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# A Note on the Homology of Signed Posets

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**Abstract.** Let $S$ be a signed poset in the sense of Reiner [4]. Fischer [2] defines the homology of $S$, in terms of a partial ordering $P(S)$ associated to $S$, to be the homology of a certain subcomplex of the chain complex of $P(S)$.
In this paper we show that if $P(S)$ is Cohen-Macaulay and $S$ has rank $n$, then the homology of $S$ vanishes for degrees outside the interval $[n/2, n]$.

**Keywords:** poset, Cohen-Macaulay, signed poset

## 1. Introduction

Let $R$ be a set of vectors in $\mathbb{R}^n$. The *positive linear closure of* $R$, denoted $\bar{R}$ is defined to be the span of all linear combinations of vectors in $R$ with non-negative real coefficients.

For each $i = 1, 2, \ldots, n$ let $e_i$ denote the $i$th unit coordinate vector in $\mathbb{R}^n$ and let $e_{-i}$ denote $-e_i$. Recall that the root system $B_n$ is the set

$$B_n = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

**Definition 1** A *signed poset* is a subset $S$ of $B_n$ such that

(a) $S \cap (-S) = \emptyset$.
(b) $\bar{S} \cap B_n = S$.

Let $(P, \leq)$ be an ordinary poset with $P = \{1, 2, \ldots, n\}$. Let $S$ be the collection of all $e_i - e_j$ such that $i < j$. Then $S$ is a subset of the root system $A_n$ which satisfies conditions (a) and (b) of Definition 1 (where $B_n$ is replaced by $A_n$ in condition (b)). Vic Reiner introduced the notion of signed poset [4] to be a $B_n$-analogue of the notion of poset.

In more recent work Steve Fischer [2] defined a homology theory for signed posets. According to Fischer's definition, the homology of a signed poset $S$ is the homology of a certain simplicial complex $C^0_*(S)$ associated to $S$. This simplicial complex is analogous to the simplicial complex of chains in a poset. Fischer showed that the Euler characteristic of this homology can be computed via a "2-Mobius function" and that analogues of Weisner's Theorem and Crapo's Complementation Theorem can be used to calculate this 2-Mobius

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function when $S$ is a "signed lattice". In view of these results on the 2-Mobius function, it would be interesting to know if there are combinatorial labelling conditions which would imply that the simplicial complex associated to $S$ is shellable.

There is an obvious analogue of EL-labelling that can be defined for signed posets, namely we say that $S$ is EL-labellable iff $P(S)$ is EL-labellable. Here $P(S)$ is a poset whose chains are used to define $C_*^0(S)$. An EL-labellable signed poset is pure in the sense that all facets of $C_*^0(S)$ have the same dimension (which we will call the dimension of $S$). Originally Fischer had hoped to show that if $S$ is EL-labellable then the homology of $C_*^0(S)$ is zero except in the top dimension. But then he constructed two EL-labellable signed posets $S_0$ and $S_1$ such that

(a) the homology of $C_*^0(S_0)$ is nonzero exactly in degree equal to half the top dimension.
(b) the homology of $C_*^0(S_1)$ is nonzero exactly in degree equal to the top dimension.

He went on to define "signed EL-labelling" to be an EL-labelling that satisfies other conditions and showed that the existence of a signed EL-labelling of $S$ implies that $C_*^0(S)$ is shellable.

The purpose of this note is to prove that the examples $S_0$ and $S_1$ above are the extreme cases, i.e., we will prove.

**Theorem 1** *Suppose $S$ is an EL-labellable signed poset of dimension $n$. Then $H_r(S)$ is 0 unless $\lfloor n/2 \rfloor \leq r \leq n$.*

## 2. Homology of a signed poset

We begin this section by defining the simplicial complex $C_*^0(S)$ that Fischer uses to compute the homology of $S$. This complex is given in terms of the chains in a certain poset $P(S)$.

**Definition 2** Let $S$ be a signed poset in $B_n$. Define the poset $P(S)$ with vertex set $\{\pm 1, \ldots, \pm n\} = V$ as follows. For $u, v \in V$ we say

$$u \leq_{P(S)} v$$

if and only if

(i) $e_u - e_v \in S$ for $|u| \neq |v|$ or
(ii) $e_u \in S$ for $v = -u$.

Fischer showed that $P(S)$ is a self dual poset.

**Definition 3** An *isotropic $r$-chain* in $P(S)$ is an $r$-chain

$$\alpha_1 < \alpha_2 < \cdots < \alpha_r$$

such that $\alpha_i$ is not equal to $-\alpha_j$ for any $i, j$. Let $\Delta_r^0(S)$ denote the collection of isotropic $r$-chains in $P(S)$ and let $C_r^0(S)$ denote the $\mathbb{C}$-span of $\Delta_r^0(S)$ (with $C_0^0(S) = \mathbb{C}$).

Note that $\Delta_r^0(S)$ is a simplicial complex in $2^{P(S)}$. This gives a boundary map $\partial_r : C_r^0(S) \to C_{r-1}^0(S)$,

$$\partial_r(\alpha_1 < \alpha_2 < \cdots < \alpha_r) = \sum_{i=1}^{r} (-1)^{i-1}(\alpha_1 < \cdots < \hat{\alpha}_i < \cdots < \alpha_r).$$

**Definition 4** Define $H_r^0(S)$ to be the $r$th homology of the complex $(C_*^0(S), \partial_*)$, i.e.

$$H_r^0(S) = \ker \partial_r / \operatorname{im} \partial_{r+1}.$$

We call $H_*^0(S)$ the *signed poset homology* of $S$.

We say $S$ is *EL-labellable* if $P(S)$ has an EL-labelling. In [2], Fischer computes $C_*^0(S)$ and $H_*^0(S)$ for a number of signed posets $S$. In particular he constructs a family of posets $\Gamma_n \subseteq B_n$ such that:

- $\Delta^0(\Gamma_n)$ is pure of dimension $n$
- $\Gamma_n$ is EL-labellable
- $\Delta^0(\Gamma_n)$ is homotopic to the $\lfloor n/2 \rfloor$-dimensional sphere.

This family of signed posets shows that an EL-labelling on $S$ does not imply that $\Delta^0(S)$ is shellable.

## 3. The main result

Let $Q$ be a finite, ranked, self-dual poset. Let $x \to x^*$ be a fixed order-reversing involution on $Q$. Split $Q = Q^L \cup Q^U$ so that $Q^L$ is an order ideal in $Q$, $(Q^L)^* \cap Q^L = \{x \in Q : x^* = x\}$ and $(Q^U)^* \subseteq Q^L$. For each chain $\gamma = \alpha_1 < \alpha_2 < \cdots < \alpha_r$ define $\omega(\gamma)$ to be the number of pairs $(\alpha_i, \alpha_j)$ with $i < j$ and $\alpha_j = \alpha_i^*$. We say $\gamma$ is *isotropic* if $\omega(\gamma) = 0$.

Let $C_r(Q)$ denote the span of all $r$-chains and $C_*^0(Q)$ the span of all isotropic $r$-chains. The boundary map $\partial_* : C_*(Q) \to C_{*-1}(Q)$ preserves $C_*^0(Q)$ and so $(C_*^0(Q), \partial_*)$ is a subcomplex of $(C_*(Q), \partial_*)$. Let $H_*^0(Q)$ denote the homology of that subcomplex. The main theorem for this section is:

**Theorem 2** *Suppose $Q$ is Cohen-Macaulay of rank $n$. Then*

$$H_d^0(Q) = 0 \quad \textit{unless } \frac{n}{2} \le d \le n.$$

**Proof:** We prove this by induction on $|Q|$. If $Q$ is the empty poset then $H_d^0(Q)$ is 0 unless $d = 0$. This agrees with the statement in Theorem 2 since $n = 0$ in this case.

Consider an arbitrary $Q$ and assume the result is true for all $Q'$ with $|Q'| < |Q|$.

Let $\gamma$ be a chain in $C_*(Q)$. We assign a non-negative integer $\rho(\gamma)$ to $\gamma$ as follows:

1) If $\gamma$ is isotropic then $\rho(\gamma) = 0$.
2) If $\gamma$ is not isotropic, write $\gamma$ as

$$\alpha_1 < \alpha_2 < \cdots < \alpha_r.$$

Then $\rho(\gamma)$ is the rank of $\alpha_i$ where $i$ is maximal subject to the condition that $\alpha_i^* = \alpha_j$ for some $j > i$. We also write $A(\gamma)$ to denote $\alpha_i$. Note that $A(\gamma) \in Q^L$.

For $r, p \in \mathbb{N}$ let $C_{r,p}(Q)$ denote the span of all $r$-chains $\gamma$ with $\rho(\gamma) = p$. Note that the boundary map $\partial$ satisfies:

$$\partial(C_{r,p}(Q)) \subseteq \bigoplus_{t \le p} C_{r-1,t}(Q).$$

Thus $(C_*(Q), \partial)$ is filtered by the parameter $\rho$. Let $(E^s, \partial^s)$ be the associated spectral sequence which abutts to $E^\infty = H_*(Q)$. Background material on spectral sequences can be found in any introductory text in homological algebra (e.g. [1] or [3]).

Our first step will be to compute the $E^1$ term in this spectral sequence.

$E^0$ is the associated graded complex. Let $\gamma$ be an $r$-chain in $E_r^0$. Write $\gamma$ as

$$\gamma = \alpha_1 < \alpha_2 < \cdots < \alpha_i < \alpha_{i+1} < \cdots < \alpha_{j-1} < \alpha_j = \alpha_i^* < \alpha_{j+1} < \cdots < \alpha_r.$$

Then

$$\partial^0 \gamma = \sum_{s=1,\ s \ne i,j}^{r} (-1)^{s-1} (\alpha_1 < \cdots < \hat{\alpha}_s < \cdots < \alpha_r). \tag{1}$$

Let $E_r^0[\alpha]$ denote the span of all $r$-chains $\gamma$ with $A(\gamma) = \alpha$ and let $E_r^0[\hat{0}]$ denote $C_r^0(Q)$. Then

1) $E_r^0 = C_r^0(Q) \oplus \bigoplus_{\alpha \in Q^L \setminus \{\hat{0}\}} E_r^0[\alpha]$
2) $\partial^0(E_r^0[\alpha]) \subseteq E_{r-1}^0[\alpha] \quad$ for all $\alpha \in Q^L \cup \{\hat{0}\}$.

So the complex $(E_r^0, \partial^0)$ splits as a direct sum of the subcomplexes

$$\bigoplus_{\alpha \in Q^L \setminus \{\hat{0}\}} (E_*^0[\alpha], \partial^0).$$

We now analyze the subcomplex $(E_*^0[\alpha], \partial^0)$. Assume $\alpha \in Q^L$ and that $\alpha^* > \alpha$. For a chain $\gamma$ to have $A(\gamma) = \alpha$, it is necessary and sufficient for $\gamma$ to consist of any chain up to $\alpha$, then an isotropic chain $\alpha$ to $\alpha^*$, and then any chain from $\alpha^*$ upward. So,

$$E_{*+2}^0[\alpha] \cong C_*(I_\alpha) \otimes C_*^0((\alpha, \alpha^*)) \otimes C_*(I^{\alpha^*}) \tag{2}$$

where $I_\alpha$ denotes the open order ideal generated by $\alpha$ in $Q$, $I^{\alpha^*}$ denotes the open order filter generated by $\alpha^*$ in $Q$ and $(\alpha, \alpha^*)$ is the open interval from $\alpha$ to $\alpha^*$ in $Q$. Moreover, (1)

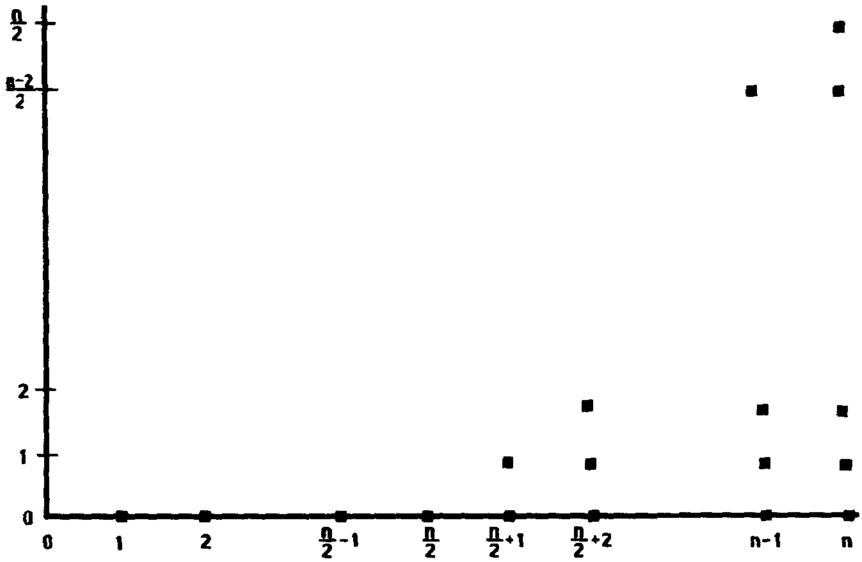


*Figure 1.*

shows that the tensor product of vector spaces given by (2) extends to a tensor product of complexes.

Let $p$ be the rank of $\alpha$ so the rank of $\alpha^*$ is $n+1-p$. Since $Q$ is Cohen-Macaulay we have

$$H_d(I_\alpha) = H_d\left(I^{\alpha^*}\right) = 0 \quad \text{unless } d = p - 1.$$

The self-dual poset $(\alpha, \alpha^*)$ is Cohen-Macaulay of rank $(n-p)-p = n-2p$. By our induction hypothesis

$$H_d^0((\alpha, \alpha^*)) = 0 \quad \text{unless } \frac{n-2p}{2} \leq d \leq n - 2p.$$

Combining these observations we find:

$$E_d^1[\alpha] = 0 \quad \text{unless } \frac{n}{2} + p \leq d \leq n.$$

At this point we know nothing about

$$E_*^1[\hat{0}] = \text{Homology of } (C_*^0(Q), \partial) = H_*^0(Q).$$

However we can draw a diagram of $E_{r,p}^1$ letting a square box denote values of $r$, $p$ where $E_{r,p}^1$ might be non-zero. This appears in Figure 1.

The $\partial^1$ differential on $E^1$ maps $E^1_{r,p}$ to $E^1_{r-1,\, p-1}$. More generally, the $\partial^s$ differential on $E^s$ maps $E^s_{r,p}$ to $E^s_{r-1,\, p-s}$. It follows by induction on $s$ that

$$E^s_{r,0} = E^1_{r,0} = E^1_r[\hat{0}] = H^0_r(Q)$$

for $0 \leq r < \frac{n}{2}$ and all $s$. Thus

$$H^0_r(Q) = E^\infty_{r,0} \subseteq H_r(Q) = 0 \quad \text{for } 0 \leq r < \frac{n}{2}.$$

This proves Theorem 2. □

Theorem 1 follows immediately from Theorem 2 by taking $Q = P(S)$.

## 4. Other problems

The question answered by Theorem 2 has an obvious generalization. Let $C$ be a simplicial complex, pure of dimension $n$, with vertex set $V$ and let $G \subseteq \mathrm{Sym}(V)$ be a group of automorphisms of $C$. Let $C^0$ be the collection of all faces of $V$ which do not contain two elements of $V$ from the same orbit.

*Question* Suppose $C$ is shellable. What can you say about the dimensions $t$ where $H_t(C^0)$ is nonzero?

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