

Algebraic Independence of Values of Elliptic Functions

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1. Introduction

In this article we give proofs of the theorems announced in [7] on the algebraic independence of values of elliptic functions. Let $\wp(z)$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3 . If $\wp(z)$ has complex multiplication we write k for the associated quadratic field; otherwise, if $\wp(z)$ has no complex multiplication, then k denotes the rational field \mathbb{Q} . For an integer $n \geq 1$ let u_1, \dots, u_n be complex numbers linearly independent over k , and for an integer $m \geq 1$ let v_1, \dots, v_m be complex numbers linearly independent over \mathbb{Q} . Our results are as follows.

Theorem 1. *If $mn \geq 2m + 4n$ then at least two of the numbers*

$$\wp(u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

are defined and are algebraically independent over \mathbb{Q} .

Theorem 2. *If $mn \geq 2m + 2n$ then at least two of the numbers*

$$u_i, \wp(u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

are defined and are algebraically independent over \mathbb{Q} .

Theorem 3. *If $mn \geq m + 4n$ then at least two of the numbers*

$$v_j, \wp(u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

are defined and are algebraically independent over \mathbb{Q} .

Theorem 4. *If $mn > m + 2n$ then at least two of the numbers*

$$u_i, v_j, \wp(u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

are defined and are algebraically independent over \mathbb{Q} .

Theorem 5. *Suppose that $m = 4$, $n = 2$ and that the numbers*

$$\wp(u_1 v_j) \quad (1 \leq j \leq m)$$

are defined and are algebraic over \mathbb{Q} . Then at least two of the numbers

$$u_i, v_j, \wp(u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

are defined and are algebraically independent over \mathbb{Q} .

Some numerically weaker results of the same form, but relating to more general functions, were obtained by the second author in [17]; for example, they imply our Theorem 2 with $m=16, n=3$ (instead of $m=6, n=3$). See also the related work [11, 12] of Smelev, and the earlier results [3] of Brownawell and Kubota on transcendence types.

We leave it to the reader to deduce the usual corollaries from our theorems by suitably specializing the u 's and v 's. But in the case of complex multiplication two of these are sufficiently interesting to be recorded here.

Corollary 1. *Suppose $\wp(z)$ has complex multiplication over $k \neq \mathbb{Q}$. Then if u is a complex number such that $\wp(u)$ is defined and is algebraic over \mathbb{Q} , and β is cubic over k , the numbers $\wp(\beta u), \wp(\beta^2 u)$ are defined and are algebraically independent over \mathbb{Q} .*

Corollary 2. *Suppose $\wp(z)$ has complex multiplication over $k \neq \mathbb{Q}$. Then at least one of the numbers $\wp(\wp(1)), \wp((\wp(1))^2)$ is defined and is transcendental over \mathbb{Q} .*

Our results are the natural elliptic analogues of theorems of Gelfond [6], Brownawell [1, 2], Smelev [10], Tijdeman [14], Waldschmidt [15, 16], and Wallisser for the exponential function. In particular Corollary 1 may be compared with Gelfond's result on the algebraic independence of $\alpha^\beta, \alpha^{\beta^2}$ for algebraic $\alpha \neq 0, 1$ and cubic β . And Corollary 2 may be compared with the theorem of Brownawell and Waldschmidt to the effect that at least one of e^e, e^{e^2} is transcendental.

An essential component in our proofs consists of some algebraic zero estimates for polynomials in elliptic functions. These replace the fundamental analytic estimates [13] of Tijdeman for the exponential case. They will be proved in Sect. 2 as a consequence of our recent work [9]. Then in Sect. 3 we prove Theorems 1, 3, and 4. Here we shall be comparatively brief in our exposition, since this part of the proof closely follows the exponential analogue, apart from the (by now routine) use of the trick known as the Baker-Coates argument. Then in Sect. 4 we prove Theorems 2 and 5. Here we shall be even briefer, and we shall leave to the reader the detailed checking of the zero estimates required.

We end this introduction by deducing Corollaries 1 and 2. Choose τ such that $k = \mathbb{Q}(\tau) \neq \mathbb{Q}$. Corollary 1 can be proved in two ways. We put $m=6, n=3$. Then either we apply Theorem 2 with $u_1=1, u_2=\beta, u_3=\beta^2$ and

$$v_1 = u, \quad v_2 = \beta u, \quad v_3 = \beta^2 u, \quad v_4 = \tau u, \quad v_5 = \tau \beta u, \quad v_6 = \tau \beta^2 u,$$

or we apply Theorem 3 with $u_1 = u, u_2 = \beta u, u_3 = \beta^2 u$ and

$$v_1 = 1, \quad v_2 = \beta, \quad v_3 = \beta^2, \quad v_4 = \tau, \quad v_5 = \tau \beta, \quad v_6 = \tau \beta^2.$$

For Corollary 2 we can apply Theorem 5 with $u_1 = \wp(1), u_2 = 1$ and

$$v_1 = 1, \quad v_2 = \wp(1), \quad v_3 = \tau, \quad v_4 = \tau \wp(1),$$

since it is well-known that $\wp(1)$ is not in k (and is even transcendental).

2. A Zero Estimate

Let $\wp(z)$ and k be defined as in Sect. 1, let \mathcal{L} be the period lattice of $\wp(z)$, and let $\mathcal{L}_0 = \mathcal{L} \otimes \mathbb{Q}$ be the set of all $x\omega$ for x in \mathbb{Q} and ω in \mathcal{L} . Then \mathcal{L}_0 is a vector space over k . For integers $m \geq 1$, $n \geq 1$ let u_1, \dots, u_n and v_1, \dots, v_m be complex numbers linearly independent over k and \mathbb{Q} respectively. Let U be the vector space over k generated by u_1, \dots, u_n , and let V be the vector space over \mathbb{Q} generated by v_1, \dots, v_m . We define integers $m_r = m_r(U, V)$ ($1 \leq r \leq n$) as follows. For an integer r with $1 \leq r \leq n$ let m_r be the maximum dimension over \mathbb{Q} of any subspace V' of V for which there exists a subspace U' of U , of dimension r over k , such that $u'v'$ lies in \mathcal{L}_0 for all u' in U' and all v' in V' . Finally let z_0 be a complex number such that for all v in V none of the numbers $z_0 + u_i v$ ($1 \leq i \leq n$) lies in \mathcal{L} .

Proposition 1. *There is a constant c depending only on m and n with the following property. Let $S \geq 0$ be a real number and let $D \geq 1$, $L \geq 1$, $T \geq 1$ be integers satisfying*

$$TS^m \geq cLD^n, \quad S^m \geq cLD^{n-2} \tag{1}$$

and

$$TS^{m-m_r} \geq cD^r, \quad S^{m-m_r} \geq cD^{r-1} \quad (1 \leq r \leq n). \tag{2}$$

Suppose $P = P(x_0, x_1, \dots, x_n)$ is a polynomial, of degree at most L in x_0 and of degree at most D in each of x_1, \dots, x_n , such that the function

$$\varphi(z) = P(z, \wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S.$$

Then P is identically zero.

Proof. This is a straightforward deduction from one of the main results of [9] for a product of k group varieties. For the convenience of the reader we reproduce this result here in the case $k=2$, which happens to suffice for our applications.

Accordingly let H and K be commutative group varieties defined over the complex field \mathbb{C} and embedded in projective spaces \mathbb{P}_M and \mathbb{P}_N for integers $M \geq 1$ and $N \geq 1$ respectively. Then $G = H \times K$ is embedded in $\mathbb{P}_M \times \mathbb{P}_N$, whose points can be described in the usual way in terms of biprojective coordinates x_0, \dots, x_M and y_0, \dots, y_N . Since G is a smooth variety, it has the natural structure of a complex manifold. Let Φ be an analytic map from \mathbb{C} to G that is a homomorphism of additive groups, and denote also by Φ the image $\Phi(\mathbb{C})$ in G . Assume Φ does not reduce to the origin of G .

For an element g of G and a bihomogeneous polynomial P in $\mathbb{C}[X_0, \dots, X_M, Y_0, \dots, Y_N]$ we define the order of vanishing of P at g along Φ as follows. It is well-known that the map T_g from G to G representing translation by g is analytic. Hence the composition $\Psi(z) = T_g(\Phi(z))$ is an analytic map from \mathbb{C} to G .

In particular, there exist functions

$$\xi_0(z), \dots, \xi_M(z), \eta_0(z), \dots, \eta_N(z), \quad (3)$$

analytic near $z=0$, such that for all z sufficiently small their values are biprojective coordinates of $\Psi(z)$. If now the function

$$f(z) = P(\xi_0(z), \dots, \xi_M(z), \eta_0(z), \dots, \eta_N(z))$$

is identically zero we write $\text{ord}_g P = \infty$, this symbol being subject to the usual conventions. Otherwise we define $\text{ord}_g P$ as the order of zero of $f(z)$ at $z=0$. It is easily seen that this definition is independent of the choice of functions (3) that represent $\Psi(z)$.

Next we assume that H and K are disjoint in the sense described in [9]. Namely, we assume that every connected algebraic subgroup C of G has the form $C = A \times B$ for algebraic subgroups A of H and B of K . Let X be a finite subset of G containing the origin. Suppose the dimensions of H and K are $p \geq 1$ and $q \geq 1$ respectively. For integers r, s with

$$0 \leq r \leq p, \quad 0 \leq s \leq q, \quad r + s \geq 1$$

we define quantities $Q_{rs}(X)$ as follows. Firstly, if either H has no algebraic subgroups of codimension r or K has no algebraic subgroups of codimension s we put

$$Q_{rs}(X) = |X|$$

the cardinality of X . Otherwise for each subgroup C of G write $|X/C|$ for the maximum number of elements of X that are distinct modulo C . Then we put

$$Q_{rs}(X) = \min_C |X/C|$$

as C runs over all connected algebraic subgroups of G of the form $C = A \times B$ with A a connected algebraic subgroup of H of codimension r and B a connected algebraic subgroup of K of codimension s . Finally for an integer $k \geq 1$ let $X^{(k)}$ denote the set of all sums $x_1 + \dots + x_k$ for x_1, \dots, x_k in X .

Now the Main Theorem (disjoint version) of [9] states the existence of a constant c' depending only on H and K with the following property. Suppose for integers $L \geq 1$, $D \geq 1$ and $T \geq 1$ there is a polynomial P' in $\mathbb{C}[X_0, \dots, X_M, Y_0, \dots, Y_N]$, homogeneous of degree at most L in X_0, \dots, X_M and homogeneous of degree at most D in Y_0, \dots, Y_N , that vanishes at each point of $X^{(p+q)}$ to order at least T along Φ . As in [9], we say that a group variety embedded in projective space is linear if its addition laws can be given by bilinear forms. Let T_0 be any integer with $T_0 \leq L$ if H is not a linear group variety and $T_0 \leq D$ if K is not a linear group variety. Suppose that

$$TQ_{rs}(X) \geq c'LD^s \quad (0 \leq r \leq p, 0 \leq s \leq q, r + s \geq 1) \quad (4)$$

and

$$T_0Q_{rs}(X) \geq c'LD^s \quad (0 \leq r \leq p, 0 \leq s \leq q, 1 \leq r + s < p + q). \quad (5)$$

Suppose further that either

$$T_0Q_{pq}(X) \geq c'L^pD^q$$

or

$$|X/G_{\text{tors}}| = |X|, \tag{6}$$

where G_{tors} is the torsion part of G . Then P' vanishes on all of $g + \Phi$ for some g in G .

We can now deduce our Proposition 1. Let $\sigma(z)$ be the Weierstrass sigma function associated with $\wp(z)$, and denote by $\psi(z)$ the point in \mathbb{P}_2 with projective coordinates

$$\wp(z_0 + z)(\sigma(z_0 + z))^3, \wp'(z_0 + z)(\sigma(z_0 + z))^3, (\sigma(z_0 + z))^3.$$

As in [8, p. 511] denote by E the elliptic curve such that ψ is a group homomorphism from \mathbb{C} to E . Thus E is embedded in \mathbb{P}_2 , and $K = E^n$ can then be embedded in \mathbb{P}_N by means of the standard Segre map λ from \mathbb{P}_2^n to \mathbb{P}_N , where $N = 3^n - 1$. Also the additive group $H = \mathbb{C}$ can be embedded in \mathbb{P}_1 as a linear group, and we put $G = H \times K$. From part (i) of Lemma 7 of [9] it is clear that H and E are disjoint, since the only relevant subquotients are H and E/F respectively, where F is a finite subgroup of E . These cannot be isomorphic, as one is complete and the other is not. Hence by repeated use of part (ii) of Lemma 7 of [9] we see that H and $K = E^n$ are also disjoint.

We define the map Φ from \mathbb{C} to $G = H \times K$ by

$$\Phi(z) = (z, \lambda(\psi(u_1 z), \dots, \psi(u_n z))),$$

and the points g_1, \dots, g_m of G by

$$g_j = \Phi(v_j) \quad (1 \leq j \leq m).$$

Let X be the set of all combinations $s_1 g_1 + \dots + s_m g_m$ for integers s_1, \dots, s_m with

$$0 \leq s_1, \dots, s_m \leq S/(n+1).$$

Thus any g in $X^{(n+1)}$ has the form $g = s_1 g_1 + \dots + s_m g_m$ for integers s_1, \dots, s_m with $0 \leq s_1, \dots, s_m \leq S$. If $v = s_1 v_1 + \dots + s_m v_m$ then $g = \Phi(v)$ and so the map $\Psi(z) = T_g(\Phi(z))$ is given by

$$\Psi(z) = \Phi(z + v) = (z + v, \lambda(\psi(u_1(z + v)), \dots, \psi(u_n(z + v)))).$$

Since by hypothesis the function

$$\wp(z + v) = P(z + v, \wp(z_0 + u_1(z + v)), \dots, \wp(z_0 + u_n(z + v)))$$

has a zero of order at least T at $z=0$, it follows without difficulty that the polynomial P of Proposition 1 gives rise to a bihomogeneous polynomial P' vanishing on $X^{(n+1)}$ to order at least T along Φ . Because H is a linear group variety, we may choose $T_0 = D$. We now verify that the conditions (1), (2) imply the conditions (4)–(6). In fact (6) is clear at once, as $G_{\text{tors}} = 0 \times K_{\text{tors}}$ and v_1, \dots, v_m are linearly independent over \mathbb{Q} . To verify (4) and (5) we have to estimate the quantities $Q_{rs}(X)$ from below.

To start with, we note that

$$Q_{1s}(X) = |X| \geq (S/(n+1))^m \quad (0 \leq s \leq n),$$

since the subgroup C of G now has the form $0 \times B$. Next, we claim that

$$Q_{0s}(X) \geq (S/(n+1))^{m-m_s} \quad (1 \leq s \leq n). \tag{7}$$

This is clear if $m_s = m$. So assume $m_s < m$. The subgroup C of G now has the form $H \times B$ for some algebraic subgroup B of K of codimension s . If \mathcal{O} is the endomorphism ring of the lattice \mathcal{L} , it follows from Lemma 11 (p. 512) of [8] that we can find s elements (t_{i1}, \dots, t_{in}) ($1 \leq i \leq s$) of \mathcal{O}^n , linearly independent over k , such that for any $(\psi(z_1), \dots, \psi(z_n))$ in B we have

$$\psi(t_{i1}z_1 + \dots + t_{in}z_n) = 0 \quad (1 \leq i \leq s);$$

that is, $t_{i1}z_1 + \dots + t_{in}z_n$ lies in \mathcal{L} ($1 \leq i \leq s$). Thus the elements

$$u_i = t_{i1}u_1 + \dots + t_{in}u_n \quad (1 \leq i \leq s)$$

of U generate a subspace U' of U of dimension s over k . Let V' be the subspace of all v' in V such that $u'v'$ lies in \mathcal{L}_0 for all u' in U' . By the definition of m_s , the dimension of V' over \mathbb{Q} is at most m_s . We may therefore find $l = m - m_s$ elements among v_1, \dots, v_m that are linearly independent over \mathbb{Q} modulo V' ; and without loss of generality we can suppose these to be v_1, \dots, v_l .

Now we observe that the elements of X given by

$$s_1g_1 + \dots + s_lg_l \quad (0 \leq s_1, \dots, s_l \leq S/(n+1))$$

are distinct modulo $C = H \times B$. For if not, we could find integers s'_1, \dots, s'_l , not all zero, such that $g' = s'_1g_1 + \dots + s'_lg_l$ lies in C . Thus if $v' = s'_1v_1 + \dots + s'_lv_l$ the element $(\psi(u_1v'), \dots, \psi(u_nv'))$ lies in B , and this would imply that $t_{i1}u_1v' + \dots + t_{in}u_nv'$ lies in \mathcal{L} ($1 \leq i \leq s$). Thus $u'v'$ lies in \mathcal{L} for all u' in U' . Hence v' lies in V' ; however, this contradicts the linear independence of v_1, \dots, v_l modulo V' . It follows that $|X/C| \geq (S/(n+1))^{m-m_s}$, and then taking the minimum over all C gives (7).

It is now easy to verify that if we define c in terms of c' by $c = c'(n+1)^m$ then the conditions (1), (2) imply (4), (5). Accordingly our bihomogeneous polynomial P' vanishes on $g + \Phi$ for some g in G . To deduce that the original polynomial P is identically zero it suffices to prove that Φ is Zariski-dense in G . But the Zariski-closure is an algebraic subgroup of G ; let C be its connected component through the origin. By disjointness $C = A \times B$ for algebraic subgroups A of H and B of K , and clearly from the form of $\Phi(z)$ we must have $A = H$. But also $B = K$, otherwise by Lemma 11 of [8] there would exist a non-zero (t_1, \dots, t_n) in \mathcal{O}^n such that $t_1u_1z + \dots + t_nu_nz$ lies in \mathcal{L} for all complex z , which is obviously impossible as $t_1u_1 + \dots + t_nu_n \neq 0$. Thus $C = G$, and Φ is indeed Zariski-dense in G . This completes the proof of Proposition 1.

Actually the calculations of [9] show that c' can be taken as $4^{3^{n+1}}$. Hence Proposition 1 holds with $c = 4^{3^{n+1}}(n+1)^m$.

To apply the proposition we need upper bounds for the numbers m_1, \dots, m_n . We record these in the following simple lemma.

Lemma 1. *We have $m_1 \leq 2$; also if $n \geq 2$ then $m_2 \leq 1$; and if $n \geq 3$ then $m_r = 0$ whenever $3 \leq r \leq n$.*

Proof. First suppose on the contrary that $m_1 \geq 3$. Then there exist elements v'_1, v'_2, v'_3 of V , linearly independent over \mathbb{Q} , and a non-zero element u'_1 of U , such that $u'_1v'_1, u'_1v'_2, u'_1v'_3$ all lie in \mathcal{L}_0 . Since \mathcal{L}_0 is a vector space over \mathbb{Q} of dimension 2,

there are rationals x_1, x_2, x_3 , not all zero, such that

$$0 = x_1 u'_1 v'_1 + x_2 u'_1 v'_2 + x_3 u'_1 v'_3 = u'_1 (x_1 v'_1 + x_2 v'_2 + x_3 v'_3).$$

This is clearly impossible under the hypotheses given, and hence indeed $m_1 \leq 2$.

Next suppose $n \geq 2$ and $m_2 \geq 2$. Then there exist elements v'_1, v'_2 of V , linearly independent over \mathbb{Q} , and elements u'_1, u'_2 of U , linearly independent over k , such that $u'_1 v'_1, u'_1 v'_2, u'_2 v'_1, u'_2 v'_2$ all lie in \mathcal{L}_0 . If V' is the vector space over \mathbb{Q} generated by v'_1, v'_2 , it follows that $\mathcal{L}_0 = u'_1 V' = u'_2 V'$. Thus u'_2/u'_1 is an endomorphism of \mathcal{L}_0 , so it must lie in k , which is not possible. Hence indeed $m_2 \leq 1$.

Finally suppose $n \geq 3$ and $m_r \geq 1$ for some r with $3 \leq r \leq n$. Then there exists a non-zero v'_1 of V , and elements u'_1, \dots, u'_r of U linearly independent over k such that $u'_1 v'_1, \dots, u'_r v'_1$ lie in \mathcal{L}_0 . In particular u'_1, u'_2, u'_3 are linearly independent over \mathbb{Q} , and now the argument of the first part applies *mutatis mutandis*. This completes the proof of Lemma 1.

It is clear that these upper bounds can be improved in the case $k \neq \mathbb{Q}$ of complex multiplication; in fact one then has $m_1 \leq 1$ and $m_r = 0$ for $r \geq 2$. But we will not need these improvements except for a very special case of Theorem 5.

3. Proof of Theorems 1, 3, 4

We shall describe in turn the proofs of Theorems 1, 3, and 4. Now that Proposition 1 is available, no new principles are involved, and we can be relatively concise.

Suppose \mathfrak{g} is transcendental over \mathbb{Q} . Then any non-zero λ in $\mathbb{Z}[\mathfrak{g}]$ can be written uniquely as $P(\mathfrak{g})$ for some non-zero polynomial $P(x)$ in $\mathbb{Z}[x]$. Recall that the (logarithmic) type of λ is defined as $\max(d, \log H)$, where d is the degree of P and H is the maximum of the absolute values of its coefficients. It is convenient also to define the type of the zero element as $-\infty$.

The first step in all the proofs is to find a complex number z_0 such that $\wp(z_0)$ is algebraic and such that for any integers s_1, \dots, s_m none of the numbers

$$z_0 + u_i(s_1 v_1 + \dots + s_m v_m) \quad (1 \leq i \leq n)$$

lie in the period lattice \mathcal{L} of $\wp(z)$. This is clearly possible.

Now assume Theorem 1 is false for some integers $m \geq 1, n \geq 1$ satisfying

$$mn \geq 2m + 4n. \tag{8}$$

It follows easily that the numbers

$$g_2, g_3, \wp(z_0), \wp'(z_0), \wp(z_0 + u_i v_i), \wp'(z_0 + u_i v_i) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

lie in a field K of transcendence degree at most 1 over \mathbb{Q} . We can therefore find complex numbers $\mathfrak{g}, \mathfrak{g}_1$ such that K is contained in $\mathbb{Q}(\mathfrak{g}, \mathfrak{g}_1)$, where \mathfrak{g} is transcendental over \mathbb{Q} , and \mathfrak{g}_1 is algebraic over $\mathbb{Q}(\mathfrak{g})$. We fix a suitably large constant C depending only on these quantities, and we choose any large integer D . We define

$$S = CD^{n/m}, \quad S_0 = C^{-2} S.$$

We use c_1, c_2, \dots for positive constants independent of D .

From Siegel's version of the Box Principle we can construct a non-zero polynomial $P(x_1, \dots, x_n)$, of degree at most D in each variable, with the following properties. Firstly, the coefficients of P are elements of $\mathbb{Z}[\mathcal{G}]$ with types at most $c_1 D S_0^2$, and secondly, the function

$$\varphi(z) = P(\wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers s_1, \dots, s_m with

$$0 \leq s_1, \dots, s_m \leq S_0.$$

Next, by applying the maximum modulus principle on circles of radius CS and $CD^{\epsilon}S$ for suitably small $\epsilon > 0$ we see that

$$\log|\varphi(s_1 v_1 + \dots + s_m v_m)| < -c_2 S_0^m \log D$$

for all integers s_1, \dots, s_m with

$$0 \leq s_1, \dots, s_m \leq S.$$

We now apply Proposition 1 with $L = T = 1$ to deduce that at least one of these numbers $\xi = \varphi(s_1 v_1 + \dots + s_m v_m)$ is non-zero. By Lemma 1, it suffices to check the conditions

$$S^m \geq cD^n, \quad S^{m-1} \geq cD^2, \quad S^{m-2} \geq cD.$$

But these follow from the definition of S and the inequality (8).

Finally, by clearing denominators and taking norms of ξ in the usual way we end up with a non-zero element $\pi = \pi_D$ of $\mathbb{Z}[\mathcal{G}]$, of type at most $c_3 D S^2$, such that

$$\log|\pi_D| < -c_4 S_0^m \log D.$$

Since

$$(DS^2)^2 \leq c_5 D^{(2m+4n)/m}, \quad S_0^m \log D \geq c_6 D^n \log D,$$

the inequality (8) shows that we can apply the well-known criterion of Gelfond to the sequence of elements π_D to obtain a final contradiction. This proves Theorem 1. We note that the critical pairs (m, n) satisfying (8) are $(5, 10)$, $(6, 6)$, $(7, 5)$, $(8, 4)$, and $(12, 3)$.

Next assume Theorem 3 is false for some integers $m \geq 1$, $n \geq 1$ satisfying

$$mn \geq m + 4n. \tag{9}$$

We note that this implies $m \geq 5$. This time the numbers

$$g_2, g_3, \wp(z_0), \wp'(z_0), v_j, \wp(z_0 + u_i v_j), \wp'(z_0 + u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

lie in a field $\mathbb{Q}(\mathcal{G}, \mathcal{G}_1)$ as before. For C, D as above we define

$$S = CD^{(n+1)/(m-2)}(\log D)^{-1/(m-2)}, \quad S_0 = C^{-2} S$$

and

$$L = [D^{(m+2n)/(m-2)}(\log D)^{-m/(m-2)}].$$

We construct a non-zero polynomial $P(x_0, x_1, \dots, x_n)$, of degree at most L in x_0 and of degree at most D in each of x_1, \dots, x_n , whose coefficients are in $\mathbb{Z}[\mathfrak{g}]$ of types at most

$$c_1(L \log D + DS_0^2) \leq c_2 DS^2,$$

such that the function

$$\varphi(z) = P(z, \wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers s_1, \dots, s_m with

$$0 \leq s_1, \dots, s_m \leq S_0.$$

The maximum modulus principle on the circles of radius CS and $CD^s S$ then gives

$$\log |\varphi(s_1 v_1 + \dots + s_m v_m)| < -c_3 S_0^m \log D$$

for all integers s_1, \dots, s_m with

$$0 \leq s_1, \dots, s_m \leq S.$$

We apply Proposition 1 with $T=1$ to deduce that at least one of these numbers $\varphi(s_1 v_1 + \dots + s_m v_m)$ is non-zero. By Lemma 1, it suffices to check the conditions

$$S^m \geq cLD^n, \quad S^{m-1} \geq cD^2, \quad S^{m-2} \geq cD.$$

These follow from the definitions and the inequality (9).

So we get a non-zero element $\pi = \pi_D$ of $\mathbb{Z}[\mathfrak{g}]$, of type at most $c_4 DS^2$, such that

$$\log |\pi_D| < -c_5 S_0^m \log D.$$

Since

$$(DS^2)^2 \leq c_6 D^{(2m+4n)/(m-2)} (\log D)^{-4/(m-2)},$$

$$S_0^m \log D \geq c_7 D^{(mn+m)/(m-2)} (\log D)^{-2/(m-2)},$$

we see again from (9) that Gelfond's criterion applies and gives our contradiction. This proves Theorem 3. The critical values of (m, n) are $(5, 5)$, $(6, 3)$, and $(8, 2)$.

Finally assume Theorem 4 is false for some integers $m \geq 1, n \geq 1$ satisfying

$$mn > m + 2n. \tag{10}$$

Then $m \geq 3$, and the numbers

$$g_2, g_3, \wp(z_0), \wp'(z_0), u_i, v_j, \wp(z_0 + u_i v_j), \wp'(z_0 + u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

lie in $\mathbb{Q}(\mathfrak{g}, \mathfrak{g}_1)$. We put

$$l = \min(m + 2n - 1, 2m) \tag{11}$$

and we define

$$S = CD^{n/m}, \quad S_0 = C^{-2} S$$

$$L = T = [D^{l/m}].$$

We construct a non-zero polynomial $P(x_0, x_1, \dots, x_n)$, of degree at most L in x_0 and of degree at most D in each of x_1, \dots, x_n , whose coefficients are in $\mathbb{Z}[\vartheta]$ of types at most

$$c_1(T \log D + L \log D + DS_0^2) \leq c_2 DS^2,$$

such that the function

$$\varphi(z) = P(z, \wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S_0.$$

We should emphasize here that in order to avoid quantities of order TS_0^2 appearing in the estimates for the types, the Baker-Coates trick should be used (see for example [4, p. 208]). The maximum modulus principle on circles of radius CS and $CD^e S$ then gives

$$\log |\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m)| < -c_3 TS_0^m \log D$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S.$$

We apply Proposition 1 to deduce that at least one of these numbers $\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m)$ is non-zero. By Lemma 1, it suffices to check the conditions

$$TS^m \geq cLD^n, \quad TS^{m-1} \geq cD^2, \quad TS^{m-2} \geq cD$$

and

$$S^m \geq cLD^{n-2}, \quad S^m \geq cD^{n-1}, \quad S^{m-1} \geq cD, \quad S^{m-2} \geq c. \tag{12}$$

Using (10) and (11), we easily verify all of these. Thus we obtain a non-zero element $\pi = \pi_D$ of $\mathbb{Z}[\vartheta]$, of type at most $c_4 DS^2$, such that

$$\log |\pi_D| < -c_5 TS_0^m \log D.$$

This time we have

$$(DS^2)^2 \leq c_6 D^{(2m+4n)/m}, \quad TS_0^m \log D \geq c_7 D^{(l+mn)/m} \log D,$$

and it can be verified from (10) and (11) that

$$l + mn \geq 2m + 4n$$

provided $m \geq 4$. Now the critical solutions of (10) are given by $(m, n) = (3, 4), (4, 3)$, and $(5, 2)$. Hence we deduce Theorem 4 in all these cases except the first. It remains thus to consider $(m, n) = (3, 4)$.

But if $k = \mathbb{Q}$ we see by interchanging the u 's and v 's that Theorem 4 with $(m, n) = (3, 4)$ is equivalent to the same theorem with $(m, n) = (4, 3)$, and is therefore also proved. Next suppose $k = \mathbb{Q}(\tau) \neq \mathbb{Q}$. Consider the vector space generated over

k by v_1, v_2, v_3 . This is also a vector space over \mathbb{Q} whose dimension d is even with $d \geq 3$. Hence $d \geq 4$, and without loss of generality we can assume that $v_4 = \tau v_1$ together with v_1, v_2, v_3 are linearly independent over \mathbb{Q} . Now since $\wp(u_i v_4)$ is algebraic over $\mathbb{Q}(\wp(u_i v_1))$ ($1 \leq i \leq 4$), we are in the situation of Theorem 4 with $(m, n) = (4, 4)$. But such a result is covered by the case $(m, n) = (4, 3)$ just established. This completes the proof of Theorem 4 in general.

4. Proof of Theorems 2, 5

For Theorem 2 we shall need the following simpler version of Proposition 1 when the polynomial $P(x_0, x_1, \dots, x_n)$ is independent of x_0 . We keep the same notation for $u_1, \dots, u_n, v_1, \dots, v_m, z_0$ and the integers m_1, \dots, m_n .

Proposition 2. *There is a constant c depending only on m and n with the following property. Let $S \geq 0$ be a real number and let $D \geq 1, T \geq 1$ be integers satisfying*

$$TS^{m-m_r} \geq cD^r \quad (1 \leq r \leq n)$$

and (if $n \geq 2$)

$$S^{m-m_r} \geq cD^{r-1} \quad (1 \leq r < n).$$

Suppose $P = P(x_1, \dots, x_n)$ is a polynomial, of degree at most D in each of x_1, \dots, x_n , such that the function

$$\varphi(z) = P(\wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S.$$

Then P is identically zero.

Proof. This can be deduced from either Main Theorem of [9] just as we proved Proposition 1. But it is simpler to appeal directly to Theorem A (p. 514) of [8] (see [9, Sect. 8]), as there is then no need to check the condition (6) above. We leave the details to the reader.

Now assume Theorem 2 is false for some integers $m \geq 1, n \geq 1$ satisfying

$$mn \geq 2m + 2n. \tag{13}$$

Then $m \geq 3, n \geq 3$, and, choosing z_0 as in Sect. 3, we see that the numbers

$$g_2, g_3, \wp(z_0), \wp'(z_0), u_i, \wp(z_0 + u_i v_j), \wp'(z_0 + u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

lie in a field $\mathbb{Q}(\wp, \wp')$ as before. For C, D as in Sect. 3 we define

$$S = CD^{(n-1)/(m+2)}(\log D)^{1/(m+2)}, \quad S_0 = C^{-2} S$$

$$T = [D^{(m+2n)/(m+2)}(\log D)^{-m/(m+2)}].$$

Using the Baker-Coates argument, we construct a non-zero polynomial $P(x_1, \dots, x_n)$, of degree at most D in each variable, whose coefficients are in $\mathbb{Z}[\mathcal{G}]$ of types at most

$$c_1(T \log D + DS_0^2) \leq c_2 DS^2,$$

such that the function

$$\varphi(z) = P(\wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S_0.$$

Next the maximum modulus principle on circles of radius CS and $CD^s S$ gives

$$\log |\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m)| < -c_3 TS_0^m \log D$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S.$$

We apply Proposition 2 to deduce that at least one of these numbers $\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m)$ is non-zero. By Lemma 1, it suffices to check the conditions

$$TS^m \geq cD^n, \quad TS^{m-1} \geq cD^2, \quad TS^{m-2} \geq cD$$

as well as

$$S^m \geq cD^{n-2}, \quad S^{m-1} \geq cD, \quad S^{m-2} \geq c. \tag{14}$$

Using (13) we easily verify all of these except the first of (14), which needs the additional inequality $m \geq 2n - 4$. We therefore temporarily assume this. Then we obtain a non-zero element $\pi = \pi_D$ of $\mathbb{Z}[\mathcal{G}]$, of type at most $c_4 DS^2$, such that

$$\log |\pi_D| < -c_5 TS_0^m \log D.$$

This time we have

$$(DS^2)^2 \leq c_6 D^{(2m+4n)/(m+2)} (\log D)^{4/(m+2)},$$

$$TS_0^m \log D \geq c_7 D^n \log D,$$

and so Gelfond's criterion applies using (13). Now the critical solutions of (13) are given by $(m, n) = (3, 6)$, $(4, 4)$, and $(6, 3)$. Since we have assumed $m \geq 2n - 4$, we deduce Theorem 2 in all these cases except the first. But we can deal with the remaining case much as in the proof of Theorem 4. Namely, if $k = \mathbb{Q}$ we see on interchanging the u 's and v 's that Theorem 2 with $(m, n) = (3, 6)$ is equivalent to Theorem 3 with $(m, n) = (6, 3)$, and is therefore also proved. If $k \neq \mathbb{Q}$ we can adjoin v_4 as before, and this puts us in the situation of Theorem 2 with $(m, n) = (4, 6)$. But such a result is covered by the case $(m, n) = (4, 4)$ of Theorem 2 just established. This completes the proof of Theorem 2 in general.

For Theorem 5 we shall need the following zero estimate, again in the notation of Sect. 2.

Proposition 3. *There is a constant c depending only on m and n with the following property. Let $S \geq 0$ be a real number and let $D_0, \dots, D_n, T \geq 1$ be integers with*

$$D_0 \geq D_1 \geq \dots \geq D_n \geq 1$$

and satisfying

$$TS^m \geq cD_0D_1 \dots D_n$$

$$TS^{m-m_r} \geq cD_0 \dots D_{r-1}, \quad D_n S^{m-m_r} \geq cD_0 \dots D_{r-1} \quad (1 \leq r \leq n).$$

Suppose $P = P(x_0, x_1, \dots, x_n)$ is a polynomial, of degree at most D_r in x_r ($0 \leq r \leq n$), such that the function

$$\varphi(z) = P(z, \wp(z_0 + u_1 z), \dots, \wp(z_0 + u_n z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_m v_m) = 0$$

for all integers t, s_1, \dots, s_m with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_m \leq S.$$

Then P is identically zero.

Proof. This can be deduced from the Main Theorem (disjoint version) of [9] just as we proved Propositions 1 and 2, using disjointness only to observe that in the notation of [9] we have

$$Q_s(X) = \min(Q_{0s}(X), Q_{1,s-1}(X)) \quad (1 \leq s \leq n)$$

and

$$Q_{n+1}(X) = Q_{1n}(X)$$

for any finite subset X of $\mathbb{C} \times E^n$. Alternatively we can appeal to Theorem ABC (p. 515) of [8]. We leave the details to the reader.

We shall also need the following special algebraic independence result for a Weierstrass elliptic function with algebraic invariants.

Lemma 2. *Suppose $\wp(z)$ has no complex multiplication, let ω_1, ω_2 be periods of $\wp(z)$ linearly independent over \mathbb{Q} , and suppose χ_1, χ_2, χ_3 are complex numbers with $1, \chi_1, \chi_2, \chi_3$ linearly independent over \mathbb{Q} . Then at least two of the numbers*

$$\omega_1/\omega_2, \wp(\omega_i \chi_j) \quad (i = 1, 2; j = 1, 2, 3)$$

are defined and are algebraically independent over \mathbb{Q} .

Proof. We only give a very rapid sketch, since the result could have been proved by classical methods without the aid of zero estimates on group varieties (see Theorem 4.2 (p. 314) of Chudnovsky [5] for a similar result). For z_0, \wp, \wp_1, C, D as before define

$$S = CD^{3/14}, \quad S_0 = C^{-2}S, \quad T = [D^{19/14}],$$

and construct a non-zero polynomial $P(x_1, x_2)$, of degree at most D in each variable, whose coefficients are in $\mathbb{Z}[\mathcal{G}]$ of types at most $c_1 DS_0^2 \leq c_1 D^{10/7}$, such that

$$\varphi(z) = P(\wp(z_0 + \omega_1 z), \wp(z_0 + \omega_2 z))$$

satisfies

$$\varphi^{(t)}(s_1 \chi_1 + s_2 \chi_2 + s_3 \chi_3) = 0$$

for all integers t, s_1, s_2, s_3 with

$$0 \leq t < T, \quad 0 \leq s_1, s_2, s_3 \leq S_0.$$

Put

$$R = D^{13/14};$$

then by periodicity we have

$$\varphi^{(t)}(s_1 \chi_1 + s_2 \chi_2 + s_3 \chi_3 + s_4) = 0$$

for all integers t, s_1, s_2, s_3, s_4 with

$$0 \leq t < T, \quad 0 \leq s_1, s_2, s_3 \leq S_0, \quad 0 \leq s_4 \leq R.$$

The maximum modulus principle on circles of radius CR and $C^2 R$ yields

$$\log |\varphi^{(t)}(s_1 \chi_1 + s_2 \chi_2 + s_3 \chi_3)| < -c_2 TS_0^3 R$$

for all integers t, s_1, s_2, s_3 with

$$0 \leq t < T, \quad 0 \leq s_1, s_2, s_3 \leq S.$$

A routine application of Theorem A of [8] shows that at least one of these values is non-zero, and this gives rise to a non-zero π_D in $\mathbb{Z}[\mathcal{G}]$ with type at most $c_3 D^{10/7}$ and

$$\log |\pi_D| < -c_4 TS_0^3 R.$$

Since $TS_0^3 R \geq c_5 D^{41/14}$, Gelfond's criterion now supplies the contradiction which proves Lemma 2.

Alternatively we can avoid the appeal to [8] by using Theorem 1 (p. 200) of [4]. Denoting by X the vector space generated over \mathbb{Q} by χ_1, χ_2, χ_3 , the distinctness condition in this result applied to $\varphi((z - z_0)/\omega_1)$ shows that there exists $\chi \neq 0$ in X with $\omega_1 \chi$ in \mathcal{L} . It is not possible to deduce a contradiction instantly; however, we observe that the content of Lemma 2 is not altered by replacing in its statement the periods ω_1, ω_2 by any two periods ω', ω'' linearly independent over \mathbb{Q} . In particular, taking $\omega'_j = \omega_1 + j\omega_2$ and suitable ω''_j ($1 \leq j \leq 4$), we see that there exists $\chi'_j \neq 0$ in X with

$$\omega'_j \chi'_j = a_j \omega_1 + b_j \omega_2 \tag{15}$$

for rational a_j, b_j ($1 \leq j \leq 4$). Since the dimension of X is 3, we can find rationals x_1, x_2, x_3, x_4 , not all zero, with $\sum_{j=1}^4 x_j \chi'_j = 0$. Hence $\tau = \omega_1/\omega_2$ satisfies the equation $R(\tau) = 0$, where

$$R(t) = \sum_{j=1}^4 x_j (a_j t + b_j) / (t + j).$$

But because there is no complex multiplication we know from Schneider's theorem that τ is transcendental; hence $R(t)$ must be identically zero. It follows easily on considering the possible poles of $R(t)$ that we have $ja_j = b_j$ for some j with $1 \leq j \leq 4$. In this case (15) shows that $\chi_j \neq 0$ is rational, which contradicts the linear independence over \mathbb{Q} of $1, \chi_1, \chi_2, \chi_3$.

Now suppose Theorem 5 is false. By Lemma 1, the corresponding integer m_2 is either 0 or 1. We assume for the moment that $m_2 = 0$. Choosing z_0 in the usual way, we find that

$$g_2, g_3, \wp(z_0), \wp'(z_0), u_i, v_j, \wp(z_0 + u_i v_j), \wp'(z_0 + u_i v_j) \quad (1 \leq i \leq n, 1 \leq j \leq m)$$

lie in $Q(\wp, \wp')$. For C, D as before define

$$S = CD^{1/2}(\log D)^{3/16}, \quad S_0 = C^{-2}S, \quad T = [D^2(\log D)^{-1/4}] \\ D_0 = [D^2(\log D)^{-1/8}], \quad D_1 = [D(\log D)^{5/8}], \quad D_2 = D.$$

We construct a non-zero polynomial $P(x_0, x_1, x_2)$, of degree at most D_r in x_r ($r=0, 1, 2$), with coefficients in $\mathbb{Z}[\wp]$, such that the function

$$\varphi(z) = P(z, \wp(z_0 + u_1 z), \wp(z_0 + u_2 z))$$

satisfies

$$\varphi^{(t)}(s_1 v_1 + \dots + s_4 v_4) = 0$$

for all integers t, s_1, \dots, s_4 with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_4 \leq S_0.$$

These coefficients are polynomials in \wp with integer coefficients of logarithmic absolute values at most

$$c_1(D_0 \log D + T \log D + D_1 S_0^2) \leq c_2 D^2 \log D,$$

but because the numbers $\wp(u_1 v_j)$ ($1 \leq j \leq 4$) are algebraic by hypothesis, the degrees in \wp are only at most

$$c_3(D_0 + T + D_2 S_0^2) \leq c_4 D^2 (\log D)^{3/8}.$$

The maximum modulus principle gives

$$\log |\varphi^{(t)}(s_1 v_1 + \dots + s_4 v_4)| < -c_5 T S_0^4 \log D$$

for all integers t, s_1, \dots, s_4 with

$$0 \leq t < T, \quad 0 \leq s_1, \dots, s_4 \leq S.$$

We use Proposition 3 to see that at least one of these values is non-zero. By Lemma 1 and our assumption $m_2 = 0$, it suffices to check the conditions

$$TS^4 \geq cD_0 D_1 D_2, \quad TS^2 \geq cD_0, \\ D_2 S^4 \geq cD_0 D_1, \quad D_2 S^2 \geq cD_0.$$

So we end up with $\pi_D \neq 0$ in $\mathbb{Z}[\wp]$ satisfying

$$\log |\pi_D| < -c_6 T S_0^4 \log D,$$

and the coefficients of π_D have logarithmic absolute values at most $c_7 D^2 \log D$, while its degree in \mathfrak{g} is at most $c_8 D^2 (\log D)^{3/8}$. Since $TS_0^4 \log D \geq c_9 D^4 (\log D)^{3/2}$, the sharpened Gelfond criterion due to Brownawell [2] and Waldschmidt [16] yields the final contradiction. This proves Theorem 5 in the case $m_2 = 0$.

It remains to consider the case $m_2 = 1$. But then the remarks at the end of Sect. 2 show that $\wp(z)$ has no complex multiplication. Furthermore, we can without loss of generality assume that $u_1 v_4 = \omega_1$, $u_2 v_4 = \omega_2$ are periods of \mathcal{L} . Now Theorem 5 is a weak consequence of Lemma 2 applied to the ratios $\chi_j = v_j/v_4$ ($1 \leq j \leq 3$). This completes the proof of Theorem 5 in general.

Note added in proof. It has kindly been pointed out to us by R. Tubbs that the applications of the maximum modulus principle on pages 8–10, 12, 14, and 15 may fail if the appropriate power of some

$$(\sigma(z_0 + u_i(s_1 v_1 + \dots + s_m v_m)))^2 \quad (1 \leq i \leq n)$$

is too small. But in this case it is not too difficult to deduce that the same power of $\wp(z_0 + u_i(s_1 v_1 + \dots + s_m v_m))$ is correspondingly large, and that a suitable denominator of this in $\mathbb{Q}(\mathfrak{g}, \mathfrak{g}_1)$ will then yield the desired element π_D for use in Gelfond's criterion.

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