RECURSIVE INTEGRAL EQUATIONS FOR THE
DETECTION OF COUNTING PROCESSES†

F. B. Dolivo

and

F. J. Beutler

Computer, Information and Control Engineering Program
The University of Michigan, Ann Arbor, Michigan 48104

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ABSTRACT. A recursive stochastic integral equation for the detection of Counting Processes is derived from a previously known formula [5] of the likelihood ratio. This is done quite simply by using a result due to Doléans-Dade [4] on the solution of stochastic integral equations.

1. INTRODUCTION. Recently modern martingale theory has been used to describe Counting Processes (hereafter abbreviated CP) in a way specially appropriate to the problems of detection and filtering. This has given rise to the notion of Integrated Conditional Rate (ICR) [5], which generalizes the notion of random rate.

Expressions for likelihood ratios (involving ICR's) for the detection of CP's have been obtained in [5] using a three-step technique introduced by Kailath [9] and Duncan ([6], [7]) in their works on detection of a stochastic signal in white noise. The three steps are the Likelihood Ratio Representation Theorem ([2], [5], [6]), the Girsanov Theorem ([5], [8], [13]) and the Innovation Theorem ([2], [5], [9]). By this method likelihood ratios for a large class of CP's can be found. These expansions represent a generalization of the formulas given in [1] and [12] in the context of Poisson processes and [2] in the context of CP's which admit a conditional rate.

The purpose of this paper is not to present a proof of the likelihood ratio formula (for that see [5]) but to derive from this formula stochastic integral equations by which the likelihood ratio can be computed recursively. This can be done quite simply using a result due to Doléans-Dade [4] on the solution of stochastic integrals equations involving semimartingales. These recursive equations are most useful in applications as they give a way of
implementing the computation of the likelihood ratio continuously in time.

2. PRELIMINARIES. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. By \((X_t)\) we denote a real valued stochastic process defined on \(\mathbb{R}_+\), the positive real line and by a Counting Process (CP) we mean

**Definition 2.1:** A CP is a stochastic process having sample paths which are zero at the time origin and consisting of right-continuous step functions with positive jumps of size one.

The time of \(n\)th jump \(J_n\) of a CP \((N_t)\) is the stopping time defined by

\[
J_n = \begin{cases} 
\inf \{t: N_t \geq n\} \\
\infty \text{ if the above set is empty.}
\end{cases}
\]

Let \((\mathcal{F}_t)\) be a right-continuous increasing family of \(\sigma\)-subalgebras of \(\mathcal{F}\) with \(\mathcal{F}_0\) containing all the \(P\) negligible sets, and suppose \((N_t)\) is a CP, adapted to \(\mathcal{F}_t\), with the sole assumption that \(EN_t\) is finite for each \(t\). Then, as a consequence of the Doob-Meyer decomposition for supermartingales we can associate to \((N_t)\) a unique natural increasing process \((A_t)\), dependent on the family \((\mathcal{F}_t)\), which makes the process \((M_t \triangleq N_t - A_t)\) a martingale (see [11]).

This decomposition \((N_t = M_t + A_t)\) is intuitively a decomposition into the part \((M_t)\) which is not predictable and \((A_t)\) which can be perfectly predicted. This unique process \((A_t)\) is called the Integrated Conditional Rate (ICR) of \((N_t)\) with respect to \((\mathcal{F}_t)\) ("the \((\mathcal{F}_t)\) ICR of \((N_t)\)"") and has been studied in [5].

The terminology ICR is motivated by the fact that when \((N_t)\) satisfies some
sufficiency conditions its ICR takes on the form \((\int_0^t \lambda_s \, ds)\) where \((\lambda_t)\) is a nonnegative process called the conditional rate (with respect to \((\mathcal{F}_t)\)) satisfying \(\lambda_t = \lim_{h \to 0} \mathbb{E}[h^{-1}(N_{t+h} - N_t) | \mathcal{F}_t]\) ([5], Section 2.5). The existence of CP's possessing a bounded conditional rate with respect to the family of \(\sigma\)-algebras generated by the process itself has been first shown in [2] and in [5]. Sufficiency conditions for the existence of a conditional rate have been given in [5]. By a change of time we can show similar results (i.e., existence (see [5], Corollary 3.1.3) and sufficiency conditions) for \((\mathcal{F}_t)\) ICR's of the form \((\int_0^t \lambda_s \, dm_s)\) where \((\lambda_t)\) is a locally bounded predictable process and \(m_t\) a deterministic increasing right-continuous function with \(m_0 = 0\).

Denote by \(\mathcal{A}(\mathcal{F}_t)\) the class of all locally bounded predictable (with respect to \((\mathcal{F}_t)\)) processes (see [3], p. 98). For example, processes adapted to \((\mathcal{F}_t)\) and having left-continuous sample paths belong to \(\mathcal{A}(\mathcal{F}_t)\).

**Remark 2.2:** Let the ICR \((A_t)\) be of the form \((\int_0^t \lambda_s \, dm_s)\) and denote by \(\Lambda\) the union of all intervals of \(\mathbb{R}_+\) on which the function \(m_t\) is constant. Observe that the ICR \((A_t)\) is not affected by a change of values of \((\lambda_t)\) for \(t \in \Lambda\) and we may well have \(\lambda_t = \infty\) for \(t \in \Lambda\). To avoid problems due to this indeterminacy we adopt the following convention: for \(t \in \Lambda\) we set \(\lambda_t\) equal to unity.

We assume here that modern martingale theory ([11], [3]) is known. Recall that a semimartingale \((X_t)\) is a process which can be written as a sum \((X_t = X_0 + L_t + A_t)\) where \(X_0\) is \(\mathcal{F}_0\)-measurable, \((L_t)\) is a \((\mathcal{F}_t)\) local martingale and \((A_t)\) is a right-continuous process adapted to \((\mathcal{F}_t)\)
having sample paths of bounded variation on every finite interval and with

\( A_0 = 0 \) a.s. (see [3]). A result basic to this study and due to Doléans-Dade [4] is the following: the stochastic integral equation

\[
Z_t = 1 + \int_0^t Z_s \, dX_s
\]

where \((X_t)\) is a semimartingale has a unique solution, which is a semimartingale given by

\[
Z_t = \exp(X_t - \frac{1}{2} <X^C>_t) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s)
\]

where the product in the right hand side converges a.s. for each \(t\). Here we define \(<X^C>_t\) as the unique natural increasing process (see [3]) associated to the continuous part of the local martingale \((L_t); \quad <X^C>_t\) is identically zero when \((X_t)\) is a semimartingale with sample paths of bounded variation on every finite interval (see [3]).

3. THE DETECTION PROBLEM. Let \(P_0\) and \(P_1\) be two measures carried on \((\Omega, \mathcal{F})\). Suppose that \((N_t)\) is a CP defined on \((\Omega, \mathcal{F})\) and denote by \(\mathcal{N}_t\) the minimal \(\sigma\)-algebra generated by \((N_t)\) up to and at time \(t\). The notation \(E_i(\cdot)\) for \(i=0, 1\) is intended for the expectation operator with respect to the measure \(P_i\).

**Definition 3.1:** For a \((\mathcal{N}_t)\) stopping time \(R\) (possibly infinite) denote by \(P_i^R\) for \(i=0, 1\) the restriction of the measure \(P_i\) to the \(\sigma\)-algebra \(\mathcal{N}_R\).

\[\text{\dag} \quad \text{When } f_t \text{ is a right-continuous function with left-hand limits } \Delta f_t \text{ denotes the jump } f_t^- - f_t^-.\]
We have the inclusion $\mathcal{N}_R \subset \mathcal{G}$ so that if $P_0 \ll P_1'$ then $P^R_0 \ll P^R_1$ and the Radon-Nikodym derivative $dP^R_0 / dP^R_1$ is well defined. We examine now the meaning of this Radon-Nikodym derivative. In the case where the stopping time $R$ is equal to a constant $a$ then $\mathcal{N}_R = \mathcal{N}_a = \sigma(N_u, 0 \leq u \leq a)$ so that $dP^a_0 / dP^a_1$ is the likelihood ratio for testing the two hypotheses $H_i$ for $i=0, 1$: $P_1$ is the probability measure on $(\Omega, \mathcal{G})$, by observations on the CP $(N_t)$ for $t \leq a$. The detection scheme then consists in selecting $H_0$ or $H_1$ according as $dP^a_0 / dP^a_1$ is above or below a given threshold. Now in the case where $R$ is a stopping time which is not a constant we know that $\mathcal{N}_R \supset \sigma(N_{u \wedge R}, 0 \leq u)$ (this follows from the fact that $N_{u \wedge R}$ is $\mathcal{N}_R$ measurable by Theorem 49-IV of [11]) but the reverse inclusion is not necessarily true. For this reason $dP^R_0 / dP^R_1$ is not the likelihood ratio for our detection problem when the time of observation is the stochastic interval $[0, R]$, as one could have conjectured. But one can interpret $dP^R_0 / dP^R_1$ as a likelihood ratio if we assume that the information accessible to the observer is $\mathcal{N}_R$ and not simply $\sigma(N_{u \wedge R}, 0 \leq u)$. For $i=0, 1$ with the measure $P_i$ carried on $(\Omega, \mathcal{G})$ suppose that the CP $(N_t)$ has the process $\left( \int_0^t \lambda^i_s dm_s \right)$ for $(\mathcal{G}^i_t)$ ICR, where $(\mathcal{G}^i_t)$ is a family of $\sigma$-algebras with $\mathcal{G}^i_t \supset \mathcal{N}_t$, $(\lambda^i_t) \in \mathcal{P}(\mathcal{G}^i_t)$ is a positive process, and $m_t$ is an increasing deterministic function with $m_0 = 0$.

It is known that we can make a change of measure under which $(N_t)$ is a CP of independent increments with mean $m_t = E N_t$ under the new measure $P$. (Theorem 2.6.1 of [5]). Using this fact and the three-step technique $P_0 \ll P$ means that the measure $P_0$ is absolutely continuous with respect to $P$ while $P_0 \sim P$ indicates that the two measures are equivalent.
of Duncan and Kailath (see Introduction) the likelihood ratio for detecting
CP's has been obtained according to

**THEOREM 3.2 (Theorem 3.4.4 of [5]):** For \(i = 0, 1\) let \((N_t)\) be,

under the measure \(P_i\), the CP described above. Assume

(a) \(P_0 \ll P\) and \(P \sim P_1\) and define for \(i = 0, 1\) the \((P, \mathcal{F}_t)\) martingale

\[
L_t^i = E\left(\frac{dP_i}{dP} \bigg| \mathcal{F}_t\right);
\]

(b) For \(i = 0, 1\), the stopping times \(T^i_n\) are such that there exists increas-

ing sequences of stopping times \((T^i_n)\) for which \(T^i_n = \lim_{n \to \infty} T^i_n\) a.s. and

\[E(\ln L_{T^i_n})^2 < \infty\]

for each \(n\). Let \(T = \bigwedge_{i=0}^1 T^i\);

(c) For \(i = 0, 1\)

\[
E \int_0^T \lambda_s^i \, dm_s < \infty.
\]

Then

\[
\frac{dP_{0 \wedge T}}{dP_{1 \wedge T}} = \prod_{n < T \wedge T} \frac{\lambda^0_n}{\lambda^1_n} \exp\left[\int_0^{T \wedge T} (\lambda^1_s - \lambda^0_s) \, dm_s\right]
\]

where \(\lambda^i_t \triangleq E \left(\lambda^i_t \big| \mathcal{F}_t\right)\) for \(i = 0, 1\) and \(J_n\) is the time of \(n^{\text{th}}\) jump of \((N_t)\). By

convention the product \(\prod(\cdot) = 1\) for \(J_{1} > T \wedge T\).

**Remark 3.3:** (a) The stopping time \(T^i\) which is the first time

after which the martingale \((L_t^i)\) can behave badly may take the value \(+ \infty\).

It is in fact desirable for \(T^i\) to be as large as possible.

(b) By our convention (Remark 2.2) condition (c) above insures that

the process \((\lambda^i_t)\) is well defined.
4. RECURSIVE INTEGRAL EQUATIONS FOR LIKELIHOOD RATIOS

We show here that the likelihood ratio (1) of our detection problem can be obtained as the unique solution of a stochastic integral equation. This stochastic integral equation can be mechanized by a feedback scheme tantamount to a recursive filter, as shown in Figure 1.

THEOREM 4.1: The likelihood ratio \( \frac{dP_{t \wedge T}}{dP_{T \wedge 1}} \) of Theorem 3.2 is the unique solution of the following stochastic integral equation:

\[
Z_t = 1 + \int_0^t Z_s \, dX_{s \wedge T}
\]

where

\[
X_t = \int_0^t \left\{ \frac{A^0_s}{A^1_s} - 1 \right\} \, dN_s + \int_0^t (A^1_s - A^0_s) \, dm_s
\]

**Proof:** By assumption \((\lambda_t^i), \ i=0,1\), is positive a.s. finite for all \(t\) (by condition (c) of Theorem 3.2 and Remark 2.2). The process \((N_t)\) has a finite number of jumps in any finite interval so that the process \(\int_0^{t \wedge T} (\frac{A^0_s}{A^1_s} - 1) \, dN_s\) has sample paths of bounded variation on any finite interval; and so does the process \(\int_0^{t \wedge T} (A^1_s - A^0_s) \, dm_s\) by assumption (c) of Theorem 3.2. Hence \((X_{t \wedge T})\) is a semimartingale with sample paths of bounded variation on any finite interval so that \(\langle X^C \rangle_{t \wedge T} = 0\) (see the remark, on p. 90, following proposition 3 of [3]). Then by Theorem 1 of [4] the unique solution of (2) is given by

\[
Z_t = \exp(X_{t \wedge T}) \prod_{s \leq t} (1 + \Delta X_{s \wedge T}) \exp(-\Delta X_{s \wedge T})
\]
Now \( \Delta X_{s \wedge T} = (\lambda_{s}^{0}/\lambda_{s}^{1} - 1)\Delta N_{s \wedge T} \) and hence the product in (4) becomes

\[
\Pi (\cdot) = \prod_{s \leq t} \left[ 1 + \left[ \frac{\lambda_{s}^{0}}{\lambda_{s}^{1}} - 1 \right] \Delta N_{s \wedge T} \right] \exp \left[ \sum_{s \leq t} \left[ \frac{\lambda_{s}^{0}}{\lambda_{s}^{1}} - 1 \right] \Delta N_{s \wedge T} \right]
\]

\[
= \prod_{J \leq t} \left[ \frac{\lambda_{J}^{0}}{\lambda_{J}^{1}} \right] \exp \left[ - \int_{0}^{t} \left[ \frac{\lambda_{s}^{0}}{\lambda_{s}^{1}} - 1 \right] dN_{s} \right]
\]

Substituting the above relation and expression (3) in (4) gives the desired result (compare with (1))

\[
Z_{t} = \prod_{J \leq t} \left[ \frac{\lambda_{J}^{0}}{\lambda_{J}^{1}} \right] \exp \left[ \int_{0}^{t} \left( \frac{\lambda_{s}^{1} - \lambda_{s}^{0}}{\lambda_{s}^{1} - \lambda_{s}^{0}} \right) dm_{s} \right] = \frac{dF_{t \wedge T}^{0}}{dF_{t \wedge T}^{1}}
\]

Observe that if under the measure \( P_{1} \) the CP \( (N_{t}) \) is a process of independent increments with mean \( m_{t} \) then \( P \equiv P_{1} \), \( \lambda_{1}^{t} = 1 \) and Eq. (3) becomes

\[
X_{t} = \int_{0}^{t} (\lambda_{s}^{0} - 1) d(N_{s} - m_{s})
\]

The process \( (M_{t} \Delta N_{t} - m_{t}) \) is a \((P, \mathcal{N}_{t})\) martingale. Hence (5) shows that the process \( (X_{t \wedge T}) \) is a local martingale. In turn, (2) then implies that the process \( (Z_{t}) \) is a local martingale. In this case we in fact have

\[
Z_{t} = E_{1}[dF_{0}^{00}/dF_{1}^{00} | \mathcal{N}_{t \wedge T}], \text{ i.e. the likelihood function is a uniformly integrable martingale.}
\]

In applications, Eqs. (2) and (3) give a way of implementing the computation of the likelihood ratio continuously in time. They represent recursive
equations if one also obtains the best estimates \( \hat{\lambda}_t \) in a recursive manner.

The block diagram of this implementation is given in Figure 1.
Recursive Scheme for Obtaining the Likelihood Function $Z_t$.

Figure 1
REFERENCES


2. P. M. Brémaud, A martingale approach to point processes, Memorandum No. ERL-M345, Electronic Research Laboratory, University of California, Berkeley, California, August 1972.


