^{2n-2\gamma} \Gamma(\gamma-j+1/2)}{j!\, (\gamma-j)!\, (n-\gamma)!\, \Gamma(\alpha+1+n-j)\, \Gamma(\alpha+\gamma+1/2)}. \tag{3.5}$$

In a fashion which we made familiar in Section II, it is possible to evaluate the sum over the index j.

To this end, we put

$$h(\gamma) = \sum_{j=0}^{\gamma} \frac{(-)^j \Gamma(\gamma-j+1/2)}{j!\, (\gamma-j)!\, \Gamma(\alpha+1+n-j)}$$

and multiply both sides of this equation by $(\gamma+1/2-n-\alpha)$. From this we get

$$h(\gamma+1) = \frac{(\gamma+1/2-n-\alpha)}{(\gamma+1)} h(\gamma)$$

and therefore

$$h(\gamma) = \frac{(1/2-n-\alpha)\dots(\gamma-1/2-n-\alpha)\,\Gamma(1/2)}{\gamma!\, \Gamma(\alpha+1+n)}$$

if $\gamma > 0$ and $h(0) = \Gamma(1/2)/\Gamma(\alpha+1+n)$. Hence (3.5) reduces to

$$\sum_{\gamma=0}^{n} \frac{(-)^\gamma b^{2n} \lambda^\gamma h(\gamma)}{(n-\gamma)!\, \Gamma(\alpha+\gamma+1/2)} = \frac{b^{2n} \Gamma(1/2)}{n!\, \Gamma(\alpha+1+n)} \cdot \frac{F(-n, 1/2-n-\alpha, \alpha+1/2, \lambda)}{\Gamma(\alpha+1/2)},$$

where F is the customary notation for the hypergeometric function and $(k/b)^2 = \lambda$. This last expression is precisely the left side of (3.4) when the quadratic transformation of GOURSAT is applied to the integral on the left side.

Now we will show that

$$\frac{1}{n!} \int_0^{\pi} \frac{\rho_1^{2n} \sin^{2\alpha-1}\theta \, d\theta}{\Gamma(\alpha)\, \Gamma(\alpha+n+1)} = \sum_{j=0}^{n} r^{2j} \int_0^{\pi} \frac{\rho_2^{2n-2j} \sin^{2\alpha+2j-1}\theta \, d\theta}{j!\, (n-j)!\, \Gamma(\alpha+j)\, \Gamma(\alpha+1+n-j)}. \tag{3.6}$$

This is true for $n=0, 1$ and 2 and we proceed by induction. Let us multiply (3.6) by x and integrate from 0 to x. Then we have

$$\frac{1}{2(n+1)!}\int_0^\pi \frac{[\rho_1^{2n+2}-(r^2+b^2-2rb\cos\theta)^{n+1}]\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\Gamma(\alpha+n+1)}$$

$$(3.7)\qquad =\sum_{j=0}^{n} r^{2j}\int_0^\pi \frac{[\rho_2^{2n-2j+2}-(b^2-r^2\cos^2\theta)^{n-j+1}]\sin^{2\alpha+2j-1}\theta\,d\theta}{2j!(n-j+1)!\Gamma(\alpha+j)\Gamma(\alpha+1+n-j)}$$

$$+\sum_{j=0}^{n} r^{2j+2}\int_0^\pi \frac{[\rho_2^{2n-2j}-(b^2-r^2\cos^2\theta)^{n-j}]\sin^{2\alpha+2j+1}\theta\,d\theta}{2j!(n-j)!\Gamma(\alpha+j+1)\Gamma(\alpha+1+n-j)}$$

and the right side may be simplified to

$$\frac{(\alpha+1+n)}{2}\sum_{j=0}^{n} r^{2j}\int_0^\pi \frac{[\rho_2^{2n-2j+2}-(b^2-r^2\cos^2\theta)^{n-j+1}]\sin^{2\alpha+2j-1}\theta\,d\theta}{j!(n+1-j)!\Gamma(\alpha+j)\Gamma(\alpha+2+n-j)}$$

But with the aid of the identity (3.4) we may reduce (3.7) to

$$\frac{1}{2(n+1)!}\int_0^\pi \frac{\rho_1^{2n+2}\sin^{2\alpha-1}\theta\,d\theta}{\Gamma(\alpha)\Gamma(\alpha+n+2)}=\frac{1}{2}\sum_{j=0}^{n+1} r^{2j}\int_0^\pi \frac{\rho_2^{2n+2-2j}\sin^{2\alpha+2j-1}\theta\,d\theta}{j!(n+1-j)!\Gamma(\alpha+j)\Gamma(\alpha+n+2-j)}$$

Hence if (3.6) is true for some positive integer n, it is also true for $n+1$ and therefore the usual induction argument demonstrates that it is true for all positive integers n. If we now examine (3.1) by the same methods we employed in Section II, we find

$$(3.8)\qquad \int_0^\pi \rho_1^{-\alpha} J_\alpha(k\rho_1)\sin^{2\alpha-1}\theta\,d\theta = r^{1-\alpha}\left(\frac{2}{k}\right)^{\alpha-1}\int_0^\pi \sin^\alpha\theta\, J_\alpha(k\rho_2)\, J_{\alpha-1}(kr\sin\theta)\,\rho_2^{-\alpha}\,d\theta$$

From (2.12) and (3.8) we then get (1.5d) and (1.5e) follows from (1.5d) by interchanging r and b. We shall discuss the properties of these representations for the fundamental solution in the following section.

IV. The case $m=2$

The case $m=2$ cannot be derived by the methods which we have employed in Sections II and III. We observe that we cannot use the axially symmetric solution for Laplace's equation in this case, since its integral representation diverges. We could employ the correspondence principle to write the odd part of the fundamental solution in the case $m=2$ ($\alpha=0$) in terms of the fundamental solution for $m=4$ ($\alpha=1$). Rather than do this which will lead us into some unnecessary numerical work, we will go directly to (1.5c, d, e) and show that (1.5c) can be evaluated. We have for $r<b$

$$\begin{aligned}(4.1)\qquad &\frac{ik}{4}\int_0^\pi \sin\theta_1\, H_1^{(1)}(k\rho_1)/\rho_1\, d\theta_1\\ &=\frac{ik}{4}\int_0^\pi \sin\theta\, J_0(kr\sin\theta)\,\frac{H_1^{(1)}[k\sqrt{(x-a+ir\cos\theta)^2+b^2}]\,d\theta}{[(x-a+ir\cos\theta)^2+b^2]^{1/2}}\end{aligned}$$

The left side of equation (4.1) may be integrated directly to give

$$\frac{i}{4rb}\{H_0^{(1)}[k\sqrt{(x-a)^2+(r-b)^2}]-H_0^{(1)}[k\sqrt{(x-a)^2+(r+b)^2}]\}.$$

That is, equation (4.1) becomes

$$\begin{aligned}(4.2)\qquad &\frac{i}{4}\{H_0^{(1)}[k\sqrt{(x-a)^2+(r-b)^2}-H_0^{(1)}[k\sqrt{(x-a)^2+(r+b)^2}]\\ &=\frac{irbk}{4}\int_0^\pi \sin\theta\, J_0(kr\sin\theta)\,\frac{H_1^{(1)}[k\sqrt{(x-a+ir\cos\theta)^2+b^2}]\,d\theta}{[(x-a+ir\cos\theta)^2+b^2]^{1/2}},\end{aligned}$$

the odd fundamental solution in the case $m=2$.

In order to find the even fundamental solution for the case $m=2$, we differentiate equation (4.2) partially with respect to r and integrate with respect to b. Then we get after some integration by parts,

$$\begin{aligned}&\frac{i}{4}\{H_0^{(1)}[k\sqrt{(x-a)^2+(r-b)^2}]+H_0^{(1)}[k\sqrt{(x-a)^2+(r+b)^2}]\}\\ (4.3)\qquad &=\frac{i}{4}\{H_0^{(1)}[k\sqrt{(x-a+ir)^2+b^2}]+H_0^{(1)}[k\sqrt{(x-a-ir)^2+b^2}]\}\\ &\quad-\frac{irk}{4}\int_0^\pi J_1(kr\sin\theta)\,H_0^{(1)}[k\sqrt{(x-a+ir\cos\theta)^2+b^2}]\,d\theta.\end{aligned}$$

It can be easily demonstrated that there is no contribution from the integration operation, once we account for the behavior of the left and right sides for $(a^2+b^2)^{1/2}\to\infty$. There is also a second representation for the right side of (4.3) when $|r|>|b|$ which is found by interchanging b and r. Equation (4.3) then supplies us with the even part of the fundamental solution in the case $m=2$, although we should mention that the left side is well known.

If we add equations (4.1) and (4.3) and divide by 2, we have the fundamental solution in the case $\alpha=0$ in the entire x, r plane. Observe that the restrictions $r<b$ and $r>b$, which arose in our earlier work, arose from the restrictions $r^2<b^2$ and $r^2>b^2$. Now, since we can deal with the entire plane, these conditions become $|r|<|b|$ and $|r|>|b|$. We also note the parallelism of the representations (4.1) and (4.3) with the non-characteristic representation in the hyperbolic case. This last remark becomes evident when we write $\cos\theta=t$ in (4.1) and (4.3) [4].

V. Final Comments

We observed that the representations (1.5d) and (1.5e) for the fundamental solution of the axially-symmetric wave equation are dependent on the inequalities $r > b$ and $r < b$. It is from these forms that we may recognize that some of the integral equations derived for exterior axially-symmetric boundary value problems by analytic continuation of axis data may be given directly from these representations. There may be, however, some further transformations necessary to bring this to pass. Suffice it to say at this point that the method of analytic continuation depends on the use of a representation whose restrictions we do not fully understand in the context of these exterior boundary value problems [2]. It has also been discussed in terms of an Ansatz by various writers in the case $\alpha = 0$ and 1/2. We shall discuss the formulation of these integral equations in a subsequent paper and show there that the representation (1.5d) and (1.5e) lead to useful equations which nevertheless are not of the classical form.

Then, there is the issue that we may encounter divergent integrals, if we try to form integral equations with (1.5d) and (1.5e) for $\alpha \geqq 1$. For $\alpha = 1/2$, there are no such problems. In the case $k = 0$, $\alpha \geqq 1$, it is possible to carry out some integration by parts to eliminate this difficulty [3]. We have not been able to do this in the case of the wave equation. Until this issure is clarified, we will not be able to examine the higher harmonics.

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