Normal Modes of Oscillation of Beams

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INTRODUCTION

The usual treatment of the vibration problems in beam theory involves the consideration of the fourth-order partial differential equation in the transverse displacement. This equation is obtained from the system of partial differential equations that expresses the basic laws of mechanics and the basic assumptions about the type of permissible deformations. These equations may or may not include such things as the effect of shear, rotary inertia of the beam, non-uniformity of the beam, and so on. If all of the above effects are included, the fourth-order partial differential equation in the displacement becomes very involved, although these same effects can be included in the basic governing system without too much complication. It therefore seems natural to consider the possibility of a direct investigation of the basic system instead of the more complicated single partial differential equation. Moreover, if the usual assumption concerning the existence of standing waves is made, i.e., vibrations harmonic in time, this alternative permits the above system to be reduced to a system of ordinary differential equations of the first-order whose simulation is direct by electronic means. Consequently, this memorandum originated in an attempt to treat certain aspects of the beam problem, to wit, the orthogonality of the eigenfunctions directly from the system obtained in the above manner.

In the course of this investigation several interesting observations were made which in modified form may be of interest in other problems of engineering mechanics. These include the lack of equivalence of the two methods outlined above as far as orthogonality properties are concerned. More specifically, while orthogonality relations are readily obtained from the system for certain boundary conditions, it is easy to see that they would not be obtainable from the displacement equation, even though the boundary conditions were appropriately expressed in terms of the displacement and its derivatives. Conversely, for the beam with both ends built in one does obtain a generalized type of orthogonality from the displacement equation which cannot be foreseen from the system. Some of the reasons underlying this lack of equivalence are discussed.

In addition to this, a critique of the standing wave assumption used above will be given, and this will suggest its limitations. In addition, a method is suggested which should, in any given uniform beam problem, permit the determination of the range of validity of the standing wave assumption. It follows as a corollary that in certain instances the distinct possibility exists that the standing wave hypothesis may well eliminate from the theory precisely those effects that are experimentally observed.

1. STATEMENT OF THE PROBLEM

The basic equations of the problem are derived in reference 1, and in the notation of reference 3 are as follows:

(1.1) $\rho \dfrac{\partial^2 y}{\partial t^2} + \dfrac{\partial V}{\partial x} = 0,$

(1.2) $-\dfrac{\partial M}{\partial x} + I' \dfrac{\partial^2 \beta}{\partial t^2} + V = 0,$

(1.3) $\dfrac{\partial \beta}{\partial x} = \dfrac{M}{EI},$

(1.4) $V = -KAG\alpha,$

(1.5) $\dfrac{\partial y}{\partial x} = \alpha + \beta.$

For the uniform beam the elimination of V, M, α and β from these equations results in the following partial differential equation in the displacement:

(1.6) $EI \dfrac{\partial^4 y}{\partial x^4} - \left(I' + \dfrac{EI\rho}{KAG}\right) \dfrac{\partial^4 y}{\partial x^2 \partial t^2} + \dfrac{I'\rho}{KAG} \dfrac{\partial^4 y}{\partial t^4} + \rho \dfrac{\partial^2 y}{\partial t^2} = 0$

In addition to this, the following expressions are obtained for the moment M and the shear V.

(1.7) $M = EI \dfrac{\partial^2 y}{\partial x^2} - \dfrac{EI\rho}{KAG} \dfrac{\partial^2 y}{\partial t^2},$

(1.8) $V + \dfrac{I'}{KAG} \dfrac{\partial^2 V}{\partial t^2} = EI \dfrac{\partial^3 y}{\partial x^3} - \left(\dfrac{EI\rho}{KAG} + I'\right) \dfrac{\partial^3 y}{\partial x \partial t^2}$

The introduction of the usual standing wave hypothesis to the effect that all vibrations are harmonic in the time requires that

(1.9) $y(x,t) = y(x)\, e^{j\omega t},$

$$(1.10) \quad V(x,t) = V(x)\, e^{j\omega t},$$

$$(1.11) \quad M(x,t) = M(x)\, e^{j\omega t},$$

$$(1.12) \quad \beta(x,t) = \beta(x)\, e^{j\omega t}.$$

The elimination of α from the system (1.1), (1.2), . . . , (1.5) and the introduction of the above standing wave hypothesis results in the following spatial system.

$$(1.13) \quad \frac{dV}{dx} = \rho\omega^2 y,$$

$$(1.14) \quad \frac{dM}{dx} - V = -I'\omega^2\beta,$$

$$(1.15) \quad \frac{d\beta}{dx} - \frac{M}{EI} = 0,$$

$$(1.16) \quad \frac{dy}{dx} - \beta + \frac{V}{KAG} = 0.$$

Similarly the introduction of this hypothesis into (1.6), (1.7) and (1.8) results in the following three expressions respectively.

$$(1.17) \quad EI\frac{d^4y}{dx^4} + \left(\frac{EI}{KAG} + \frac{I'}{\rho}\right)\rho\omega^2\frac{d^2y}{dx^2} - \left(1 - \frac{I'\omega^2}{KAG}\right)\rho\omega^2 y = 0,$$

$$(1.18) \quad M = EI\left(\frac{d^2y}{dx^2} + \frac{\rho\omega^2}{KAG}\, y\right)$$

$$(1.19) \quad V = 1 - \frac{I'\omega^2}{KAG}\left[EI\frac{d^3y}{dx^3} + \left(\frac{EI}{KAG} + \frac{I'}{\rho}\right)\rho\omega^2\frac{dy}{dx}\right]$$

As an historical remark, it should be said that equation (1.6) appears to have been derived by Timoshenko in reference 4 and to have been treated by means of the standing wave hypothesis by Goens in reference 5.

2. THE LAGRANGE IDENTITY FOR THE SPATIAL SYSTEM

The starting point for the investigation of the orthogonality relations is the same here as in the more usual Sturm-Liouville theory and consists in the derivation of an appropriate Lagrange identity or Green's theorem. Because of the relative simplicity of the spatial system (1.13), (1.14), (1.15) and (1.16), this will be done directly instead of by means of the general theory developed by Bôcher, Bliss and others. A general reference to this theory is to be found in Kamke (Ref. 6).

Let y_1, V_1, M_1 and β_1 represent a solution of the system of equations (1.13), (1.14), (1.15) and (1.16) which correspond to ω_1 and to a certain set of appropriate boundary conditions, and let y_2, V_2, M_2 and β_2 represent another solution of the system corresponding to ω_2 and to the same set of boundary conditions. Multiplication of the left-hand sides of equations (1.13), (1.14), (1.15) and (1.16) by y_2, $-\beta_2$, M_2 and $-V_2$ respectively and integration of the resulting expression from the end x = 0 to the end x = L of the beam yields the following identities.

$$\int_0^L y_2 \frac{dV_1}{dx}\, dx = -\int_0^L V_1 \frac{dy_2}{dx}\, dx + \Big[y_2 V_1\Big]_{x=0}^{x=L} ,$$

$$\int_0^L -\beta_2 \left(\frac{dM_1}{dx} - V_1\right) dx = \int_0^L M_1 \frac{d\beta_2}{dx} + \beta_2 V_1 \, dx - \Big[\beta_2 M_1\Big]_{x=0}^{x=L} ,$$

$$\int_0^L M_2 \left(\frac{d\beta_1}{dx} - \frac{M_1}{EI}\right) dx = -\int_0^L \left(\beta_1 \frac{dM_2}{dx} + \frac{M_2 M_1}{EI}\right) dx + \Big[M_2 \beta_1\Big]_{x=0}^{x=L} ,$$

$$\int_0^L -V_2 \left(\frac{dy_1}{dx} - \beta_1 - \frac{V_1}{KAG}\right) dx = \int_0^L \left(y_1 \frac{dV_2}{dx} + V_2 \beta_1 + \frac{V_1 V_2}{KAG}\right) dx - \Big[V_2 y_1\Big]_{x=0}^{x=L}$$

The addition of this last set of identities yields, after some re-arrangement of terms, the following Lagrange identity.

$$\int_0^L \left[y_2 \frac{dV_1}{dx} - \beta_2 \left(\frac{dM_1}{dx} - V_1 \right) + M_2 \left(\frac{d\beta_1}{dx} - \frac{M_1}{EI} \right) - V_2 \left(\frac{dy_1}{dx} - \beta_1 + \frac{V_1}{KAG} \right) \right] dx -$$

$$(2.1) \quad - \left\{ \int_0^L \left[y_1 \frac{dV_2}{dx} - \beta_1 \left(\frac{dM_2}{dx} - V_2 \right) + M_1 \left(\frac{d\beta_2}{dx} - \frac{M_2}{EI} \right) - V_1 \left(\frac{dy_2}{dx} - \beta_2 + \frac{V_2}{KAG} \right) \right] dx \right\} =$$

$$= y_2(L)\, V_1(L) - \beta_2(L)\, M_1(L) + M_2(L)\, \beta_1(L) - V_2(L)\, y_1(L) -$$

$$- y_2(0)\, V_1(0) + \beta_2(0)\, M_1(0) - M_2(0)\, \beta_1(0) + V_2(0)\, y_1(0)$$

This identity is obviously of the form

$$\sum_{i=1}^{n} \int_0^1 \left[v_i L_i(u) - u_i L_i(v) \right] dx \quad = \text{values on the boundary}$$

in which the same differential operators L_i operate on the sets y_1, V_1, M_1, β_1 and y_2, V_2, M_2, β_2 so that the set of operators defined by the left-hand side of equations (2.1) is obviously self-adjoint. If the same set of boundary conditions is satisfied by the two solutions corresponding to the two different eigenvalues ω_1 and ω_2, and if these boundary conditions cause the right-hand side of the above Lagrange identity to vanish, the problem is self-adjoint.

3. ORTHOGONALITY RELATIONS FOR THE SPATIAL SYSTEM

Let it be assumed that the boundary conditions of the problem make the right-hand side of the Lagrange identity vanish. Then if the right-hand members of equations (1.13), (1.14), (1.15) and (1.16) are substituted into this identity with the appropriate subscripts, the Lagrange identity reduces to the relation

$$(3.1) \quad \left(\omega_1^2 - \omega_2^2 \right) \int_0^L \left(\rho y_1 y_2 + I' \beta_1 \beta_2 \right) dx = 0$$

Thus it is apparent that unless the rotary inertia term I' is neglected there can be no orthogonality in the usual sense. It should be stressed that the above derivation is valid whether the beam is uniform or not.

It remains to verify which of the usually assumed boundary conditions for beams will cause the right-hand side of the Lagrange identity to vanish.

a. The Free-free Beam

Here the boundary conditions are that the moment and shear should vanish at both ends of the beam: $M = V = 0$ at $x = 0$ and $x = L$. Thus condition (3.1) holds, and if $I' = 0$, one has the usual orthogonality. This provides a more transparent proof of this same orthogonality relation which was derived by G. Hess in reference 2.

b. The Cantilever Beam (The End $x = 0$ Built in and the End $x = L$ Free)

There exists some doubt as to the correct boundary conditions at $x = 0$. Some authors take $y = y' = 0$ at $x = 0$, while others take $y = \beta = 0$ at $x = 0$. At $x = L$ the conditions are $M = V = 0$. The first set of boundary conditions fails to make (3.1) valid while the second set does render (3.1) true. This can probably be taken as more indirect evidence of validity of the second set of boundary conditions.

c. The Hinged-hinged Beam

Here the boundary conditions are $y = M = 0$ at $x = 0$ and $x = L$; so that equation (3.1) again follows and orthogonality again results if $I' = 0$.

d. The Beam with Both Ends Built In.

If the boundary conditions are taken as $y = y' = 0$ at both ends of the beam there is no orthogonality, since (3.1) does not hold. On the other hand, if the boundary conditions are taken as $y = \beta = 0$ at both ends, then again if I' is zero orthogonality does result.

e. The Built In Hinged Beam

If the boundary conditions are taken as $y = \beta = 0$ at $x = 0$ and $y = M = 0$ at $x = L$, then (3.1) holds; so that if $I' = 0$ orthogonality results. If the condition $\beta = 0$ at $x = 0$ is replaced by the condition that $y' = 0$ at $x = 0$, there is no orthogonality.

In resumé, then, one sees that if $I' = 0$, weighed orthogonality in the sense that

$$(3.2) \qquad \int_0^1 \rho y_1 y_2 \, dx = 0$$

results for non-uniform beams, if the condition $\beta = 0$ is used as one of the boundary conditions at any built in end. In short, if this is done, the functions

$$\left\{\sqrt{\rho}\, y_i\right\}, \qquad i = 1, 2, \ldots, n$$

are orthogonal in the usual sense, and all of the above problems are self-adjoint. For more general boundary conditions the possibility of biorthogonality may exist, in that one may be able to pick two solutions of the system (1.13), (1.14), (1.15) and (1.16) which satisfy adjoint boundary conditions; so that (3.1) will still hold.

4. THE UNIFORM BEAM WITH NEGLIGIBLE ROTARY MOMENT OF INERTIA: ORTHOGONALITY RELATION FOR THE SPATIAL DISPLACEMENT EQUATION: DISCUSSION OF THE LACK OF EQUIVALENCE OF THE SYSTEM AND THE DISPLACEMENT EQUATION

If I' is taken to be zero and if the beam is assumed to be uniform, equation (1.6) reduces to

$$(4.1) \qquad EI \frac{\partial^4 y}{\partial x^4} - \frac{EI\rho}{KAG} \frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho \frac{\partial^2 y}{\partial t^2} = 0$$

while the expressions for the moment and shear become respectively,

$$(4.2) \qquad M = EI \left(\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{KAG} \frac{\partial^2 y}{\partial t^2} \right)$$

(4.3) $$V = EI\left(\frac{\partial^3 y}{\partial x^3} - \frac{\rho}{KAG}\frac{\partial^3 y}{\partial x\,\partial t^2}\right)$$

(The assumption of uniformity is not necessary for this discussion, but without it the equations would be more cumbersome without essentially modifying the results.)

For simplicity let $a = \frac{\rho}{KAG}$ and $b = \frac{\rho}{EI}$; so that they become respectively

(4.4) $$\frac{\partial^4 y}{\partial x^4} - a\frac{\partial^4 y}{\partial x^2 \partial t^2} + b\frac{\partial^2 y}{\partial t^2} = 0$$

(4.5) $$M = EI\left(\frac{\partial^2 y}{\partial x^2} - a\frac{\partial^2 y}{\partial t^2}\right)$$

(4.6) $$V = EI\left(\frac{\partial^3 y}{\partial x^3} - a\frac{\partial^3 y}{\partial x\,\partial t^2}\right)$$

The usual separation of variables process assumes that

(4.7) $y(x,t) = X(x)\,T(t)$;

so that (4.4) implies that

$$X^{iv}T - aX''T'' + bXT'' = 0$$

or that

$$X^{iv}T + T''(bX - aX'') = 0;$$

from this it follows that

$$\frac{X^{iv}}{aX'' - bX} = \frac{T''}{T} = -\omega^2.$$

Thus one obtains the two equations

(4.8) $X^{iv} + \omega^2(aX'' - bX) = 0$ and

(4.9) $T'' + \omega^2 T = 0.$

The solution of the equation for T(t) is, of course

$$T(t) = A \sin \omega t + B \cos \omega t,$$

in accordance with the standing wave hypothesis.

In exactly the same way the substitution of (4.7) into the expressions for the moment and shear yield the following:

$$M = EI\left(X''T - aXT''\right) = EI\,T\left(X'' - aX\frac{T''}{T}\right) = EI\,T\left(X'' + a\omega^2 X\right)$$

$$V = EI\left(X'''T - aX'T''\right) = EI\,T\left(X''' - aX'\frac{T''}{T}\right) = EI\,T\left(X''' + a\omega^2 X'\right);$$

so that, for boundary conditions requiring the moment and shear to be zero, we have the conditions (i) M = 0 implies that

(4.10) $X'' + a\omega^2 X = 0,$

while (ii) V = 0 implies that

(4.11) $X''' + a\omega^2 X' = 0.$

In order to investigate orthogonality relations for equation (4.8), it is once again necessary to obtain an appropriate Lagrange identity. For this purpose consider

(4.8) $X^{iv} + \omega^2 (aX'' - bX) = 0,$

and let Z (x) be any other function of x possessing the necessary integral and differential properties which are used in the sequel. From (4.8) it follows that

$$\int_0^L Z\left[X^{iv} + \omega^2(aX'' - bX)\right] dx = 0$$

However, by integration by parts,

$$\int_0^L Z X^{IV} dx = \left[ZX''' - Z'X''\right]_{x=0}^{x=L} + \int_{x=0}^{x=L} Z''X''dx ,$$

and

$$\int_0^L ZX'' dx = \left[Z X'\right]_0^L - \int_0^L Z'X' dx ;$$

so that

$$\int_0^L Z\left[X^{IV} + \omega^2\left(aX'' - bX\right)\right] dx = \int_0^L \left[Z''X'' + \omega^2 Z\left(aX'' - bX\right)\right] dx +$$

$$+ \left[ZX''' - Z'X''\right]_0^L =$$

$$= \int_0^L \left[Z''X'' - \omega^2\left(aZ'X' + bZX\right)\right] dx + \left[ZX''' - Z'X'' + a\omega^2 ZX'\right]_0^L ,$$

or

$$\omega^2 \int_0^L \left(aZ'X' + bZX\right) dx = \int_0^L Z''X''dx + \left[ZX''' - Z'X'' + a\omega^2 ZX'\right]_0^L \tag{4.12}$$

Now suppose that X (x) corresponds to a solution of (4.8) with $\omega = \omega_1$, and that Z (x) corresponds to a solution of (4.8) with $\omega = \omega_2$. It is then possible to write down the identity (4.12) for both ω_1 and ω_2; so that by subtraction one obtains the following Lagrange identity.

$$\left(\omega_1^2 - \omega_2^2\right) \int_0^L \left(aZ'X' + bZX\right) dx = \left[ZX''' - Z'''X - Z'X'' + Z''X' + a\omega_1^2 ZX' - a\omega_2^2 Z'X\right]_0^L \tag{4.13}$$

Orthogonality will follow then in the self-adjoint case if the right-hand side of the above expression vanishes. In particular, if the boundary conditions for the beam with both ends built in are assumed to be X = X' = 0 at x = 0 and x = L, one sees immediately the generalized orthogonality relation

$$(4.14) \qquad \left(\omega_1^2 - \omega_2^2\right) \int_0^L \left(aZ'X' + bZX\right) dx = 0$$

Since this differs quite radically from the relation obtained by the assumption that y = β = 0 at x = 0 and x = L, which was

$$(3.2) \qquad \left(\omega_1^2 - \omega_2^2\right) \int_0^L \rho y_1 y_2 \, dx = 0$$

it should be possible to decide on the basis of experimental data which of these two boundary conditions represents a closer approximation to the true situation.

A little investigation also makes it clear that orthogonality with self-adjoint boundary conditions will not necessarily exist for the other beam cases treated previously. For example, for the free-free beam one must use the boundary conditions (4.10) and (4.11). Their substitution into (4.13) is readily seen to yield the following unsymmetrical expression; so that there is no orthogonality for self-adjoint boundary conditions.

$$\left(\omega_1^2 - \omega_2^2\right) \int_0^L \left(aZ'X' + bZX\right) dx = Z\left(X''' + a\omega_1^2 X'\right) - X\left(Z''' + a\omega_2^2 Z'\right) -$$

$$- Z'\left(-a\omega_1^2 X\right) + X'\left(-a\omega_2^2 Z\right)\Big]_{x=0}^{x=L} =$$

$$= \left[a\omega_1^2 XZ' - a\omega_2^2 X'Z\right]_{x=0}^{x=L}$$

In fact, orthogonality relations of the form of (4.14) seem to be possible if, and only if, the second derivative is paired with the first derivative or the third with the second. Thus, if X = 0 and $X'' + a\omega_1^2 X$ at x = 0 and x = L, and if one supposes the same boundary conditions for Z with $\omega = \omega_2$ instead of ω_1, one finds a relation of the form

$$(\omega_1^2 - \omega_2^2)\left[\int_0^L (aX'Z' + bXZ)\,dx - aZ'(L)X'(L) + aZ'(0)X'(0)\right] = 0$$

while a similar relation exists for the other cases indicated above.

At first glance the above lack of equivalence between the system (1.13), (1.14), (1.15) and (1.16) with I' = 0 and the single equation (4.8), as far as orthogonality relations are concerned, seems paradoxical. This situation can be at least partially resolved by the following considerations. It is first of all obvious that the differential equation (4.8) is not equivalent to the system (1.13), (1.14), (1.15) and (1.16) with I' set equal to zero, since one cannot reverse the steps leading to the differential equation from the system without introducing constants of integration. Thus, the differential equation contains many more solutions than those compatible with the system. Certainly there is no a priori reason why these additional solutions should exhibit the same orthogonality properties as those possessed by the more limited class of solutions of the system. That is, without returning to the system, there is no immediately apparent way of choosing the constants of integration of the single eouation, so as to insure their compatibility with the additional constraints imposed by the system. For this purpose one must again return to the considerations of the system.

On the other hand, the fact that one can derive a generalized orthogonality relation from the single differential equations for the doubtful boundary conditions X = X' = 0 at both ends of the beam is not so surprising. Recall, for this purpose, that the relation so obtained involves the derivatives X' (x), and that its counterpart does not occur in a symmetric fashion in the Lagrange identity for the system. Thus, the possibility of this type of relationship is merely left open.

For a more complete discussion of self-adjoint equations of the fourth-order by similar means, a general reference may be made to Boerner (Ref. 7).

5. THE STANDING WAVE ASSUMPTION

As has already been seen, the partial differential equation (1.6), as well as the expressions for the moment and shear, admit a separation of variables for the uniform beam with I' neglected. Moreover, it has been shown that this separation process does in fact lead to a time function which is harmonic. However, in spite of this, the system (1.1), (1.2), . . . ,(1.5) with I' = 0 from which the partial differential equation (4.1) is derivable by elimination does not separate into space and time factors directly because of the single equation

$$\frac{\partial y}{\partial x} = \beta - \frac{V}{KAG} = \beta + \alpha.$$

Of course, the system does separate under the assumptions (1.9), (1.10), (1.11), (1.12). Thus, it is entirely possible even here that the system (1.1), (1.2), . . . ,(1.5) with I' = 0 admits a more general class of solutions than those which are of the form assumed by (1.9), (1.10), (1.11) and (1.12). Other plausibility considerations which will shortly be discussed seem to indicate, however, that the standing wave assumption is adequate for the initial value problem here.

The situation is far less certain, however, if the term I' is not neglected. Not only does the system still not separate in space and time, but the partial differential equation in the displacements (1.6) as well as the expression (1.8) in the shear fails to separate.

It is to be noted, however, that both the system and the partial differential equation do separate under the more restrictive standing wave hypothesis as given by (1.9). In fact, to obtain this type of separation it is sufficient to assume that $T'' = \mu T$, or even that $y''(x) = \lambda y$. Under this latter assumption one obtains the following differential equation in the independent variable t.

$$\frac{I'\rho}{KAG} T^{iv} - \lambda \left(I' + \frac{EI\rho}{KAG} \right) T'' + \rho T'' + EI\lambda^2 T = 0.$$

This equation certainly does not admit simple harmonic functions of the time.

Now the following plausibility arguments can be advanced for the adequacy of the standing wave assumption for the case of equations (1.1), (1.2), . . . ,(1.5) where I' is neglected and for its inadequacy when I' is different from zero.

The system (1.1), (1.2), . . . , (1.5) with I' = 0 and the partial differential equation (4.1) are of the second-order in time and of the fourth-order in the displacements. Thus, the initial-boundary value problem will be completely determined by the specification of four boundary conditions and two initial conditions, say $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$. The standing wave hypothesis, which for the differential equation (4.1) amounts to separation of variables, will result in an expansion for the displacement $y(x,t)$ of the form

$$(5.1) \quad y(x,t) = \Sigma A_n y_n(x) \cos \omega_n t + \Sigma B_n y_n(x) \sin \omega_n t,$$

where the $y_n(x)$ and the ω_n have been chosen to satisfy the four boundary conditions. The solution will then be completely determined if the initial conditions can be satisfied. This requires that

$$f(x) = \Sigma A_n y_n(x);$$

and that

$$g(x) = \Sigma B_n \omega_n y_n(x).$$

One must investigate the possibility of expansions of this type as well as the completeness of the set of eigenfunctions. However, for this case orthogonality relations do follow from (3.1) for all of the usual boundary conditions; so that it is possible to calculate the coefficients formally. Moreover, the standing wave hypothesis does furnish in this case the two denumerable sets of constants $\{A_n\}$ and $\{B_n\}$; so that a complete solution does seem possible.

In contrast to this case, if $I' \neq 0$ both the system (1.1), (1.2), . . . , (1.5) and the partial differential equation (1.6) are of the fourth-order in both the time and the displacements; so that a complete specification of the initial-value boundary value problem for either of them requires the use of four boundary conditions and four initial conditions. Now the standing wave hypotheses will still only produce a solution of the form of (5.1); so that there will still be only two sets of denumerable quantities $\{A_n\}$ and $\{B_n\}$ which are to be determined by four initial conditions. Consequently, unless certain compatability relations should happen to exist between the four initial values of, for example, y and its first three derivatives with respect to the time, a solution of the initial value problem would be impossible. In short, if the system (1.1), (1.2), . . . , (1.5) and the partial differential equation (1.6) are correct, then the standing wave hypothesis does not take full account of all possible free motions of the beam if I' is not neglected. In all probability, the part neglected by the standing wave assumption will not separate into space and time unless it just happens to correspond to the other possibility outlined above, in which it was assumed that $y'' = \lambda y$.

In any case, granted the validity of the original equations, certain possible effects or states of vibration appear to be excluded by the use of the standing wave assumption. Just what these states of motion are, as well as a detailed check on the validity of the above plausibility arguments, can probably be established by a rather tedious application of the Laplace transformation, either to the system (1.1), (1.2), . . . , (1.5) or to the differential equation (1.6) for the two cases of $I' = 0$ and $I' \neq 0$. Unfortunately there appears to be no direct way of using the Laplace transformation method to investigate the eigenvalue problem; the eigenvalue problem in the displacements in the transformed space that one always gets after taking the Laplace transform on t is not the eigenvalue problem of interest here. However, if the above initial value problem were to be solved by the transformation method, what eigenfunctions there are should be extractable from the expression for the initial value of y and its derivatives furnished by this method of solution. That is, since the system, even with $I' \neq 0$, does admit solution according to the standing wave assumption, the solution should be extractable from the general solution obtained by the transformation method. The remainder would then represent those states which are not describable in terms of the standing wave hypothesis. Even if this process were to be carried through, it is clear that any eigenfunctions so obtained would not be orthogonal if I' were not zero.

REFERENCES

NO.	TITLE
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