Math. Ann. 303, 579–597 (1995)

Mathematische Annalen

Holomorphic embeddings of planar domains into C^2

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Received: 4 October 1994

1 Introduction and the main result

We say that a map F from a complex manifold E to C^q is a proper holomorphic embedding of E into C^q if it is a holomorphic immersion which is one to one and such that the preimage of every compact set is compact. If $F: E \to C^q$ is a proper holomorphic embedding then $F(E)$ is a closed submanifold of C^q.

By a theorem of Remmert-Narasimhan-Bishop [Re, Na, Bi] for every p-dimensional Stein manifold E there is a proper holomorphic embedding of E into C^q where $q = 2p + 1$. Forster [Fo] showed that one can take $q = 2p$ if $p \geqq 2$. Recently Eliashberg and Gromov [EG] proved that one can take any $q > (3p + 1)/2$ and showed that this is sharp for even p.

In the case $p = 1$ the question remains open: Given an open Riemann surface E does there exist a proper holomorphic embedding of E into C^2? Trying to answer this question it is natural to begin with planar domains, i.e. open connected subsets of C. Very few results are known and even in the simplest cases the construction of such an embedding is not easy. Kasahara and Nishino [KN, St] used a technique involving the Fatou-Bieberbach map from C^2 to C^2 to prove that the unit disc can be properly holomorphically embedded into C^2. Their method can be used to prove that for each $M \in \mathcal{N}$ there is an M-connected domain in C which can be properly holomorphically embedded into C^2. Laufer [La] showed that every annulus can be properly holomorphically embedded into C^2. Alexander [Al] used elliptic modular functions to construct such an embedding of the punctured disc into C^2 which, after a slight modification, becomes a proper holomorphic embedding of the unit disc into C^2.

Our main result is the following

Theorem 1.1 *Every bounded, finitely connected domain $D \subset C$ without isolated points in the boundary can be properly holomorphically embedded into C^2.*

Given M we shall start out by finding a large class of M-connected domains which can be properly holomorphically embedded into C^2. Then we shall show that every M-connected domain as in the theorem is conformally equivalent to one of these embeddable domains which will imply our theorem.

2 A metric on the space of M-connected domains bounded by Jordan curves

We denote by Δ the open unit disc in C and by $\bar{C}$ the Riemann sphere. With no loss of generality we work with domains in $\bar{C}$ which contain ∞. Fix $M \in \mathcal{N}$, the connectivity of the domains to be studied. For our purposes it will suffice to consider domains whose boundaries consist of M pairwise disjoint Jordan curves contained in C, i.e. domains of the form $\Omega = \bar{C}\setminus\bigcup_{j=1}^{M} \bar{D}_j$ where for each j, D_j is a bounded domain in C whose boundary is a Jordan curve J_j and where $\bar{D}_i \cap \bar{D}_j = \emptyset$ for $i \neq j$. Thus such a domain is uniquely determined by an M-tuple $J = (J_1, \dots, J_M)$ of Jordan curves as above and any permutation of curves J_j will produce the same domain which we call the domain bounded by J. For our purposes, rather than with domains we prefer to work with M-tuples $(J_1, \dots, J_M)$ of Jordan curves as above. Obviously, such an M-tuple can be identified with the domain it determines together with an enumeration of its boundary curves. If all curves J_j above are circles then Ω is called a circular domain. Again, an M-connected circular domain together with an enumeration of its boundary components can be identified with an M-tuple of the form $((a_1, r_1), \dots, (a_M, r_M))$ where for each $j, 1 \leqq j \leqq M$, $a_j \in C$ is the center and $r_j > 0$ is the radius of the j'th bounding circle. The obvious condition to be satisfied is

$$r_i + r_j < |a_i - a_j| \quad (i \neq j, 1 \leqq i,j \leqq M)\,. \tag{2.1}$$

We denote by $\mathcal{J}_M$ the set of all M-tuples of Jordan curves as above and by $\mathcal{M}_M$ the set of all M-tuples $((a_1, r_1), \dots, (a_M, r_M))$ satisfying (2.1).

Clearly $\mathcal{M}_M$ inherits a metric from $(C \times R)^M$. To define a metric on $\mathcal{J}_M$ we first define the distance between two Jordan curves J_1 and J_2 in C. Let $H(J_1, J_2)$ be the set of all homeomorphisms from J_1 to J_2. Define

$$\rho(J_1, J_2) = \inf_{\varphi \in H(J_1, J_2)} \max_{z \in J_1} |\varphi(z) - z|$$

and if $J = (J_1, \dots, J_M)$, $J' = (J'_1, \dots, J'_M)$ belong to $\mathcal{J}_M$ define

$$d(J, J') = \max_{1 \leqq j \leqq M} \rho(J_j, J'_j)\,.$$

It is easy to see that d is a metric on $\mathcal{J}_M$. Without a possibility of confusion we identify $((a_1, r_1), \dots, (a_M, r_M)) \in \mathcal{M}_M$ with $(\{\zeta \in C: |\zeta - a_1| = r_1\}, \dots, \{\zeta \in C: |\zeta - a_M| = r_M\}) \in \mathcal{J}_M$.

3 The idea of the proof

To explain the first part of the proof suppose that we want to get a holomorphic embedding of Δ into C^2 in a constructive way, that is, to begin with an injective immersion and then, in an inductive process, push the values near $b\Delta$ out to get the desired embedding in the limit. Here is a way to do this:

Let $\{T_n\}$ be a sequence of positive real numbers and let $\{N_n\}$ be a sequence of positive integers. The maps

$$\begin{aligned}
F_1(\zeta) &= (T_1\zeta, 0)\\
F_2(\zeta) &= (T_1\zeta, T_2\zeta^{N_2})\\
F_3(\zeta) &= (T_1\zeta + T_3(\zeta^{N_2})^{N_3}, T_2\zeta^{N_2})\\
F_4(\zeta) &= \left(T_1\zeta + T_3(\zeta^{N_2})^{N_3}, T_2\zeta^{N_2} + T_4\left((\zeta^{N_2})^{N_3} + \frac{T_1}{T_3}\zeta\right)^{N_4}\right)\\
&\ \vdots
\end{aligned}$$

are injective immersions of $\bar{\Delta}$ to C^2. As we see we keep adding functions of the form $T_k\varphi_k^{N_k}$ where $|\varphi_k|$ approximately equals 1 on $b\Delta$. We will choose our constants T_k so that they will converge to infinity since we want our functions F_k to get larger and larger on the boundary and thus achieve that the limit of F_k blows up when we approach the boundary. Since $|\varphi_k|$ are not exactly equal to 1 on $b\Delta$ we loose control over $|\varphi_k^{N_k}|$ on $b\Delta$ when N_k have to become larger and larger if we want to achieve uniform convergence of F_k on compact sets. To get good control over $|\varphi_k^{N_k}|$ we would need $|\varphi_k| \equiv 1$ on $b\Delta$. As this is not the case we pass from $\Delta = \Omega_0$ to the domain Ω_k bounded by $\{\zeta \in C: |\varphi_k(\zeta)| = 1\}$ which, provided that the sequences $\{T_k\}$ and $\{N_k\}$ converge to infinity fast enough, is a small perturbation of Δ. In fact, each Ω_k is a small perturbation of Ω_{k-1} and, if we choose the sequences $\{T_k\}$ and $\{N_k\}$ carefully, the domain $\Omega = \lim \Omega_k$ is, by the Riemann mapping theorem, conformally equivalent to Δ, and the functions F_k converge, uniformly on compact subsets of Ω, to a proper holomorphic embedding of Ω into C^2. So if $\omega: \Delta \to \Omega$ is a conformal map then $F \circ \omega$ is the desired holomorphic embedding of Δ into C^2.

In the case of multiply connected domains we begin by a circular domain Ω_0, that is, with $u \in \mathscr{M}_M$ and perform a similar procedure. Again, if done carefully, the process will produce a holomorphic embedding of $\Omega = \lim \Omega_k$, a domain bounded by M Jordan curves, into C^2. Now, in general Ω will not be conformally equivalent to Ω_0 and we will have to use additional arguments to prove that Ω_0 embeds holomorphically into C^2. Starting with $u \in \mathscr{M}_M$ we shall denote by $\Theta_m(u) \in \mathscr{J}_M$ the result of such a process where each $m \in \mathscr{N}$ denotes a different choice of sequences $\{T_k\}$ and $\{N_k\}$. The first part of our proof will consist of proving

Lemma 3.1 *Let $M \in \mathcal{N}$. Given $u_0 \in \mathcal{M}_M$ there are a neighbourhood $U \subset \mathcal{M}_M$ of u_0 and a sequence of continuous maps $\Theta_n : U \to \mathcal{J}_M$ such that*

$$\text{as } n \to \infty,\ \Theta_n(u) \in \mathcal{J}_M \text{ converges to } u \in \mathcal{M}_M \subset \mathcal{J}_M, \text{ uniformly on } U, \tag{3.1}$$

and such that for every $n \in \mathcal{N}$ and $u \in U$ the domain bounded by $\Theta_n(u)$ can be properly holomorphically embedded into C^2.

The second part of the proof depends on a different circle of ideas, the ones related to the "continuity method" of Koebe and to the convergence theorem for conformal maps of variable domains. One proves that to prove Theorem 1.1 it is enough to prove the existence of holomorphic embeddings for the domains in Lemma 3.1:

Lemma 3.2 *Let $M \in \mathcal{N}$ and let $U \subset \mathcal{M}_M$ be a neighbourhood of $((a_1, r_1), \ldots, (a_M, r_M)) \in \mathcal{M}_M$. Let $D = \bar{C} \setminus \bigcup_{j=1}^{M}(a_j + r_j\Delta)$. Suppose that there is a sequence of continuous maps $\Theta_n : U \to \mathcal{J}_M$ satisfying (3.1). Then there are $\nu \in \mathcal{N}$ and $\tilde{u} \in U$ such that D is conformally equivalent to the domain bounded by $\Theta_\nu(\tilde{u})$.*

That Theorem 1.1 now follows from Lemmas 3.1 and 3.2 is a consequence of the fact that every N-connected domain $D \subset \bar{C}$, $\infty \in D$, without isolated points in the boundary, is conformally equivalent to an N-connected circular domain [Go].

4 Conformal maps of multiply connected domains

In this section we list some facts that we need in our proofs. Their proofs can be found in [Go].

4.1 Let $\varphi : D_1 \to D_2$ be a conformal map where $D_1, D_2 \subset \bar{C}$ are M-connected domains, both containing ∞ and each of them bounded by M pairwise disjoint Jordan curves. Then φ extends to a homeormorphism from $\bar{D}_1$ to $\bar{D}_2$.

If D_1, D_2 are two domains both containing ∞ and if $\varphi : D_1 \to D_2$ is a conformal map then φ is called *normalized* if $\varphi(\infty) = \infty$ and if in a neighbourhood of ∞ the map φ has the form

$$\varphi(z) = z + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots .$$

4.2 Given an M-connected domain $D \subset \bar{C}$, $\infty \in D$, without isolated points in the boundary, there is a unique normalized conformal map $\varphi : D \to \varphi(D)$ such that $\varphi(D)$ is a circular domain.

We will not need the fact that 4.2 still holds if D has isolated points in the boundary. In this case $\varphi(D)$ also has isolated points in the boundary.

Let $B_n \subset \bar{C}$ be a sequence of domains, all containing ∞. The *kernel* of B_n is the largest nonempty domain $B \subset \bar{C}$ which contains ∞ and has the property that for every compact set $E \subset \bar{C}$ contained in B there is an $n(E) < \infty$ such that $E \subset B_n$ for all $n \in \mathcal{N}$, $n \geqq n(E)$. We say that a sequence of domains $B_n \subset \bar{C}$, $\infty \in B_n$ $(n \in \mathcal{N})$, *converges* to a domain B and write $B_n \to B$ if every subsequence of B_n has a kernel which coincides with B.

4.3 Let $f_n: A_n \to B_n$ be normalized conformal maps where $A_n \subset \bar{C}$, $\infty \in A_n$ $(n \in \mathcal{N})$, is a sequence of domains converging to a domain A. The following are equivalent

(a) f_n converges uniformly on closed sets $E \subset C, E \subset A$, to a one-to-one map f

(b) B_n has a kernel B and converges to B.

If either (a) or (b) holds then f is the normalized conformal map from A to B.

There is a well defined map $\Phi: \mathcal{J}_M \to \mathcal{M}_M$ where $((a_1, r_1), \ldots, (a_M, r_M)) = \Phi((J_1, \ldots, J_M))$ where the circular domain bounded by $\Gamma_j = \{\zeta \in C: |\zeta - a_j| = r_j\}$, $1 \leqq j \leqq M$, is the image of the domain bounded by $J_j, 1 \leqq j \leqq M$, under the unique normalized map and that this map maps J_j to Γ_j, $1 \leqq j \leqq M$.

5 Proof of Lemma 3.2

Part 1. We first prove that Φ is continuous. Suppose that $Q_n \in \mathcal{J}_M$ converges to $Q_0 \in \mathcal{J}_M$. For each $n \in \mathcal{N}$ let D_n be the domain bounded by Q_n and let D_0 be the domain bounded by Q_0. For each $n \in \mathcal{N}$ write $\Phi(Q_n) = ((a_{n1}, r_{n1}), \ldots (a_{nM}, r_{nM}))$ and $\Gamma_{nj} = \{\zeta \in C: |\zeta - a_{nj}| = r_{nj}\}$ $(1 \leqq j \leqq M)$ and let Ω_n be the M-connected domain bounded by $\Gamma_{nj}, 1 \leqq j \leqq M$. It is clear that $D_n \to D_0$; in particular, all D_n contain a fixed neighbourhood of ∞ so there is an $R < \infty$ such that $bD_n \subset R\Delta$ for all n. If $\varphi_n: D_n \to \Omega_n$ are normalized conformal maps Koebe's covering theorem [Go, p. 178, Hilfssatz 2] implies that $b\Omega_n \subset 2R\bar{\Delta}$ $(n \in \mathcal{N})$. In particular,

$$\{\Phi(Q_n): n \in \mathcal{N}\} \text{ is a bounded subset of } (C \times R)^M . \tag{5.1}$$

Assume that $\Phi(Q_n)$ does not converge. By (5.1) there are two subsequences Q_{n_k}, Q_{m_k} such that $\Phi(Q_{n_k}) \to (b, \rho) = ((b_1, \rho_1), \ldots, (b_M, \rho_M))$ and $\Phi(Q_{m_k}) \to (c, \mu) = ((c_1, \mu_1), \ldots, (c_M, \mu_M))$ where $(b, \rho) \neq (c, \mu)$. It is easy to see that Ω_{n_k} converges to A, the unbounded component of $\bar{C} \setminus \bigcup_{j=1}^{M} (b_j + \rho_j \bar{\Delta})$ and Ω_{m_k} converges to B, the unbounded component of $\bar{C} \setminus \bigcup_{j=1}^{M} (c_j + \mu_j \bar{\Delta})$. By 4.3 the sequence φ_{n_k} converges on D_0 uniformly on each closed set $E \subset C$, $E \subset D_0$, to the normalized conformal map from D_0 to A and φ_{m_k} converges on D uniformly on each closed set $E \subset C, E \subset D_0$, to the normalized conformal map from D_0 to B. By the nature of the sets D_0, A, B it follows that both A, B are M-connected circular domains and so the uniqueness part of 4.2 implies that $A = B$. In particular, $((b_1, \rho_1), \ldots, (b_M, \rho_M))$ differs from $((c_1, \mu_1), \ldots, (c_M, \mu_M))$ only for a possible permutation of entries.

With no loss of generality assume that the first components of $(b, \rho), (c, \mu)$ do not coincide. This means that there are two different boundary circles Λ_1, Λ_2

of $\varphi(D)$ such that the circle $\varphi_{n_k}(Q_{n_k 1})$ is arbitrarily close to Λ_1 and the circle $\varphi_{m_j}(Q_{m_j 1})$ is arbitrarily close to Λ_2 provided that k, j are large enough. Let $\Gamma \subset D_0$ be a Jordan curve separating Q_{01} from $Q_{02}, \ldots, Q_{0M}$ and ∞. Since $Q_n \to Q_0, D_n \to D_0$ it follows that for sufficiently large n, $\Gamma \subset D_n$ separates Q_{n1} from $Q_{n2}, \ldots, Q_{nM}$ and ∞. Since $\Omega_n \to \varphi(D)$ there is a neighbourhood $U \subset \varphi(D)$ of $\varphi(\Gamma)$ such that $U \subset \Omega_n$ for all sufficiently large n. Now, $\varphi_n(\Gamma)$ separates $\varphi_n(J_{n1})$ from $\varphi_n(J_{n2}), \ldots, \varphi_n(J_{nM})$ and ∞. Since $\varphi_n \to \varphi$ uniformly on Γ it follows that for n large enough, $\varphi_n(\Gamma) \subset U$ is arbitrarily small perturbation of $\varphi(\Gamma)$. Since U does not meet $b(\varphi(D_n)), b(\varphi(D_0))$ if n is large enough it follows that $\varphi(\Gamma)$ separates $\varphi_n(Q_{n1})$ from $\varphi_n(Q_{n2}), \ldots, \varphi_n(Q_{nM})$ and ∞ if n is large enough. This implies that Λ_1 and Λ_2 are both in the bounded component of $C \setminus \varphi(\Gamma)$ contradicting the fact that Γ separates one boundary component of D from all the others. This proves that $\Phi(Q_n)$ converges in $\mathscr{M}_M$. Now 4.3 together with a separation argument similar to the one used above shows that the limit of $\Phi(Q_n)$ must be $\Phi(Q_0)$. This completes the proof.

Part 2. Let $\tilde{U} \subset U$ be a compact neighbourhood of $((a_1, r_1), \ldots, (a_M, r_M))$. We prove that for each $\delta > 0$ there is an $n(\delta)$ such that

$$|\Phi(\Theta_{n(\delta)}(u)) - u| < \delta \quad (u \in \tilde{U}) .$$

To see this, assume that there are $\delta_0 > 0$, a sequence $u_k \in \tilde{U}$ and $n_k \to \infty$ such that

$$|\Phi(\Theta_{n_k}(u_k)) - u_k| \geqq \delta_0 \quad (k \in \mathscr{N}) . \tag{5.2}$$

By the compactness of $\tilde{U}$ we may, passing to a subsequence if necessary, assume that $u_k \to u_0 \in \tilde{U}$. By (3.1), $\Theta_{n_k}(u_k) - u_k \to 0$. Since $u_k \to u_0$ in $\mathscr{M}_M \subset \mathscr{J}_M$ it follows that $\Theta_{n_k}(u_k) \to u_0$ which, by Part 1, implies that $\Phi(\Theta_{n_k}(u_k)) \to \Phi(u_0)$. Since $\Phi(u_0) = u_0$ this contradicts (5.2).

Part 3. Let $G \subset \tilde{U}$ be a closed ball centered at $(a, r) = ((a_1, r_1), \ldots, (a_M, r_M))$ and let μ be its radius. By Part 2 there is a $\nu \in \mathscr{N}$ such that $|\Phi(\Theta_\nu(u)) - u| < \mu \ \ (u \in G)$ which, by the continuity of Φ and Θ_ν implies that

$$u \mapsto H(u) = (a, r) + u - \Phi(\Theta_\nu(u))$$

is a continuous map from G to G. By Brouwer's fixed point theorem there is a $\tilde{u} \in G$ such that $H(\tilde{u}) = \tilde{u}$, that is, $\Phi(\Theta(\tilde{u})) = (a, r)$ which, in particular, implies that D is conformally equivalent to the domain bounded by $\Theta_\nu(\tilde{u})$. This completes the proof of Lemma 3.2.

6 Proof of Lemma 3.1, Part 1

Composing by a fractional linear map we may assume that the initial domain Ω_0 is the open unit disc from which we remove $M - 1$ pairwise disjoint closed discs contained in Δ:

$$\Omega_0 = \Delta \setminus \bigcup_{j=2}^{M} (z_j + r_j \bar{\Delta}) .$$

We understand that $z_1 = 0$ and $r_1 = 1$. Choose $\gamma > 0$ so small that $8\gamma < r_j$ $(1 \leqq j \leqq M)$ and that the sets

$$z_j + (r_j + 8\gamma)\bar{\Delta},\ 2 \leqq j \leqq M, \text{ and } \{\zeta \in C\colon\ |\zeta| \geqq 1 - 8\gamma\}$$

are pairwise disjoint. Define

$$\Omega = (1 + 6\gamma)\Delta \setminus \bigcup_{j=2}^{M} (z_j + 2\gamma\bar{\Delta}) .$$

For each j, $1 \leqq j \leqq M$, define the annulus $\mathscr{A}_j = \{\zeta \in C\colon r_j - 6\gamma < |\zeta - z_j| < r_j + 6\gamma\}$. The closed convex hulls of $\mathscr{A}_j$, $2 \leqq j \leqq M$, are pairwise disjoint and all contained in the bounded component of $C \setminus \overline{\mathscr{A}_1}$. Also, $\mathscr{A}_j \subset \Omega$ $(1 \leqq j \leqq M)$.

We now define small perturbations of Ω_0 that we shall need in our proofs.

Definition 6.1 *Let* $p = (p_1, \ldots, p_M)\colon b\Delta \to (-5\gamma, 5\gamma)^M$ *be a continuous function. The domain*

$$\begin{aligned} D = \{&z_1 + \lambda(r_1 + p_1(\zeta))\zeta\colon \zeta \in b\Delta, 0 \leqq \lambda < 1\} \\ &\setminus \bigcup_{j=2}^{M} \{z_j + \lambda(r_j + p_j(\zeta))\zeta\colon \zeta \in b\Delta, 0 \leqq \lambda \leqq 1\} \end{aligned}$$

is called a **standard domain** *and* p *is called the* **parametrization** *of* bD.

By the properties of γ and $p_j, 1 \leqq j \leqq M$, every standard domain is M-connected. Its boundary bD consists of M Jordan curves $\Gamma_j = \{z_j + (r_j + p_j(\zeta))\zeta\colon \zeta \in b\Delta\} \subset \mathscr{A}_j$, $1 \leqq j \leqq M$. Note that each curve Γ_j consists of points obtained by moving, for each $\zeta \in b\Delta$, the point $z_j + \zeta r_j$ from the circle $z_j + r_j b\Delta$ for $p_j(\zeta)$ along the ray $\{z_j + t\zeta\colon t > 0\}$. Since p_j is sufficiently small, Γ_j is a small perturbation of $z_j + r_j b\Delta$. Under radial projection $z \to z_j + r_j(z - z_j)/|z - z_j|$, Γ_j is homeomorphic to the circle $z_j + r_j b\Delta$ and the domain bounded by Γ_j is a domain which is starlike with respect to z_j.

Proposition 6.1 *Suppose that* $\Phi\colon \Omega \to C$ *is a nonconstant holomorphic function. If there is a standard domain* D *such that* $|\Phi| = 1$ *on* bD *then there is only one such domain* D.

Proof. Assume that there are two such domains, D and D', and denote by $p = (p_1, \ldots, p_M)$, $p' = (p'_1, \ldots, p'_M)$ the parametrizations of their boundaries. If $2 \leqq j \leqq M$ then the maximum principle on D implies that for each $\zeta \in b\Delta$, $|\Phi(z_j + \zeta t)| < 1$ for each t, $r_j + p_j(\zeta) < t < r_j + 5\gamma$, and the maximum principle on D' implies that for each $\zeta \in b\Delta$, $|\Phi(z_j + \zeta t)| < 1$ for each t, $r_j + p'_j(\zeta) < t < r_j + 5\gamma$. Since $|\Phi(z_j + (r_j + p_j(\zeta))\zeta)| = |\Phi(z_j + (r_j + p'_j(\zeta))\zeta| =$

1 ($\zeta \in b\Delta$) it follows that $p_j'(\zeta) = p_j(\zeta)$ ($\zeta \in b\Delta$). A similar argument gives $p_1' = p_1$. This completes the proof.

Definition 6.2 *Given a standard domain whose boundary Γ is parametrized by $(p_1, \ldots, p_M)$ and given ε, $0 < \varepsilon < \gamma$, we call the set*

$$\bigcup_{j=1}^{M} \{z_j + (r_j + p_j(\zeta) + \lambda)\zeta\colon\ \zeta \in b\Delta,\ -\varepsilon < \lambda < \varepsilon\}$$

the ε-**belt** *along Γ.*

Thus the ε-belt along Γ is a particularly nice neighbourhood of Γ.

Assume that $f \in \mathscr{C}^1(\mathscr{A})$ where $\mathscr{A} = \bigcup_{j=1}^{M} \mathscr{A}_j$. We now define the *radial derivative* $D_r f$ of f, which, in each annulus $\mathscr{A}_j$, will be the radial derivative with respect to the center z_j :

Definition 6.3 *For each $j, 1 \leqq j \leqq M$,*

$$(D_r f)(\zeta) = \lim_{s \in R\setminus 0, s \to 0} \frac{1}{s}\left(f\left(\zeta + s\frac{\zeta - z_j}{|\zeta - z_j|}\right) - f(\zeta)\right) \quad (\zeta \in \mathscr{A}_j).$$

Denote by $\mathscr{T} \subset C^M \times R^M$ the closed ball of radius γ centered at $(z_1, \ldots, z_M, r_1, \ldots, r_M)$.

We shall call a **special shear** *on C^2 a map from C^2 to C^2 which is either of the form $(z,w) \mapsto (z + Tw^n, w)$ or of the form $(z,w) \mapsto (z, w + Tz^n)$ with $n \in \mathscr{N}$ and $T \in R$. Note that the special shears are particularly nice examples of automorphisms of C^2.*

To prove Lemma 3.1 we shall use an induction process. It will begin by a choice of a large positive integer m and by defining, for $t = (\xi_1, \ldots, \xi_M; \rho_1, \ldots, \rho_M) \in \mathscr{T}$ the functions

$$\varphi_1(t,\zeta) = \left(\frac{\zeta - \xi_1}{\rho_1}\right)^m + \sum_{j=2}^{M}\left(\frac{\rho_j}{\zeta - \xi_j}\right)^m, \quad F_1(t,\zeta) = (\varphi_1(t,\zeta), \zeta).$$

The maps at each step of the induction process will be of the form $\zeta \mapsto A(F_1(t,\zeta))$ where A is a composition of a finite number of special shears. We have

Proposition 6.2 *Let $H\colon\ C^2 \to C^2$ be a composition of a finite number of special shears. The map $(t,\zeta) \mapsto H(F_1(t,\zeta))$ is continuous on $\mathscr{T} \times \Omega$ and for each $t \in \mathscr{T}$ the map $\zeta \mapsto H(F_1(t,\zeta))$ is one to one on Ω. Moreover, there are constants M_1, M_2, $0 < M_1 < M_2 < \infty$, depending only on m and H such that*

$$|H(F_1(t,\zeta))| \leqq M_2, \quad M_1 \leqq \left|\frac{\partial}{\partial \zeta} H(F_1(t,\zeta))\right| \leqq M_2 \quad (\zeta \in \Omega,\ t \in \mathscr{T}).$$

7 A perturbation lemma

Given a domain $\Omega \subset C$ and $M < \infty$ we denote by $\mathscr{H}(\Omega, M)$ the set of all holomorphic functions h on Ω which satisfy $|h(\zeta)| < M$ $(\zeta \in \Omega)$ and $|h'(\zeta)| < M$ $(\zeta \in \Omega)$.

To construct lots of domains holomorphically embeddable into C^2 in order to prove Lemma 3.1 we shall, given $t = (\xi_1, \dots, \xi_M, \rho_1, \dots \rho_M) \in \mathscr{T}$, start with the circular domain $\Omega_0(t) = (\xi_1 + t_1\Delta) \setminus \bigcup_{j=2}^{M}(\xi_j + t_j\bar{\Delta})$ and then, in an inductive process, construct a sequence $\Omega_n(t)$ of standard domains each of which is a small perturbation of $\Omega_0(t)$ such that this sequence has a limit, a standard domain $\Omega(t)$, which is again a small perturbation of $\Omega_0(t)$ and has the additional property that it is holomorphically embeddable into C^2. On each step the boundary $b\Omega_n(t)$ will be of the form $\{\zeta \in \mathscr{A}: |a_n(\zeta)| = 1\}$ where a_n is a holomorphic function on $\mathscr{A}$ with nonzero derivative on the set where its modulus equals 1, and the boundary $b\Omega_{n+1}(t)$ will have the form $\{\zeta \in \mathscr{A}: |a_n(\zeta)^N + b_n(\zeta)| = 1\}$ where b_n is holomorphic and small on $\mathscr{A}$ and N is a large positive integer. To get the precise control over the changes of these boundaries in the induction process we prove a perturbation lemma. Note that in order to prove Lemma 3.1 we need uniform control over the changes for all $t \in \mathscr{T}$ and this is why the lemma has to be more complicated than what one would expect.

Lemma 7.1 *Let $\gamma > 0$, $R > 8\gamma$, $0 < r' < \gamma$, $\eta > 0$, $M < \infty$ and $0 < \mu < \gamma$. Write $\Omega = \{\zeta \in C: R - 6\gamma < |\zeta| < R + 6\gamma\}$. There are r, $0 < r < r'$, $\rho > 0$, and sequences $r_n, \nu_n, 0 < r_n < \gamma, \nu_n > 0$ such that the following holds:*

Let $\Phi \in \mathscr{H}(\Omega, M)$ and assume that there is a (smooth) function d: $b\Delta \to (-4\gamma, 4\gamma)$ such that $|\Phi| = 1$ on $\Gamma = \{(R + d(\zeta))\zeta$: $\zeta \in b\Delta\}$ and such that $|D_r|\Phi|^2| > \eta$ on $B(r') = \{(R + d(\zeta) + s)\zeta$: $\zeta \in b\Delta$, $-r' < s < r'\}$. Then for every $\Psi \in \mathscr{H}(\Omega, \rho)$ and for every $N \in \mathscr{N}$, $N \geqq 2$, there is a (smooth) function $f_N(\Psi)$: $b\Delta \to \mathscr{R}$ such that

(i) *if $B(r) = \{(R + d(\zeta) + s)\zeta$: $\zeta \in b\Delta, -r < s < r\}$ then $\{(R + f_N(\Psi)(\zeta))\zeta$: $\zeta \in b\Delta\} = \{\zeta \in B(r)$: $|\Phi(\zeta)^N + \Psi(\zeta)| = 1\}$*

(ii) *$|D_r|\Phi^N + \Psi|^2| > \nu_N$ on $\{(R + f_N(\Psi)(\zeta) + s)\zeta$: $s \in b\Delta$, $-r_N < s < r_N\}$.*

If A is a bounded domain in R^2 we denote by $\mathscr{C}^1(A)$ the space of all real valued functions on A which are, together with their first order partial derivatives, uniformly continuous on A and write

$$\|f\|_{\mathscr{C}^1(A)} = \sup_{\zeta \in A} |f(\zeta)| + \sup_{\zeta \in A} \left| \frac{\partial f}{\partial x}(\zeta) \right| + \sup_{\zeta \in A} \left| \frac{\partial f}{\partial y}(\zeta) \right| .$$

To prove Lemma 7.1 we first prove

Sublemma 7.1 *Let $\gamma > 0$, $R > 8\gamma$, $M < \infty$, $0 < r < \gamma$, $\alpha > 0$ and $0 < \mu < \gamma$. There are constants $\rho > 0$ and $L < \infty$ such that the following holds: Let d: $b\Delta \to (-4\gamma, 4\gamma)$ be a $\mathscr{C}^1$ function. Write $A = \{(R + d(\zeta) + s)\zeta$: $-r < s < r$, $\zeta \in b\Delta\}$ and assume that $f \in \mathscr{C}^1(A)$, $\|f\|_{\mathscr{C}^1(A)} < M$, $f((R + d(\zeta))\zeta) = 0$*

($\zeta \in b\Delta$) and $|(D_r f)(\zeta)| > \alpha$ ($\zeta \in A$), and let $\omega \in \mathscr{C}^1(A)$ satisfy $\|\omega\|_{\mathscr{C}^1(A)} < \rho$. Then there is an $F(\omega) \in \mathscr{C}^1(b\Delta)$ such that

(i) $\{\zeta \in A\colon\ f(\zeta) + \omega(\zeta) = 0\} = \{\zeta(R + F(\omega)(\zeta))\colon\ \zeta \in b\Delta\}$

(ii) $|F(\omega)(\zeta) - d(\zeta)| < \mu \quad (\zeta \in b\Delta)$

(iii) $\left|\dfrac{d}{d\theta}F(\omega)(e^{i\theta})\right| < L \quad (\theta \in R.)$

Proof. Choose $\rho > 0$ so small that $\rho < \alpha r$, $\rho < \mu\alpha$, $\rho < \alpha/2$ and assume that d, f and ω satisfy the assumptions above. With no loss of generality assume that $(D_r f)(\zeta) > \alpha$ ($\zeta \in A$.) Since $f(R + d(1)) = 0$ the mean value theorem for $t \mapsto f(R + d(1) + t)$ on $(-r, r)$ implies that $f(R + d(1) + r) > \alpha r$, $f(R + d(1) - r) < -\alpha r$. Since $|\omega(R + d(1) + t)| < \rho < \alpha r$ $(-r < t < r)$ it follows that $(f + \omega)(R + d(1) - r) < 0$, $f(R + \omega(R + d(1) + r) > 0$ so there is at least one t, $-r < t < r$, such that $(f + \omega)(R + d(1) + t) = 0$. Since $\rho < \alpha/2$ we have $\dfrac{d}{dt}(f(t) + \omega(t)) > \alpha - \alpha/2$ on $(R + d(1) - r, R + d(1) + r)$ which implies that $t \mapsto f(t) + \omega(t)$ strictly increases on $(R + d(1) - r, R + d(1) + r)$ so there is precisely one point $t \in (-r, r)$ such that $(f + \omega)(R + d(1) + t) = 0$. The preceding discussion also implies that $(f + \omega)(R + d(1) + t) > \alpha t - \rho$ $(0 < t < r)$ and $(f + \omega)(R + d(1) + t) < -\alpha t + \rho$ $(-r < t < 0.)$ Thus, $-r < t < r$ and $(f + \omega)(R + d(1) + t) = 0$ implies that $-\rho/\alpha < t < \rho/\alpha$ and since $\rho/\alpha < \mu$ it follows that $|t| < \mu$.

Repeating the preceding discussion for $t \mapsto (f + \omega)(\zeta(R + d(\zeta) + t))$ for $\zeta \in b\Delta$ and $|t| < r$ we prove, for each ω as above, the existence of a function $F(\omega)\colon b\Delta \to R$ satisfying (i) and (ii).

To prove the smoothness of $F(\omega)$ and (iii) assume that f and ω satisfy the assumptions above and consider $(\theta, t) \mapsto Q(\theta, t) = f(te^{i\theta}) + \omega(e^{i\theta})$, a $\mathscr{C}^1$ function on $K = \bigcup_{\theta \in \mathscr{R}} \{\theta\} \times (d(e^{i\theta}) - r, d(e^{i\theta}) + r)$ which satisfies

$$\frac{\partial}{\partial t}Q(\theta, t) > \alpha - \rho > \alpha/2 > 0\,. \tag{7.1}$$

Thus, if $(\theta_0, t_0) \in K$ and $Q(\theta_0, t_0) = 0$ then by (7.1) the implicit mapping theorem implies that $\theta \mapsto F(\omega)(e^{i\theta})$ is smooth near θ_0. Further, by the properties of f and ω there is a constant $c < \infty$ depending only on R and γ such that

$$\left|\frac{\partial Q}{\partial \theta}(\theta, t)\right| \leqq \left|\frac{\partial}{\partial \theta}f(t(e^{i\theta})\right| + \left|\frac{\partial}{\partial \theta}\omega(te^{i\theta})\right| \leqq c(M + \rho)\,.$$

Now, $Q(\theta, F(\omega)(e^{i\theta}) \equiv 0$ and (7.1) give

$$\left|\frac{d}{d\theta}F(\omega)(e^{i\theta})\right| = \frac{\left|\dfrac{\partial Q}{\partial \theta}(\theta, F(\omega)(e^{i\theta}))\right|}{\left|\dfrac{\partial Q}{\partial t}(\theta, F(\omega)(e^{i\theta}))\right|} \leqq \frac{2c(M + \rho)}{\alpha}\,.$$

This completes the proof.

Proof of Lemma 7.1.Part 1. Note that if $g: \Omega \to C$ is a holomorphic function then in the usual identification $R^2 \equiv C$ we have

$$\operatorname{grad}|g(\zeta)|^2 = 2\frac{\partial}{\partial\bar{\zeta}}[g(\zeta)\overline{g(\zeta)}] = 2g(\zeta)\overline{g'(\zeta)}\,.$$

There is an $M' < \infty$ such that $|D_r|\Phi|^2| < M'$ on Ω for all $\Phi \in \mathscr{H}(\Omega, M)$. It follows that there are r, $0 < r < r'$, and $\beta > 0$ such that if $\Phi \in \mathscr{H}(\Omega, M)$ satisfies $|\Phi| = 1$ on Γ then $|\Phi| > \beta$ on the r-belt $B(r)$ along Γ. This implies that there is a constant $M'' < \infty$ such that

$$|D_r\varphi| < M'' \quad \text{on } B(r) \tag{7.2}$$

whenever $\zeta \mapsto \Phi(\zeta) = S(\zeta)e^{i\varphi(\zeta)}$, $S(\zeta)$ real, and Γ satisfy the assumptions in the lemma.

We now show that there is a $\rho_0 > 0$ such that for every Φ as in the lemma we have

$$\left\{\begin{array}{l} |\operatorname{grad}|\Phi^N + \Psi|^2(\zeta)| > \eta/3 \text{ for every } N, N \geqq 2, \text{ for every } \Psi \in \mathscr{H}(\Omega, \rho_0) \\ \text{and for every } \zeta \in B(r) \text{ which satisfies } |\Phi(\zeta)^N + \Psi(\zeta)| = 1\,. \end{array}\right\} \tag{7.3}$$

To see this, choose $\rho_0 > 0$ so small that $\rho_0 < 1/3$, $\rho_0 < \eta/3$. Assume that $\zeta \in B(r)$. Our assumption implies that $|\operatorname{grad}|\Phi|^2(\zeta)| > \eta$, that is, $2|\Phi(\zeta)||\Phi'(\zeta)| > \eta$. If, in addition, $|\Phi(\zeta)^N + \Psi(\zeta)| = 1$ then $|\Psi(\zeta)| < \rho_0$ implies that $|\Phi(\zeta)|^N > 2/3$. Thus, $|\operatorname{grad}|\Phi^N + \Psi|^2(\zeta)| = |\Phi(\zeta)^N + \Psi(\zeta)| \cdot |N\Phi(\zeta)^{N-1}\Phi'(\zeta) + \Psi'(\zeta)| \geqq N(2/3)^{(N-2)/N}\eta/2 - \rho_0 > 2(2/3)(\eta/2) - (\eta/3)$. This proves (7.3).

Part 2. Let Φ be a holomorphic function on Ω which satisfies the assumptions of the lemma. Write $\Phi(\zeta) = S(\zeta)e^{i\varphi(\zeta)}$ with $S(\zeta)$ real and $\Psi(\zeta) = u(\zeta) + iw(\zeta)$ with $u(\zeta), w(\zeta)$ real. Now, $|\Phi(\zeta)^N + \Psi(\zeta)| = 1$ is equivalent to $[S(\zeta)^N \cos N\varphi(\zeta) + u(\zeta)]^2 + [S(\zeta)^N \sin N\varphi(\zeta) + w(\zeta)]^2 = 1$, that is, to $S(\zeta)^{2N} + 2S(\zeta)^N[u(\zeta)\cos N\varphi(\zeta) + w(\zeta)\sin N\varphi(\zeta)] + u(\zeta)^2 + w(\zeta)^2 - 1 = 0$ which, if we assume that $u(\zeta), w(\zeta)$ are small enough (which we may with no loss of generality), is equivalent to

$$S(\zeta)^2 - 1 + \omega(\zeta) = 0$$

where

$$\omega(\zeta) = -P(\zeta)^{2/N} + 1 \text{ and } P(\zeta) = -Q(\zeta) + [1 - u(\zeta)^2 - w(\zeta)^2 + Q(\zeta)^2]^{1/2}$$

where $Q(\zeta) = u(\zeta)\cos N\varphi(\zeta) + w(\zeta)\sin N\varphi(\zeta)$. Observe first that if $\sup_\Omega |\Psi|$ is small enough then $\sup_\Omega |Q|$ is arbitrarily small, uniformly with respect to $N \in \mathscr{N}$, so $\sup_\Omega |P - 1|$ is arbitrarily small, uniformly with respect to $N \in \mathscr{N}$, which implies that $\sup_\Omega |\omega|$ is arbitrarily small, uniformly with respect to $N \in \mathscr{N}$ provided that $\sup_\omega |\Psi|$ is small enough. Further, $D_r\omega = -2(N^{-1}D_rP)/P^{1-2/N}$ and

$$\frac{1}{N}D_rP = \frac{1}{N}\Big[D_rQ + \frac{1}{2(1 - u^2 - w^2 + Q^2)^{1/2}}(-2uD_ru - 2wD_rw + 2QD_rQ)\Big].$$

Since $N^{-1}D_rQ=N^{-1}(D_ru\cos N\varphi - Nu\sin N\varphi\cdot D_r\varphi + D_rw\sin N\varphi + Nw\cos N\varphi \cdot D_r\varphi)$ it follows by (7.2) that $\sup_{B(r)}|N^{-1}D_rQ|$ is arbitrarily small, uniformly with respect to $N\in\mathcal{N}$, provided that $\sup_\Omega|\Psi|$, $\sup_\Omega|\Psi'|$ are small enough.

Apply Sublemma 7.1 with $f(\zeta)=S(\zeta)^2-1$ to get $\rho>0$, $\rho<\rho_0$, such that for each $N\in\mathcal{N}$, $N\geqq 2$, we get a constant $L_N>0$, and for each Φ,Ψ satisfying the assumptions of the lemma we get a function $f_N(\Psi)$ satisfying (i) and

$$\left|\frac{d}{d\theta}f_N(\Psi)(e^{i\theta})\right| < L_N \quad (\theta\in R)\,.$$

This means that for each $N\in\mathcal{N}, N\geqq 2$, there is a constant $Q_n>0$ such that whenever Φ,Ψ satisfy the assumptions of the lemma the angle $\psi(w)$ between the tangent line at w to $\{(R+f_N(\Psi(\zeta))\zeta\colon \zeta\in b\Delta\}$ containing w and the ray through w emanating from the origin satisfies $|\sin\psi(w)|\geqq 3q_N$. By (7.3) it follows that $|D_r|\Phi^N+\Psi|^2(w)| = |\mathrm{grad}|\Phi^N+\Psi|^2(w)|\cdot|\sin\psi(W)|\geqq \eta q_N$ so

$$|D_r|\Phi^N+\Psi|^2(w)|\geqq \eta q_N \quad (w\in\{(R+f_N(\Psi)(\zeta))\zeta\colon\ \zeta\in b\Delta\})\,. \tag{7.4}$$

Fix $N\in\mathcal{N},N\geqq 2$. By Cauchy's inequalities there are a slightly smaller domain $\Omega'\subset\subset\Omega$ and a constant $P_N<\infty$ such that $|(\Phi^N+\Psi)''|<P_N$ on Ω' whenever $\Phi\in\mathcal{H}(\Omega,M)$, $\Psi\in\mathcal{H}(\Omega,\rho)$ and consequently there is a constant $P_N'<\infty$ such that for all such Φ and Ψ we have

$$|D_r(D_r|\Phi^N+\Psi|^2)|<P_N' \quad \text{on } \Omega'. \tag{7.5}$$

Put $v_N=\eta q_N/2$. For every $w\in\Omega'$ such that the segment $[w-tw/|w|, w+tw/|w|]$ belongs to Ω', (7.5) implies that $|D_r|\Phi^N+\Psi|^2(w\pm tw/|w|) - D_r|\Phi^N+\Psi|^2(w)| = |\int_0^{\pm t}D_r(D_r|\Phi^N+\Psi|^2)(w+\lambda w/|w|)d\lambda|\leqq |t|\cdot P_N'$ so putting $r_N=\min\{(\eta q_N)/(2P_N'),\gamma/2\}$ and using (7.4) we get $|D_r|\Phi^N+\Psi|^2(w+tw/|w|)|\geqq \eta q_N-|t|P_N'\geqq \eta q_N - r_NP_N' = v_N$ $(|t|<r_N, w\in\{(R+f_N(\Psi)(\zeta))\zeta\colon\ \zeta\in b\Delta\})$. This completes the proof.

8 A lemma of Narasimhan

In our induction process we will need the fact that sufficiently small holomorphic perturbations of holomorphic functions that are one to one and regular on an open set are still one to one and regular on a compact subset. Such a lemma was proved and used by Narasimhan in [Na, p. 426]. Here we need a somewhat more general lemma of this sort as we want to perturb an entire family of functions depending on a parameter $t\in\mathcal{T}$:

Lemma 8.1 *Let $\mathcal{T}\subset\mathbb{R}^k$ be a compact set, let $\Omega\subset C$ be a bounded domain and let $(t,\zeta)\mapsto f(t,\zeta)$ be a continuous function on $\mathcal{T}\times\Omega$ with values in C^2 such that for each $t\in\mathcal{T}$, $\zeta\mapsto f(t,\zeta)$ is holomorphic on Ω. Assume that there* *are constants $m > 0$ and $M < \infty$ such that*

$$m \leqq \left|\frac{\partial f}{\partial \zeta}(t,\zeta)\right| \leqq M \quad (t \in \mathscr{T},\ \zeta \in \Omega),$$

and that for each $t \in \mathscr{T}$ the function $\zeta \mapsto f(t,\zeta)$ is one to one on Ω. For each $R > 0$ there is an $\varepsilon(r) > 0$ with the following property:

If $K \subset \Omega$ is a compact set such that $K + r\Delta \subset \Omega$ and if g: $K + r\Delta \to C^2$ is a holomorphic function such that for some $t \in \mathscr{T}$ we have

$$|g(\zeta) - f(t,\zeta)| < \varepsilon(r) \quad (\zeta \in K + r\Delta)$$

then g is one-to-one and regular on K.

Sublemma 8.1 *Let $R > 0$ and let $h: R\Delta \to C^2$ be a holomorphic function. Assume that $|h'(0)| \geqq a > 0$ and $|h'(\zeta)| \leqq M$ $(\zeta \in R\Delta)$. Then h is regular and one-to-one on $r\Delta$ where $r = (Ra)/(2^{1/2}M)$.*

Proof. The assumptions imply that for one component, say for h_1, we have $|h_1'(0)| \geqq 2^{-1/2}a$ and $|h_1'(\zeta)| \leqq M$ $(\zeta \in R\Delta)$. By a classical result of Landau [Ha, p.11] this implies that h_1 is one-to-one and regular on $r\Delta$ which completes the proof.

Proof of Lemma 8.1. Since Ω is bounded the set $P = \{\zeta \in \Omega : \mathrm{dist}(\zeta, b\Omega \geqq r\}$ is a compact set which contains every compact set K such that $K + r\Delta \subset \Omega$. Cauchy's inequalities show that there is an $\varepsilon(r) > 0$ with the following property:

Whenever K is a compact set, $K + r\Delta \subset \Omega$, and whenever g is holomorphic on $K + r\Delta$ such that for some $t \in \mathscr{T}$

$$|g(\zeta) - f(t,\zeta)| < \varepsilon(r) \quad (\zeta \in K + r\Delta) \tag{8.1}$$

then $\frac{m}{2} < |g'(\zeta)| < 2M$ $\left(\zeta \in K + \frac{r}{2}\Delta\right)$ and consequently, by Sublemma 8.1,

$$\text{for each } \zeta \in K,\ g \text{ is one-to-one and regular on } \zeta + (2^{1/2}rM)(16M)^{-1}\Delta\,. \tag{8.2}$$

Denote by D the diagonal in $\Omega \times \Omega$. The preceding discussion implies that there is an open neighbourhood $U(r)$ of $(P \times P) \cap D$ such that

$$\left\{\begin{array}{l}\text{whenever } K \text{ is a compact set, } K + r\Delta \subset \Omega \text{ and } g \text{ is a holomorphic} \\ \text{function on } K + r\Delta \text{ satisfying (8.1) for some } t \in \mathscr{T}, \text{ and} \\ (x,y) \in K \times K, (x,y) \in U(r)\backslash D, \text{ then } g(x) \neq g(y)\,.\end{array}\right\} \tag{8.3}$$

Let

$$\mu = \inf_{(x,y)\in(P\times P)\backslash U(r),\ t\in\mathscr{T}} |f(t,x) - f(t,y)|\,.$$

We claim that $\mu > 0$. If not then there are $t_n \in \mathscr{T}$, $(x_n, y_n) \in (P \times P)\backslash U(r)$ such that $|f(t_n, x_n) - f(t_n, y_n)| \to 0$. By compactness of $\mathscr{T}$ and $(P \times P)\backslash U(r)$ we may assume that $t_n \to t_0 \in \mathscr{T}$ and $(x_n, y_n) \to (x_0, y_0) \in (P \times P)\backslash U(r)$. In particular, $x_0 \neq y_0$. By the continuity of f we get $f(t_0, x_0) = f(t_0, y_0)$ contradicting the fact that $\zeta \mapsto f(t_0, \zeta)$ is one to one on Ω.

Now shrink $\varepsilon(r)$ if necessary to get $0 < \varepsilon(r) < \mu/4$. Thus, if K is a compact set satisfying $K + r\Delta \subset \Omega$ and if g is holomorphic on $K + r\Delta$ and satisfies (8.1) for some $t \in \mathscr{T}$ then if $(x, y) \in (K \times K) \setminus U(r)$ we have $|g(x) - g(y)| \geqq |f(t,x) - f(t, y)| - |g(x) - f(t,x)| - |g(y) - f(t, y)| \geqq \mu - \mu/4 - \mu/4 = \mu/2 > 0$. Together with (8.3) this proves that g is one-to-one on K. This completes the proof.

9 Proof of Lemma 3.1 continued

Part 1. Let $z_1, \ldots, z_M, r_1, \ldots, r_M, \gamma, \Omega, \mathscr{T}$ and $\mathscr{A}_1, \ldots, \mathscr{A}_M$, have the same meaning as in Sect. 6. Note that $\gamma < 1$. For each $t = (\xi_1, \ldots, \xi_M, \rho_1, \ldots, \rho_M) \in \mathscr{T}$ let $\Omega_0(t) = (\xi_1 + \rho_1\Delta)\backslash\bigcup_{j=2}^{M}(\xi_j + \rho_j\bar{\Delta})$. Clearly $\Omega_0(t)$ is a standard domain; let $p_0(t)$: $b\Delta \to \mathscr{R}^M$ be the parametrization of $b\Omega_0$. By the properties of $\mathscr{T}$ we have

$$|p_0(t)(\zeta)| < \gamma \quad (\zeta \in b\Delta,\ t \in \mathscr{T})\,. \tag{9.1}$$

Let $0 < \mu < \gamma$. Lemma 7.1 implies that there are $m \in \mathscr{N}$, e_1', $0 < e_1' < \mu/2$, and $\delta_1 > 0$ such that if $t = (\xi_1, \ldots, \xi_M, \rho_1, \ldots, \rho_M) \in \mathscr{T}$ and if

$$\varphi_1(t, \zeta) = \left(\frac{\zeta - \xi_1}{\rho_1}\right)^m + \sum_{j=2}^{M}\left(\frac{\rho_j}{\zeta - \xi_j}\right)^m$$

then there is a standard domain $\Omega_1(t)$ such that if $p_1(t)\colon b\Delta \to \mathscr{R}^M$ is the parametrization of $b\Omega_1(t)$ then

$$|p_1(t)(\zeta) - p_0(t)(\zeta)| < \mu \quad (\zeta \in b\Delta) \tag{9.2}$$

$$|\varphi_1(t, \zeta)| = 1 \quad (\zeta \in b\Omega_1(t))\,, \tag{9.3}$$

and

$$|D_r|\varphi_1(t, \zeta)|^2| > \delta_1 \quad \text{on } e_1'\text{-belt along } b\Omega_1(t)\,. \tag{9.4}$$

The point here is that if m is large enough then $|\varphi_1(t, \zeta)| = 1$ determines the boundary of a standard domain which is a slight perturbation of the boundary of $\Omega_0(t)$. Note that by Proposition 6.1, (9.3) implies that $\Omega_1(t)$ is uniquely determined by t. Put $\psi_1(t, \zeta) = \zeta$ $(t \in \mathscr{T})$ and $S_1 = T_1 = N_1 = 1$.

We shall show that one can choose sequences $\{S_n\}$, $\{T_n\}$ of positive numbers converging to ∞, a sequence $\{N_n\}$ of positive integers, and a sequence $\{e_n\}$ of positive real numbers, $e_n < \mu/2^n$ $(n \in \mathscr{N})$ such that if we put

$$\psi_{n+1}(t, \zeta) = \varphi_n(t, \zeta)^{N_{n+1}} + \frac{T_n}{T_{n+1}}\psi_n(t, \zeta) \text{ and } \varphi_{n+1}(t, \zeta) = \varphi_n(t, \zeta) \text{ if } n \text{ is odd} \tag{9.5}$$

and

$$\varphi_{n+1}(t,\zeta) = \psi_n(t,\zeta)^{N_{n+1}} + \frac{S_n}{S_{n+1}}\varphi_n(t,\zeta) \text{ and } \psi_{n+1}(t,\zeta) = \psi_n(t,\zeta) \text{ if } n \text{ is even} \tag{9.6}$$

then for each $n \in \mathcal{N}$ and for each $t \in \mathcal{T}$ there is a smoothly bounded standard domain $\Omega_n(t)$ such that

$$b\Omega_{n+1}(t) \text{ is contained in the } e_n\text{-belt along } b\Omega_n(t) \tag{9.7}$$

and such that

$$|\psi_n(t,\zeta)| = 1 \quad (\zeta \in b\Omega_n(t)) \text{ if } n \text{ is even} \tag{9.8}$$

$$|\varphi_n(t,\zeta)| = 1 \quad (\zeta \in b\Omega_n(t) \text{ if } n \text{ is odd}\,. \tag{9.9}$$

Note that if T_n/T_{n+1} and S_n/S_{n+1} above are very small then by Lemma 7.1 it is clear that $b\Omega_{n+1}(t)$ is a slight perturbation of $b\Omega_n(t)$. Note that by Proposition 6.1, (9.8) and (9.9) imply that for each $n \in \mathcal{N}$, $\Omega_n(t)$ is uniquely determined by $t \in \mathcal{T}$. For each $t \in \mathcal{T}$ and $n \in \mathcal{N}$ let $p_n(t)$ be the parametrization of $b\Omega_n(t)$. Then (9.7) and the fact that $e_n < \mu/2^n$ $(n \in \mathcal{N})$ imply that for each $t \in \mathcal{T}$, $p_n(t)$ converge on $b\Delta$ uniformly to a function $p(t)$: $b\Delta \to \mathcal{R}^M$ and by (9.2) we have

$$|p(t)(\zeta) - p_0(t)(\zeta)| < 2\mu \quad (\zeta \in b\Delta)\,. \tag{9.10}$$

For each $t \in \mathcal{T}$ denote by $\Omega(t)$ the (not necessarily smoothly bounded) standard domain whose boundary is parametrized by $p(t)$. Then $\Omega_n(t) \to \Omega(t)$ $(t \in \mathcal{T})$. We shall show that if one chooses m, and the sequences $\{T_n\}, \{S_n\}, \{N_n\}, \{e_n\}$ in the right way then for each $t \in \mathcal{T}$ the maps

$$\zeta \mapsto F_n(t,\zeta) = (S_n\varphi_n(t,\zeta), T_n\psi_n(t,\zeta))$$

converge, uniformly on compact subsets of $\Omega(t)$, to a holomorphic embedding of $\Omega(t)$ into C^2.

Note that after fixing m and the sequences $\{S_n\}, \{T_n\}, \{N_n\}$ the domain $\Omega(t)$ is uniquely determined by $t \in \mathcal{T}$. We shall also show that if $m, \{S_n\}, \{T_n\}, \{N_n\}$ are chosen in the right way then $t \mapsto \Omega(t)$ is a continuous map from $\mathcal{T}$ to $\mathcal{J}_M$.

Assume that we have already proved all above. To conclude the proof of Lemma 3.1 choose a sequence $\{\mu_j\}$ of positive numbers converging to 0. For $j \in \mathcal{N}$ put $\mu = \mu_j$ and choose m and the sequences $\{S_n\}, \{T_n\}, \{N_n\}, \{e_n\}$ as above and for each $t \in \mathcal{T}$ denote the domain $\Omega(t)$ obtained by the process above (or more precisely, the M-tuple $(J_1(t), \ldots, J_M(t))$ where for each j, $J_j(t)$ is the component of $b\Omega(t)$ contained in $\mathcal{A}_j$) by $\Theta_j(t)$. Now (9.10) implies that the sequence Θ_j has all the properties required in Lemma 3.1.

Part 2. Let $0 < \mu < \gamma$ and choose $m, \varphi_1, \psi_1, S_1, T_1, N_1$ and e_1' as above. Put $e_0 = \gamma$. In an induction process we shall now construct sequences $\{S_n\}, \{T_n\}$, $\{N_n\}$ and a decreasing sequence $\{e_n\}$, of positive numbers, together with decreasing sequences $\{e_n'\}, \{\varepsilon_n\}$, of positive numbers, sequences $\{\delta_n\}, \{\kappa_n\}$ of positive numbers, and sequences $\Omega_n(t)$, $t \in \mathcal{T}$, of standard domains such that if the sequences $\varphi_n(t,\zeta)$, $\psi_n(t,\zeta)$ are defined by (9.5) and (9.6) and if for $n \in \mathcal{N}$, $t \in \mathcal{T}$,

$K_n(t) = \Omega_n(t)\backslash(3e_n)$-belt along $b\Omega_n(t)$,

$\mathcal{S}_n(t) = \Omega_n(t)\backslash(4e_n)$-belt along $b\Omega_n(t)$,

$\tilde{\Omega}_n(t) = \Omega_n(t) \cup (3e_n)$-belt along $b\Omega_n(t)$

then the following holds for each $t \in \mathcal{T}$ and each $n \in \mathcal{N}$:

(A1') $S_{n+1} = S_n \geqq n$ if n is odd

(A1") $T_{n+1} = T_n \geqq n$ if n is even

(A2) $4e_n < e_n' \leqq \mu/2^n, e_n < e_{n-1}/4$

(A3) $b\Omega_{n+1}(t)$ is contained in e_n-belt along $b\Omega_n(t)$

(A4') $|\varphi_n(t,\zeta)| = 1(\zeta \in b\Omega_n(t))$ if n is odd

(A4") $|\psi_n(t,\zeta)| = 1(\zeta \in b\Omega_n(t))$ if n is even

(A5') $|D_r|\varphi_n(t,\zeta)|^2| > \delta_n$ on the e_n'-belt along $b\Omega_n(t)$ if n is odd

(A5") $|D_r|\psi_n(t,\zeta)|^2| > \delta_n$ on the e_n'-belt along $b\Omega_n(t)$ if n is even

(A6) $\mathcal{S}_n(t) + \kappa_n\Delta \subset K_n(t)$

(A7') $|S_n\varphi_n(t,\zeta)| > n-1(\zeta \in \tilde{\Omega}_n(t)\backslash K_n(t))$ if n is odd

(A7") $|T_n\psi_n(t,\zeta)| > n-1(\zeta \in \tilde{\Omega}_n(t)\backslash K_n(t))$ if n is even

(A8') $|S_{n+1}\psi_n(t,\zeta)^{N_{n+1}}| < \varepsilon_n/2^n(\zeta \in K_n(t))$ if n is odd

(A8") $|T_{n+1}\varphi_n(t,\zeta)^{N_{n+1}}| < \varepsilon_n/2^n(\zeta \in K_n(t))$ if n is even

(A9) if a holomorphic function h: Int $K_n(t) \to C^2$ satisfies $|h(\zeta)| < \varepsilon_n$ ($\zeta \in$ Int $K_n(t)$) and if $F_n(t,\zeta) = (S_n\varphi_n(t,\zeta), T_n\psi_n(t,\zeta))$ then $\zeta \mapsto F_n(t,\zeta) + h(\zeta)$ is one-to-one and regular on $\mathcal{S}_n(t)$.

Note that by (A1) and by (9.5) and (9.6) we have

$(S_{n+1}\varphi_{n+1}, T_{n+1}\psi_{n+1}) = (S_n\varphi_n, T_n\psi_n + T_{n+1}\varphi_n^{N_{n+1}})$ if n is odd, and

$(S_{n+1}\varphi_{n+1}, T_{n+1}\psi_{n+1}) = (S_n\varphi_n + S_{n+1}\psi_n^{N_{n+1}}, T_n\psi_n)$ if n is even.

Note also that if $p_n(t)$ is the parametrization of $b\Omega_n(t)$, $n \in \mathcal{N}$, $t \in \mathcal{T}$, then (A3) and (A2) imply that

$$|p_{n+1}(t)(\zeta) - p_n(t)(\zeta)| < \mu/2^n \quad (\zeta \in b\Delta)\,. \tag{9.11}$$

Part 3. Assume for a moment that we have proved the existence of all the functions, sets and numbers in Part 2 with the properties (A1)–(A9).

Let $t \in \mathcal{T}$. Clearly (9.11) implies that $p_n(t)$ converge uniformly on $b\Delta$ to a (continuous) function $p(t)$: $b\Delta \to \mathcal{R}^M$. By (9.2) and (9.11) we have $|p(t)(\zeta) - p_0(t)(\zeta)| < 2\gamma$ ($\zeta \in b\Delta$) so $p(t)$ is the parametrization of the boundary of a standard domain that we denote by $\Omega(t)$. Note also that by (A3) and (A2), the sequences $K_n(t)$ and $\mathcal{S}_n(T)$ are increasing and we have $\bigcup_{n=1}^\infty K_n(t) = \bigcup_{n=1}^\infty \mathcal{S}_n(t) = \Omega(t)$. If $n \in \mathcal{N}$ and $\zeta \in K_n(t)$ then for $j \geqq n$

$|F_{j+1}(t,\zeta) - F_j(t,\zeta)| = T_{j+1}\varphi_j(t,\zeta)^{N_{j+1}}$ if j is odd and

$|F_{j+1}(t,\zeta) - F_j(t,\zeta)| = S_{j+1}\psi_j(t,\zeta)^{N_{j+1}}$ if j is even

so, since ε_n is decreasing (A8) implies that

$$|F_{j+1}(t,\zeta) - F_j(t,\zeta)| < \frac{\varepsilon_j}{2^j} \leqq \frac{\varepsilon_n}{2^j} \quad (\zeta \in K_n(t),\ j \geqq n)\,.$$

It follows that the sequence of functions $\zeta \mapsto F_n(t,\zeta)$ converges, uniformly on compact subsets of $\Omega(t)$, to a function $\zeta \mapsto F(t,\zeta)$ holomorphic on $\Omega(t)$ and that

$$|F(t,\zeta) - F_n(t,\zeta)| < \varepsilon_n \quad (\zeta \in K_n(t))\,. \tag{9.12}$$

By (A9) it follows that for each $n \in \mathcal{N}$, $\zeta \mapsto F(t,\zeta)$ is one-to-one and regular on $\mathcal{S}_n(t)$ and consequently it is one-to-one and regular on $\Omega(t)$. Further, since $S_{n+1}\varphi_{n+1} = S_n\varphi_n$ for odd n and $T_{n+1}\psi_{n+1} = T_n\psi_n$ for even n, (A7) implies that $|F_{n+1}(\zeta,t)| > n-1$ $(\zeta \in \tilde{\Omega}_n(t)\backslash K_n(t))$ which, together with (9.12) with n replaced by $n+1$, implies that for each $n \in \mathcal{N}$,

$$|F(t)(\zeta)| > n - 1 - \varepsilon_{n+1} \quad (\zeta \in K_{n+1}(t)\backslash K_n(t)).$$

Since the sequence ε_n is decreasing it follows that $F(t)$ is a proper map from $\Omega(t)$ to C^2.

10 The conclusion of the proof of Lemma 3.1

Part 1. Given $\mu > 0$ we now prove the existence of the functions, sets and numbers in Sect. 9, Part 2, which satisfy (A1)–(A9).

We have already shown that there are $\varphi_1, \psi_1, S_1, T_1, N_1, e_1', \delta_1$ and $\Omega_1(t)$ $(t \in \mathcal{T})$ such that e_1' satisfies the portion $e_1' \leqq \mu/2$ of (A2) for $n=1$, such that (A4) and (A5) hold for $n=1$ and such that $S_1 \geqq 1$. Put $e_0 = \gamma$, $\varepsilon_0 = 1$.

Suppose that $k \in \mathcal{N}$ is *odd* and that we have already shown that for $j \leqq k$ there are $\varphi_j, \psi_j, S_j, T_j$, N_j, e_j', δ_j and $\Omega_j(t)$ $(t \in \mathcal{T})$ and that for $j \leqq k-1$ there are $e_j, \kappa_j, \varepsilon_j$ such that e_j, e_j', ε_j decrease with j and such that the portion $\varepsilon_k' \leqq \mu/2^k$ of (A2) for $n=k$ holds, such that (A4) and (A5) hold for $n=k$, and such that $S_k \geqq k$. Put $S_{k+1} = S_k$ so that (A1) is satisfied for $n=k$.

Using Sublemma 7.1 in the trivial case when $\omega = 0$ together with Proposition 6.2, we see that (A4) and (A5) for $n=k$ imply that there is a constant $l_k < \infty$ such that

$$\left|\frac{d}{d\theta} p_k(t)(e^{i\theta})\right| < l_k \quad (\theta \in \mathcal{R}, t \in \mathcal{T})\,. \tag{10.1}$$

Further, Proposition 6.2 implies that there is a constant $l_k' < \infty$ such that $|\mathrm{grad}|\varphi_k(t,\zeta)|^2| < l_k'$ $(\zeta \in \Omega,\ t \in \mathcal{T})$ which, by (A4) for $n=k$, and by the fact that $S_k \geqq k$, implies that there is a number e_k, $0 < e_k < e_{k-1}/4$, $4e_k < e_k'$ (so that (A2) is satisfied for $n=k$), such that for each $t \in \mathcal{T}$

$$|S_k\varphi_k(t,\zeta)| > n-1 \quad (\zeta \in ((3e_k)\text{-belt along } b\Omega_k(t))\,. \tag{10.2}$$

Define $K_k(t), \mathcal{S}_k(t), \tilde{\Omega}_k(t)$ as in Sect. 9, Part 2. Now, (10.2) implies that (A7) is satisfied for $n=k$. Further, (10.1) together with a simple geometric argument implies that there is a $\kappa_k > 0$ such that (A6) is satisfied for $n=k$. The point here is that κ_k-neighbourhood of $\mathcal{S}_k(t)$ is contained in $K_k(t)$ with κ_k independent of t so that we can apply Lemma 8.1. Now, Lemma 8.1 together with Proposition 6.2 implies that there is an ε_k, $0 < \varepsilon_k < \varepsilon_{k-1}$, such that (A9) holds for $n = k$.

We now apply Lemma 7.1 together with Proposition 6.2 to the function $\Phi(\zeta) = \varphi_k(t,\zeta)$, $r' = e_k$ and $\eta = \delta_k$ in each of the annuli $\mathscr{A}_j$, $1 \leqq j \leqq M$, to get r_k, $0 < r_k < e_k$, $\rho_k > 0$, and sequences $\beta_N > 0, \gamma_N > 0$ such that the following holds:

Choose $T_{k+1} \geqq k+1$ so large that for each $t \in T$ the function $\zeta \mapsto (T_k/T_{k+1})\psi_k(t,\zeta)$ belongs to $\mathscr{H}(\Omega, \rho_k)$ (this is possible by Proposition 6.2). Then for every $N \geqq 2$ and for every $t \in \mathscr{T}$ the set

$$\left\{\zeta \in r_k\text{-belt along } b\Omega_n(t)\colon \left|\varphi_k(t,\zeta)^N + \frac{T_k}{T_{k+1}}\psi_k(t,\zeta)\right| = 1\right\}$$

is the boundary of a standard domain $bD(N,t)$ and we have

$$\left|D_r\left|\varphi_k(t,\zeta)^N + \frac{T_k}{T_{k+1}}\psi_k(t,\zeta)\right|^2\right| > \gamma_N \text{ on } \beta_N\text{-belt along } bD(N,t)\,.$$

Note that we have not chosen $N = N_{k+1}$ yet. Since $4e_k < e_k'$, (A5) for $n = k$ implies that $|D_r|\varphi_k(t,\zeta)|^2| > \delta_k$ on $(4e_k)$-belt along $b\Omega_k(t)$ which, together with the maximum principle implies that there is a constant $\nu_k < 1$ such that $|\varphi_k(t,\zeta)| \leqq \nu_k$ $(\zeta \in bK_k(t),\ t \in \mathscr{T})$ and that the maximum principle further gives $|\varphi_k(t,\zeta)| \leqq \nu_k$ $(\zeta \in K_k(t), t \in \mathscr{T})$. Choose $N = N_{k+1} \in \mathscr{N}$ so large that $T_{k+1}\nu^{N_{k+1}} < \varepsilon_k/2^k$. Then (A8) is satisfied for $n = k$. In the preceding discussion put $N = N_{k+1}$, $\Omega_{k+1}(t) = D(N_{k+1}, t)$, $e'_{k+1} = \min\{\beta_{N_{k+1}}, \mu/2^{k+1}\}$, $\delta_{k+1} = \gamma_{N_{k+1}}$. Then (A3) is satisfied for $n = k$ and (A4), (A5) are satisfied for $n = k+1$.

If $k \in \mathscr{N}$ is even then we repeat the procedure with roles of φ, S and ψ, T interchanged. This completes the proof of the induction step.

Part 2. Given $\mu > 0$ and having chosen the sequences S_n, T_n and N_n as above we know by Proposition 6.1 that given $n \in \mathscr{N}$ and $t \in \mathscr{T}$ the domain $\Omega_n(t)$ is the unique standard domain such that $|\varphi_n(t,\zeta)| = 1$ $(\zeta \in b\Omega_n(t))$ if n is odd and $|\psi_n(\zeta)| = 1$ $(\zeta \in b\Omega_n(t))$ if n is even. If for each $n \in \mathscr{N}$ the function $p_n(t)$ parametrizes $b\Omega_n(t)$ then $\Omega(t)$ was defined as the standard domain whose boundary is parametrized by $p(t)$, the limit in $\mathscr{C}(b\Delta)^M$ of the sequence $p_n(t)$. Thus $t \in \mathscr{T}$ determines $\Omega(t)$ uniquely. It remains to prove that $t \to \Omega(t)$ is a continuous map from $\mathscr{T}$ to $\mathscr{J}_M$. It is enough to prove that $t \mapsto p(t)$ is a continuous map from $\mathscr{T}$ to $\mathscr{C}(b\Delta)^M$. By (9.11) $p_n(t)$ converges to $p(t)$ in $\mathscr{C}(b\Delta)^M$, uniformly with respect to $t \in \mathscr{T}$. Thus it is enough to prove that for each $n \in \mathscr{N}$, $t \mapsto p_n(t)$ is a continuous map from $\mathscr{T}$ to $(b\Delta)^M$.

Assume that $n \in \mathscr{N}$ is odd. We know that $\Omega_n(t)$ is the unique standard domain such that $|\varphi_n(t,\zeta)| = 1$ $(\zeta \in b\Omega_n(t))$. For $t, t' \in \mathscr{T}$ consider the function $\zeta \mapsto h(t,t')(\zeta) = |\varphi_n(t',\zeta)|^2 - |\varphi_n(t,\zeta)|^2$. It is easy to see that given $\lambda > 0$ there is a $\delta > 0$ such that $t, t' \in \mathscr{T}$, $|t - t'| < \delta$, implies that $\|h(t,t')\|_{\mathscr{C}^1(\Omega)} < \lambda$. Since $|D_r|\varphi_n(t,\zeta)|^2| > \delta_n$ on e_n-belt along $b\Omega_n(t)$, Sublemma 7.1 applies to show that given $\varepsilon > 0$ there is a $\delta > 0$ such that if $t, t' \in \mathscr{T}$, $|t' - t| < \delta$, then $b\Omega_n(t')$ is contained in ε-belt along $b\Omega_n(t)$. This proves that $t \mapsto p_n(t)$ is continuous. If n is even then the same reasoning applies with φ_n replaced by ψ_n. This completes the proof of Lemma 3.1. Theorem 1.1 is proved.

Acknowledgement. The first author was supported in part by a grant from the Ministry of Science and Technology of the Republic of Slovenia.

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