Power Series Spaces $\Lambda(\alpha)$ and Associated $\Lambda(\alpha)$-Nuclearity*

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Introduction

The class of strongly nuclear spaces was introduced by Martineau [8] and also rediscovered by Brudovskii [1, 2]. The definition of strongly nuclear maps is related to the space $s$ of rapidly decreasing sequences (see [9]). In this paper we introduce the class of $\Lambda(\alpha)$-nuclear maps, where $\Lambda(\alpha)$ is a general power series space, known to include as special cases the familiar spaces $s$ and $\Gamma$, the space of entire functions (see [3]). We define then the notion of $\Lambda(\alpha)$-nuclear spaces and obtain a criterion for the $\Lambda(\alpha)$-nuclearity of sequence spaces $\lambda(P)$. Köthe [7] has proved such a criterion for the $s$-nuclearity of $\lambda(P)$. Some applications of this criterion are given and finally we obtain a product of several copies of $[\Lambda(\alpha)]_b$ as a universal $\Lambda(\alpha)$-nuclear space.

Preliminaries

For terminology not explained here, see Köthe [5].

Suppose $a = (a_n)$ and $b = (b_n)$ are two sequences of scalars. Write $a < b$ if there exists a $M > 0$ such that $|a_n| \leq M |b_n|$ for all $n$. Let $P$ be a set of sequences $a = (a_n)$ such that (i) $a_n \geq 0$ for each $n$, (ii) for each $n$ there exists an $a \in P$ such that $a_n \to 0$ and (iii) $P$ is directed by $<$. Define now the sequence space $\lambda(P) = \{x = (x_n) : p_a(x) = \sum |x_n| a_n < \infty \text{ for each } a \in P \}$. The (normal) topology on $\lambda(P)$ is generated by the seminorms $\{p_a\}$.

We recall here the Grothendieck-Pietsch criterion for the nuclearity of $\lambda(P)$ with the above topology (see Pietsch [9], p. 88).

Lemma 1. $\lambda(P)$ is nuclear if and only if for each $a \in P$ there exists a $b \in P$ and a $c \in l^1$ such that $a_n \leq b_n c_n$ for all $n$.

Consider next a sequence $\alpha = (\alpha_n)$ such that $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \to \infty$. Let $P$ be the set of sequences $(\varrho^{\alpha_n})$, $0 < \varrho < \varrho_0$ (fixed). The sequence space $\lambda(P)$ obtained from this set $P$ is called the power series space $\Lambda(\alpha)$. Throughout this work $\varrho_0 = \infty$ and $\Lambda(\alpha)$ is provided with the topology indicated earlier and generated by the seminorms $p_R(x) = \sum |x_n| R^{\alpha_n}$, $R = 1, 2, \ldots$.

The case $\alpha_n = \log(n + 1)$ gives the space $s$ of rapidly decreasing sequences and $\alpha_n = n$ yields the space $\Gamma$ of entire functions, discussed extensively by

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Ganapathy Iyer [3]. The space $A(\alpha)$ has received brief mention in the unpublished thesis of Somasundaram [10]. Observe that $A(\alpha) \subset l^1$.

Condition G. The sequence $(\alpha_n)$ is defined to satisfy condition G if the following growth condition holds: there exists a $R > 1$ such that $\sum R^{-\alpha_n} < \infty$.

The spaces $s$ and $I$ mentioned above meet this requirement. If $\alpha_n = \log \log(n + k)$, $k$ fixed, then the condition G is not met.

We shall list now some simple consequences of condition G.

In what follows we shall adopt the convention that if $\alpha_n = 0$ for $n < N$ and $\alpha_n \neq 0$ for $n \geq N$, then any statement involving the sequence $(1/\alpha_n)$ is to be interpreted as the corresponding statement on the sequence starting from the $N$-th term onward. Our assumption that $\alpha_n \uparrow \infty$ guarantees that $\alpha_n \neq 0$ for all $n \geq N$, for some $N$.

**Lemma 2.** If $\alpha = (\alpha_n)$ satisfies condition G then $A(\alpha) = \{a = (a_n) : |a_n|^{1/\alpha_n} \to 0\}$.

**Proof.** Let $a = (a_n) \in A(\alpha)$ and if possible, let $|a_n|^{1/\alpha_n} \to 0$. Then there exists some $\varepsilon > 0$ and a sequence of integers $(n_i)$ such that $|a_{n_i}|^{1/\alpha_{n_i}} > \varepsilon$ for $n = n_1, n_2, \ldots$. Choose now $R$ such that $\varepsilon R > 1$. Then $\sum |a_{n_i}|R^{\alpha_{n_i}} > \sum \varepsilon^{\alpha_{n_i}}R^{\alpha_{n_i}} = \infty$ and this violates that $a \in A(\alpha)$.

Conversely, suppose that $|a_n|^{1/\alpha_n} \to 0$ and that condition G is fulfilled. Thus we have that $\sum u^{\alpha_n} < \infty$ for some $0 < u < 1$. Now given any $R$, choose $\varepsilon > 0$ such that $\varepsilon R = u$. Since $|a_n|^{1/\alpha_n} \to 0$, we can find $N$ such that $|a_n|^{1/\alpha_n} < \varepsilon$ for $n \geq N$. Then $\sum |a_{n_i}|R^{\alpha_{n_i}} < \sum \varepsilon^{\alpha_{n_i}}R^{\alpha_{n_i}} < \infty$ and this completes the proof of the lemma.

**Lemma 3.** If $\alpha = (\alpha_n)$ satisfies condition G and $\beta = (\beta_n)$ is such that $\beta_n \geq \alpha_n$ for all $n$ then $\beta$ also satisfies condition G and $A(\beta) \subset A(\alpha)$.

**Lemma 4.** If $\alpha$ satisfies condition G then $A(\alpha)$ is nuclear.

**Lemma 5.** If $\alpha$ satisfies condition G then $\frac{\log(n + 1)}{\alpha_n}$ is bounded.

The proof of Lemma 3 is trivial while Lemma 4 follows from Lemma 1. Lemma 5 follows easily from the inequalities $\sum_{i=0}^n R^{-\alpha_i} \leq M$ for some $R > 1$ and for some $M > 1$ and therefore $(n + 1)R^{-\alpha_n} \leq M$ for all $n$.

**Lemma 6.** If $\alpha$ satisfies condition G and the sequence $b = (b_n)$ is such that $|b_n|^{1/\alpha_n}$ is bounded then $b \in [A(\alpha)]'$, the topological dual of $A(\alpha)$.

Lemma 6 is an immediate consequence of the following result of Köthe [6] and of Lemma 4.

**Lemma 7.** If $\lambda(P)$ is nuclear then the strong topological dual $\lambda_b' = \lambda'(Q)$ where $Q = \{a \in \lambda(P), a_n \geq 0\}$ and $\lambda'(Q)$ has its normal topology.
A(\alpha)-Nuclearity

Throughout the rest of the paper we assume that (\alpha_n) satisfies condition G. Let \(E\) and \(F\) be two normed linear spaces and \(T\) be a linear map on \(E\) into \(F\). Suppose \(T\) admits the representation

\[Tx = \sum_{n=0}^{\infty} \lambda_n \langle x, a_n \rangle y_n, \quad \text{for each } x \in E \]

(1)

where \((\lambda_n) \in \Lambda(\alpha), a_n \in E', \|a_n\| \leq 1\) and \(y_n \in F\) with \(\|y_n\| \leq 1\). Then \(T\) is said to be a \(\Lambda(\alpha)\)-nuclear map on \(E\) into \(F\). It is easy to see that each \(\Lambda(\alpha)\)-nuclear map is continuous, nuclear and therefore precompact.

We define next a linear map \(T\) on \(E\) into \(F\) to be quasi-\(\Lambda(\alpha)\)-nuclear if there exist \(a_n \in E'\) such that \((\|a_n\|) \in \Lambda(\alpha)\) and \(\|Tx\| \leq \sum |\langle x, a_n \rangle|\) for each \(x \in E\).

It is trivial that each \(\Lambda(\alpha)\)-nuclear map is quasi-\(\Lambda(\alpha)\)-nuclear. In the opposite direction we prove the following result.

**Proposition 1.** Suppose the Banach space \(F\) has the property that if \(S\) is an arbitrary closed subspace of \(\Lambda(\alpha)\) and \(L\) is an arbitrary quasi-\(\Lambda(\alpha)\)-nuclear map of \(S\) into \(F\) then \(L\) has a continuous linear extension to the whole of \(\Lambda(\alpha)\). Then for arbitrary normed linear spaces \(E\), each quasi-\(\Lambda(\alpha)\)-nuclear map of \(E\) into \(F\) is indeed \(\Lambda(\alpha)\)-nuclear.

**Proof.** Let \(T\) be a quasi-\(\Lambda(\alpha)\)-nuclear map on \(E\) into \(F\). Determine \((a_n), a_n \in E', (\|a_n\|) \in \Lambda(\alpha)\) such that \(\|Tx\| \leq \sum |\langle x, a_n \rangle|\). Define the maps \(T_1 : E \to l_\infty\), by \(T_1 x = (\langle x, a_n/\|a_n\| \rangle)\) and \(T_2 : l_\infty \to \Lambda(\alpha)\) by \(T_2 [(y_n)] = (\sqrt{\|a_n\| y_n})\). Then \(T_1\) is linear and continuous and \(T_2\) is a \(\Lambda(\alpha)\)-nuclear map since \((\sqrt{\|a_i\|}) \in \Lambda(\alpha)\), by

Lemma 2 and \(T_2 [(y_n)] = \sum_{i=1}^{\infty} \langle (y_n), \sqrt{\|a_i\| e_i} \rangle e_i\), where \(e_i\) is the usual unit vector with 1 in the \(i\)-th coordinate. Consider the subspace \(S = T_2 T_1 (E)\) of \(\Lambda(\alpha)\) and on this space define \(\Phi : S \to F\) by \(\Phi (T_2 T_1 x) = T x\). Then \(\Phi\) is quasi-\(\Lambda(\alpha)\)-nuclear since \(\|\Phi (T_2 T_1 x)\| = \|Tx\| \leq \sum |\langle x, a_n \rangle| = \sum |\langle T_2 T_1 x, \sqrt{\|a_n\| e_n} \rangle|\) and \(\Phi\) is a quasi-\(\Lambda(\alpha)\)-nuclear map on \(S\) into \(F\). Use next the hypothesis to obtain a continuous linear extension of \(\Phi\) to the whole of \(\Lambda(\alpha)\) and call this extension \(T_3\). Now \(T = T_3 T_2 T_1\), where \(T_2\) is \(\Lambda(\alpha)\)-nuclear and the other two are linear and continuous and therefore \(T\) is \(\Lambda(\alpha)\)-nuclear.

**Corollary.** If \(F\) has the bounded extension property then each quasi-\(\Lambda(\alpha)\)-nuclear map of a normed linear space \(E\) into the Banach space \(F\) is \(\Lambda(\alpha)\)-nuclear.

For any linear, continuous map \(T\) on \(E\) into \(F\) define the \(n\)-th approximation number \(\alpha_n(T)\) by \(\alpha_n(T) = \inf \|T - A_n\|\), where the infimum is taken over all linear maps \(A_n\) of \(E\) into \(F\) which have a range of dimension at most \(n\). Obviously \(\alpha_n(T) \geq \alpha_{n+1}(T)\) for each \(n\). If \((\alpha_n(T))\) belongs to a specified sequence space \(\lambda\) we say that \(T\) is of type \(\lambda\).

**Proposition 2.** Each \(\Lambda(\alpha)\)-nuclear map of a normed linear space \(E\) into a normed linear space \(F\) is of type \(\Lambda(\alpha)\).
Proof. Let $T$ be $\Lambda(\alpha)$-nuclear and have the representation (1). Consider maps $A_i$ defined by $A_i x = \sum_{n=0}^{i-1} \lambda_n \langle x, a_n \rangle y_n$. Now $\|T - A_i\| \leq \sum_{n=i}^{\infty} |\lambda_n|$ and thus $\alpha_i(T) \leq \sum_{n=i}^{\infty} |\lambda_n|$. For $R \geq 1$, $\sum_{i=0}^{\infty} \alpha_i(T) R^x_i \leq \sum_{i=0}^{\infty} \left( \sum_{n=i}^{\infty} |\lambda_n| \right) R^x_i = \sum_{n=0}^{\infty} |\lambda_n| \sum_{i=0}^{n} R^x_i \leq \sum_{n=0}^{\infty} |\lambda_n| (n+1)^x R^{x n}$. Consider now the sequence $\{(n+1)^{1/x_n}\}$. From Lemma 5 it follows that this sequence is bounded and thus $\sum_{n=0}^{\infty} |\lambda_n| (n+1)^x R^{x n} < \infty$ since $(\lambda_n) \in \Lambda(\alpha)$.

Consider next a bounded sequence $(d_n)$, $d_n > 0$ and the diagonal transformation $D$ on $l^1$ into itself defined by $D[(x_n)] = (d_n x_n)$. Let $\Pi$ denote a permutation of the positive integers $I^+$. Köthe [7] proves the following result.

Lemma 8. The map $D$ is $s$-nuclear on $l^1$ into $l^1$ if and only if there exists a permutation $\Pi$ of $I^+$ such that $(d_{\Pi(n)}) \in s$.

The proof of the sufficiency in the above lemma is immediate. In proving the necessity, Köthe observes that a $s$-nuclear map on $l^1$ into $l^1$ is compact and consequently $d_n \to 0$. Since a permutation $\Pi$ can be found so that $d_{\Pi(n)}$ is monotone decreasing, it suffices to prove that for a positive monotone decreasing sequence $(d_n)$ converging to zero, the corresponding diagonal transformation $D$ is $s$-nuclear implies $(d_n) \in s$. This last statement is achieved by showing $\alpha_n(D) = d_{n+1}$.

In view of the above analysis of the proof of Köthe's theorem and proposition 2 above we state below a criterion for the $\Lambda(\alpha)$-nuclearity of $D$ and omit its proof.

Proposition 3. A diagonal transformation $D = (d_n)$, $d_n \geq 0$ is a $\Lambda(\alpha)$-nuclear map on $l^1$ into $l^1$ if and only if the $d_n$ which are different from zero can be rearranged into a sequence in $\Lambda(\alpha)$.

The above proposition provides the proof of the existence of maps which are nuclear but not $\Lambda(\alpha)$-nuclear. The sequence $(2^{-n})$ provides a diagonal map of $l^1$ into $l^1$ which is nuclear but not $\Gamma$-nuclear (see also proposition 7).

$\Lambda(\alpha)$-nuclear Spaces

Let $E$ be a locally convex space and $U$ be an absolutely convex neighbourhood, with $p_U$ being its gauge. Set $E(U) = E/N(U)$, where $N(U)$ is the null space of $p_U$. $E(U)$ is normed by $p_U$. For another such neighbourhood $V$ which is absorbed by $U$ (written $V \prec U$), the canonical map $\tilde{K}(V, U)$ is the map on $\tilde{E}(V)$ into $\tilde{E}(U)$ which associates to the equivalence class $x(V)$ the element $x(U)$. The map $\tilde{K}$ is continuous.
Now define the locally convex space $E$ to be a $\Lambda(\alpha)$-nuclear space if a fundamental system $\mathcal{U}$ of absolutely convex neighbourhoods in $E$ (or equivalently, each such system) has the property that for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V < U$ and the canonical map $\tilde{K}(V, U)$ is $\Lambda(\alpha)$-nuclear. The equivalence stated above results from the fact that the composition of linear continuous maps with $\Lambda(\alpha)$-nuclear maps will be $\Lambda(\alpha)$-nuclear.

We postpone giving examples of $\Lambda(\alpha)$-nuclear spaces. First we obtain a criterion for a sequence space $\lambda(P)$ to be $\Lambda(\alpha)$-nuclear. The result is the analogue of the Grothendieck-Pietsch criterion for the nuclearity of $\lambda(P)$ and is motivated by the Köthe's corrected version of a result of Brudovskii [1, 2] on the s-nuclearity of $\lambda(P)$.

**Proposition 4.** The sequence space $\lambda(P)$ is $\Lambda(\alpha)$-nuclear if and only if to each $a \in P$, there exists a $b \in P$ such that $a \prec b$ and such that the sequence $(a_k/b_k)$ with $a_k \neq 0$ can be rearranged into a member of $\Lambda(\alpha)$.

**Proof** (Sketch). The sets $U_a = \{x: p_a(x) \leq 1\}$, $a \in P$, form a fundamental system of neighbourhoods in $\lambda(P)$. Let $M(a)$ denote the set of indices $k$ for which $a_k \neq 0$. If now $E = \lambda(P)$, then as is well-known, $\tilde{E}(U_a)$ is norm isomorphic to $l^1[M(a)]$, $a \prec b$ is equivalent to $U_b < U_a$ and the canonical map $\tilde{K}(U_b, U_a)$ can be identified with the diagonal transformation $(y_a) \rightarrow (a_k b_k^{-1} y_a)$ of $l^1[M(b)]$ into $l^1[M(a)]$. Now the desired result is a consequence of Proposition 3.

**Corollary.** The power series space $\Lambda(\gamma)$ is $\Lambda(\alpha)$-nuclear if and only if there exists a $R > 1$ and a permutation $\Pi$ of $1^+$ such that $\Pi(R^{-\gamma}) \in \Lambda(\alpha)$.

**Proof.** Suppose $\Lambda(\gamma)$ is $\Lambda(\alpha)$-nuclear. Since $\Lambda(\gamma)$ is generated by the stufen system $(R^{-\gamma})$, $R = 1, 2, \ldots$ it follows from Proposition 4 that given $R$, there exists $S$ and a permutation $\Pi$ such that $\Pi(R^{-\gamma}/S^{-\gamma}) \in \Lambda(\alpha)$. But $S^{-\gamma} \leq R^{-\gamma} S^{-\gamma}$ and also $\Lambda(\alpha)$ is a normal sequence space. Thus $\Pi(S^{-\gamma}) \in \Lambda(\alpha)$.

Conversely, let there exist $R$ and $\Pi$ such that $\Pi(R^{-\gamma}) \in \Lambda(\alpha)$. Now, given any $(T^{-\gamma})$ belonging to the stufen system, let $S$ denote the max$(R, T)$. Then $S^{-\gamma} \leq R^{-\gamma}$ and therefore $\Pi(S^{-\gamma}) \in \Lambda(\alpha)$. Consider now $(T^{-\gamma} S^{-\gamma}) = (T^{-\gamma} S^{-\gamma}) (S^{-\gamma}) \leq (S^{-\gamma})$ and therefore $\Pi(T^{-\gamma} S^{-\gamma}) \in \Lambda(\alpha)$. Since $(S^2^{-\gamma})$ is also a member of the stufen system, Proposition 4 now gives the desired result.

For a result of this type for Schwartz spaces, see Terzioglu [11].

Examples of $\Lambda(\alpha)$-nuclear spaces are provided by means of the following two simple propositions both of which are consequences of Proposition 4.

**Proposition 5.** If $(\alpha_n/\beta_n) \rightarrow 0$ then $\Lambda(\beta)$ is $\Lambda(\alpha)$-nuclear.

The proof follows immediately from Lemma 2 and the corollary to Proposition 4 since $(2^{-\beta_n/\alpha_n}) \rightarrow 0$.

**Proposition 6.** The strong dual $[\Lambda(\alpha)]_b$ of $\Lambda(\alpha)$ is $\Lambda(\alpha)$-nuclear.

**Proof.** If $a = (a_n) \in \Lambda(\alpha)$ and $a_n \geq 0$, it follows easily that $\sqrt{a} = (\sqrt{a_n})$ is also in $\Lambda(\alpha)$. Now the result is a consequence of Lemma 7 and Proposition 4.
It is well known that \( l^1 \) is not nuclear; Brudovskii [1] states that \( s \) is not \( s \)-nuclear. We conjecture that in general it is true that \( \Lambda(\alpha) \) is not \( \Lambda(\alpha) \)-nuclear. The following proposition supplements the two known results quoted above and the proof is also of some interest.

**Proposition 7.** If \( \alpha_n = n^m, m > 0 \) and fixed, then \( \Lambda(\alpha) \) is not \( \Lambda(\alpha) \)-nuclear.

**Proof.** After the corollary to Proposition 4 we need only show that whatever be \( R > 1 \), there exists no permutation \( \Pi \) such that \( \Pi(R^{-\alpha_n}) \in \Lambda(\alpha) \); i.e., for each permutation \( \Pi, (R^{-\alpha_n}/\Pi(n)) \) does not converge to zero. This is equivalent to showing that there exists no permutation \( \Pi \) of \( I^+ \) such that \( (\alpha_n/\Pi(n)) \to \infty \). If possible let there exist a \( \Pi \) such that \( (\alpha_n/\Pi(n)) = (n/\Pi(n))^m \to \infty \) or equivalently \( (\Pi(n)/n) \to 0 \). Then \( \Pi(n)/n < 1/k \), for \( n \geq N \), where we pick \( k \) to be an integer larger than 2 and choose \( N \) such that \( N \equiv 0(\text{mod } k) \). Then

\[
\Pi(N + i) < (N/k) + 1, \quad i = 0, 1, 2, \ldots, k;
\]

\[
\Pi(N + j) < (N/k) + 2, \quad j = k + 1, k + 2, \ldots, 2k,
\]

and proceeding thus we get,

\[
\Pi(N + l) < (N/k) + p, \quad l = (p - 1)k + 1, \ldots, pk.
\]

Thus, since \( \Pi \) is a \( 1 - 1 \), onto map of \( I^+ \) to \( I^+ \), we have to find \( pk \) positive integers less than \( (N/k) + p \). Choosing \( p = N/k \), we get \( N < 2(N/k) \) or \( k < 2 \), setting up a contradiction.

**Remark.** It follows from Proposition 5 and 7 that the space \( \Lambda(\alpha), \alpha_n = n^m \), while being not \( \Lambda(\alpha) \)-nuclear, is \( \Lambda(\gamma) \)-nuclear for \( \gamma_n = n^{m-\epsilon}, m \geq \epsilon > 0 \); in particular, the space \( \Gamma \) is \( \Lambda(\gamma) \)-nuclear for \( \gamma_n = n^\delta, 0 \leq \delta < 1 \).

Following Köthe [7], we say that \( \lambda(P) \) is uniformly \( \Lambda(\alpha) \)-nuclear if there exists a “universal” permutation \( \Pi \) such that for each \( a \in P \) there exists a \( b \in P \) such that \( b > a \) and \( a_{\Pi(n)} \leq c_n b_{\Pi(n)} \) for a suitable \( c = (c_n) \in \Lambda(\alpha) \). Terzioglu [12] proves that the strong dual of a uniformly \( s \)-nuclear space is nuclear and therefore, by a theorem of Köthe [7], strong dual of a metrizable \( s \)-nuclear space is nuclear. Martineau [8] and Brudovskii [1] announce that the strong dual of a metrizable \( s \)-nuclear space is \( s \)-nuclear. In this context we prove now the following result which is analogous to that of Terzioglu and Köthe.

**Proposition 8.** The strong dual of a uniformly \( \Lambda(\alpha) \)-nuclear sequence space \( \lambda(P) \) is \( \Lambda(\beta) \)-nuclear for each \( \beta = (\beta_n) \) such that \( (\alpha_n/\beta_n) \to \infty \).

**Proof.** Since \( \lambda(P) \) is uniformly \( \Lambda(\alpha) \)-nuclear, to each \( a \in P \) there exists a \( b \in P \) such that \( b > a \) and \( a_{\Pi(n)} \leq c_n b_{\Pi(n)} \) for a suitable \( c = (c_n) \in \Lambda(\alpha) \). Consider now the sequence \( (2^{-\alpha_n}) \). By Lemma 2 it is in \( \Lambda(\beta) \); also, by Lemma 6, \( (2^{\alpha_n}) \) is in \( [\Lambda(\alpha)]' \). Therefore \( (c_n 2^{\alpha_n}) \in l^1 \).

Consider next an arbitrary \( x = (x_n) \in \lambda(P) \), with \( x_n \geq 0 \). Let \( a \in P \) be arbitrary and determine \( b \) and \( c \) as above. Since \( x \in \lambda(P) \) and \( b \in P \), we have \( \sum x_n b_n = \sum x_{\Pi(n)} b_{\Pi(n)} < \infty \). But \( \sum x_{\Pi(n)} a_{\Pi(n)} 2^{\alpha_n} \leq \sum x_{\Pi(n)} b_{\Pi(n)} c_n 2^{\alpha_n} < \infty \). Thus for each \( a \in P \), we have \( \sum x_n 2^{\alpha_n - (1-n)} a_n < \infty \) and consequently \( (x_n 2^{\alpha_n - (1-n)}) \in \lambda(P) \).
Also \( \Pi \left( \frac{x_n}{x_n 2^{a_{n+1}^{(n)}}} \right) = (2^{-a_n}) \) in \( \Lambda(\beta) \). The proof is now complete by appealing to Lemma 7 and Proposition 4.

**Remark.** The above result yields in particular that the strong dual of a uniformly \( \Gamma \)-nuclear space is \( \Lambda(n^{1-\varepsilon}) \)-nuclear for each \( \varepsilon \) such that \( 1 \geq \varepsilon > 0 \), and also \( s \)-nuclear.

**Universal \( \Lambda(\alpha) \)-nuclear Spaces**

Komura and Komura [4] exhibit \( [s]^A \) as a universal nuclear space and Martineau [8] states that \( [s']^A \) is a universal \( s \)-nuclear space. We prove the following general imbedding theorem.

**Proposition 9.** The locally convex space \( E \) is \( \Lambda(\alpha) \)-nuclear if and only if it is (topologically) isomorphic to a subspace of an \( I \)-fold product \( ([\Lambda(\alpha)])_b \).

**Proof.** In an (as yet) unpublished work P. Spuhler proves that subspaces and arbitrary products of \( \Lambda(\alpha) \)-nuclear spaces are \( \Lambda(\alpha) \)-nuclear and since \( [\Lambda(\alpha)]_b \) is \( \Lambda(\alpha) \)-nuclear the proof of the sufficiency is complete.

Suppose now that \( E \) is \( \Lambda(\alpha) \)-nuclear. Let \( \mathcal{U} = [U_i, i \in I] \) be a fundamental system of absolutely convex neighbourhoods in \( E \), where \( I \) is a suitably determined index set. Since \( E \) is \( \Lambda(\alpha) \)-nuclear, to each \( U_i \) we can find a \( V_i \in \mathcal{U} \) such that \( V_i < U_i \) and the canonical map \( \tilde{K}_i(V_i, U_i) \) of \( \tilde{E}(V_i) \) onto \( \tilde{E}(U_i) \) is \( \Lambda(\alpha) \)-nuclear and therefore can be written as

\[
x(U_i) = \tilde{K}_i[x(V_i)] = \sum_n \lambda_n^{(i)} \langle x(V_i), a_n^{(i)} \rangle y_n^{(i)},
\]

with \( \lambda_n^{(i)} = (\lambda_n^{(i)}) \in \Lambda(\alpha) \), \( a_n^{(i)} \in [E(V_i)]', \|a_n^{(i)}\| \leq 1 \) for each \( n \) and \( y_n^{(i)} \in \tilde{E}(U_i) \) with \( \|y_n^{(i)}\| \leq 1 \); thus we can get \( b_n^{(i)} \in V_i^0 \) for each \( n \), so that

\[
P_{V_i}(x) \leq \sum_n |\lambda_n^{(i)}| \|\langle x, b_n^{(i)} \rangle\| \leq P_{V_i}(x) \sum |\lambda_n^{(i)}|.
\]

Define now the map \( \Phi_i : E \to [\Lambda(\alpha)]_b \) by \( \Phi_i(x) = (\langle x, b_n^{(i)} \rangle) \). Since for each \( z = (z_n) \in \Lambda(\alpha) \) with \( z_n \geq 0 \), we have \( \sum |\langle x, b_n^{(i)} \rangle| z_n \leq P_{V_i}(x) \sum |z_n| \) it follows from Lemma 7 that \( \Phi_i(x) \in [\Lambda(\alpha)]' \).

Next define \( \Phi : E \to \Pi [\Lambda(\alpha)]_b = [\Lambda'] \) by the relation \( \Phi x = [\Phi_i x, i \in I] \). Since \( \Phi \) and \( \Phi_i \) are linear, it follows from (2) above that \( \Phi \) is \( 1 - 1 \). Now given an arbitrary \( \gamma = (\gamma_n) \in \Lambda(\alpha) \) or equivalently, an arbitrary seminorm generating the topology of \( [\Lambda(\alpha)]' \), we have, for each \( i \in I \), \( \sum |\gamma_n| |\langle x, b_n^{(i)} \rangle| \leq \varrho P_{V_i}(x) \) where \( \varrho = \sum |\gamma_n| \). This proves the continuity of the linear maps \( \Phi_i \) and since \( \Phi_i = p_i \circ \Phi \), where \( p_i \) is the projection of \( [\Lambda'] \) onto \( \Lambda' \) we get that \( \Phi \) is continuous. To prove the continuity of \( \Phi^{-1} \) consider the neighbourhood

\[
N_i = \{ z = (z_n) \in \Lambda' : \sum |z_n| |\lambda_n^{(i)}| \leq 1 \}
\]

in \( \Lambda' \). Then \( \tilde{N}_i = p_i^{-1}(N_i) \) is a neighbourhood in the product space. It is easy to verify, using (2), that \( \Phi^{-1}(\tilde{N}_i) \cap E \subset U_i \) and the proof is complete.
Added in proof. (Oct. 1, 1970). Mr. P. Spuhler has now informed the author that the product of even two \( A(\alpha) \)-nuclear spaces need not be \( A(\alpha) \)-nuclear. In view of this remark, Proposition 9 needs to be revised as follows: If \( E \) is \( A(\alpha) \)-nuclear then it is topologically isomorphic to a subspace of \( ([A(\alpha)])' \); if arbitrary products of \( A(\alpha) \)-nuclear spaces is \( A(\alpha) \)-nuclear, then the converse of the above statement holds.

References


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