Math. Ann. 334, 1–36 (2006) **Mathematische Annalen**

DOI: 10.1007/s00208-005-0638-2

Beurling primes with large oscillation

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Received: 16 September 2003 / Revised version: 12 January 2004
Published online: 8 November 2005 – © Springer-Verlag 2005

Abstract. A Beurling generalized number system is constructed having integer counting function $N_B(x) = \kappa x + O\big(x^\theta\big)$ with $\kappa > 0$ and $1/2 < \theta < 1$, whose prime counting function satisfies the oscillation estimate $\pi_B(x) = \mathrm{li}(x) + \Omega\big(x \exp(-c\sqrt{\log x})\big)$, and whose zeta function has infinitely many zeros on the curve $\sigma = 1 - a/\log t$, $t \ge 2$, and no zero to the right of this curve, where a is chosen so that $a > (4/e)(1-\theta)$. The construction uses elements of classical analytic number theory and probability.

Mathematics Subject Classification (2000): 11M41, 11N80, 11M26, 11N05

1. Introduction

The first proofs of the Prime Number Theorem (PNT) relied on the analytic continuability of the Riemann zeta function and its Hadamard product representation. In this way C. J. de la Vallée Poussin [18] proved in 1899 that

$$\zeta(s) \neq 0 \qquad \text{for } \sigma > 1 - \frac{c}{\log t}, \ t \ge 2, \tag{1}$$

where c is a suitable positive constant, which in this paper may be different from one occurrence to another, and $s = \sigma + it$. This is the so-called 'classical' zero-free region of the zeta function, from which the PNT was deduced with a quantitative error term

$$\pi(x) = \mathrm{li}(x) + O\Big(x \exp\big(-c\sqrt{\log x}\,\big)\Big) \tag{2}$$

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⋆ The author was supported in part by NSF grants DMS-0070720 and DMS-0244660.

⋆⋆ The author was supported in part by NSF grant DMS-0244660.

where $\mathrm{li}(x) = \int_2^x du/\log u$. Conversely, P. Turán [17] showed in 1950 that (2) is the best estimate that can be deduced from the zero-free region (1) (apart from the values of the constants).

Early in the twentieth century, E. Landau made a number of discoveries that simplified and extended the scope of prime number theory. He showed [9], [10], [11] that the classical zero-free region (1) could be derived by means of local analytic lemmas (e.g., Jensen's inequality and the Borel–Carathéodory lemma), and that essentially the same arguments used in proving the PNT could be applied to establish the Prime Ideal Theorem. Let K be a fixed algebraic number field, and let $N(x)$ denote the number of integral ideals in K with norm not exceeding x . Weber [19] had shown that

$$N(x) = \kappa x + O\big(x^{\theta}\big)$$

with $\kappa > 0$ and $\theta < 1$. In 1903, Landau [9] used this and the multiplicative structure of ideals to prove the Prime Ideal Theorem, which asserts that the number of prime ideals in K with norm not exceeding x is asymptotic to $x/\log x$ as x tends to infinity.

Developing Landau's ideas further, A. Beurling [3] (see also Bateman and Diamond [2]) gave the following more abstract formulation of prime number theory, in which a sequence $\mathcal{P} = \{\lambda_j\}$ of real numbers $\lambda_1 \le \lambda_2 \le \dots$, with $\lambda_1 > 1$ and $\lambda_j \to \infty$, is taken to be a set of generalized primes and the finite products $\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_r^{k_r}$ are considered to be the generalized integers $\mathcal{N}$ arising from these primes. Let $N_B(x)$ denote the number of such products not exceeding x (counted with appropriate multiplicity in case some real numbers are represented as Beurling integers in more than one way), and let $\pi_B(x)$ denote the number of Beurling primes λ_j not exceeding x. Beurling proved that if $N_B(x) = \kappa x + O\big(x/(\log x)^{\tau}\big)$ with $\kappa > 0$ and $\tau > 3/2$, then $\pi_B(x) \sim x/\log x$, which is to say the PNT holds in this setting. In his proof, Beurling established properties of an associated generalized zeta function which for $\sigma > 1$ is given by the formulæ

$$\zeta_B(s) = \sum_{\boldsymbol{k}} \frac{1}{\big(\lambda_1^{k_1}\lambda_2^{k_2}\cdots\big)^s} = \int_{1^-}^{\infty} x^{-s}\, dN_B(x) = \prod_{j=1}^{\infty}\big(1-\lambda_j^{-s}\big)^{-1}$$

where $\boldsymbol{k} = (k_1, k_2, \dots)$ is a vector with non-negative integer components, all but a finite number of which are zero.

Although the work of Landau antedated that of Beurling, and hence was not cast in this language, Landau's reasoning in fact provides a proof that if

$$N_B(x) = \kappa x + O\big(x^{\theta}\big) \tag{3}$$

with $\kappa > 0$ and $0 \le \theta < 1$, then $\zeta_B(s)$ satisfies (1) and $\pi_B(x)$ satisfies (2), as in the classical situation. Landau exhibited even the dependence of the constants in (1) and (2) on θ. From his arguments we see that there exists a positive absolute constant c such that if (3) holds, then

$$\zeta_B(s) \neq 0 \qquad \text{for } \sigma > 1 - \frac{c(1-\theta)}{\log t}, \ \ t \geq t_0, \tag{4}$$

and

$$\pi_B(x) = \mathrm{li}(x) + O\Big(x \exp\big(-c\sqrt{(1-\theta)\log x}\,\big)\Big) \qquad \text{for } x \geq 2. \tag{5}$$

The goal of this paper is to establish the optimality of (4) and (5), apart from the numerical value of the constant c, for a Beurling generalized number system satisfying (3).

Our method is first to construct (in §§ 3–5) an example of a *continuous* generalized prime counting function $\Pi_C(x)$ whose associated zeta function has the desired properties. Then (in §§ 6–9) we use a probabilistic construction to show that there exists a discrete measure $\pi_B(x)$ that is sufficiently close to $\Pi_C(x)$ to ensure that the associated Beurling primes and integers have the desired properties. An earlier attempt at such a construction along somewhat similar lines was made by R. S. Hall [8], but it did not succeed because the associated zeta function was excessively large near certain points. Indeed, it was subsequently shown by W.-B. Zhang [20] that no construction of the type given by Hall could succeed in generating a large oscillation in the prime counting function.

We recall from classical prime number theory that $\pi(x)$ is the counting function of the primes, that

$$\Pi(x) = \sum_{p^k \leq x} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{\pi\left(x^{1/k}\right)}{k},$$

and that

$$\psi(x) = \sum_{p^k \leq x} \log p = \int_1^x \log u \, d\Pi(u).$$

For $\sigma > 1$ the Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \int_{1^-}^{\infty} x^{-s} \, d[x].$$

Moreover, by taking the logarithm of the Euler product for $\zeta(s)$ we see that

$$\log \zeta(s) = \sum_{p,k} \frac{1}{kp^{ks}} = \int_1^{\infty} x^{-s} \, d\Pi(x)$$

for $\sigma > 1$, and by differentiating we see further that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p,k} \frac{\log p}{p^{ks}} = \int_1^\infty x^{-s}\, d\psi(x)$$

for $\sigma > 1$. These relations all extend to the Beurling situation, with $[x]$ replaced by $N_B(x)$, and the rational primes replaced by the λ_j. The integer counting measure dN_B is expressed in terms of the weighted prime counting measure $d\Pi_B$ by the relation

$$dN_B = \exp d\Pi_B \tag{6}$$

where $\exp d\Pi_B$ is defined as

$$\delta + d\Pi_B + d\Pi_B \star d\Pi_B/2! + d\Pi_B \star d\Pi_B \star d\Pi_B/3! + \dots .$$

Here $\star$ denotes the multiplicative convolution of measures supported in $[1, \infty)$, and δ denotes a unit point mass at 1.

Theorem 1. *Let θ and a be fixed with $\frac{1}{2} < \theta < 1$ and $a > (4/e)(1-\theta)$. Then, in the above notation, there is a system of Beurling primes $\mathcal{P}$ such that*

- (i) *the resulting Beurling integers satisfy* (3) *with $\kappa > 0$;*
- (ii) *the associated zeta function $\zeta_B(s)$ is analytic for $\sigma \ge \theta$, apart from a simple pole at $s = 1$ with residue κ;*
- (iii) *the function $\zeta_B(s)$ has infinitely many zeros on the curve $\sigma = 1 - a/\log t$, $t \ge 2$, and no zero to the right of this curve;*
- (iv) $\psi_B(x) - x = \Omega_{\pm}\big(x \exp(-2\sqrt{a \log x}\,)\big)$;
- (v) $\psi_B(x) = x + O\big(x \exp(-2\sqrt{a \log x}\,)\big)$.

By adjusting finitely many primes we could arrange to have $\kappa = 1$ in (3) without affecting the other assertions.

As discussed at the end of Section 2, the factor $4/e$ in the constraint on a is the best that our construction admits.

Examples have been given previously by Malliavin [13], pp. 296–297, and Diamond [5], p. 24, of continuous non-negative ‘prime counting’ measures $d\Pi$ for which the associated ‘integer counting’ measure $dN = \exp d\Pi$ satisfies (3), but for which the corresponding zeta function has a zero at any desired location in the interval $(0, 1)$. Thus for such systems, the relation (3) does not imply RH. However, it seems that in Theorem 1 we have for the first time a proof that (3) does not imply RH for discrete Beurling primes. On the other hand, it may still be the case that (3) with $\theta < 1/2$ *does* imply RH for discrete Beurling generalized numbers.

Subsequent to the proof of the zero-free region (1), estimates of exponential sums which depend on the additive structure of the integers were used to establish wider zero-free regions, first by Littlewood, and later by Vinogradov and Korobov. Of course, the results obtained so far in this direction fall far short of what we believe to be true. Informed by the above theorem, we see that a zeta function need not have a wider zero-free region if the only assumptions made are the Beurling conditions that the integers have a multiplicative structure and satisfy (3).

The critical insight as to how the zeros of $\zeta_B(s)$ should be arranged for our construction was provided by an unpublished analysis of D. R. Heath–Brown concerning the behaviour of the Dirichlet L-function $L(s, (\frac{\cdot}{p}))$ under the assumption that the least quadratic nonresidue modulo p is as large as the bound $p^{1/(4\sqrt{e})+\varepsilon}$ of D. A. Burgess [4]. We include our reconstruction of this reasoning in an Appendix at the end of this paper. For an independent study of this topic, see Granville and Soundararajan [7].

It is a noteworthy feature of our construction that zeros of $\zeta_B(s)$ near the line $\sigma = 1$ do not occur in isolation but rather in clusters. We show below that this is not just an artifact of our construction, but must occur in any situation of this type.

Theorem 2. *Suppose that $d\Pi_B$ is a non-negative measure supported on $(1,\infty)$, that dN_B is determined by* (6), *that $N_B(x)$ satisfies* (3) *with $\kappa > 0$ and $0 \le \theta < 1$, and that the associated zeta function $\zeta_B(s)$ is given by $\int_{1^-}^{\infty} x^{-s}\, dN_B(x)$. Then $(s-1)\zeta_B(s)$ is analytic for $\sigma > \theta$. Let $n_B(r,t)$ denote the number of zeros of $\zeta_B(s)$ in the disk of radius r centered at $1+it$. Then*

$$n_B(r,t) \ll \frac{r\log t}{1-\theta} \qquad \text{for } \ 0 \le r \le \frac{1}{2}(1-\theta), \ \ t \ge t_0\,. \tag{7}$$

If $\beta_0 + i\gamma_0$ is a zero of $\zeta_B(s)$ with

$$\beta_0 \, > \, 1 - \frac{1-\theta}{\sqrt{\log \gamma_0}}\,, \quad \gamma_0 \ge t_0\,, \tag{8}$$

then

$$n_B(r,\gamma_0) + n_B(r, 2\gamma_0) \gg \frac{r(1-\theta)}{(1-\beta_0)^2 \log \gamma_0} \tag{9}$$

uniformly for

$$\frac{C(1-\beta_0)^2 \log \gamma_0}{1-\theta} \le r \le \frac{1}{2}(1-\theta) \tag{10}$$

where C is a large absolute constant.

The lower bound (9) is valid also for

$$1-\beta_0 \le r \le C(1-\beta_0)^2(\log\gamma_0)/(1-\theta)\,,$$

but in this range (9) is weaker than the trivial bound $n_B(r,\gamma_0)\ge 1$.

Note that if $1-\beta_0 \asymp (1-\theta)/\log\gamma_0$, which can happen, in view of Theorem 1, then (7) and (9) imply that

$$n_B(r,\gamma_0)+n_B(r,2\gamma_0) \asymp \frac{r\log\gamma_0}{1-\theta} \qquad \text{for } 1-\beta_0 \le r \le \tfrac{1}{2}(1-\theta)\,.$$

Thus in this extreme situation, we see not only that there is a clump of zeros, but furthermore that the density of the zeros in the cluster is approximately the same as in our construction.

An estimate similar to (9) was established by Montgomery [14], pp. 85–94, for the Riemann zeta function, but his analysis depended on its global analyticity. We obtain the above by using the Landau local lemmas.

The authors are happy to thank Paul T. Bateman, Enrico Bombieri, Roger Heath–Brown, Walter Philipp and K. Soundararajan for their helpful comments and remarks.

2. Sketch of the proof of Theorem 1

Let

$$G(z) = 1 - \frac{e^{-z}-e^{-2z}}{z}\,. \tag{11}$$

This is an entire function with $G(0)=0$. Note that

$$G(z) = \int_1^2 (1-e^{-uz})\,du\,.$$

We write $z=x+iy$. If $x>0$, then the integrand has positive real part, and hence $\Re\, G(z) > 0$. Similarly, $\Re\, G(iy) > 0$ if $y\ne 0$. Thus $G(z)\ne 0$ for $x\ge 0$ except for the simple zero $z_0=0$. The function has infinitely many zeros z_j with $\Re z_j < 0$, which will be shown in § 3 to satisfy

$$z_{\pm j} = -\tfrac{1}{2}\log(\pi j) \pm \big(j+\tfrac{1}{4}\big)\pi i + O\big(j^{-1/2}\big) \tag{12}$$

for $j=1,2,3,\ldots$ These zeros, suitably rescaled and translated, become the zeros of our zeta function.

By the hypothesis that $a>(4/e)(1-\theta)$, we have $2/(ea) < 1/(2(1-\theta))$. Let α be a number chosen so that $\max(1,2/(ea)) < \alpha < 1/(2(1-\theta))$. From this it follows that

$$a\alpha > \frac{2}{e}\,, \tag{13}$$

and that

$$\frac{1}{2} < 1 - \frac{1}{2\alpha} < \theta\,. \tag{14}$$

Let $1 < \gamma_1 < \gamma_2 < \ldots$ be a sequence of real numbers tending to infinity sufficiently rapidly. Later, we find that it suffices to take

$$\gamma_k = \gamma_1^{A^{k-1}} \tag{15}$$

if γ_1 and A are sufficiently large. For $k < 0$ let $\gamma_k = -\gamma_{-k}$. For $k \neq 0$ put $\beta_k = 1 - a/\log|\gamma_k|$, and for brevity let $\rho_k = \beta_k + i\gamma_k$ and $\ell_k = \alpha \log|\gamma_k|$. Then set

$$\zeta_C(s) = \frac{s}{s-1}\prod_{k\neq 0} G\big(\ell_k(s-\rho_k)\big) \qquad \text{for } \sigma > 1\,. \tag{16}$$

The first factor, $s/(s-1)$, will give the main contribution to the 'integer' and 'prime' counting functions, and the product of G's will provide the desired zeta zeros and fluctuation in the prime count. Since $G(0) = 0$, it follows that $\zeta_C(s)$ has zeros at the points ρ_k. Of course, $\zeta_C(s)$ also has other zeros, at points of the form $\rho_k + z_j/\ell_k$. Since

$$G(z) = 1 + O\Big(\frac{1+e^{-2x}}{1+|z|}\Big)\,, \tag{17}$$

the product (16) converges locally uniformly for $\sigma \geq 1 - 1/(2\alpha)$. Thus $\zeta_C(s)$ is continuous in this closed half-plane, and $(s-1)\zeta_C(s)$ is analytic for $\sigma > 1 - 1/(2\alpha)$.

Thus far we have a zeta function but no primes. To address this, we show that there is a positive measure $\Pi_C(x)$ such that

$$\log\zeta_C(s) = \int_1^\infty v^{-s}\, d\Pi_C(v) \qquad \text{for } \sigma > 1\,. \tag{18}$$

To establish the existence of such a measure and to understand its behavior, we first express $\log G(z)$ as a Mellin transform:

$$\log G(z) = -\int_1^\infty f(u)\, u^{-z-1}\, du \qquad \text{for } x > 0\,.$$

In § 4 we show that $f(u)$ is non-negative and calculate it explicitly for $1 \leq u \leq e^3$. For larger u the direct method becomes unwieldy, but for such u we express $f(u)$ in terms of zeros of $G(z)$:

$$f(u)\log u = \sum_j u^{z_j} \qquad \text{for } u > e^2\,. \tag{19}$$

Since $z_0 = 0$, from (12) and (19) it follows that $f(u)$ is quite close to $1/\log u$ when u is large.

In § 5 we show that (18) holds with

$$d\Pi_C(v) = \Big(\frac{1-1/v}{\log v} - 2\sum_{k=1}^{\infty} \frac{f(v^{1/\ell_k})}{\ell_k}\; v^{\beta_k - 1} \cos(\gamma_k \log v)\Big)\, dv \qquad (20)$$

for $v \geq 1$. From this representation we shall find that $d\Pi_C$ is a positive measure if (13) holds and if A and γ_1 are sufficiently large. Also, we see from (20) that the measure $d\Pi_C$ is absolutely continuous (hence the subscript C in Π_C and ζ_C).

To complete our construction, we define a sequence $1 = v_0 < v_1 < v_2 < \cdots$ tending to infinity very slowly. For $k \geq 1$ we take v_k to be a generalized prime with probability

$$p_k = \int_{v_{k-1}}^{v_k} 1\, d\Pi_C(v)\,.$$

This gives rise to a probability space of random Beurling primes. In § 6 we develop the necessary probabilistic tools, and in § 7 we show that there is a sequence of Beurling primes with counting function $\pi_B(x)$ for which

$$\int_1^{\infty} v^{-s}\, d\Big(\Pi_C(v) - \pi_B(v)\Big) \ll \sqrt{\log(|t|+2)} \qquad \text{for } \sigma \geq 1/2 + \varepsilon\,.$$

Thus $\zeta_B(s)$ has the same pole $(s = 1)$ and zeros as $\zeta_C(s)$ in $\sigma > 1 - 1/(2\alpha)$. Moreover,

$$\zeta_B(s) \ll |\zeta_C(s)|\, \exp\left(c\sqrt{\log(|t|+2)}\,\right) \qquad \text{for } \sigma > 1 - 1/(2\alpha)\,. \qquad (21)$$

In § 8 we use (15)–(17) to show that

$$\int_T^{2T} |\zeta_C(\sigma + it)|\, dt \ll T \qquad \text{for } \sigma \geq 1 - 1/(2\alpha)\,.$$

From this and (21) we thus have

$$\int_T^{2T} |\zeta_B(\sigma + it)|\, dt \ll T^{1+\varepsilon} \qquad \text{for } \sigma \geq 1 - 1/(2\alpha)\,,$$

and then it is a simple matter to use an inverse Mellin transform to show that $N_B(x)$ satisfies the estimate (*i*) of Theorem 1.

Finally in § 9 we analyze how the zeros of $\zeta_B(s)$ affect the error term $\psi_B(x) - x$.

The constant $4/e$ in the constraint on a is the best that our construction allows, based on $G(z)$. If we were to use instead a function more closely resembling the function $H(z)$ defined in (A.2), it should be possible to reduce the $4/e$

to something closer to $-x_1' = 0.5651\ldots$. However, the use of $G(z)$ makes our argument simpler in several ways. First, $G(\sigma + it)$ tends to 1 as t tends to infinity with σ bounded (recall (17)), which allows the use of $G(z)$ as a factor in an infinite product. Second, it is rather easy to express $\log G(z)$ as a Mellin transform, while for $\log H(z)$ this takes more work (see Granville and Soundararajan [7]). Third, the function f is supported on $[e, \infty)$, with the consequence that it is easy to show that the various γ_k do not interfere with each other. With $\log H(z)$, on the other hand, the inverse Mellin transform has support $[1, \infty)$, and the various γ_k may have a cumulative effect.

3. The zeros of $G(z)$

Let $G(z)$ be the entire function defined in (11). In the preceding section we observed that $G(z) \neq 0$ for $x \geq 0$ except when $z = 0$. The following information concerning the zeros with negative real part will be essential later. Let z_j denote the zeros of G, ordered by increasing imaginary part, with $z_0 = 0$. Since G is real on the real axis, it is clear by the reflection principle that $z_{-j} = \overline{z_j}$. Hence it is enough to consider those zeros z_j for positive integers j.

Lemma 1. *Let $G(z)$ be as in* (11), *and let $z_j = x_j + iy_j$ denote the zeros of $G(z)$ as described above. Then $\pi j < y_j < \pi(j+1)$ for each positive integer j, and*

$$x_j < -\frac{1}{2}\log\left(\frac{\pi j}{2}\right). \tag{22}$$

Proof. We find it convenient to work with $G_1(z) = zG(z) = z - e^{-z} + e^{-2z}$. This function has the same zeros as G, except that it has a double zero at the origin. Let j denote a positive integer. We note that $\arg G_1(x + i\pi j)$ tends to 0 as x tends to $+\infty$ and also as x tends to $-\infty$. Moreover, $\Im G_1(x + i\pi j) = \pi j > 0$ for all real x. Hence the total change of the argument of G_1 on this line is exactly 0. If x is large and positive, then the change of argument of G_1 as one moves from $x + iy$ up to $x + i(y+\pi)$ is approximately 0. If x is large and negative, then the change of argument of G_1 as one moves from $x + i(y+\pi)$ down to $x + iy$ is approximately 2π. Thus it follows that the open strip $\pi j < y < \pi(j+1)$ contains exactly one zero, which of course must be simple. Moreover, the open strip $-\pi < y < \pi$ contains exactly two zeros of G_1. Since G_1 has a double zero at the origin, it follows that G has no zero in this strip other than its simple zero at the origin. Thus $\pi j < y_j < \pi(j+1)$ for each positive integer j, as claimed.

By considering real and imaginary parts separately, we see that the equation $G_1(x_j + iy_j) = 0$ implies that

$$e^{-x_j}\cos y_j - e^{-2x_j}\cos 2y_j = x_j, \tag{23}$$

$$-e^{-x_j}\sin y_j + e^{-2x_j}\sin 2y_j = y_j. \tag{24}$$

Since $j \ge 1$, it follows that $x_j < 0$ and $y_j > 0$. Therefore, the left hand side of (24) is $\le 2e^{-2x_j}$. Hence $2e^{-2x_j} \ge y_j > \pi j$, which gives (22), so the proof is complete. □

The remainder of this section is concerned with a more precise description of the zeros of $G(z)$. This material is not required for the proof of Theorem 1, but it enables us to appreciate better the distribution of the zeros of $\zeta_B(s)$.

Lemma 2. *As described above, let z_j denote the j^{th} zero of the function $G(z)$ defined in* (11). *Then*

$$z_{\pm j} = -\tfrac{1}{2}\log \pi j \pm i\pi(j+1/4) + O\big(j^{-1/2}\big) \qquad \textit{for } j = 1, 2, 3, \ldots .$$

Proof. From (22) we see that if j is large, then x_j is large and negative. This makes e^{-2x_j} far larger than either of the other two terms in (23), and thus this equation can hold only because $\cos 2y_j$ is close to 0. Then $\sin 2y_j$ is near ± 1. From (24) we see that the negative sign is excluded, so $\sin 2y_j$ is near 1. That is, $2y_j$ is near $\pi/2 \pmod{2\pi}$, which is to say that y_j is near $\pi/4 \pmod{\pi}$. Thus $y_j = \pi(j+1/4)+o(1)$. From (24) we see that $e^{-2x_j} \sim \pi j$. It follows that $e^{-x_j} = O\big(j^{1/2}\big)$, and hence (23) implies that $j\cos 2y_j = O\big(j^{1/2}\big)$. This in turn implies that $y_j = \pi(j+1/4)+O\big(j^{-1/2}\big)$. Thus $\sin 2y_j = 1+O(1/j)$, and hence (24) gives $e^{-2x_j} = \pi j + O\big(j^{1/2}\big)$. Consequently $-2x_j = \log \pi j + O(j^{-1/2})$. This completes the proof. □

With somewhat more effort one can show that

$$z_{\pm j} = -\frac{1}{2}\log \pi j - \frac{(-1)^j}{2\sqrt{2\pi j}} \pm i\Big(\pi(j+\tfrac{1}{4}) - \frac{(-1)^j}{2\sqrt{2\pi j}} - \frac{\log j}{4\pi j}\Big) + O\Big(\frac{1}{j}\Big)$$

for $j = 1, 2, 3, \ldots$. Here the $(-1)^j$ produces a wobble that is already apparent in the values displayed below. By the generalized Lindemann theorem we know that α and e^α are both algebraic only when $\alpha = 0$. Thus the numbers z_j are transcendental, except when $j = 0$. Numerical values of the zeros may be determined by applying Newton's method to the function $G_1(z) = zG(z)$.

The first twenty (rounded) zeros with positive imaginary part are given in Table 1. In Figure 1, the curves of constant modulus and curves of steepest ascent/descent of $G(z)$ are depicted.

From (22) it follows that $x_j < -1$ if $j \ge 5$. In conjunction with the data in Table 1, we deduce that $x_j \ge -1$ only when $j = -1, 0$ or 1.

Let $N(\sigma, T)$ denote the number of zeros $\beta+i\gamma$ of the Riemann zeta function in the rectangle $\sigma \le \beta \le 1$, $0 \le \gamma \le T$, and define $N_B(\sigma, T)$ similarly for $\zeta_B(s)$. Estimates for $N(\sigma, T)$ play an important role in prime number theory. From Lemma 2, our definition (16) of $\zeta_C(s)$, and the fact that $\zeta_B(s)$ and $\zeta_C(s)$ have the same zeros for $\sigma \ge 1 - 1/(2\alpha)$, we see that there exist arbitrarily large numbers T such that

Table 1. Zeros of $G(z)$

$z_1 = -0.5127 + 4.0256i$	$z_{11} = -1.7227 + 35.3765i$
$z_2 = -1.1148 + 6.8662i$	$z_{12} = -1.8831 + 38.4045i$
$z_3 = -1.0510 + 10.2630i$	$z_{13} = -1.8092 + 41.6579i$
$z_4 = -1.3954 + 13.2077i$	$z_{14} = -1.9544 + 44.6942i$
$z_5 = -1.3140 + 16.5368i$	$z_{15} = -1.8833 + 47.9396i$
$z_6 = -1.5705 + 19.5184i$	$z_{16} = -2.0166 + 50.9826i$
$z_7 = -1.4883 + 22.8153i$	$z_{17} = -1.9480 + 54.2216i$
$z_8 = -1.6985 + 25.8182i$	$z_{18} = -2.0718 + 57.2702i$
$z_9 = -1.6187 + 29.0955i$	$z_{19} = -2.0054 + 60.5037i$
$z_{10} = -1.7995 + 32.1129i$	$z_{20} = -2.1213 + 63.5570i$

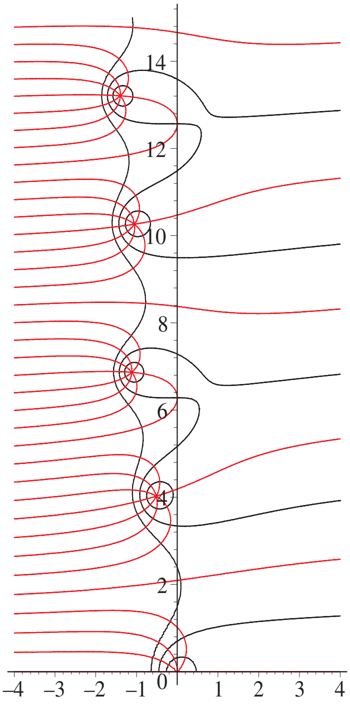

Fig. 1. Contour plot of the modulus and phase of $G(z)$

$$N_B(\sigma, T) \gg T^{2\alpha(1-\sigma)}$$

uniformly for $1 - 1/(2\alpha) \le \sigma \le 1 - a/\log T$.

4. The representation of $\log G(z)$ as a Mellin transform

Suppose that $x > 0$. Then

$$\left| e^{-z} - e^{-2z} \right| = \left| \int_z^{2z} e^{-u}\, du \right| < |z| ,$$

and hence we may write

$$\log G(z) = \log\Big(1 - \frac{e^{-z} - e^{-2z}}{z}\Big) = -\sum_{n=1}^{\infty} \frac{1}{n}\Big(\frac{e^{-z} - e^{-2z}}{z}\Big)^n.$$

Clearly

$$\frac{e^{-z} - e^{-2z}}{z} = \int_1^{\infty} \chi(u)\, u^{-z-1}\, du$$

where χ is the characteristic function of the interval $[e, e^2]$. On the other hand, if $\mathcal{M}_f(z) = \int_1^{\infty} f(u) u^{-z-1}\, du$ denotes the Mellin transform of f, then $\mathcal{M}_{f\star g}(z) = \mathcal{M}_f(z)\, \mathcal{M}_g(z)$ where $f \star g$ is the multiplicative convolution of f and g, namely $(f \star g)(u) = \int_1^u f(v) g(u/v)\, dv/v$. Hence, from the above we see that

$$\Big(\frac{e^{-z} - e^{-2z}}{z}\Big)^n = \int_1^{\infty} \chi^{\star n}(u) u^{-z-1}\, du$$

where $\chi^{\star n}$ denotes the n-fold convolution of χ with itself. Therefore

$$\log G(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \chi^{\star n}(u)\, u^{-z-1}\, du = -\int_1^{\infty} f(u)\, u^{-z-1}\, du \tag{25}$$

where

$$f(u) := \sum_{n=1}^{\infty} \frac{1}{n}\, \chi^{\star n}(u)\,. \tag{26}$$

In (25), the exchange of integration and summation is justified by absolute convergence: since $|u^{-z-1}| = u^{-x-1}$, it follows that the sum there is majorized by

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \chi^{\star n}(u) u^{-x-1}\, du = -\log G(x) < \infty$$

for $x > 0$. Thus we have proved

Lemma 3. *For $G(z)$, defined as in* (11)*, we have*

$$\log G(z) = -\int_e^{\infty} f(u)\, u^{-z-1}\, du \qquad \textit{for} \quad x > 0 \tag{27}$$

where f is the non-negative function defined in (26).

The function f has support $[e, \infty)$, and indeed for small u we can calculate $f(u)$ explicitly:

$$f(u) = \begin{cases} 0 & \text{for} \quad 1 \le u < e\,, \\ 1 & \text{for} \quad e \le u \le e^2\,, \\ \frac{1}{2}\log u - 1 & \text{for} \quad e^2 < u \le e^3\,. \end{cases} \tag{28}$$

Clearly $\chi^{\star n}$ is supported on $[e^n, e^{2n}]$, but for $n > 2$ the explicit computation of such convolution powers is complicated, so for larger u we take a different approach.

Lemma 4. *Let* $f(u)$ *be defined as in* (26). *If* $u > e^2$, *then*

$$f(u)\log u = \sum_{j=-\infty}^{\infty} u^{z_j} \tag{29}$$

where the z_j *are the zeros of* $G(z)$, *as discussed in the preceding section.*

Proof. We differentiate (27) to see that

$$\frac{G'}{G}(z) = \int_e^\infty u^{-z-1} f(u)\log u\, du \qquad \text{for} \quad x > 0\,. \tag{30}$$

Now, $f(u)\log u$ is continuous for $1 \le u < \infty$ except at $u = e$ and $u = e^2$, where it has jumps, and f is locally of bounded variation because f' is piecewise continuous. Thus for $u > e^2$ and $c > 0$ the Mellin inversion formula can be applied to (30). This formula arises by means of the change of variable $v = \log u$ from the inversion formula for the Laplace transform (see Theorem 15–34 on p. 498 of Apostol [1]). Thus we find that

$$f(u)\log u = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{G'}{G}(z)\, u^z\, dz\,.$$

We evaluate this integral in terms of the zeros of G by integrating around a rectangle with vertices $-K \pm J\pi i,\ c \pm J\pi i$, for J an odd positive integer and c and $K,\ \ K \ge J$, positive numbers. Thus

$$\frac{1}{2\pi i}\int_{c-J\pi i}^{c+J\pi i} \frac{G'}{G}(z)\, u^z\, dz \;=\; \sum_{|j|<J} u^{z_j} + I_T + I_B + I_L \tag{31}$$

where $I_T,\ I_B,\ I_L$ denote the integrals of $1/(2\pi i)(G'/G)(z)u^z$ along the top, bottom and left sides respectively (in suitable directions).

The integrand in (31) is

$$\frac{G'}{G}(z)\, u^z \;=\; \frac{e^{-z} - 2e^{-2z} + e^{-z}/z - e^{-2z}/z}{z - e^{-z} + e^{-2z}}\, u^z\,.$$

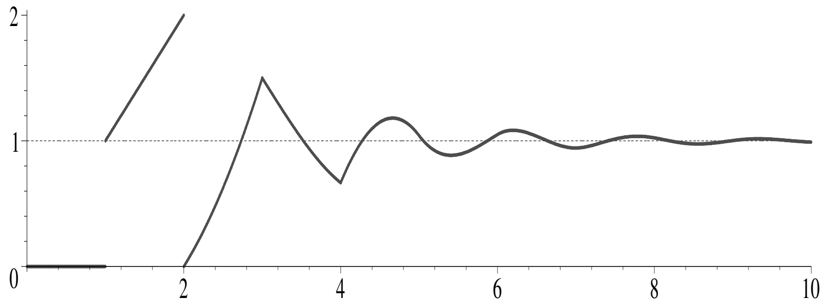

Fig. 2. Graph of $f(e^v)v$ for $0 \le v \le 10$

The integral I_L goes to zero as $K \to \infty$, because $G'/G \sim -2$ on this line and u^z is exponentially small here. On the top line, the denominator has imaginary part $J\pi$, and the numerator is $\ll \exp(-2x)$. Thus for fixed $u > e^2$,

$$I_T \ll \frac{1}{J} \int_{-\infty}^{c} \exp\big(x(\log u - 2)\big)\, dx\,,$$

which tends to 0 for $J \to \infty$. Since $I_B = -\overline{I_T}$, this estimate suffices for the bottom also, so the proof is complete. □

From the estimate (22) of Lemma 1 it follows that the sum (29) is absolutely convergent when $u > e^2$. By combining Lemma 1 with the data in Table 1 we find that $x_j < -1/2$ for $j \neq 0$. Thus we see that

$$f(u)\log u \;=\; 1 + O\big(u^{-1/2}\big) \qquad \text{for} \quad u \ge 1\,.$$

For our present purposes we require a much weaker estimate, but one with an explicit numerical constant.

Lemma 5. *Let* $f(u)$ *be defined as in* (26). *Then* $f(u) \le 4/\log u$ *for all* $u > 1$.

With more work it can be shown that this inequality holds with 4 replaced by 2. The constant 2 is best possible, since equality is attained with $u = e^2$.

Proof. For $1 < u \le e^3$ this inequality is clear from (28). For $u > e^3$ we apply (29) and (22) to see that

$$\begin{aligned} f(u)\log u &\le 1 + 2\sum_{j=1}^{\infty} u^{x_j} \le 1 + 2\sum_{j=1}^{\infty} \exp\big(-\tfrac{3}{2}\log(\pi j/2)\big) \\ &= 1 + 2\big(\tfrac{\pi}{2}\big)^{-3/2}\zeta(3/2) = 3.653907\ldots < 4\,. \end{aligned}$$

□

We conclude with a still weaker estimate for f that is tailored to our needs in the next section.

Lemma 6. *Let* $\Delta(b) = 2/(eb)$. *If* $b \ge 2/e$, *then*

$$f(u) \le \Delta(b)\,\frac{u^b}{2\log u} \qquad \textit{for} \quad u > 1\,.$$

This inequality holds also for b somewhat smaller than $2/e$, but this is not relevant since the above is useful for our purposes only when $\Delta(b) < 1$. When $1/2 \le b \le 1$, equality holds at $u = e^{1/b}$.

Proof. The function $u^b/\log u$ is strictly decreasing for $1 < u < e^{1/b}$, and is strictly increasing for $u \ge e^{1/b}$, and thus we see that $u^b/\log u \ge eb$ for all $u > 1$. This suffices when $1 < u \le e^3$, since $f(u) \le 1$ in this interval. For $u > e^3$ we appeal to the bound of the preceding lemma, and thus it suffices to show that $\Delta(b)u^b \ge 8$ when $u \ge e^3$. That is, we must show that $e^{3b} \ge 4eb$. Put $g(w) = e^{3w} - 4ew$. This function is increasing for $w \ge (1 + \log 4/3)/3 = 0.4292\ldots$. Thus if $b \ge 2/e$, then $g(b) \ge g(2/e) = 1.0909\ldots > 0$, and the proof is complete. □

5. The measure $d\Pi_C(x)$

We now express $\log \zeta_C(s)$ as a Mellin transform. From (16) it follows that

$$\log \zeta_C(s) \;=\; \log \frac{s}{s-1} + \sum_{k\neq 0} \log G\big(\ell_k(s-\rho_k)\big)$$

for $\sigma > 1$. We first show that

$$\log \frac{s}{s-1} \;=\; \int_1^\infty \frac{1-1/v}{\log v}\, v^{-s}\, dv \qquad \text{for} \quad \sigma > 1\,. \tag{32}$$

To see this, let $h(s)$ denote the right hand side. Then

$$h'(s) = -\int_1^\infty \Big(1-\frac{1}{v}\Big) v^{-s}\, dv = \frac{1}{s} - \frac{1}{s-1}\,.$$

Hence $h(s) = \log(s/(s-1)) + C$ for some constant C. On the other hand, $h(s) \ll \int_1^\infty v^{-\sigma}\, dv = 1/(\sigma-1)$. Thus by comparing $h(\sigma)$ with $\log \sigma/(\sigma-1)$ as $\sigma \to \infty$, we see that $C = 0$, and hence we have (32).

By Lemma 3 we see that

$$\log G\big(\ell_k(s-\rho_k)\big) = -\frac{1}{\ell_k}\int_{\exp(\ell_k)}^\infty f(v^{1/\ell_k})\, v^{-s+\rho_k-1}\, dv$$

for $\sigma > 1$. Thus

$$\sum_{k\neq 0} \log G\big(\ell_k(s-\rho_k)\big)$$
$$= -2\int_1^\infty \Big(\sum_{\substack{k\geq 1\\ \exp(\ell_k)\leq v}} \frac{f(v^{1/\ell_k})}{\ell_k}\, v^{\beta_k-1}\cos(\gamma_k \log v)\Big) v^{-s}\, dv\,. \tag{33}$$

To justify the exchange of summation and integration, we show that the integral above is absolutely convergent, which is accomplished under the assumption that

$$\gamma_k = \gamma_1^{A^{k-1}} \qquad \text{for} \quad k = 2,\ 3,\ 4,\ \ldots \tag{34}$$

where $\gamma_1 \geq 2$ and $A > 1$. (Later, we will need A to be sufficiently large.) For $v \geq \gamma_1^\alpha$, choose K so that $\gamma_K^\alpha \leq v < \gamma_{K+1}^\alpha$. Since $f(u) \ll 1/\log u$, and $\alpha \log \gamma_K \leq \log v$, the last sum above is

$$\ll \frac{1}{\log v}\sum_{k=1}^K v^{\beta_k-1} \leq \frac{1}{\log v}\sum_{k=1}^K e^{-a\alpha A^{K-k}} \ll \frac{1}{\log v}\,.$$

Thus the integral in (33) is absolutely convergent, since

$$\int_2^\infty \frac{v^{-\sigma}}{\log v}\, dv \ < \ \infty$$

for $\sigma > 1$. On combining our results we find that

$$\log \zeta_C(s) = \int_1^\infty v^{-s}\, d\Pi_C(v)$$

for $\sigma > 1$ where

$$d\Pi_C(v) \ = \ \Big(\frac{1-1/v}{\log v} - 2\sum_{\substack{k\geq 1\\ \exp(\ell_k)\leq v}} \frac{f(v^{1/\ell_k})}{\ell_k}\, v^{\beta_k-1}\cos(\gamma_k\log v)\Big)\, dv \tag{35}$$

for $v \geq 1$. We now show that this measure is positive if the parameters are chosen appropriately.

Lemma 7. *Suppose that α is fixed, with $a\alpha > 2/e$. If the γ_k are given by* (34) *with γ_1 and A sufficiently large, then not only is $d\Pi_C(v)$ a positive measure, but*

$$\frac{dv}{\log 2v} \ll d\Pi_C(v) \ \leq \ \frac{2\,dv}{\log 2v}$$

uniformly for $v \geq 1$.

By integrating the above bounds we see immediately that

$$\frac{x}{\log x} \ll \Pi_C(x) \le \frac{3x}{\log x} \qquad \text{for } x \ge x_0 . \tag{36}$$

Proof. Suppose first that $1 \le v < \gamma_1^\alpha$. Then

$$d\Pi_C(v) = \frac{1 - 1/v}{\log v}\, dv ,$$

and the stated estimates are clear in this case. Now suppose that $v \ge \gamma_1^\alpha$. In view of (13) we may choose b so that $2/e < b < a\alpha$. Choose K so that $\gamma_K^\alpha \le v < \gamma_{K+1}^\alpha$. Then by Lemma 6, we see that

$$2\sum_{k=1}^{K} \frac{f(v^{1/\ell_k})}{\ell_k} v^{-a\alpha/\ell_k} \le \frac{\Delta(b)}{\log v} \sum_{k=1}^{K} v^{(b-a\alpha)/\ell_k} .$$

Now $\ell_K = \alpha \log \gamma_K \le \log v$ and so $\ell_k = \alpha \log \gamma_k \le A^{k-K} \log v$ for $1 \le k \le K$, by (34). Hence the last expression above is at most

$$\frac{\Delta(b)}{\log v} \sum_{k=1}^{K} e^{(b-a\alpha)A^{K-k}} .$$

Here $\Delta(b) < 1$, and the sum tends to 1 as A tends to infinity. Thus there is a $\delta > 0$ such that the above is at most $(1-\delta)/\log v$ if A is large enough. Hence by (35) we conclude that

$$\frac{\delta - 1/v}{\log v}\, dv \le d\Pi_C(v) \le \frac{2-\delta}{\log v}\, dv$$

uniformly for $v \ge \gamma_1^\alpha$. This gives the stated estimates if γ_1 is sufficiently large. □

6. A probabilistic lemma

We begin with a familiar inequality of Kolmogorov (see § 18.1A.(i) of Loève [12]), for which we provide a simple proof.

Lemma 8. *For* $1 \le k \le K$ *let* Y_k *be independent random variables such that* $E(Y_k) = 0$ *and* $|Y_k| \le 1$ *. Also, for* $1 \le k \le K$ *let* r_k *be real numbers such that* $|r_k| \le 1$, *and let* $Y = \sum_{k=1}^K r_k Y_k$ *. Finally, set* $\sigma^2 = \mathrm{Var}(Y) = \sum_{k=1}^K r_k^2 \,\mathrm{Var}(Y_k)$ *. Then*

$$P(Y \ge v) \le \begin{cases} \exp\left(\dfrac{-v^2}{4\sigma^2}\right) & \text{if } \; 0 \le v \le 2\sigma^2 , \\ \exp(-v/2) & \text{if } \; v \ge 2\sigma^2 . \end{cases} \tag{37}$$

Proof. Let $\lambda \ge 0$ be a parameter to be determined later. Then $P(Y \ge v) \le E(e^{\lambda(Y-v)})$. Since the Y_k are independent, this is equal to

$$e^{-\lambda v} E\Big(\prod_{k=1}^{K} e^{\lambda r_k Y_k}\Big) = e^{-\lambda v} \prod_{k=1}^{K} E\big(e^{\lambda r_k Y_k}\big)\,.$$

It is not hard to show that $e^u \le 1 + u + u^2$ when $-1 \le u \le 1$. Indeed, this inequality holds for $-\infty < u \le u_0$ where $u_0 = 1.79328\ldots$. Thus if $0 \le \lambda \le 1$, then

$$E\big(e^{\lambda r_k Y_k}\big) \le E\big(1 + \lambda r_k Y_k + \lambda^2 r_k^2 Y_k^2\big) = 1 + \lambda^2 r_k^2 E\big(Y_k^2\big) \le \exp\big(\lambda^2 r_k^2 E(Y_k^2)\big)$$

by the inequality $1 + u \le e^u$, which holds for all real u. Thus we see that

$$P(Y \ge v) \;\le\; \exp(-\lambda v + \lambda^2 \sigma^2)\,.$$

We obtain (37) by taking $\lambda = v/(2\sigma^2)$ in the first case, and $\lambda = 1$ in the second. Thus $0 \le \lambda \le 1$ in either case, and the proof is complete. □

In our contemplated application of the above, the size of the r_k is unclear, which makes it impossible to assess the size of the variance σ^2, beyond the fact that it does not exceed the estimate $\sigma_e^2 := \sum_{k=1}^K \mathrm{Var}(Y_k)$. We note that (37) asserts that

$$P(Y \ge v) \le \max\Big(\exp\Big(\frac{-v^2}{4\sigma^2}\Big),\ \exp(-v/2)\Big)\,.$$

Since $\sigma^2 \le \sigma_e^2$, the above is at most

$$\max\Big(\exp\Big(\frac{-v^2}{4\sigma_e^2}\Big),\ \exp(-v/2)\Big)\,,$$

which is to say that

$$P(Y \ge v) \;\le\; \begin{cases} \exp\Big(\dfrac{-v^2}{4\sigma_e^2}\Big) & \text{if } 0 \le v \le 2\sigma_e^2\,, \\ \exp(-v/2) & \text{if } v \ge 2\sigma_e^2\,. \end{cases} \tag{38}$$

Here the bound in the first case is weaker than in the first case of (37), but (38) has the advantage that the first case applies over an interval of assured length, while the first case in (37) may apply only in a much shorter range, since σ^2 may be much smaller than σ_e^2.

Let X_k be independent Bernoulli variables with parameter p_k, and set $X = \sum_{k=1}^K r_k X_k$ where $|r_k| \le 1$ for all k. Then $E(X) = \sum_{k=1}^K r_k p_k$. Take $Y_k = X_k - p_k$. Thus $E(Y_k) = 0$ and $\mathrm{Var}(Y_k) = p_k(1 - p_k)$, so we find from the above that if

$$0 \le v \le 2\sum_{k=1}^K p_k(1 - p_k)\,, \tag{39}$$

then

$$P\big(X \geq E(X) + v\big) \leq \exp\left(\frac{-v^2}{4\sum_{k=1}^{K} p_k(1-p_k)}\right). \tag{40}$$

7. From continuous to discrete measures

In this section we use Lemma 7, (36), and Lemma 8 to show that there exists a set of Beurling primes $\mathcal{P}$ whose counting function $\pi_B(x)$ resembles $\Pi_C(x)$ in crucial ways.

Lemma 9. *There exists a sequence $\mathcal{P} = \{\lambda_j\}$ of Beurling primes such that*

$$\int_1^x v^{-it}\, d\pi_B(v) = \int_1^x v^{-it}\, d\Pi_C(v) + O\Big(\sqrt{x \log(|t|+2)}\Big) \tag{41}$$

uniformly for real t and $x \geq 1$.

In particular, for $t = 0$ this lemma asserts that

$$\pi_B(x) = \Pi_C(x) + O\big(\sqrt{x}\big) \tag{42}$$

uniformly for $x \geq 1$.

Proof. Let $1 = v_0 < v_1 < v_2 < \ldots$ be a sequence of real numbers tending very slowly to infinity. We shall specify later how slowly this must be. For $k = 1, 2, \ldots$ let X_k be independent Bernoulli variables with parameters

$$p_k = \int_{v_{k-1}}^{v_k} 1\, d\Pi_C(v).$$

The v_k must increase sufficiently slowly to ensure that $p_k \leq 1/2$. At any given point ω of our probability space, let $\mathcal{P}(\omega)$ be the set of those v_k for which $X_k = 1$. We show that 'most' of the sets $\mathcal{P}(\omega)$ determine a measure $d\pi_B(v)$ for which (41) holds.

We turn our attention first to (42). Suppose that $x \geq 2$, and let K be determined by $v_K \leq x < v_{K+1}$. Then $\pi_B(x) = \sum_{k=1}^K X_k = X$, and $E(X) = \sum_{k=1}^K p_k = \Pi_C(v_K) \leq \Pi_C(x)$. To verify that condition (39) is satisfied with $v = 5\sqrt{x}$, note that

$$\sum_{k=1}^{K} p_k = \Pi_C(v_K) = \Pi_C(v_{K+1}) - p_{K+1} \geq \Pi_C(x) - 1.$$

Since $p_k \leq 1/2$ for all k, condition (39) holds if $5\sqrt{x} \leq \Pi_C(x) - 1$, and this inequality is a consequence of (36) for all sufficiently large x. Then by (40) we see that

$$P\big(\pi_B(x) \geq \Pi_C(x) + 5\sqrt{x}\big) \leq P\big(X \geq \Pi_C(v_K) + 5\sqrt{x}\big) \leq \exp\left(\frac{-6x}{\Pi_C(v_K)}\right).$$

Now

$$\Pi_C(v_K) \;\le\; \Pi_C(x) \;\le\; \frac{3x}{\log x}$$

by (36), and hence $P\big(\pi_B(x) \ge \Pi_C(x) + 5\sqrt{x}\big) \le x^{-2}$. By a similar argument with $r_k = -1$ we find that $P\big(\pi_B(x) \le \Pi_C(x) - 5\sqrt{x}\big) \le x^{-2}$ for all sufficiently large x.

For $m = 1, 2, \ldots$ let A_m denote the event that $|\pi_B(m) - \Pi_C(m)| \ge 5\sqrt{m}$. Thus $P(A_m) \ll m^{-2}$. Since $\sum_m P(A_m) < \infty$, it follows by the easier implication of the Borel–Cantelli lemma that $P\big(\bigcup_{m=M}^{\infty} A_m\big) < \varepsilon$ if $M \ge m_0(\varepsilon)$. From now on we restrict our attention to those points ω that lie in the event $\bigcap_{m=m_0}^{\infty} A_m^c$. We know that this event has probability $> 1 - \varepsilon$ if m_0 is large. Suppose that $x \ge m_0$, and take $m = [x]$. Thus $|\pi_B(m) - \Pi_C(m)| < 5\sqrt{m}$ and $|\pi_B(m+1) - \Pi_C(m+1)| < 5\sqrt{m+1}$. Hence

$$\begin{aligned}\pi_B(x) \le \pi_B(m+1) \;&\le\; \Pi_C(m+1) + 5\sqrt{m+1}\\ &= \Pi_C(x) + 5\sqrt{m+1} + \int_x^{m+1} 1\, d\Pi_C(v)\,.\end{aligned}$$

Thus far we have used (36), but not Lemma 7, which is stronger, apart from the values of constants. Now we invoke Lemma 7 to see that the above is at most

$$\Pi_C(x) + 5\sqrt{x+1} + 2/\log x \;\le\; \Pi_C(x) + 6\sqrt{x}$$

if m_0 is large. Similarly,

$$\begin{aligned}\pi_B(x) \ge \pi_B(m) \;&\ge\; \Pi_C(m) - 5\sqrt{m}\\ &\ge \Pi_C(x) - 5\sqrt{x} - \int_m^x 1\, d\Pi_C(v)\\ &\ge \Pi_C(x) - 5\sqrt{x} - 2/\log(x-1) \;\ge\; \Pi_C(x) - 6\sqrt{x}\end{aligned}$$

by Lemma 7 and m_0 sufficiently large. Consequently, for the remainder of our argument we may suppose that

$$|\pi_B(x) - \Pi_C(x)| \le 6\sqrt{x} \qquad \text{for } x \ge m_0\,.$$

Note that this inequality implies (42).

We now complete the proof of (41). Let

$$S_B(x;t) = \int_1^x v^{-it}\, d\pi_B(v)\,, \qquad S_C(x;t) = \int_1^x v^{-it}\, d\Pi_C(v)\,.$$

By the reflection principle it suffices to prove (41) when $t \ge 0$. By integrating by parts and using (42) we see that

$$S_B(x;t) = S_C(x;t) + O\big((t+1)\sqrt{x}\big)\,.$$

This gives (42) for $0 \le t \le t_0$. We suppose henceforth that $t \ge t_0$. From (36) we have the trivial estimate $|S_C(x;t)| \le S_C(x;0) \ll x/\log x$, and from (42) we see similarly that $|S_B(x;t)| \le S_B(x;0) \ll x/\log x$. These estimates suffice to give (41) when $2 \le x \ll (\log t)(\log\log t)^2$. Thus we may suppose that

$$x \ge C_0(\log t)(\log\log t)^2 \tag{43}$$

where C_0 is a suitably large absolute constant. Note that

$$S_B(x;t) = \sum_{\substack{p \le x \\ p \in \mathcal{P}}} p^{-it} = \sum_{k=1}^{K} v_k^{-it} X_k$$

since $v_K \le x < v_{K+1}$. By taking $r_k = \cos(t \log v_k)$ in (40) we find that if $t \ge 5$, then

$$\begin{aligned} P\Big(\Re\, S_B(x;t) \ge E\big(\Re\, S_B(x;t)\big) + 5\sqrt{x \log t}\,\Big) & \\ \le \exp\Big(\frac{-6x \log t}{\Pi_C(v_K)}\Big) \le \exp\Big(\frac{-6x\log t}{\Pi_C(x)}\Big) & \\ \le \exp\big(-2(\log x)(\log t)\big) & \end{aligned} \tag{44}$$

by (36). As before, we find that $\sum_{k=1}^{K} p_k(1-p_k) \gg x/\log x$, and hence (43) implies the hypothesis (39), if C_0 is sufficiently large.

It remains to consider how close $E\big(\Re\, S_B(x;t)\big)$ is to $\Re\, S_C(x;t)$. In this connection we show that

$$\big|E\big(S_B(x;t)\big) - S_C(x;t)\big| \le \sqrt{x} \tag{45}$$

if the v_k are chosen to increase sufficiently slowly. By the triangle inequality we see that

$$\begin{aligned} \big|E\big(S_B(x;t)\big) - S_C(x;t)\big| &\le \int_{v_K}^{x} 1\, d\Pi_C(v) + \sum_{k=1}^{K} \Big| \int_{v_{k-1}}^{v_k} \big(v_k^{-it} - v^{-it}\big)\, d\Pi_C(v) \Big| \\ &\le p_{K+1} + \sum_{k=1}^{K} \int_{v_{k-1}}^{v_k} \big|v_k^{-it} - v^{-it}\big|\, d\Pi_C(v)\,. \end{aligned}$$

Here $p_{K+1} \le 1/2$. Choose L so that $v_L \le \sqrt{x} < v_{L+1}$. Since the integrand above is uniformly bounded by $2,$ it follows that the contribution of the $k \le L$ is at most

$$2\,\Pi_C(v_L) \ll \frac{\sqrt{x}}{\log x}\,.$$

Now suppose that $L < k \le K$. From the inequality $\big|e^{i\alpha} - e^{i\beta}\big| \le |\alpha - \beta|$ we see that the integrand is at most

$$t(\log v_k - \log v) \;\le\; t(\log v_k - \log v_{k-1}) \;\le\; \frac{t(v_k - v_{k-1})}{v_{k-1}} .$$

It follows from Lemma 7 that the contribution of such k is at most

$$\sum_{k=L+1}^{K} \frac{t(v_k - v_{k-1})p_k}{v_{k-1}} \;\ll\; \sum_{k=L+1}^{K} \frac{t(v_k - v_{k-1})^2}{v_{k-1}\log v_{k-1}} .$$

We now set $v_k = \sqrt{\log k}$ for $k \ge k_0$. This ensures that $p_k \le 1/2$, as was needed in the proof of Lemma 9. From the definition of L it follows that $L = [e^x]$, and from (43) we see that $t \le e^x$. Thus

$$\sum_{k=L+1}^{K} \frac{t(v_k - v_{k-1})^2}{v_{k-1}\log v_{k-1}} \;\ll\; t \sum_{k=L+1}^{K} k^{-2} \;\ll\; tL^{-1} \;\ll\; te^{-x} \;\ll\; 1 .$$

On collecting our estimates we obtain (45).

By combining (45) with (44) we conclude that

$$P\Big(\Re S_B(x;t) \ge \Re S_C(x;t) + 6\sqrt{x\log t}\Big) \;\le\; \exp\big(-2(\log x)(\log t)\big) .$$

By three similar applications of (40), with $r_k = -\cos(t\log v_k)$ and with $r_k = \pm\sin(t\log v_k)$, we deduce that

$$P\Big(|S_B(x;t) - S_C(x;t)| \ge 9\sqrt{x\log t}\Big) \;\le\; 4\exp\big(-2(\log x)(\log t)\big) .$$

For positive integers m, n let A_{mn} denote the event that $|S_B(m;n) - S_C(m;n)| \ge 9\sqrt{m\log n}$. Since

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} P(A_{mn}) < \infty ,$$

it follows that we may choose a point ω in our probability space so that not only (42) holds but also so that $|S_B(m;n) - S_C(m;n)| < 9\sqrt{m\log n}$ for $m \ge m_0$ and $n \ge n_0$. If $m \le x < m+1$ where $m \ge m_0$, then $S_B(x;n) = S_B(m;n) + O\big(\sqrt{x}\big)$ by (42) combined with Lemma 7. Similarly, from Lemma 7 we see that $S_C(x;n) = S_C(m;n) + O(1)$. Hence it follows that

$$S_B(x;n) = S_C(x;n) + O\Big(\sqrt{x\log n}\Big) \tag{46}$$

for all real $x \ge m_0$ and all integers $n \ge n_0$. Suppose that $t \ge n_0$, and let n denote the integer nearest t. By integration by parts we see that

$$S_B(x;t) = x^{i(n-t)}S_B(x;n) + i(t-n)\int_1^x S_B(v;n)v^{i(n-t)-1}\,dv$$

and similarly for S_C. Thus (41) follows from (46), and the proof is complete. □

Lemma 10. *Let the Beurling primes be chosen as in Lemma* 9*, and for* $\sigma > 1$ *set*

$$\log\zeta_B(s) = -\int_1^\infty \log\big(1 - v^{-s}\big)\,d\pi_B(v) .$$

Then $\log\big(\zeta_B(s)/\zeta_C(s)\big)$ *has an analytic continuation to* $\sigma > 1/2$ *satisfying*

$$\log \frac{\zeta_B(s)}{\zeta_C(s)} = O\Big(\sqrt{\log(|t|+2)}\Big) \tag{47}$$

uniformly for $\sigma \geq 1/2+\varepsilon$ *for any fixed* $\varepsilon > 0$.

From the above it follows that $\zeta_B(s)$ is analytic for $\sigma > 1-1/(2\alpha)$, and that $\zeta_B(s)$ has the same zeros and poles as $\zeta_C(s)$ in this half-plane. Hence $\zeta_B(s)$ has a single pole at $s = 1$ and zeros at the points $\rho_k = 1 - a/\log|\gamma_k| + i\gamma_k$. Therefore, we have the assertions (*ii*) and (*iii*) of Theorem 1.

Proof. Clearly

$$\begin{aligned}\log \frac{\zeta_B(s)}{\zeta_C(s)} = &-\int_1^\infty \Big(v^{-s} + \log\big(1-v^{-s}\big)\Big)\, d\pi_B(v) \\ &+ \int_1^\infty v^{-s}\, d\big(\pi_B(v) - \Pi_C(v)\big).\end{aligned}$$

Since $\log\big(1-v^{-s}\big) = -v^{-s} + O\big(v^{-2\sigma}\big)$ for $v \geq 1+\delta$, and since $\pi_B(v)$ has no support near $v = 1$, it follows that the first integral above is analytic and uniformly bounded for $\sigma \geq 1/2+\varepsilon$. By using integration by parts and (41), we see that the second integral above is analytic and $O\big(\sqrt{\log(|t|+2)}\big)$ for $\sigma \geq 1/2+\varepsilon$. Thus we have (47). □

Exponential bounds of the type considered in the preceding section form one of the primary tools used to prove the classical Law of the Iterated Logarithm for sums of independent uniformly bounded random variables (see Petrov [15], Theorem 7.1, p. 239). Indeed, by applying this more powerful tool we could show that

$$\pi_B(x) = \Pi_C(x) + O\left(\frac{\sqrt{x\log\log x}}{\sqrt{\log x}}\right) \qquad \text{almost surely,}$$

which is somewhat stronger than (42). The Law of the Iterated Logarithm could also be applied to bound the difference between the integrals in (41) for any fixed t, or even for t in any fixed compact set. However, we need an estimate in which the dependence on t is explicit, and for this purpose we make recourse to the more basic exponential bounds.

8. Asymptotics of $N_B(x)$

In view of (14) we may choose σ_1 so that $1-1/(2\alpha) < \sigma_1 < \theta$. From the definition (16) of $\zeta_C(s)$ and the estimate (17) of $G(z)$ we see that if $\sigma \geq \sigma_1$, then $\zeta_C(s) \ll 1$ for $|t| \geq 1$ unless $|t-\gamma_k| \leq |t|/2$ for some k, in which case

$$\zeta_C(s) \ll \frac{|t|^{2\alpha(1-\sigma)}+1}{1+|t-\gamma_k|\log|\gamma_k|} .$$

From these estimates we see that in any case

$$\int_T^{2T} \left|\zeta_C(\sigma_1+it)\right| dt \ll T .$$

Hence by Lemma 10 it follows that

$$\int_T^{2T} \left|\zeta_B(\sigma_1+it)\right| dt \ll T^{1+\varepsilon} . \tag{48}$$

Suppose that $\sigma_0 > 1$. By a familiar Mellin transform formula we know that

$$\sum_{\substack{n\in\mathcal{N}\\ n\le x}}(x-n) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta_B(s)\frac{x^{s+1}}{s(s+1)}\,ds .$$

Let h be a positive parameter to be chosen later in such a way that $1 \le h \le x$. By applying the above formula twice, and differencing, we see that

$$\begin{aligned} N_B(x) &\le \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x+h}}(x+h-n) - \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x}}(x-n)\\ &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta_B(s)\frac{(x+h)^{s+1}-x^{s+1}}{hs(s+1)}\,ds . \end{aligned}$$

Let κ denote the residue of $\zeta_B(s)$ at $s=1$. On moving the path of integration to the line $\sigma=\sigma_1$, we see that the above is

$$= \kappa(x+h/2) + \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \zeta_B(s)\frac{(x+h)^{s+1}-x^{s+1}}{hs(s+1)}\,ds .$$

Now

$$\frac{(x+h)^{s+1}-x^{s+1}}{s+1} \ll \begin{cases} hx^{\sigma} & \text{for } |t|\le x/h ,\\ x^{\sigma+1}/|t| & \text{for } |t|\ge x/h . \end{cases} \tag{49}$$

Thus the last integral is

$$\ll \int_0^{x/h} \left|\zeta_B(\sigma_1+it)\right| \frac{x^{\sigma_1}}{t+1}\,dt + \int_{x/h}^{\infty} \left|\zeta_B(\sigma_1+it)\right| \frac{x^{\sigma_1+1}}{ht^2}\,dt .$$

By (48) this is $\ll x^{\sigma_1+\varepsilon} \ll x^{\theta}$. The $h/2$ in the main term is negligible if we take $h=1$, for example. We observe also that

$$N_B(x) \ge \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x}}(x-n) - \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x-h}}(x-h-n) ,$$

so by arguing similarly we may derive the corresponding lower bound. Thus we have (i) of Theorem 1. □

9. Asymptotics of $\pi_B(x)$

Let

$$\psi_B(x) = \sum_{\substack{n\in\mathcal{N}\\ n\le x}} \Lambda_B(n)$$

where $\Lambda_B(n) = \log\lambda_j$ if $n = \lambda_j^k$ for some j and k, and $\Lambda_B(n) = 0$ otherwise. Let h be a parameter to be chosen later such that $1 \le h \le x$. Then for $\sigma_0 > 1$,

$$\begin{aligned}\psi_B(x) &\le \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x+h}}(x+h-n)\Lambda_B(n) - \frac{1}{h}\sum_{\substack{n\in\mathcal{N}\\ n\le x}}(x-n)\Lambda_B(n)\\ &= -\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{\zeta_B'}{\zeta_B}(s)\,\frac{(x+h)^{s+1}-x^{s+1}}{hs(s+1)}\,ds\,.\end{aligned}$$

Choose ϕ so that $1-\theta < \phi < 1/(2\alpha)$, and choose T , $(1/4)x^\phi \le T \le x^\phi$, in such a way that there is no k for which $|T-\gamma_k| \le T/2$. Move the path of integration so that it runs from $1-i\infty$ to $1-iT$, to $\theta-iT$, to $\theta+iT$ to $1+iT$ to $1+i\infty$. In each case the path follows a line segment, except that in running from $\theta-iT$ to $\theta+iT$ detours should be made to the left so that the path is at all times at a distance $\gg 1/\log x$ from the nearest zero. Choose σ_1 and σ_2 so that $1-1/(2\alpha) < \sigma_1 < \sigma_2 < \theta$. Then by Landau's analysis (as presented, for example, in Lemma α of Titchmarsh [16], p. 56),

$$\frac{\zeta_B'}{\zeta_B}(s) \;=\; \sum_\rho \frac{1}{s-\rho} + O\big(\log(|t|+2)\big)$$

for $\sigma \ge \sigma_2$ where the sum is over those zeros ρ of ζ_B, if any, for which $\Re\rho \ge \sigma_1$ and $|\Im\rho - t| \le 1$. Since the number of summands is $\ll \log(|t|+2)$, it follows that $\big(\zeta_B'/\zeta_B\big)(s) \ll \big(\log x(|t|+2)\big)^2$ on our contour. Thus by (49) we see that

$$\begin{aligned}\int_{1-i\infty}^{1-iT} \cdots \;&\ll\; \frac{x^2(\log x)^2}{hT}\,,\\ \int_{1-iT}^{\theta-iT} \cdots \;&\ll\; \frac{x^2\log x}{hT^2}\,,\\ \int_{\theta-iT}^{\theta+iT} \cdots \;&\ll\; x^\theta(\log x)^3\,,\end{aligned}$$

and similarly for the remaining portions of the contour. Hence, if we take $h = x^{1-\phi/2}$, then we find that

$$\psi_B(x) \le x - \sum_\rho \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} + O\big(x^{(1+\theta)/2}\big)$$

where the sum is over all zeros ρ of $\zeta_B(s)$ for which $\Re\rho \ge \theta$ and $|\Im\rho| \le T$. Choose K so that $\gamma_K < T < \gamma_{K+1}$. Then by (49) again we see that the sum over zeros is

$$\ll \sum_\rho \frac{x^{\Re\rho}}{|\rho|}\,.$$

A typical zero is of the form

$$1 - \frac{a\alpha - x_j}{\ell_k} \pm i\gamma_k \pm i\frac{y_j}{\ell_k}$$

where $j \ge 0$ and $k \ge 1$. Suppose $j \ge 1$. By (22) we know that $x_j < (-1/2)\log j$, and hence

$$x^{1-(a\alpha - x_j)/\ell_k} < x^{1-a\alpha/\ell_k} j^{-(\log x)/(2\ell_k)}\,.$$

But $1/(2\alpha) > \phi$ and $\gamma_k < T \le x^\phi$, so the exponent of j is less than $-1/(2\alpha\phi) < -1$, uniformly in k. Hence the sum over j is absolutely convergent, and so

$$\sum_\rho \frac{x^{\Re\rho}}{|\rho|} \ll x\sum_{k=1}^K \frac{x^{-a/\log\gamma_k}}{\gamma_k}\,,$$

which is the same as for $j = 0$. The function $f(u) = x^{-a/\log u}/u$ is largest when $u = U = \exp\big(\sqrt{a\log x}\big)$, where it takes the value $\exp\big(-2\sqrt{a\log x}\big)$. Suppose that $\gamma_L \le U < \gamma_{L+1}$. For γ_L and γ_{L+1} we use the estimate $f(\gamma) \le \exp\big(-2\sqrt{a\log x}\big)$. For $1 \le k \le L-1$, we have

$$f(\gamma_k) < x^{-a/\log\gamma_k} < \exp\Big(-\sqrt{a\log x}\,\tfrac{\log\gamma_L}{\log\gamma_k}\Big) = \exp\Big(-A^{L-k}\sqrt{a\log x}\Big)$$

by (15). For $k \ge L+2$,

$$\begin{aligned} f(\gamma_k) < \frac{1}{\gamma_k} &= \exp\Big(-\sqrt{a\log x}\,\frac{\log\gamma_k}{\sqrt{a\log x}}\Big) < \exp\Big(-\sqrt{a\log x}\,\frac{\log\gamma_k}{\log\gamma_{L+1}}\Big)\\ &= \exp\big(-A^{k-L-1}\sqrt{a\log x}\big)\,. \end{aligned}$$

Thus the combined contribution of all zeros is $\ll x\exp\big(-2\sqrt{a\log x}\big)$, provided that we take $A \ge 2$. Since lower bounds for $\psi_B(x)$ may be derived similarly, we have (v) of Theorem 1.

To study the oscillation of $\psi_B(x) - x$, note first that

$$\frac{(x+h)^{\rho+1} - x^{\rho+1}}{h\rho(\rho+1)} = \big(1 + o(1)\big)\,x^\rho/\rho$$

for $\rho h = o(x)$. Choose $K \in \mathbb{N}$, and set $X_K = \exp\big((\log \gamma_K)^2/a\big)$. If $X_K \le x \le 2X_K$, then the combined contribution of the zeros near $1 - a/\log \gamma_K \pm i\gamma_K$ to the error term $\psi_B(x) - x$ is

$$-2\Re \frac{x^{1-a/\log \gamma_K + i\gamma_K}}{1 - a/\log \gamma_K + i\gamma_K} = \big(-2\sin(\gamma_K \log x) + o(1)\big)x \exp\big(-2\sqrt{a \log x}\big).$$

As x passes from X_K to $2X_K$, the sine above takes the values $+1$ and -1, each one $\gg \gamma_K$ times. On the other hand, from the preceeding estimates we see that the combined contribution of all other zeros is $\ll x \exp\big(-c\sqrt{\log x}\big)$ with $c > 2\sqrt{a}$ for x in this range. Thus we have (*iv*) of Theorem 1, and the proof is complete. □

10. Proof of Theorem 2

The proof of the upper bound proceeds in the same way as the proof for the Riemann zeta function: Let $t \ge t_0$. Since we have a zero-free region of width $c(1-\theta)/\log t$, we may assume that $(1-\theta)/\log t \le r \le (1-\theta)/2$. By Jensen's inequality and the Borel–Carathéodory lemma as used by Landau (see Lemma α of Titchmarsh [16], p. 56), we know that

$$\frac{\zeta_B'}{\zeta_B}(s) = \sum_\rho \frac{1}{s-\rho} + O\Big(\frac{\log t}{1-\theta}\Big) \tag{50}$$

where the sum is over those zeros ρ of $\zeta_B(s)$ in the disk of radius $(1-\theta)/2$ centered at $1 + it$, and $\sigma \ge 1 - (1-\theta)/4$. Note that the zeta function $\zeta_B(s)$ here is not necessarily the same as the one constructed in proving Theorem 1. Take $s = 1 + r + it$. Then

$$\begin{aligned}
\Big|\frac{\zeta_B'}{\zeta_B}(s)\Big| &= \Big|\int_1^\infty x^{-s} \log x \, d\Pi_B(x)\Big| \\
&\le \int_1^\infty x^{-1-r} \log x \, d\Pi_B(x) \\
&= -\frac{\zeta_B'}{\zeta_B}(1+r) \ll \frac{1}{r} \le \frac{\log t}{1-\theta}.
\end{aligned}$$

In (50) the contribution of each zero has positive real part, and if the zero lies in the disk of radius r centered at $1 + it$, then that contribution is $\gg 1/r$. Hence the number of such zeros is $\ll r(\log t)/(1-\theta)$, which establishes (7).

On differencing (50) it is clear that

$$\frac{\zeta_B'}{\zeta_B}(s+\delta) - \frac{\zeta_B'}{\zeta_B}(s) = \sum_\rho \Big(\frac{1}{s+\delta-\rho} - \frac{1}{s-\rho}\Big) + O\Big(\frac{\log t}{1-\theta}\Big). \tag{51}$$

We now argue that the error term can, with a little more care, be reduced by a factor of $\delta/(1-\theta)$. To see this, let $g(s) = \zeta_B'(s)/\zeta_B(s) - \sum_\rho 1/(s-\rho)$. Thus by (50), we know that $g(s) \ll (\log t)/(1-\theta)$. By two applications of Cauchy's integral formula we see that

$$\begin{aligned} g(s+\delta) - g(s) &= \frac{1}{2\pi i}\oint \frac{g(z)}{z-s-\delta}\,dz - \frac{1}{2\pi i}\oint \frac{g(z)}{z-s}\,dz \\ &= \frac{\delta}{2\pi i}\oint \frac{g(z)}{(z-s-\delta)(z-s)}\,dz\,. \end{aligned}$$

Here the path of integration may be taken to be the circle $|z-s| = (1-\theta)/4$, and we assume that $0 \le \delta \le (1-\theta)/5$. In the integrand on the right, the numerator is $\ll (\log t)/(1-\theta)$, and the denominator is $\gg (1-\theta)^2$. Since the path of integration has length $\ll 1-\theta$, it follows that the above is $\ll \delta(\log t)/(1-\theta)^2$. Thus

$$\frac{\zeta_B'}{\zeta_B}(s+\delta) - \frac{\zeta_B'}{\zeta_B}(s) = \sum_\rho \Big(\frac{1}{s+\delta-\rho} - \frac{1}{s-\rho}\Big) + O\Big(\frac{\delta\log t}{(1-\theta)^2}\Big) \tag{52}$$

when $\sigma \ge 1$ and $0 \le \delta \le (1-\theta)/5$. If $\delta > (1-\theta)/5$, then this is already implied by (51), and hence the above holds for all $\delta \ge 0$.

Let $h(s)$ denote the left hand side of (52). Then for $\sigma > 1$,

$$h(s) = \int_1^\infty x^{-s}\big(1-x^{-\delta}\big)(\log x)\,d\Pi_B(x)\,.$$

Hence

$$\begin{aligned} &\Re\big(3h(\sigma) + 4h(\sigma+it) + h(\sigma+2it)\big) \\ &\quad = \int_1^\infty x^{-\sigma}(1-x^{-\delta})(\log x)C(t\log x)\,d\Pi_B(x) \end{aligned} \tag{53}$$

where $C(\alpha) = 3 + 4\cos\alpha + \cos 2\alpha = 2(1+\cos\alpha)^2$. Since $C(\alpha) \ge 0$ for all α, we see that the expression above is non-negative. From (52) we know that

$$h(s) = -\delta\sum_\rho \frac{1}{(s+\delta-\rho)(s-\rho)} + O\Big(\frac{\delta\log t}{(1-\theta)^2}\Big).$$

Suppose that $\beta_0 + i\gamma_0$ is a zero of $\zeta_B(s)$ with β_0 near 1. We take $\sigma = 1 + 4(1-\beta_0)$ and $t = \gamma_0$. When we form the expression (53) we find that the combined contribution of the pole at $s=1$ and the zero ρ_0 is

$$\frac{3}{\sigma-1} - \frac{3}{\sigma+\delta-1} - \frac{4}{\sigma-\beta_0} + \frac{4}{\sigma+\delta-\beta_0}\,. \tag{54}$$

This expression is monotonically decreasing in δ for $\delta \ge 3(1-\beta_0)$. Thus we see that if

$$\delta \ge 32(1-\beta_0)\,, \tag{55}$$

then the quantity (54) is at most $-1/(40(1-\beta_0))$. Since

$$\Big|\frac{1}{(s+\delta-\rho)(s-\rho)}\Big| \le \frac{1}{|1+i\gamma_0-\rho|^2}\,,$$

it follows from the upper bound (7) already proved that

$$\sum_{\substack{\rho \\ |1+i\gamma_0-\rho|>r}} \frac{\delta}{(s+\delta-\rho)(s-\rho)} \ll \frac{\delta \log\gamma_0}{r(1-\theta)}\,.$$

On the other hand,

$$\Re\frac{-\delta}{(s+\delta-\rho)(s-\rho)} = \Re\frac{1}{s+\delta-\rho} - \Re\frac{1}{s-\rho} \le \Re\frac{1}{s+\delta-\rho} \le \frac{1}{\delta}\,,$$

and hence

$$\Re \sum_{\substack{\rho\ne\rho_0 \\ |1+i\gamma_0-\rho|\le r}} \frac{-\delta}{(s+\delta-\rho)(s-\rho)} \le \frac{n_B(r,\gamma_0)}{\delta}\,,$$

and similarly when γ_0 is replaced by $2\gamma_0$.

On combining these estimates, we deduce that

$$\begin{aligned}0 &\le \Re\big(3h(\sigma)+4h(\sigma+i\gamma_0)+h(\sigma+2i\gamma_0)\big) \\ &\le \frac{-1}{40(1-\beta_0)} + \frac{4\,n_B(r,\gamma_0)+n_B(r,2\gamma_0)}{\delta} + O\Big(\frac{\delta\log\gamma_0}{r(1-\theta)}\Big)\,. \end{aligned} \tag{56}$$

The other error term, of size $O\big(\delta(\log\gamma_0)/(1-\theta)^2\big)$, is absorbed in the error term displayed above, since by hypothesis $r \le (1-\theta)/2$. Suppose now that

$$\delta = \frac{cr(1-\theta)}{(1-\beta_0)\log\gamma_0}\,. \tag{57}$$

If the constant c is taken to be sufficiently small, then the error term in (56) is $<1/(80(1-\beta_0))$, and then from (56) it follows that

$$\frac{n(r,\gamma_0)+n(r,2\gamma_0)}{\delta} \ge \frac{1}{320(1-\beta_0)}\,,$$

which is the desired lower bound (9). Note that our choice (57) of δ is consistent with the constraint (55) if $r \ge 32(1-\beta_0)^2(\log\gamma_0)/(c(1-\theta))$. Thus we take $C = 32/c$ in (10). □

11. Appendix. Large Least Quadratic Nonresidues

We present here our reconstruction of an unpublished analysis of Heath–Brown concerning the behaviour of the Dirichlet L-function $L(s, (\frac{\cdot}{p}))$ under the assumption that the least quadratic nonresidue modulo p is as large as the bound $p^{1/(4\sqrt{e})+\varepsilon}$ of D. A. Burgess [4].

Let p be an odd prime, and let r be a positive integer. The basic estimate of Burgess [4], as refined by Friedlander and Iwaniec [6], asserts that

$$\sum_{n=M+1}^{M+N} \left(\frac{n}{p}\right) \ll N^{1-1/r} p^{(r+1)/(4r^2)} r^{1/2} (\log p)^{3/(2r)} . \tag{58}$$

If $N \le p^{1/4}$, then the above is trivial since the right hand side is $\gg N$. Suppose that $N > p^{1/4}$, and define δ by the relation $N = p^{1/4+\delta}$. The parameter r is a little troublesome, so our first task is to cast the above into a more readily applicable form. If r could be allowed to take arbitrary real values, then we would take $r = 1/(2\delta)$, which would give the bound

$$\sum_{n=M+1}^{M+N} \left(\frac{n}{p}\right) \ll N p^{-\delta^2} \delta^{-1/2} (\log p)^{3\delta} .$$

Since r is restricted to integral values, we obtain an estimate that is slightly larger than this:

$$\sum_{n=M+1}^{M+N} \left(\frac{n}{p}\right) \ll N p^{-\delta^2/2} \log p \tag{59}$$

for $0 < \delta \le 3/4$.

If $\delta > 3/4$, then $N > p$, and we would instead appeal to the Pólya–Vinogradov inequality, which asserts that

$$\sum_{n=M+1}^{M+N} \left(\frac{n}{p}\right) \ll p^{1/2} \log p . \tag{60}$$

To prove (59), we first consider the powers of N and of p in (58) and (59). Let r be an integer in the interval

$$\Big[\frac{1 - 1/\sqrt{2}}{\delta}, \ \frac{1 + 1/\sqrt{2}}{\delta} \Big].$$

The existence of such an integer is guaranteed by the fact that the length of the above interval is $\sqrt{2}/\delta \ge 4\sqrt{2}/3 > 1$. Moreover, this r is positive since the lower endpoint is positive. Since the numbers $1 \pm 1/\sqrt{2}$ are the roots of the polynomial $2z^2 - 4z + 1$, and $r\delta$ lies between these roots, it follows that $2\delta^2 r^2 - 4\delta r + 1 \le 0$. On dividing by $4r^2$, it follows that

$$-\frac{\delta}{r} + \frac{1}{4r^2} \le -\frac{\delta^2}{2}.$$

Hence

$$N^{1-1/r} p^{(r+1)/(4r^2)} = Np^{-\delta/r+1/(4r^2)} \le Np^{-\delta^2/2}.$$

Next we consider the factor $r^{1/2}$ and the power of $\log p$. Put

$$\delta_0 = \sqrt{\frac{2\log\log p}{\log p}}. \tag{61}$$

If $\delta < \delta_0$, then the right hand side of (59) is $\gg N$. Thus we may suppose that $\delta \ge \delta_0$, and hence $r \ll \sqrt{\log p}$. Consequently if $r \ge 2$, then

$$r^{1/2}(\log p)^{3/(2r)} \ll (\log p)^{1/4}(\log p)^{3/4} = \log p.$$

If $r = 1$, then we use (60) instead of (58), so $\log p$ appears to at most the first power in all cases. This completes our proof of (59).

Burgess [4] deduced from his character sum estimates that if $p > p_0(\varepsilon)$, then there is an integer n, $1 < n < p^{1/(4\sqrt{e})+\varepsilon}$, such that $\left(\frac{n}{p}\right) = -1$. We now assume that this estimate is, for some prime p, essentially best-possible, and explore the consequences. Specifically, from now on we assume that

$$\left(\frac{n}{p}\right) = 1 \qquad \text{for } 1 \le n \le N \tag{62}$$

where $N = p^{1/(4\sqrt{e})}$. Set

$$S(x) = \sum_{n \le x} \left(\frac{n}{p}\right), \qquad S(x, y) = \sum_{\substack{n \le x \\ p'|n \Rightarrow p' \le y}} \left(\frac{n}{p}\right).$$

Put $X = p^{1/4+2\delta_0}$ where δ_0 is given by (61). Then from (59) it follows that $S(X) \ll X(\log p)^{-3}$. Since the Legendre symbol is totally multiplicative, by classifying integers according to their greatest prime factor we find that

$$S(x, y) = 1 + \sum_{p' \le y} \left(\frac{p'}{p}\right) S(x/p', p').$$

Thus

$$S(x) = S(x, N) + \sum_{N < p' \le x} \left(\frac{p'}{p}\right) S(x/p', p'). \tag{63}$$

Since $\left(\frac{n}{p}\right) = 1$ for all integers n counted in $S(x, N)$, it follows that $S(x, N) = \psi(x, N)$ where $\psi(x, y)$ denotes the number of positive integers not exceeding x, composed entirely of primes not exceeding y. Suppose that $x \le N^2$. Then

$$\begin{aligned}\psi(x, N) &= [x] - \sum_{N<p'\le x} [x/p'] \\ &= x\Big(1 - \log \frac{\log x}{\log N}\Big) + O\Big(\frac{x}{\log x}\Big).\end{aligned}$$

In (63) we have $x/p' < N < p'$, so that $S(x/p', p') = [x/p'] = x/p' + O(1)$. On combining these estimates we find that

$$S(x) = x\Big(1 - 2\log \frac{\log x}{\log N}\Big) + x \sum_{N<p'\le x} \frac{1 + \left(\frac{p'}{p}\right)}{p'} + O\Big(\frac{x}{\log x}\Big) \tag{64}$$

for $x \le N^2$. By taking $x = X$ we discover from the above that

$$\sum_{N<p'\le X} \frac{1 + \left(\frac{p'}{p}\right)}{p'} \ll \delta_0 . \tag{65}$$

Thus $\left(\frac{p'}{p}\right) = -1$ for almost all primes $p' \in (N, X]$ in the above sense. Since the summands are nonnegative, the above still holds when the sum is restricted to $N < p' \le x$ for $x \le X$. Thus from (64) we find that

$$S(x) = x\Big(1 - 2\log \frac{\log x}{\log N}\Big) + O(\delta_0 x)$$

uniformly for $N \le x \le X$.

Although we do not use this information later, we remark that by taking $x = N^2$ in (64) we see that

$$1 - \log 2 + \sum_{N<p'\le N^2} \frac{\left(\frac{p'}{p}\right)}{p'} \ll \frac{1}{\log p}.$$

On combining this with (65) we deduce that $\left(\frac{p'}{p}\right) = 1$ for almost all primes $p' \in (X, N^2]$ in the sense that

$$\sum_{X<p'\le N^2} \frac{\left(\frac{p'}{p}\right) - 1}{p'} \ll \delta_0 .$$

Clearly $S(x) = x + O(1)$ for $x \le N$. The range $x \ge X$ is dealt with in (59).

We are now in a position to derive asymptotic estimates for $L(s, (\frac{\cdot}{p}))$ when s is sufficiently near 1, under the hypothesis (62). For $\sigma > 0$ we have

$$L(s, (\tfrac{\cdot}{p})) = s \int_1^\infty S(x) x^{-s-1}\, dx.$$

Since

$$S(x) = \begin{cases} x + O(1) & \text{for } 1 \le x \le N\,, \\ x\big(1 - 2\log \frac{\log x}{\log N}\big) + O(\delta_0 x) & \text{for } N \le x \le p^{1/4}\,, \\ O(\delta_0 x) & \text{for } p^{1/4} \le x \le X\,, \\ O\big(x p^{-\delta^2/2} \log p\big) & \text{for } X \le x \le p\,, \\ O\big(p^{1/2}\log p\big) & \text{for } x \ge p, \end{cases}$$

it follows that

$$\begin{aligned} L(s, (\tfrac{\cdot}{p})) = s \int_1^N x^{-s}\, dx \;+\; &O\Big(|s| \int_1^N x^{-\sigma-1}\, dx\Big) \\ +s \int_N^{p^{1/4}} &\Big(1 - 2\log \frac{\log x}{\log N}\Big) x^{-s}\, dx + O\Big(|s|\delta_0 \int_N^X x^{-\sigma}\, dx\Big) \\ +O\Big(|s| \log p \int_X^p &p^{-\delta^2/2} x^{-\sigma} dx\Big) + O\Big(|s|\, p^{1/2} \log p \int_p^\infty x^{-\sigma-1} dx\Big). \end{aligned} \tag{66}$$

As far as the main terms are concerned, the first integral is easily computed. The second main term integral we integrate by parts, and then we make the substitution $x = p^{u/4}$ to see that the sum of the main terms is $\frac{1}{4} s H(\frac{1}{4}(s-1)\log p) \log p$ where

$$H(z) = \frac{2}{z} \int_{1/\sqrt{e}}^1 \big(1 - e^{-zu}\big) \frac{du}{u} = 2 \sum_{k=0}^\infty \frac{(-1)^k (1 - e^{-(k+1)/2})}{(k+1)!(k+1)} z^k\,. \tag{67}$$

As for the error terms, the first error term is $\ll |s|$ uniformly for $\sigma \ge 1/2$. The second error term is

$$\ll \begin{cases} |s|\delta_0/(\sigma - 1) & \text{for } \sigma \ge 1 + 1/\log p\,, \\ |s|\delta_0 \log p & \text{for } 1 - 1/\log p \le \sigma \le 1 + 1/\log p\,, \\ |s|\delta_0 X^{1-\sigma}/(1-\sigma) & \text{for } 1/2 \le \sigma \le 1 - 1/\log p\,. \end{cases}$$

In the third error term we make the change of variables $x = p^{1/4+\delta}$, so that $dx/x = (\log p)\, d\delta$. Thus the third error term is

$$|s|(\log p)^2 \int_{2\delta_0}^{3/4} p^{-\delta^2/2} p^{(1/4+\delta)(1-\sigma)}\, d\delta\,.$$

Since $\int_c^\infty e^{-au^2}\,du \asymp e^{-ac^2}/(ac)$ if $c \ge 1/\sqrt{a}$, it follows that the third error term is

$$\ll |s|(\log p)^{-5/2}X^{1-\sigma} \qquad \text{for } \sigma \ge 1-\delta_0\,.$$

Finally, the fourth error term is $\ll |s|p^{-1/2+\varepsilon}$. The second error term is largest. On combining our estimates we find that

$$\begin{aligned} L\big(s,(\tfrac{\cdot}{p})\big) &= \tfrac{1}{4}sH\big(\tfrac{1}{4}(s-1)\log p\big)\log p \\ &\quad + O\Big(\delta_0\big(1+X^{1-\sigma}\big)\min\big(\log p, \frac{1}{|\sigma-1|}\big)\Big) \end{aligned} \tag{68}$$

for $1/2 \le \sigma \le 2$, $|t| \le 1$.

In order to understand how the error term above compares with the main term, we must examine $H(z)$ more closely. We first note that if $x > 0$, then $e^{-xu} < 1$, so that the real part of the integrand in (67) is positive, and thus $H(z)$ has no zero in the half-plane $x > 0$. By the same reasoning, $\Re zH(z) > 0$ when $z = iy$, $y \ne 0$. Finally, $H(0) = 2(1-1/\sqrt{e}) \ne 0$. As for asymptotics, by integration by parts we find that

$$H(z) = \frac{1}{z} + \frac{2e^{-z}}{z^2} - \frac{2e^{1/2-z/\sqrt{e}}}{z^2} + \frac{2}{z^2}\int_{1/\sqrt{e}}^1 \frac{e^{-zu}}{u^2}\,du\,.$$

Thus for $|z| \ge 1$,

$$H(z) = \frac{1}{z} + \frac{2e^{-z}}{z^2} - \frac{2e^{1/2-z/\sqrt{e}}}{z^2} + O\Big(\frac{e^{-x}}{|z|^3}\Big) + O\Big(\frac{e^{-x/\sqrt{e}}}{|z|^3}\Big).$$

In particular if $x \ge 0$, then $|H(z)| \asymp 1/|z|$ when $|z| \ge 1$. When $x < 0$,

$$H(z) \ll \frac{1}{|z|} + \frac{e^{-x}}{|z|^2}\,.$$

In general, $|H(z)|$ is comparable to the larger of these two terms, but if these terms have the same size, then they may cancel and $|H(z)|$ may be smaller. It is under these circumstances that $H(z)$ has zeros.

Let $z'_j = x'_j + iy'_j$ denote the zeros of $H(z)$ with y'_j in increasing order. The first zeros of $H(z)$ are given in Table 2. A contour plot of $zH(z)$, which resembles $G(z/2)$ is given in Figure 3.

The main term in (68) has order of magnitude at least as great as the error term when s lies in the rectangle

$$1 - \frac{1}{\log p} \le \sigma \le 1 + \frac{1}{\sqrt{(\log p)\log\log p}}\,,\; |t| \le \frac{1}{\sqrt{(\log p)\log\log p}}\,.$$

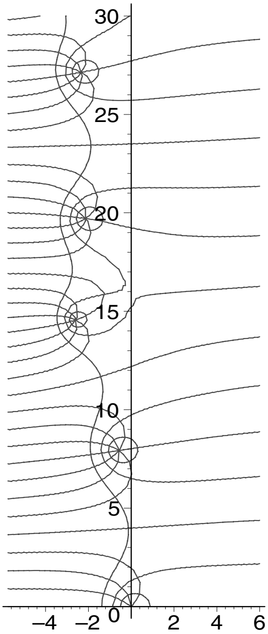

Fig. 3. Contour plot of the modulus and phase of $zH(z)$

Table 2. Zeros of $H(z)$

$z'_1 = -0.5651 + 7.9302i$	$z'_{11} = -3.2112 + 70.4879i$
$z'_2 = -2.5773 + 14.5727i$	$z'_{12} = -3.8108 + 77.3009i$
$z'_3 = -2.1164 + 19.7226i$	$z'_{13} = -3.8001 + 82.8398i$
$z'_4 = -2.3111 + 27.1456i$	$z'_{14} = -3.8175 + 89.7209i$
$z'_5 = -3.3218 + 32.6819i$	$z'_{15} = -4.2417 + 95.7863i$
$z'_6 = -2.5064 + 39.1175i$	$z'_{16} = -3.6587 + 101.8711i$
$z'_7 = -3.4176 + 45.9090i$	$z'_{17} = -4.0638 + 108.6965i$
$z'_8 = -3.2368 + 51.3297i$	$z'_{18} = -4.1826 + 114.3271i$
$z'_9 = -3.0899 + 58.3976i$	$z'_{19} = -3.8171 + 121.0647i$
$z'_{10} = -3.8990 + 64.2680i$	$z'_{20} = -4.4777 + 127.2776i$

The same assertion applies also for s in the rectangle

$$1 - \frac{1}{\sqrt{(\log p)\log\log p}} \le \sigma \le 1 - \frac{1}{\log p}\,,\ |t| \le \frac{(\log\log p)^{1/4}}{(\log p)^{3/4}}$$

with the exception of those s for which $\frac{1}{4}(s-1)\log p$ is near one of the zeros z'_j of $H(z)$. Thus by Rouché's theorem, if

$$0 < |j| \le c(\log p)^{1/4}(\log\log p)^{1/4}$$

where c is a suitably small positive absolute constant, then $L(s, (\frac{\cdot}{p}))$ has a zero ρ_j such that

$$\rho_j = 1 + \frac{4z_j'}{\log p} + O\Big(\frac{j^2\,(\log\log p)^{1/2}}{\log(2|j|)\,(\log p)^{3/2}}\Big).$$

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