

Oka's inequality for currents and applications

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1 Introduction

In a 1962 paper Oka [Ok] proved that given a family of varieties in an open set $\Omega \subset \mathbb{C}^2$ the set $G \subset \Omega$ where the family is normal is pseudoconvex in Ω . The proof is based on the following remarkable inequality.

Let

$$H = \{(|z| < 1, |w| < r) \cup (r_1 < |z| < 1, |w| < 1) \mid r_1, r < 1\}$$

and

$$A_\rho = \{|z| < \rho, |w| < \rho\}.$$

If V is a closed complex curve in A_1 , then for any $\rho < 1$ there exists C_ρ such that

$$\text{vol}(V \cap A_\rho) \leq C_\rho \text{vol}(V \cap H). \quad (*)$$

Here C_ρ is independent of V .

The result was generalized to varieties of codimension 1 in $\Omega \subset \mathbb{C}^n$ by Fujita [Fu]. Riemenschneider [Ri] proved the analogue of inequality (*) for varieties of dimension p in $\Omega \subset \mathbb{C}^n$, in which case the standard Hartogs figure has to be replaced by the right analogue. It turns out that the domain G where a family of analytic varieties of dimension p , has locally bounded mass is p -pseudoconvex (see below for a precise definition) in Ω . This notion was introduced by Rothstein [Ro], see also [Siu2], and developed by Andreotti and Grauert [A-G]. Observe that when a sequence of analytic varieties has bounded volume, then any limit in the Hausdorff metric is an analytic variety, as follows from Bishop's theorem [Bi].

The second author has given in [Si] some estimates on currents from which it is easy to deduce an inequality generalizing (*) to closed positive currents of bidimension (p, p) . The purpose in [Si] was to extend the domain of definition of the Monge Ampere operator $(dd^c)^k$ to some unbounded plurisubharmonic functions. It was not realized there that the given condition for defining $(dd^c u)^k$

was that the set where u is locally unbounded is in the $n - k$ pseudoconvex envelope of its complement.

We start here by proving a version of Oka’s inequality for currents of the form uT , where T is a positive closed current of bidimension (p, p) in Ω and u is a negative plurisubharmonic function in Ω . It follows from this inequality that $G := \{z \in \Omega; uT \text{ has bounded mass in a neighborhood of } z\}$ is p -pseudoconvex in Ω (usual pseudoconvexity coincides with $n - 1$ pseudoconvexity). If the current (uT) has locally bounded mass in Ω one can define $dd^c u \wedge T := dd^c(uT)$.

We then prove a convergence result for the operator $(u_1, \dots, u_q) \mapsto dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$. The main idea is that if we control the mass or convergence on an open set, then we have the same type of control in the envelope of l -pseudoconvexity for the right l . When the (u_j) are bounded, this operator was studied by Chern, Levine and Nirenberg [CLN] and Bedford and Taylor [BT]. The case where the (u_j) are unbounded has been considered in Griffiths [Gr], Siu [Siu1], Sibony [Si] and more recently by Demailly [De].

Under pseudoconvexity assumptions we can define $T \wedge R_1 \wedge \dots \wedge R_q$ where T is a current of bidegree $(n - p, n - p)$ and R_j are currents of bidegree $(1, 1)$. When the currents are in \mathbb{P}^k we prove a Bezout type theorem i.e. express the mass of $T \wedge \dots \wedge R_q$ in terms of the mass of the factors.

Let T be a positive closed current in \mathbb{P}^k of bidegree $(1, 1)$. So locally T can be written as $dd^c u$. Assuming that u is continuous, we show that support T^l is connected, provided $2l \leq k$.

In the last paragraph we apply the results on currents to holomorphic dynamics in \mathbb{P}^k . This was the main motivation to try to develop some tools in order to understand closed currents which are not analytic varieties. To a holomorphic surjective map $f : \mathbb{P}^k \mapsto \mathbb{P}^k$ of degree $d > 1$, one associates a positive closed current T of bidegree $(1, 1)$. The support of $T^l := T \wedge \dots \wedge T$ (l factors) are of dynamical interest. In particular the support of T coincides with the Julia set J_0 of f : the sequence (f^n) is equicontinuous precisely on $\mathbb{P}^k \setminus J_0$. In this context we show that if $2l \leq k$, then support T^l is connected.

2 Oka’s inequality

We first define the notion of k -pseudoconvexity, see [Ri]. Let $0 < r'_1 < r'$ and $0 < r_1 < r$.

Definition 2.1 *An $(n - k, k)$ Hartogs figure H is defined as*

$$H = \{(z, w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, \|z\| < 1, \|w\| < r\} \\ \cup \{(z, w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, r_1 < \|z\| < 1, \|w\| < 1\}$$

where $0 < r_1, r < 1$. We set

$$\hat{H} = \{(z, w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, \|z\| < 1, \|w\| < 1\} = \mathcal{D}^n.$$

Here $\|z\| = \max |z_j|$, $\|w\| = \max |w_j|$.

Definition 2.2 (*k*-pseudoconvexity) *Let $\Omega_0 \subset \Omega$ be open subsets of $\mathbb{C}^n, 0 < k < n$. Then Ω_0 is **k-pseudoconvex** in Ω if it satisfies the *Kontinuitätssatz* with respect to $(n - k)$ polydiscs. More precisely, whenever H is an $(n - k, k)$ Hartogs figure, $\Phi : \hat{H} \mapsto \Omega$ is a $1 - 1$ holomorphic map and $\Phi(H) \subset \Omega_0$, then $\Phi(\hat{H}) \subset \Omega_0$.*

Usual pseudoconvexity is the same as $(n - 1)$ pseudoconvexity.

For $\rho > 0, \Delta_\rho$ will denote the polydisc of radius ρ .

Theorem 2.3 [Ri] *Let X be a pure k -dimensional closed complex analytic subvariety of the unit polydisc $\Delta \subset \mathbb{C}^n, 0 < k < n$. Then if $0 < \rho < 1$, and H is an $(n - k, k)$ Hartogs figure, $\text{vol}(X \cap \Delta_\rho) \leq C_\rho \text{vol}(X \cap H)$ for C_ρ independent of X .*

It follows that if (X_i) is a family of closed analytic varieties of pure dimension k in $\Omega \subset \mathbb{C}^n$, then if $\Omega' := \{z; \exists \delta > 0; \sup_i \text{vol}(X_i \cap B(z, \delta)) < \infty\}$ then Ω' is *k*-pseudoconvex in Ω .

We want to prove a version of the previous theorem for positive closed currents or even for currents of the form uT where T is a positive closed current and u is a plurisubharmonic function.

For the fundamental results on currents we refer to [de Rh], [Le1] or [LG]. We recall a few facts.

Let Ω be an open set in \mathbb{C}^n . Denote by $D^{p,q}(\Omega)$ the space of smooth differential forms of bidegree (p, q) with compact support in Ω . The space of currents of bidimension (p, q) , hence of bidegree $(n - p, n - q)$ is the dual space of $D^{p,q}(\Omega)$. A current T of bidimension (p, p) and bidegree $(n - p, n - p)$ is positive if for all $\alpha_1, \dots, \alpha_p \in D^{1,0}(\Omega)$ the current $T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p$ is a positive distribution. A current U is negative if $-U$ is positive.

If α is a *k*-covector in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, let $|\alpha|$ denote the usual Hilbertian norm of α . If T is a current of bidimension (p, p) of order zero, i.e. with measure coefficients, we define the measure $M_V[T]$ on Ω as follows. If V is open in Ω , let $M_V[T] = \sup\{|T(\Phi)|; \Phi \in D^{p,p}(V), |\Phi(x)| \leq 1, x \in V\}$. This is called the mass norm of T .

Let $d^c := (i/2\pi)(\bar{\partial} - \partial)$, then $dd^c = (i/\pi)\partial\bar{\partial}$. Let $\beta := dd^c|z|^2$. If T is a positive current of bidimension (p, p) , then the trace measure σ_T is defined as $\sigma_T := T \wedge \beta^p/p!$. It is easy to show that there exists a constant C , depending only on n and p such that for every open set $V \subset \Omega$,

$$(1/C)\sigma_T(V) \leq M_V[T] \leq \sigma_T(V).$$

An upper semicontinuous function $u : \Omega \mapsto [-\infty, \infty)$ is plurisubharmonic if and only if $u \in L^1_{\text{loc}}(\Omega)$ and $dd^c u \geq 0$, for short we will say that u is p.s.h.

Theorem 2.4 *Let H be an $(n - l, l)$ Hartogs figure. Assume $\hat{H} = \Delta$. For every $\rho < 1$, there exists a constant C_ρ such that for every negative current U in a neighborhood of $\bar{\Delta}$, of bidimension (l, l) , such that $dd^c U \geq 0$ we have*

$$M_{\Delta_\rho}[U] + M_{\Delta_\rho}[dd^c U] \leq C_\rho M_H[U].$$

Proof. Let Ω be an open set in \mathbb{C}^n . Assume $\omega \Subset \tilde{\omega} \Subset \Omega$, and that $\omega = \{z; z \in \tilde{\omega}, \psi(z) < 0, \nabla\psi|_{\partial\omega} \neq 0\}$ for some smooth function ψ defined on $\tilde{\omega}$. Let M be a closed subset in Ω . Let $x \in M \cap \omega$ and let $\phi \in C^\infty(\Omega)$ such that

- (i) $\phi(x) > 0$
- (ii) $\phi \equiv 0$ on a neighborhood V of $\partial\omega \cap M$.

Let h be a smooth function in Ω , $0 \leq h \leq 1$, $h = 1$ in a neighborhood of ω .

We will need the following lemma. See also Lemma 4.1 in [Si].

Lemma 2.5 *Let ω, Ω, ϕ, h be as defined above. Let S and U be currents with smooth coefficients in Ω . Assume S is closed of bidegree (p, p) and U is of bidegree (q, q) with $p + q = n - 1$. Then we have*

$$(1) \quad \int_{\omega} \phi(1 - h) dd^c U \wedge S + \int_{\omega} (-U) \wedge dd^c \phi \wedge S = \int_{\omega} (-U) \wedge dd^c(\phi h) \wedge S.$$

If $U \leq 0$, $dd^c U \geq 0$ then relation (1) holds without smoothness assumption on U .

Proof. When χ is a smooth function with compact support then since S is closed

$$\int \chi dd^c U \wedge S = \int dd^c \chi \wedge U \wedge S.$$

So we just apply the above formula with $\chi = \phi(1 - h)$ which is of compact support since $h = 1$ near $\partial\omega$.

If U is not smooth, let U_ε be regularized currents, U_ε is just convolution of U with an approximation of identity. Relation (1) holds for U_ε . We can just let $\varepsilon \mapsto 0$.

The formula of integration by part is written in the form (1) to emphasize that if $h = 0$ near x and if $dd^c \phi \wedge S$ is positive then it is possible to control the mass of $dd^c U$ near x and of $(-U) \wedge dd^c \phi \wedge S$ by what happens near the support of $dd^c(\phi h)$.

We continue the proof of the Theorem. Let Ω be a neighborhood of $\bar{\Delta}$ where U is defined. Recall that $H = \{(z, w), z \in \mathbb{C}^{n-l}, w \in \mathbb{C}^l, \|z\| < 1, \|w\| < r_1\} \cup \{r'_1 < \|z\| < 1, \|w\| < 1\}$. Let $M = \Delta \setminus H$. Let M_1 be the set of points in $\Delta_{\rho_1} \setminus H$ for some $\rho < \rho_1 < 1$, such that exactly one of the w -coordinates is $\geq r_1$. Let $x \in M_1$. For simplicity assume that $|x_{n-l+1}| = |w_1| \geq r_1$ and $|x_{n-l+2}| = |w_2| < r_1, \dots, |x_n| = |w_l| < r_1$. Let $\phi = 1/|w_1|^2$. Let H_1 be the $(n - l, 1)$ Hartogs figure obtained by fixing the coordinates $w_2 = x_{n-l+2}, \dots, w_l = x_n$, with the same numbers r_1, r'_1 . We can construct an open set ω_1 in \mathbb{C}^{n-l+1} such that $\phi(x) > \max_{\partial\omega_1 \cap [\bar{H}_1 \setminus H_1]} \phi$. Fatten ω_1 to obtain an open set ω in \mathbb{C}^n and extend ϕ to Φ_1 such that $\Phi_1(x) > \max_{\partial\omega \cap [\bar{H} \setminus H]} \Phi_1$, by adding to the trivial extension a function θ of w_2, \dots, w_n only.

Let $\Phi_2 := \Phi_1 + \delta|(z, w)|^2$. For $\delta > 0$ small enough, $\Phi_2(x) > \sup_{\partial\omega \cap M} \Phi_2 := c$. But also $dd^c \Phi_2 \wedge (dd^c|w'|^2)^{l-1} > 0$ where $w' = (w_2, \dots, w_l)$. This is

because the negativity in $dd^c\Phi_2$ introduced by the function $\theta(w')$ is cancelled by $(dd^c|w'|^2)^{l-1}$.

Let λ be a smooth increasing convex function on $[0, \infty)$, vanishing on a neighborhood of $[0, c]$ and then strictly increasing, we can assume that $\lambda(\Phi_2(x)) > 0$. Let $\Phi = \lambda \circ \Phi_2$. We apply Lemma 2.5 with $S = (dd^c|w'|^2)^{l-1}$ and $\phi = \Phi$ and h a function vanishing near x , h has value 1 near $\partial\omega$, but $M \cap \text{support } h$ is contained in $(\phi = 0)$.

Relation (1) in Lemma 2.5 gives an estimate of $dd^cU \wedge (dd^c|w'|^2)^{l-1}$ and of $(-U) \wedge dd^c\phi \wedge S$ near x in terms of the mass of U on a compact set of H . If we apply the same argument to suitable perturbations of the coordinates w , we finally get an estimate of the mass of $(-U)$ and dd^cU near x in terms of quantities supported in a compact set of H . We have proved the theorem for M_1 instead of Δ_ρ . Let M_j be the set of points in $\Delta_{\rho_j} \setminus H$ such that exactly j of the w -coordinates are $\geq r_1$ where we have chosen $1 > \rho_1 > \dots > \rho_l = \rho$. We will prove the theorem by induction in j . For simplicity of notation we prove it for $j = 2$. Let $x \in M_2$. We can assume that $|x_{n-l+1}| = |w_1| \geq r_1$ and $|x_{n-l+2}| = |w_2| \geq r_1$ while $|w_3| < r_1, \dots, |w_l| < r_1$.

Let $\phi_2 = 1/|w_2|^2$. We can construct an open set ω_2 in \mathbb{C}^{n-l+1} such that $\phi_2(x) > \max_{\partial\omega_2 \cap (\hat{H}_2 \setminus H_2)} \phi$. We fatten ω_2 to obtain an open set ω in \mathbb{C}^n and proceed as above. We will obtain an estimate of the mass of U and dd^cU near x in term of the mass in $H \cup M_1$ which in turn is controlled by the mass in H . So by induction we get the estimate in the theorem.

Corollary 2.6 *Let (U_i) be a sequence of negative currents of bidimension (p, p) in a complex manifold Ω . Assume for all $i, dd^cU_i \geq 0$. Then the domain Ω_0 on which the family has locally bounded mass is p -pseudoconvex. In particular, if U is a negative current of bidimension (p, p) in Ω , such that $dd^cU \geq 0$, then $\Omega \setminus (\text{supp } U)$ is p -pseudoconvex in Ω . If T is a positive closed current of bidimension (p, p) on Ω , then $\Omega \setminus \text{supp } T$ is p -pseudoconvex.*

Proof. Let H be an $(n - p, p)$ Hartogs figure and $f : \hat{H} \mapsto \Omega$ be an injective holomorphic map. If $f(H) \subset \Omega_0$, Theorem 2.4 shows that $f(\hat{H}) \subset \Omega_0$. The estimate in the theorem is just given with $f = \text{Id}$. So Ω_0 is p -pseudoconvex in Ω . To show that $\Omega \setminus \text{supp } U$ is p -pseudoconvex we just apply the first part of the corollary to the sequence of currents $U_k = kU, k \in \mathbb{N}$. If T is positive and closed, we apply the previous result to $U = -T$.

Let T be a positive closed current of bidimension (p, p) in Ω . Let u be a plurisubharmonic function on Ω . We consider the closed set $M(T, u) = \{q; q \in \Omega, uT \text{ is not of finite mass in any neighborhood of } q\}$. For simplicity, since the results are of semi local nature (in neighborhoods of compacts) we will assume that $u < 0$.

Proposition 2.7 *Let T, u, Ω be as above. Then $\Omega \setminus M(T, u)$ is p -pseudoconvex. And on $\Omega \setminus M(T, u)$ the current uT is negative and satisfies $dd^c(uT) \geq 0$.*

Proof. Let H be an $(n - p, p)$ Hartogs figure, assume $\hat{H} = \Delta \in \Omega$. Let $u_j \searrow u$ be a decreasing sequence of smooth plurisubharmonic functions in a neighborhood of $\bar{\Delta}$. If $U_j := u_j T$ we clearly have $U_j \leq 0$ for large j and $dd^c U_j \geq 0$. Theorem 2.4 implies that

$$\int_{\Delta_p} |u_j| \sigma_T \leq C_\rho \int_H |u_j| \sigma_T \leq C_\rho \int_H |u| \sigma_T .$$

Here σ_T denotes the trace measure for the current T . Hence if $\int_H |u| \sigma_T < \infty$ we get that

$$\int_{\Delta_p} |u| \sigma_T \leq C_\rho \int_H |u| \sigma_T ,$$

by Lebesgue’s dominated convergence theorem. The p -pseudoconvexity of $\Omega \setminus M(T, u)$ follows easily from the above estimate. Since locally in $\Omega \setminus M(T, u)$, $u_j T \mapsto u T$ in the sense of currents and since $dd^c(u_j T) = dd^c u_j \wedge T \geq 0$ we get that $dd^c(u T) \geq 0$.

3 The operator $(u_1, \dots, u_q) \mapsto dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$

It is very useful, in many questions in Algebraic Geometry and Complex Analysis, to define an expression such that $dd^c u \wedge T$ where T is a closed (p, p) current and u is an unbounded plurisubharmonic function. The case where u is bounded is studied in [BT].

Here we want to extend the approach in [Si] to define

$$dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge T$$

under quite general assumptions on u_1, \dots, u_k and T .

Let Ω be an open subset in \mathbb{C}^n . If u is a plurisubharmonic function in Ω , define $M(u) := \{q; q \in \Omega, u \text{ is unbounded in any neighborhood of } q\}$. Recall that if T is a positive closed (p, p) current we define $M(T, u) := \{z; z \in \Omega, u \text{ is not } \sigma_T \text{ integrable in a neighborhood of } z\}$. In $\Omega \setminus M(T, u)$ we will define $dd^c u \wedge T := dd^c(u T)$.

If X is a closed set in Ω , we say that X is in the envelope of p -pseudoconvexity of $\Omega \setminus X$ with respect to Ω if all points in X can be reached by pushing polydiscs of dimension $(n - p)$ using biholomorphic images of $(n - p, p)$ Hartogs figures with hulls in Ω . So we use the same procedure as for obtaining the envelope of holomorphy by pushing one dimensional discs.

Proposition 3.1 *Let u be a plurisubharmonic function in Ω . And let T be a positive closed bidimension (p, p) current on Ω . If $M(u) \cap \text{supp } T := X$ is in the envelope of p -pseudoconvexity of $\Omega \setminus X$ with respect to Ω , then u is locally σ_T integrable.*

Proof. Without loss of generality we can assume $u < 0$ on Ω . It is clear that $M(T, u) \subset M(u) \cap \text{supp } T$. Since as proved in Proposition 2.7 $\Omega \setminus M(T, u)$ is p -pseudoconvex it follows that $M(T, u)$ is empty.

Proposition 3.2 *Let u_j, u be nonpositive plurisubharmonic functions in Ω . Let T_j, T be positive closed currents of bidimension (p, p) in Ω . Assume*

- (1) $T_j \mapsto T$,
- (2) $(u_j T_j)$ has uniformly bounded mass on every compact,
- (3) $u_j \mapsto u$ in L^1_{loc} .

If L is any weak limit of $u_j T_j$, then $L \leqq uT$. If $u_j \geqq u$ and $T_j \leqq T$, then $u_j T_j \mapsto uT$.

Proof. Let $\gamma \geqq 0$ be a smooth form with compact support $X \subset \Omega$ of bidegree (p, p) . Let f be a continuous function on X such that $f \geqq u$ on X . By Hartogs lemma, see [H],

$$\overline{\lim}_j \sup_X (u_j - f) \leqq \sup_X (u - f) \leqq 0.$$

Hence given $\varepsilon > 0$, there is a j_0 so that for $j \geqq j_0, u_j \leqq f + \varepsilon$. Since $\gamma \geqq 0$ and $T_j \geqq 0$ we have

$$u_j T_j \wedge \gamma \leqq (f + \varepsilon) T_j \wedge \gamma.$$

Hence

$$L \wedge \gamma \leqq (f + \varepsilon) T \wedge \gamma.$$

Since ε and f are arbitrary we get that $L \leqq uT$. If $u_j \geqq u$ and $T_j \leqq T$, we can assume using again Hartogs lemma that $u_j \leqq 0$ on support γ , we have

$$\int uT \wedge \gamma \leqq \int u_j T \wedge \gamma \leqq \int u_j T_j \wedge \gamma \mapsto L \wedge \gamma.$$

So $L = uT$.

Corollary 3.3 *Let $u_n \leqq 0$ be plurisubharmonic in Ω and assume that $u_n \mapsto u$ in L^1_{loc} and that $u_n \geqq u$. Then for any positive closed current of bidimension (p, p) , we have in $\Omega \setminus M(T, u)$ that $u_n T \mapsto uT$ and that $dd^c u_n \wedge T \mapsto dd^c u \wedge T$.*

Proof. Since $u_j \leqq 0, |u_j| \leqq |u|$ so given a compact K in $\Omega \setminus M(T, u)$,

$$\int_K |u_j| d\sigma_T \leqq \int_K |u| d\sigma_T.$$

Hence we can apply Proposition 3.2.

Let A^α denote the Hausdorff measure of dimension α . We have the following convergence result.

Corollary 3.4 *Let (u_j) be a sequence of plurisubharmonic functions in Ω . Assume $u_j \mapsto u$ in L^1_{loc} and $u_j \geqq u$. Let T be a positive, closed current of bidimension (p, p) in Ω . If $A^{2p}(M(u) \cap \text{supp } T) = 0$, then $u_j T \mapsto uT$ in Ω .*

Proof. We just have to check that $M(T, u)$ is empty. It is clear that $M(T, u) \subset M(u) \cap \text{Supp } T := E$. The idea is to show that E is in the p envelope of the complement. By the results of Federer [Fe], for almost all $n - p$ planes say parallel to a given one the intersection with E is empty, hence we can construct Hartogs figures of the right type. More precisely, fix $x \in E$. Since

$A^{2p}(E \setminus \{x\}) = 0$, almost every $n - p$ complex plane through x does not intersect $E \setminus \{x\}$. Assume that $x = 0$ and let $L = \{w_1 = \dots w_p = 0\}$ be such a plane. Let (z_1, \dots, z_{n-p}) be the coordinates in L . Fix a polydisc $\Delta^{n-p} \Subset \Omega$ centered at the origin. Since $\partial\Delta^{n-p} \cap E = \emptyset$, we can find a polydisc Δ^p in the orthogonal complement L^\perp centered at the origin such that $(\partial\Delta^{n-p} \times \Delta^p) \cap E = \emptyset$. We can then complete the Hartogs figure $H \subset \Omega \setminus E$ such that $0 \in \hat{H}$.

Theorem 3.5 *Let T be a closed positive current of bidimension (p, p) in $\Omega, 0 < p < n$. Let u_1, \dots, u_q be plurisubharmonic ≤ 0 functions in Ω . If for all $j_1, \dots, j_m, M(u_{j_1}) \cap \dots \cap M(u_{j_m}) \cap \text{supp } T$ is in the $p - m + 1$ envelope of pseudoconvexity of the complement, then $u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ has locally bounded mass in Ω and similarly for $dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$. Moreover the mass on a compact set of Ω is majorized by the mass on a compact where all the u_j are bounded. We also have $u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T \mapsto u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ provided $u_k^j \mapsto u_k$ in L^1_{loc} and $u_k^j \geq u_k$, moreover $dd^c u_1^j \wedge \dots \wedge dd^c u_q^j \wedge T$ converge weakly to $dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$.*

Proof. Case $q = 1$. We assume $M(u) \cap \text{supp } T$ is in the p -convex envelope of the complement relative to Ω . Then uT is well defined in Ω and if L is a compact in Ω , Theorem 2.4 shows that

$$\int_L |u| d\sigma_T \leq c \int_K |u| d\sigma_T$$

where K is a compact of $\Omega \setminus M(u)$. Let χ be a nonnegative test function with value 1 in a neighborhood of L . If $\beta = dd^c |z|^2$, then

$$M_L[dd^c u \wedge T] \leq \int \chi dd^c u \wedge T \wedge \beta^s = \int uT \wedge \beta^s \wedge dd^c \chi.$$

So we also get that $M_L[dd^c u \wedge T] \leq c' M_{K'}[uT]$. For $q = 1$ the convergence result is just Corollary 3.3.

So assume the theorem has been proved for $(q - 1)$ functions (v_j) . Let $S = dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ which is, by induction, a well defined current on Ω of bidimension $(p - q + 1, p - q + 1)$. We want to show that $\Omega \setminus M(u_1, S)$ contains the complement of $M(u_1) \cap \dots \cap M(u_q) \cap \text{supp } T$. Since $\Omega \setminus M(u_1, S)$ is $p - q + 1$ pseudoconvex, the hypotheses of the theorem imply that $M(u_1, S)$ is empty.

Fix $z_0 \notin M(u_1) \cap \dots \cap M(u_q) \cap \text{supp } T$. If u_1 is bounded near z_0 we are done. Assume $z_0 = 0$ and that u_2 is bounded on a neighborhood of $\bar{B} := \bar{B}(0, r)$. Replacing u_2 by $\max(u_2, A(|z|^2 - r^2))$ we can assume that u_2 is unchanged on $B_1 = B(0, r/4)$, and is equal to $A(|z|^2 - r^2)$ in a neighborhood of ∂B . Let $T' = dd^c u_3 \wedge \dots \wedge dd^c u_q \wedge T$. We will show that

$$\int_B -u_1 dd^c u_2 \wedge T' \wedge \beta^s \tag{*}$$

is bounded by the mass of $u_1 T'$ on B .

Let h be a smooth function supported near ∂B with value 1 near ∂B . Let $v_1^j \searrow u_1, v_1^j$ smooth plurisubharmonic functions. We can apply Lemma 2.5 with $U = v_1^j dd^c u_2 \wedge T'$ and $S = \beta^{s-1}$ (in this case $\omega = B, \phi = |z|^2$ and M is empty). We have

$$\begin{aligned} & \int_B \phi(1-h) dd^c v_1^j \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} \\ & - \int_B v_1^j dd^c \phi \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} \\ & = - \int_B v_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} . \end{aligned}$$

Observe that $-v_1^j \geq 0$ and $-A'\beta \leq dd^c(\phi h) \leq A'\beta$ and that on the support of $h, dd^c u_2 = A\beta$. So the last integral is smaller in absolute value than

$$-AA' \int_B v_1^j T' \wedge \beta^{s+1} .$$

Using Fatou's lemma and the induction hypothesis we get

$$\int_B |u_1| dd^c \phi \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} \leq AA' \int_B |u_1| T' \wedge \beta^{s+1} .$$

Hence we have shown (*). As a consequence, also the mass of $dd^c u_1 \wedge dd^c u_2 \wedge T'$ on B_1 is bounded by the mass of $u_1 T'$ on B .

We now turn to the convergence question. Let $u'_1 \geq u_1, \dots, u'_q \geq u_q$. We know that if R is any limit point of $u'_1 dd^c u'_2 \wedge \dots \wedge T$ then $R \leq u_1 dd^c u_2 \wedge \dots \wedge T$ (by Proposition 3.2). In order to prove equality, it is enough to prove that for any x_0 there exists an open set $\omega \in \Omega, x_0 \in \omega$, a smooth function ϕ in a neighborhood of $\bar{\omega}$, a positive closed form γ such that $dd^c \phi \wedge \gamma$ is strictly positive in a neighborhood of x_0 and $\int_{\omega} u_1 dd^c u_2 \wedge \dots \wedge T \wedge \gamma \wedge dd^c \phi \leq \liminf \int_{\omega} u'_1 dd^c u'_2 \wedge \dots \wedge T \wedge \gamma \wedge dd^c \phi$.

If $x_0 \notin \text{supp } T$ this is clear. Assume $x_0 \notin M(u_1) \cap \dots \cap M(u_q)$. Then there exists a ball B such that (u'_k) is a bounded sequence in a neighborhood of \bar{B} for some $1 \leq k \leq q$. Let $B = \{\psi < 0\}, \psi$ plurisubharmonic. Since the result we want is local we can assume that on a neighborhood ω_1 of $\partial B, u'_1 = A\psi$ where A is a large constant. Let h be a function in $C_0^\infty(\omega_1)$ with value 1 in a neighborhood of ∂B . We are going to use repeatedly the identity of Lemma 2.5 with $\phi = |z|^2$ and M empty.

Assume that all the u'_k s are smooth.

Let T' denote $T \wedge \beta^{p-q}$, we have

$$\begin{aligned} & \int_B u_1 dd^c u_2 \wedge \dots \wedge T' \wedge dd^c \phi \\ & \leq \int_B u'_1 dd^c u_2 \wedge \dots \wedge T' \wedge dd^c \phi \\ & = \int_B \phi(1-h) dd^c u'_1 \wedge dd^c u_2 \wedge \dots \wedge T' \\ & \quad + \int_B u'_1 dd^c(\phi h) \wedge dd^c u_2 \wedge \dots \wedge T' \end{aligned}$$

$$\begin{aligned}
 &= \int_B u_2 dd^c \phi \wedge dd^c u_1^j \wedge \dots \wedge T' \\
 &\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \dots \wedge T' \\
 &\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \dots \wedge T' \\
 &\leq \int_B u_2^j dd^c \phi \wedge dd^c u_1^j \wedge T' \dots \\
 &\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \dots \wedge T' \\
 &\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \dots \wedge T' \dots =
 \end{aligned}$$

(where we do not write terms containing $dd^c(\phi h)$)

$$\begin{aligned}
 &= \int_B \phi(1-h) dd^c u_1^j \wedge dd^c u_2^j \wedge dd^c u_3 \wedge \dots \\
 &\quad + \int_B u_2^j dd^c(\phi h) \wedge dd^c u_1^j \wedge \dots \wedge T' + \dots \\
 &= \int_B u_1^j dd^c u_2^j \wedge dd^c u_3 \wedge \dots \wedge T' \wedge dd^c \phi \\
 &\quad - \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2^j \wedge \dots \wedge T' \\
 &\quad + \int_B u_2^j dd^c(\phi h) \wedge dd^c u_1^j \wedge \dots \wedge T' \\
 &\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \dots \wedge T' \\
 &\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \dots \wedge T' \\
 &\leq \int_B u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T \wedge dd^c \phi +
 \end{aligned}$$

sum of integrals involving $dd^c(\phi h)$.

Recall that if positive measures $\mu_k \mapsto \mu$ weakly on an open Ω set then $\overline{\lim}_{k \rightarrow \infty} \mu_k(K) \leq \mu(K)$ for each compact $K \subset \Omega$ and $\mu(V) \leq \underline{\lim} \mu_k(V)$ for each open set. Hence, if V is an open set such that $\mu_k(\partial V) = \mu(\partial V) = 0$ for all k , then $\mu(V) = \lim_k \mu_k(V)$. We can assume that all the measures, which are coefficients of the currents we work with, have no mass on ∂B .

On support of $dd^c(\phi h), u_l^j = A\psi$, so we have only $q - 1$ nonconstant sequences (u_m^j) so for every integral involving $dd^c(\phi h)$ we have convergence, but as it is shown from the above development the limit of the sum of integrals involving $dd^c(\phi h)$ is zero, (observe that the manipulations are purely algebraic) so

$$\int u_1 dd^c u_2 \wedge \dots \wedge T' \wedge dd^c \phi \leq \underline{\lim}_j \int u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T' \wedge dd^c \phi .$$

Now assume $x \in M(u_1) \cap \dots \cap M(u_q) \cap \text{supp } T := M$. Since this set is in the envelope of $p - q + 1$ pseudoconvexity of the complement we can construct finitely many Hartogs figures to absorb successively all of M . Given a Hartogs

figure H and $x \in \hat{H} \cap M$, we can construct a smoothly bounded neighborhood $\omega \in \hat{H}$ of x and smooth functions $(\phi_i)_{i \leq s}$ vanishing in a neighborhood of $\partial\omega \cap M$ such that $dd^c \phi_i \wedge \gamma_i \geq 0$ and $\sum_i dd^c \phi_i \wedge \gamma_i > 0$ in a neighborhood of x , where the γ_i 's are closed forms of bidegree $(p - q, p - q)$. We compute the integrals in the same way as above, where now T' denotes $T \wedge \gamma_i$. Since the support of ϕh is contained in the set where we have already proved convergence, the sum of integrals involving $dd^c(\phi h)$ is going to converge to zero as $j \rightarrow \infty$, and then we add up with respect to $1 \leq i \leq s$. Hence,

$$\int u_1 dd^c u_2 \wedge \dots \wedge \gamma_1 \wedge dd^c \phi \leq \liminf \int u_1^j dd^c u_2^j \wedge \dots \wedge \gamma_1 \wedge dd^c \phi.$$

This finishes the convergence proof for general q . The last part follows immediately.

Corollary 3.6 *Let Ω , T , u_1, \dots, u_q and $u_j \geq u_k$ be as in theorem. Assume that $A^{2(p-m+1)}(M(u_{j_1}) \cap \dots \cap M(u_{j_m}) \cap \text{supp } T) = 0$ for all $j_1, \dots, j_m \leq q$. Then $u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ has locally bounded mass and $u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T \rightarrow u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ in Ω .*

Proof. The assumption on the Hausdorff dimension of $X := M(u_{j_1}) \cap \dots \cap M(u_{j_m}) \cap \text{supp } T$, implies that X is in the envelope of $p - m + 1$ pseudoconvexity of $\Omega \setminus X$, as in the proof of Corollary 3.4 so the result follows from Theorem 3.5.

A weaker form of the corollary was proved recently by Demailly [De]. He requires that $A^{2p-2m+1}(M(u_{j_1}) \cap \dots \cap M(u_{j_m}) \cap \text{supp } T) = 0$. The case when $M(u_j)$ is empty, $1 \leq j \leq q$ is due to Bedford and Taylor [BT].

Remark 3.7 Theorem 3.5 gives a Thullen type extension for currents of type uT . More precisely, let T be a positive closed current in Ω , of bidimension (p, p) . Let u be a plurisubharmonic function in Ω and V an analytic subvariety of dimension p in Ω . If $M(T, u)$ is contained in V and $\Omega \setminus M(T, u)$ intersects every irreducible branch of V then $M(T, u) = \emptyset$. The example of Shiffman–Taylor in [Siu] is a case where V is of codimension 1 in \mathbb{C}^2 and $udd^c u$ has locally bounded mass in $\mathbb{C}^2 \setminus V$ and $M(T, u) = V$.

A natural question about the operator $(dd^c)^k$ is the following: Let (u_j) be a sequence of plurisubharmonic functions in Ω . Find the right notion of convergence such that $u_j \rightarrow u$ implies $(dd^c u_j)^k \rightarrow (dd^c u)^k$ in the sense of currents. Cegrell [Ce] and Lelong [Le] observed that the convergence of (u_j) to u in L_{loc}^p , $p < \infty$, is not enough. We start with a refinement of their example.

Proposition 3.8 *There exists a uniformly bounded sequence (v_p) of plurisubharmonic functions in the unit disc $\Delta \subset \mathbb{C}^2$ such that:*

- (i) $v_p(z) \rightarrow v(z)$ for every $z \in \Delta$ except on a set of Hausdorff dimension 2. More precisely, given any $\varepsilon > 0$ and $\alpha > 0$, there are balls $B(x_i, r_i)$ with $\sum r_i^{2+\varepsilon} < \infty$ such that for $p > p(\varepsilon)$, $v_p > v - \varepsilon$ on $B \setminus \bigcup B(x_i, r_i)$.
- (ii) $(dd^c v_p)^2$ does not converge to $(dd^c v)^2$.

Proof. Let $v_p(z, w) := |z|^{2p} + |w|^{2p}$ and $v(z, w) = \max(|z|, |w|)$. Clearly $0 \leq v_p \leq 2^{1/p}v$. Let $\omega := \{|z| \neq |w|\}$. We have uniform convergence on compact subsets of ω . Since through every point q in \mathbb{C}^2 there exists a disc Δ_q on which v_p is harmonic, it follows that $(dd^c v_p)^2 = 0$, see [FS1, Lemma 6.9]. On the other hand one computes easily that $(dd^c v)^2 \neq 0$ since $v = 1$ on the boundary of the unit polydisc but we do not have that $1 \leq v$ as should be the case by maximum principle if $(dd^c v)^2 = 0$, see [BT].

We now study for which points $w = ze^{i\theta}$, $v_p(z, w)$ does not converge to $v(z, w)$. We have $v_p(z, e^{i\theta}z) = |z||1 + e^{2ip\theta}|^{1/(2p)} = 2^{1/2p}|z| |\cos p\theta|^{1/(2p)}$.

Let $E = \{e^{i\theta}; \liminf_p |\cos p\theta|^{1/p} < 1\}$. Let $\varepsilon_k = 1 - 1/k$. Then $E \subset \bigcup_k \bigcap_N \bigcup_{p>N} \{|\cos p\theta| < \varepsilon_k^p\} =: \bigcup E_k$. The set $\{|\cos p\theta| < \varepsilon_k^p\}$ is contained in a union of $2p$ intervals of length at most $3\varepsilon_k^p/p$ for $p \geq p(k)$. Given $\alpha > 0$ we have $\Lambda^\alpha(E) \leq \sum_k \Lambda^\alpha(E_k)$, and $\Lambda^\alpha(E_k) \leq c \sum_{p>N} p(\varepsilon_k^p)^\alpha = o(1)$ as $N \mapsto \infty$ so for every $k, \Lambda^\alpha(E_k) = 0$. Hence $\Lambda^\alpha(E) = 0$. It follows easily that $v_p(z, w) \mapsto v(z, w)$ except on a set of Hausdorff dimension 2. Indeed, given any $\varepsilon > 0$ and $\alpha > 0$, there are balls $B(x_i, r_i)$ with $\sum r_i^{2+\alpha} < \varepsilon$ and such that for $p > p(\varepsilon), v_p \geq v - \varepsilon$ on $B \setminus \bigcup_i (B(x_i, r_i))$.

The previous example is quite sharp as the following result * shows.

Theorem 3.9 *Let T_j be a sequence of positive closed currents of bidimension (p, p) in an open set Ω in \mathbb{C}^n . Let A be a subset in Ω with $\Lambda^{2p}(A) = 0$. Let (u_j) be a bounded sequence of plurisubharmonic functions and u a plurisubharmonic function in Ω , whose restriction to $\Omega \setminus A$ is continuous. Assume*

- (i) $T_j \mapsto T$ as currents.
- (ii) $u_j \mapsto u$ in L^1_{loc} .
- (iii) For every compact $K \subset \Omega \setminus A$ and $\varepsilon > 0$ there exists $j(K)$ such that for $j \geq j(K) u_j \geq u - \varepsilon$.

Then $u_j T_j \mapsto uT$ and $dd^c u_j \wedge T_j \mapsto dd^c u \wedge T$.

Proof. Observe that we do not assume that A is closed.

Let S be a limit point of the bounded sequence of currents $u_j T_j$. By Proposition 3.2 we know that $S \leq uT$. Let γ be a positive form with compact support in $\omega \Subset \omega_1 \Subset \Omega$. Since (u_j) is bounded, we can assume that $0 \leq u_j \leq M$. We have to show that $\int uT \wedge \gamma \leq \liminf \int u_j T_j \wedge \gamma$.

Since the mass of (T_j) is bounded by a constant C on ω_1 , for every ball $B(x, r) \subset \omega_1$, we have [Le1]

$$\sigma_{T_j} B(x, r) \leq Cr^{2p}. \tag{1}$$

Fix $\varepsilon > 0$ and let $B(x_j, r_j)$ be a sequence of balls such that $A \subset \bigcup B(x_i, r_i)$ and $\sum_i r_i^{2p} < \varepsilon$. It follows from (1) that there exists an open set $U \supset A \cap \omega$ such that $\sigma_{T_j}(U) < \varepsilon$ independently of j . Let $K = \text{supp } \gamma \cap (\Omega \setminus U)$. Since $u_j T_j \geq 0$,

*Note. We thank Russakovski for pointing out that a weaker version of Theorem 3.9 has been proved by Ronkin [Ro].

we have for $j \geq j(K)$

$$\begin{aligned} \int u_j T_j \wedge \gamma &= \int_{\omega \setminus U} (u_j - u) T_j \wedge \gamma + \int_{\omega \setminus U} u T_j \wedge \gamma + \int_U u_j T_j \wedge \gamma \\ &\geq -C\varepsilon + \int_{\omega \setminus U} u T_j \wedge \gamma. \end{aligned}$$

Let \tilde{u} be a continuous extension of u from $\text{supp } \gamma \setminus U$ to $\text{supp } \gamma$, $0 \leq \tilde{u} \leq M$. Since for all j , $\sigma_{T_j}(U) < \varepsilon$, the last integral is close to $\int_{\omega} \tilde{u} T_j \wedge \gamma$ which converges to $\int_{\omega} u T \wedge \gamma$. We have then proved that

$$\int u T \wedge \gamma \leq \liminf_j \int u_j T_j \wedge \gamma$$

which is what we wanted to show.

4 Bezout theorem for currents in \mathbb{P}^k

Let ω be the standard Kahler form on \mathbb{P}^k , normalized such that $\int_{\mathbb{P}^k} \omega^k = 1$. If $\pi : \mathbb{C}^{k+1} \rightarrow \mathbb{P}^k$ is the canonical projection, then $\pi^* \omega = dd^c \log \|z\|$ where $z = (z_0, z_1, \dots, z_k)$ and $\|\cdot\|$ is the Euclidean norm in \mathbb{C}^{k+1} . Given a positive closed current T of bidimension (p, p) we define $\|T\| := \int T \wedge \omega^p$. When T corresponds to integration on an analytic manifold V , then $\|T\|$ is just the $2p$ -volume of V . We will need the following standard result.

Proposition 4.1 *Let R be a closed positive $(1, 1)$ current on \mathbb{P}^k .*

- (i) *There exists a plurisubharmonic function v on \mathbb{C}^{k+1} such that $\pi^* R = dd^c v$, $v(\lambda z) = c \log |\lambda| + v(z)$ for $\lambda \in \mathbb{C}, z \in \mathbb{C}^k$.*
- (ii) *If v is a plurisubharmonic function on \mathbb{C}^{k+1} , satisfying the previous homogeneity condition, then there is a decreasing sequence (v_ε) of plurisubharmonic functions on \mathbb{C}^{k+1} satisfying the homogeneity condition, such that $v_\varepsilon \in C^\infty(\mathbb{C}^{k+1} \setminus \{0\})$ and $v_\varepsilon \searrow v$.*

Proof. A proof of (i) can be found in Theorem 4.1 of [FS1], see also [LG].

For simplicity assume $c = 1$. If α is an approximation of the identity in \mathbb{C}^{k+1} depending only on $\|z\|$, we define

$$v_\varepsilon(z) := \frac{1}{\|z\|^{2(k+1)}} \int v(w) \alpha_\varepsilon \left(\frac{z-w}{\|z\|} \right) d\lambda(w) = \int v(z - \|z\|w) \alpha_\varepsilon(w) d\lambda(w)$$

where λ denotes Lebesgue measure. The proof is then the same as in Theorem 7.6 of Lelong and Gruman [LG].

Proposition 4.2 *Let R be a positive closed current of bidegree $(1, 1)$ on \mathbb{P}^k . Let v be a plurisubharmonic function on \mathbb{C}^{k+1} satisfying:*

- (i) $\pi^* R = dd^c v$
- (ii) $v(\lambda z) = c \log |\lambda| + v(z), \lambda \in \mathbb{C}$.

Then $\|R\| = c$.

Proof. We can assume $c = 1$. Let (v_ϵ) be the sequence of plurisubharmonic functions constructed in Proposition 4.1. Consider the positive closed current on $\mathbb{P}^k, (R_\epsilon)$, defined by the relation $\pi^*R_\epsilon = dd^c v_\epsilon$. Let $u_\epsilon[z_0 : \dots : z_k] = v_\epsilon(z_0, \dots, z_k) - \log \|z\|$. The function u_ϵ is smooth and well defined on \mathbb{P}^k . Moreover $R_\epsilon - \omega = dd^c u_\epsilon$. Since $v_\epsilon \rightarrow v$ in L^1_{loc} , then $R_\epsilon \rightarrow R$ in the sense of currents. Using Stoke's theorem we get:

$$\begin{aligned} \|R\| &= \int R \wedge \omega^{k-1} = \lim_{\epsilon \rightarrow 0} \int R_\epsilon \wedge \omega^{k-1} \\ &= \lim_{\epsilon \rightarrow 0} \int \omega^k + dd^c u_\epsilon \wedge \omega^{k-1} = \int \omega^k = 1 . \end{aligned}$$

Remark 4.3 Let P be a homogeneous polynomial of degree d in \mathbb{C}^{k+1} . Let $V = \{z \in \mathbb{P}^k, P(z) = 0\}$ and $[V]$ the current of integration on V . Since $\pi^*[V] = dd^c \log |P|$, we get $\text{vol}(V) = \|[V]\| = \int [V] \wedge \omega^{k-1} = d$.

Let T be a positive closed current of bidimension (p, p) in \mathbb{P}^k . Let R_1, \dots, R_q be closed positive currents of bidegree $(1, 1)$. Locally in \mathbb{P}^k each R_j can be written as $dd^c u_j$ where u_j is a plurisubharmonic function. We will say that (T, R_1, \dots, R_q) admit a wedge product if for every $x \in \mathbb{P}^k$, there exists an open set ω containing x , and plurisubharmonic functions (u_1, \dots, u_q) on ω such that for every $j, 1 \leq j \leq q, R_j = dd^c u_j$, and moreover for all $i_1, \dots, i_m \in [1, q]$, the set

$$X := \text{Supp } T \cap M(u_{i_1}) \cap \dots \cap M(u_{i_m}) \cap \omega$$

is in the envelope of $p - m + 1$ pseudoconvexity of $\omega \setminus X$.

Under this assumption and using Theorem 3.5 we can define the wedge product $T \wedge R_1 \wedge \dots \wedge R_q$.

Theorem 4.4 *Let T be a positive closed current of bidimension (p, p) on \mathbb{P}^k . Let R_1, \dots, R_q be positive closed currents of bidegree $(1, 1)$ on \mathbb{P}^k . Assume that (T, R_1, \dots, R_q) admit a wedge product. Then*

$$\|T \wedge R_1 \dots \wedge R_q\| = \|T\| \|R_1\| \dots \|R_q\| .$$

In particular $T \wedge R_1 \wedge \dots \wedge R_q$ is nonzero and $\text{supp } T \cap \text{supp}(R_1) \cap \dots \cap \text{supp}(R_q) \neq \emptyset$.

Proof. Without loss of generality we can assume that $\|T\| = \|R_1\| = \dots = \|R_q\| = 1$. Assume $\pi^*R_q = dd^c v, v(\lambda z) = \log |\lambda| + v(z)$. Then as in the proof of Proposition 4.2, $R_q = \lim_{\epsilon \rightarrow 0} R_q^\epsilon$ where $\pi^*R_q^\epsilon = dd^c v_\epsilon$. The assumption that (T, R_1, \dots, R_q) admit a wedge product and Theorem 3.5 imply that $T \wedge R_1 \wedge \dots \wedge R_q = \lim_{\epsilon \rightarrow 0} T \wedge R_1 \wedge \dots \wedge R_{q-1} \wedge R_q^\epsilon$. (Indeed convergence of currents has to be checked locally, and $v_\epsilon \searrow v$ in a given chart in \mathbb{P}^k .)

We can also assume that $p + q = k$. Hence using Stokes theorem we get if $u_\epsilon[z] = v_\epsilon(z) - \log \|z\|$

$$\begin{aligned} \int T \wedge R_1 \wedge \dots \wedge R_q &= \lim_{\epsilon \rightarrow 0} \int T \wedge R_1 \wedge \dots \wedge R_{q-1} \wedge R_q^\epsilon \\ &= \lim_{\epsilon \rightarrow 0} \int T \wedge R_1 \wedge \dots \wedge R_{q-1} \wedge (\omega + dd^c u_\epsilon) \\ &= \int T \wedge R_1 \wedge \dots \wedge R_{q-1} \wedge \omega . \end{aligned}$$

Repeating the procedure we then show that

$$\int T \wedge R_1 \wedge \dots \wedge R_q = \int T \wedge \omega^q = 1 .$$

Remark 4.5 Let P_1, \dots, P_q be homogeneous polynomials of degree d_1, \dots, d_q in \mathbb{C}^{k+1} . Let $V_j = \{z \in \mathbb{P}^k; P_j(z) = 0\}$. Assume $\text{codim}(V_{j_1} \cap \dots \cap V_{j_m}) = m$. Then the currents $([V_1], \dots, [V_q])$ admit a wedge product. Indeed if u_j is such that on a given chart on $\mathbb{P}^k, dd^c u_j = [V_j]$, then $M(u_{i_1}) \cap \dots \cap M(u_{i_m})$ has dimension $k - m$, so is in the $k - m + 1$ envelope of the complement. So Theorem 4.4 and Remark 4.3 gives in this case Bezout's theorem. See Demailly [De] who shows that $[V_1] \wedge \dots \wedge [V_q]$ is the current of integration of the intersection with multiplicity.

Theorem 4.6 *Let T be a nonzero, positive closed current of bidegree (p, p) in \mathbb{P}^k . Let $X = \text{supp } T$. For any algebraic variety $V = \{h_1 = \dots = h_l = 0\}, X \cap V \neq \emptyset$ provided $p + l \leq k$ and provided V is a geometric complete intersection.*

Proof. Let ω be the Kahler form on \mathbb{P}^k . Replacing T by $T \wedge \omega^{k-(p+l)}$ we can assume that $p + l = k$. We assume that $X \cap V = \emptyset$.

Define

$$u[z_0 : \dots : z_k] = \log \frac{\|z\|^{2s}}{|h_1|^2 + \dots + |h_l|^2}$$

where we have assumed that the polynomials h_j are homogenous of degree s .

The function u is smooth in $\mathbb{P}^k \setminus V$. Let S be the current in \mathbb{P}^k defined by the relation:

$$\pi^* S = (dd^c \log(|h_1|^2 + \dots + |h_l|^2))^{l-1} .$$

We show at first that in $\mathbb{P}^k \setminus V$, we have

$$dd^c u \wedge S = 2s\omega \wedge S .$$

In local coordinates

$$dd^c u \wedge S = dd^c \log \|z\|^{2s} \wedge S - (dd^c \log \|h\|^2)^l$$

and the last term is zero outside V . Indeed, if for example $h_1 \neq 0, \log \|h\|^2 = \log |h_1|^2 + \log \left(1 + \frac{|h_2|^2}{|h_1|^2} + \dots + \frac{|h_l|^2}{|h_1|^2} \right)$. If we consider the variety $\frac{h_m}{h_1} = c^{te}$, we see that $dd^c \log \|h\|^2$ has at least $k - l + 1$ zero eigenvalues, the result follows.

Let θ be a nonnegative test function with support in $\mathbb{P}^k \setminus V$ and with value 1 in a neighborhood of X . Since S is smooth and closed in a neighborhood of $\text{supp } T$, the current $T \wedge S$ is well defined and

$$\langle T \wedge S, \theta dd^c u \rangle = \langle T \wedge S, dd^c(u\theta) \rangle = \langle dd^c T \wedge S, u\theta \rangle = 0$$

and hence $T \wedge S \wedge \omega = 0$. We can vary slightly the h_j 's to obtain varieties V_i close to V and still not intersecting X . We obtain for the corresponding forms

S_i , that $T \wedge S_i \wedge \omega = 0$. It is clear that if we take enough perturbations of V , this implies that $T = 0$, a contradiction.

Theorem 4.7 *Let T be a positive, closed $(1, 1)$ current on \mathbb{P}^k . Assume $\pi^*T = dd^c v$ with v plurisubharmonic and continuous on $\mathbb{C}^{k+1} \setminus \{0\}$. Let $X_l = \text{supp } T^l$. Then for $2l \leq k$, X_l is connected.*

Proof. We first observe that a positive closed current S of bidegree (l, l) is a limit of smooth positive closed currents. Indeed it is enough to take an approximation of identity in $\text{Aut}(\mathbb{P}^k)$ and average the current S with respect to the approximation of identity. Let $\rho_\varepsilon(g)$ be an approximation of identity on the group $U(k)$ which acts transitively on \mathbb{P}^k . Let μ be the Haar measure for $U(k)$. For a positive (l, l) current S define S_ε by

$$S_\varepsilon = \int_{U(k)} \rho_\varepsilon(g)(g_*S) d\mu(g).$$

If $\rho_\varepsilon \geq 0$, S_ε is positive, closed and smooth [deRh]. The previous formula has to be understood as follows: If ϕ is a test form,

$$\langle S_\varepsilon, \phi \rangle = \int \rho_\varepsilon(g) \langle S, g^* \phi \rangle d\mu(g).$$

It is also known, Dold [Do] or de Rham [deRh], that there exists a constant c such that $S - c\omega^l = dd^c H$ where H has integrable coefficients. Observe that since $S \geq 0$ then $c \geq 0$.

Since T has locally a continuous potential u , i.e. $dd^c u = T$, we can define T^l for any $l \leq k$. Assume there exists two disjoint open sets U_1, U_2 each one intersecting $X_l = \text{supp } T^l$ and such that $X_l \cap \partial U_1 = X_l \cap \partial U_2 = \emptyset$. Let $S_1 = T^l|_{U_1}$ and $S_2 = T^l|_{U_2}$. Clearly S_1 is a positive closed current. Let (S_1^n) be a sequence of smooth positive closed currents $S_1^n \rightarrow S_1$. Let B be a ball where $T = dd^c u$. Since all the currents are positive, hence with measure coefficients, we have $uS_1^n \rightarrow uS_1$, on B hence $dd^c u \wedge S_1^n \rightarrow dd^c u \wedge S_1$. Inductively we get $(dd^c u)^l \wedge S_1^n \rightarrow (dd^c u)^l \wedge S_1$, hence $T^l \wedge S_1^n \rightarrow T^l \wedge S_1$. Since T^l has no mass on ∂U_2 we also have $S_2 \wedge S_1^n \rightarrow S_2 \wedge S_1$.

Let $s = k - 2l$ and let $S_2 = c_2\omega^l + dd^c H_2$. We have

$$\begin{aligned} \int S_2 \wedge S_1 \wedge \omega^s &= \lim_n \int S_2 \wedge S_1^n \wedge \omega^s \\ &= \lim_n \int c_2 S_1^n \wedge \omega^{l+s} + \int dd^c H_2 \wedge S_1^n \wedge \omega^s. \end{aligned}$$

Since S_1^n are smooth, the last integral is zero. Hence

$$\int S_2 \wedge S_1 \wedge \omega^s = c_2 \int \omega^{l+s} \wedge S_1.$$

But $S_2 \neq 0$ implies $c_2 \neq 0$ and $S_1 \neq 0$ implies that $S_1 \wedge \omega^{l+s} \neq 0$. Consequently the previous relation shows that $S_1 \wedge S_2 \neq 0$, contradicting the assumption that $\text{supp } S_1 \cap \text{supp } S_2 = \emptyset$.

5 Examples from holomorphic dynamics in \mathbb{P}^k

For background on this section we refer to [FS1], [FS2] and [HP].

Let f be a holomorphic surjective map on \mathbb{P}^k . Let $F = [F_0 : \dots : F_k]$ be a lifting of f to \mathbb{C}^{k+1} . Let d be the common degree of the polynomials F_j . The fact that f is holomorphic implies that $F^{-1}(0) = (0)$ and hence, there exists a constant $c > 0$ such that $1/c\|z\|^d \leq \|F(z)\| \leq c\|z\|^d$. Hence out of 0 we have

$$|1/d^{n+1} \log \|F^{n+1}\| - 1/d^n \log \|F^n\|| \leq c_1/d^n$$

and the function

$$G(z) := \lim_{n \rightarrow \infty} 1/d^n \log \|F^n(z)\|$$

is continuous on $\mathbb{C}^{k+1} \setminus \{0\}$ since the convergence is uniform. Clearly we have:

- (i) $G(\lambda z) = \log |\lambda| + G(z), \lambda \in \mathbb{C}$
- (ii) G is plurisubharmonic in \mathbb{C}^{k+1}
- (iii) $G((F(z))) = dG(z)$.

Definition 5.1 Let $f: \mathbb{P}^k \mapsto \mathbb{P}^k$ be in the space H_d of holomorphic self maps on \mathbb{P}^k given by polynomials of degree d . For $0 \leq l \leq k - 1$, a point $p \in \mathbb{P}^k$ belongs to the Fatou set F_l if there exists a neighborhood $U(p)$ such that for every $q \in U(p)$ there exists a complex variety X_q of codimension l such that $\{f^n|X_q\}$ is equicontinuous. Observe that F_0 is the largest open set where f^n is equicontinuous. We call F_0 the Fatou set. We have $F_0 \subset F_1 \subset \dots \subset F_{k-1}$.

Let $J_l := \mathbb{P}^k \setminus F_l$, we call J_l the Julia set of order l .

Let T be the $(1, 1)$ positive closed current defined on \mathbb{P}^k by the relation $\pi^* T = dd^c G$. Since G is continuous on $\mathbb{C}^{k+1} \setminus \{0\}$, T has a continuous potential on any chart in \mathbb{P}^k , for example if $z_0 \neq 0, T = dd^c G(1, z_1, \dots, z_k)$. Hence we can define the closed positive currents of bidegree (l, l) by the relation $\pi^*(T^l) = (dd^c G)^l$ for any $1 \leq l \leq k$. We define $J'_{l-1} = \text{supp } T^l, 1 \leq l \leq k$, and $F'_{l-1} = \mathbb{P}^k \setminus J'_{l-1}$.

Theorem 5.2 Let $f: \mathbb{P}^k \mapsto \mathbb{P}^k$ be a surjective holomorphic map of degree $d \geq 2$ on \mathbb{P}^k . Let T be the positive closed $(1, 1)$ current associated to f . Then

- (i) J'_{k-1} is nonempty and for every $0 \leq l < k, f(J'_l) = f^{-1}(J'_l) = J'_l$. Moreover $J'_l \subset J_l$ for $0 \leq l < k$.
- (ii) $\text{Support } T = J'_0 = J_0$ and F_0 is a domain of holomorphy. For $0 \leq l < k, F'_l$ is $(k - l - 1)$ pseudoconvex.
- (iii) J'_{l-1} is connected if $2l \leq k$.
- (iv) If X is an algebraic variety of dimension r , which is a complete intersection, then $T^r \wedge [X]$ is a positive measure of total mass $\text{vol}(X)$. Moreover X intersects every component of $\text{supp } T^r$.

Proof. It follows from Proposition 4.2 that $\|T\| = 1$ since $G(\lambda z) = \log |\lambda| + G(z)$. Theorem 4.4 implies that for every $1 \leq l \leq k \|T^l\| = 1$, in particular T^k is a probability measure, hence $J'_{k-1} = \text{supp } T^k$ is nonempty.

Let $G_n := 1/d^n \log \|F^n\|$ and observe that G_n are smooth and converge uniformly to G on $\mathbb{C}^{k+1} \setminus \{0\}$. If $(dd^c G_n)^l \mapsto 0$, then $(dd^c G_n \circ F)^l = (dd^c (G_{n+1}d))^l \mapsto 0$, hence F_l^j and J_l^j are totally invariant.

We prove now that $J_l^j \subset J_l, 0 \leq l < k$. We have to show that if $p \in F_l$ then T^{l+1} vanishes in a neighborhood of p . Let U be a neighborhood of $p, q \in U$, and let X_q be an analytic variety of codimension l through q such that $f^n|_{X_q}$ is equicontinuous. Let $f^{n_i}|_{X_q} \mapsto h$. This means that we can find non zero holomorphic functions λ_{n_i} on X_q such that $\frac{F^{n_i}}{\lambda_{n_i}}$ converge to a holomorphic function H . We can write

$$G_{n_i} = 1/d^{n_i} \log \|F^{n_i}/\lambda_{n_i}\| + 1/d^{n_i} \log |\lambda_{n_i}|.$$

The first sequence converges uniformly to zero on $\pi^{-1}(X_q)$ and the sequence $1/d^{n_i} \log |\lambda_{n_i}|$ is harmonic on any analytic disc in $\pi^{-1}(X_q)$. So G is harmonic on any analytic disc in $\pi^{-1}(X_q)$. Hence if G_1 is a local section of G in U , G_1 is harmonic on any analytic disc contained in X_q . Assume that U is a polydisc $\Delta^{k-l-1} \times \Delta^{l+1}$. For every $a \in \Delta^{k-l-1}$ and every q in $a \times \Delta^{l+1}, X_q^a := a \times \Delta^{l+1} \cap X_q$ is at least of dimension one, and G_1 is harmonic on any analytic disc contained in X_q^a . Therefore Lemma 6.9 in [FS2] implies that $(dd^c G_1(a, w))^{l+1} = 0$ in $a \times \Delta^{l+1}$. Since a was arbitrary, standard results in slicing theory applied to the current T^{l+1} [HS] or [Fe], imply that T^{l+1} is zero on U .

(ii) We show that $\text{supp } T = J_0$. We have that $\text{supp } T \subset J_0$, in this case no slicing is needed since the first part of the proof shows that G is pluriharmonic on $\pi^{-1}(F_0)$. Let U be an open set disjoint from $\text{supp } T$. We want to prove that $U \subset F_0$. We can assume U is a ball. We know that G is pluriharmonic on $\pi^{-1}(U)$, hence $G = \log |h|$ where h is holomorphic on $\pi^{-1}(U)$. Since

$$|G - G_n| \leq c/d^n$$

we get

$$e^{-c} \leq \frac{\|F^n\|}{|h^{d^n}|} \leq e^c.$$

So the family (f^n) is normal on U .

The fact that $F_0 = \mathbb{P}^k \setminus J_0$ is pseudoconvex follows from Corollary 2.6. Since the Levi problem has a positive solution for domains in \mathbb{P}^k , it follows that F_0 is a domain of holomorphy. Corollary 2.6 also implies that F_{l-1}^j , the complement of $\text{supp } T^l$ is $(k - l)$ -pseudoconvex.

(iii) Theorem 4.7 shows that J_{l-1}^j is connected if $l \leq 2k$.

(iv) Since T has locally a continuous potential, T^r and $[X]$ admit a wedge product. The current $[X]$ is given locally as $dd^c \log |P_1| \wedge \dots \wedge dd^c \log |P_{k-r}|$ where (P_l) are homogeneous polynomials. So we can apply Theorem 4.4 and we get that $\|T^r \wedge [X]\| = \|[X]\|$. Let Y be a component of $\text{supp } T^r$. Assume $Y \cap X = \emptyset$. Let U be a neighborhood of Y such that $U \cap X = \emptyset$. Let $S = T^r|_U, S$ is a positive closed current of bidegree (r, r) to which we can apply Theorem 4.6, hence $\text{supp } S$ intersects X , a contradiction.

We next show that f cannot be hyperbolic on J'_l for $l < k - 1$. Let $f \in H_d, f : \mathbb{P}^k \mapsto \mathbb{P}^k$. We define first hyperbolicity, see Ruelle [Ru]. Let $K \subset \mathbb{P}^k$ be a compact set. We assume that K is *surjectively forward invariant*, that is $f(K) = K$. The space $\hat{K} = K^{\mathbb{N}}$ of orbits $\{x_n\}_{n=-\infty}^0, f(x_n) = x_{n+1}$ is compact in the product topology. By the tangentbundle T_k of \hat{K} we mean the space of (x, ξ) where $x = \{x_n\} \in \hat{K}$ and $\xi \in T_{\mathbb{P}^2}(x_0)$ is a tangent vector. We give this tangentbundle the obvious topology. Then f lifts to a homeomorphism $\hat{f} : \hat{K} \mapsto \hat{K}$ and f' lifts to a map f' on T_k in the obvious way.

Definition 5.3 Let $K \subset \mathbb{P}^k$ be a compact surjectively forward invariant set. Then f is **hyperbolic** on K if there exists a continuous splitting $E^u \oplus E^s$ of the tangent space bundle of \hat{K} such that \hat{f}' preserves the splitting and for some constants $C, c > 0, \lambda > 1, \mu < 1$ depending on the choice of a Hermitian metric on \mathbb{P}^2 ,

$$|D\hat{f}'^n(\xi)| \geq c\lambda^n|\xi|, \xi \in E^u$$

$$|D\hat{f}'^n(\xi)| \leq C\mu^n|\xi|, \xi \in E^s, \quad n = 1, 2, \dots$$

Theorem 5.4 Let $f: \mathbb{P}^k \mapsto \mathbb{P}^k, f \in H_d$. If $\text{supp } \mu = J'_{k-1}$ is hyperbolic, then the unstable dimension of J'_{k-1} is equal to k .

Proof. Suppose the stable dimension is equal to $l \geq 1$. Then through every point $x \in J'_{k-1}$ there is an analytic disc Δ_x on which the family (f^n) is equicontinuous. Hence if G_1 is a local potential for μ , i.e. $(dd^c G_1)^k = \mu$, then G_1 is harmonic on Δ_x . The theorem will then be a consequence of the following lemma.

Lemma 5.5 Let u be a continuous plurisubharmonic function in a ball $\bar{B} \subset \mathbb{C}^k$. Let $(dd^c u)^k = \nu$. Assume that through every point x of $X := \text{supp } \nu$ there is an analytic disc Δ_x such that $u|_{\Delta_x}$ is harmonic, then $\nu = 0$.

Proof. We are going to prove that u is maximal and hence $\nu = 0$. Let v be a continuous function on \bar{B} which is plurisubharmonic on B , with $v \leq u$ on ∂B . Let $M := \sup_B v - u$, assume $M > 0$. Let $K = \{z; z \in B, (v - u)(z) = M\}$. Let p be a peak point for a function $h \in C(K)$ which is a uniform limit of K of holomorphic polynomials, i.e., $h(p) = 1$ and $|h| < 1$ on $K \setminus \{p\}$. If $p \in X$, then since $v \leq u + M$ and $u|_{\Delta_p}$ is harmonic then $v - u = M$ on Δ_p , hence $\Delta_p \subset K$, contradicting that p is a peak point for h . So $p \in B \setminus X$. Let $B(p, r) \Subset B$ be a ball such that $B(p, r) \cap X = \emptyset$. Since $(dd^c u)^k = 0$ on $B(p, r)$, it follows from [GS] that given any $z \in B(p, r)$, there exists a probability measure σ_z supported on $\partial B(p, r)$ such that $u(z) = \int u d\sigma_z$, and moreover for any continuous function on $\bar{B}(p, r)$, plurisubharmonic on $B(p, r)$

$$\phi(z) \leq \int \phi d\sigma_z .$$

Since $M = v(p) - u(p) \leq \int (u - v) d\sigma_p \leq M$, we have that support σ_p is contained in K , and p cannot be a peak point for h , since $h(p) = \int h d\sigma_p$, hence K is empty and $\nu = 0$.

Theorem 5.6 *Let $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$ be a holomorphic map of degree $d \leq 2$. Then f cannot be hyperbolic on \mathbb{P}^k nor on J'_l for $l < k - 1$.*

Proof. Assume f is hyperbolic on \mathbb{P}^k . Since the critical set is nonempty the fibre dimension of E^u is $\leq k - 1$. If $\dim E^s = k$ then all periodic orbits are attractive. Pick one, p , with immediate basin of attraction Ω . Since f is surjective, $\partial\Omega$ is a non empty, compact, forward invariant subset of \mathbb{P}^k . Hyperbolicity implies that orbits of points $q \in \Omega$ close to $\partial\Omega$ converge to $\partial\Omega$ contradicting that they are in the attractive basin of p . Hence $1 \leq \dim E^s \leq k - 1$. Let $\dim E^s = l$. Then we have a foliation of \mathbb{P}^k by stable manifolds of dimension l , and on each manifold the family (f^n) is equicontinuous, so $\mathbb{P}^k \subset F_{k-l}$. Since $F_{k-l} \subset F_{k-1}$, we get $J_{k-1} = \emptyset$ which is a contradiction.

Let $l < k - 1$ and assume J'_l is hyperbolic. Theorem 5.2(iv) shows that every component of J'_l intersects C , the critical set. Hence the fibre dimension of $E^u \leq k - 1$, so $\dim E^s \geq l$. This implies that through every point p in $J'_l \supset \text{supp } \mu$ there exists an analytic disc Δ_p on which G_1 is harmonic, G_1 is a local solution of $dd^c G_1 = T$.

So Lemma 5.6 applies and $\mu = 0$, a contradiction.

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