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The Julia Set of Hénon Maps*

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Abstract. In this paper we investigate the support of the unique measure of maximal entropy of complex Hénon maps, J^*. The main question is whether this set is the same as the analogue of the Julia set J.

Key words. Complex Dynamics, Julia sets, Hyperbolic Hénon maps

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1. Introduction

Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be an automorphism of the form $H(z, w) = (P(z) + aw, bz)$ where $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree d at least 2 and a, b are nonzero complex constants. More generally, we define a (complex) Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ as $H = H_1 \circ \cdots \circ H_n$ where each $H_i = (P_i(z) + a_i w, b_i z)$ is of the above form and $n \geq 1$.

Associated to each Hénon map there is a natural invariant measure μ with compact support J^*. There is also a natural notion of Julia set, J. We recall the precise definitions in Section 2. The following is one of the basic questions in the theory of complex dynamics.

Problem. *Is $J = J^*$?*

It was proved in [2] that if H is uniformly hyperbolic when restricted to J then $J = J^*$. This leaves open the following interesting special case of the Problem:

Problem. *If H is uniformly hyperbolic on J^*, is $J = J^*$?*

For motivation, we recall that this question arose naturally in the author's investigation of sustainable complex Hénon maps, see [5] which also contains several references to the dynamics of complex Hénon maps and also includes

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an exposition of background material for this article. The author showed that complex Hénon maps which are uniformly hyperbolic on J are sustainable and that sustainable maps are uniformly hyperbolic on J^*. The last step in the characterization of sustainable complex Hénon maps was to show the equality of J and J^*, however, this used sustainability. We are left with the equivalent problem:

Problem. *If H is uniformly hyperbolic on J^*, is H sustainable?*

We also note in passing that it is an interesting open question whether there is a similar characterization of sustainable real Hénon maps.

Main Theorem. *Let H be a complex Hénon map which is hyperbolic on J^*. If H is not volume preserving, then $J = J^*$.*

In Section 2 we introduce notation and review background results. We prove the Main Theorem in Section 3. The author would like to thank Eric Bedford for valuable comments.

2. Notation and background results

We recall some standard notation for Hénon maps which can be found in many sources, see for example [5]. Let H be a complex Hénon map. We denote by H^n the n- fold composition $H \circ \cdots \circ H$ for any positive integer $n \geq 1$. If $n < 0$ we write $H^n := (H^{-1})^{|n|}$ where H^{-1} denotes the inverse map. Also $H^0 = \mathrm{Id}$. We define the sets of bounded orbits and their boundaries.

$$\begin{aligned} K^+ &:= \{(z, w) \in \mathbb{C}^2; \{H^n(z, w)\}_{n \geq 1} \text{ is a bounded sequence}\} \\ K^- &:= \{(z, w) \in \mathbb{C}^2; \{H^n(z, w)\}_{n \leq -1} \text{ is a bounded sequence}\} \\ K &:= K^+ \cap K^- \\ J^+ &:= \partial K^+ \\ J^- &:= \partial K^- \\ J &:= J^+ \cap J^- \end{aligned}$$

We let d denote the degree of the highest order term in the polynomial mapping H, $d = \Pi_{i=1}\deg(P_i)$. We define the escape functions $G^{\pm} : \mathbb{C}^2 \to \mathbb{R}$ by

$$\begin{aligned} G^+(z, w) &:= \lim_{n\to\infty} \frac{\log^+ \|H^n(z,w)\|}{d^n} \\ G^-(z, w) &:= \lim_{n\to-\infty} \frac{\log^+ \|H^n(z,w)\|}{d^n} \end{aligned}$$

We remark that the sets $K^{\pm}$ are closed and the functions $G^{\pm}$ are continuous and plurisubharmonic. Moreover $G^{\pm}$ vanishes on $K^{\pm}$ and is strictly positive and pluriharmonic on $\mathbb{C}^2 \setminus K^{\pm}$.

We define $\mu^{\pm} := dd^c G^{\pm}$ and $\mu := \mu^+ \wedge \mu^-$. The $(1, 1)$ currents $\mu^{\pm}$ are supported on $J^{\pm}$ and hence μ is a positive measure supported inside J. We set J^* equal to the support of μ. So we get immediately that $J^* \subset J$.

We recall the notion of uniform hyperbolicity. Let $F : M \to M$ be a holomorphic automorphism of a complex manifold M of dimension m with a metric d. Suppose that the compact set S, $S \subset M$ is completely invariant, i.e. $F(S) = S$.

We say that F is uniformly hyperbolic on S if there exists a continuous splitting $E^s_x \oplus E^u_x = T_x, x \in S$ of complex subspaces of the tangentspaces. Moreover $F'(E^s_x) = E^s_{F(x)}$ and $F'(E^u_x) = E^u_{F(x)}$. Also there exist constants $0 < c, \lambda < 1$ so that $\|(F^n)'(v)\| \leq \frac{\lambda^n}{c}\|v\|$ for every $v \in E^s_x, x \in S, n \geq 1$ and $\|(F^n)'(v)\| \geq \frac{c}{\lambda^n}\|v\|$ for every $v \in E^u_x, x \in S, n \geq 1$.

For $p \in S$, let $W^s_p := \{x \in M; d(F^n(x), F^n(p)) \to 0, n \to \infty\}$ and $W^u_p := \{x \in M; d(F^n(x), F^n(p)) \to 0, n \to -\infty\}$. Then W^s_p and W^u_p are immersed complex manifolds of the same dimension as E^s_p and E^u_p respectively. Similarly, we define $W^s_S := \{x \in M; d(F^n(x), S) \to 0, n \to \infty\}$ and $W^u_S := \{x \in M; d(F^n(x), S) \to 0, n \to -\infty\}$, the stable and unstable sets of S.

Lemma 1. *[3] Let $\tilde{M}$ be a one dimensional complex manifold in $\mathbb{C}^2$. Let $M \subset\subset \tilde{M}$ be smoothly bounded. If $\mu^+_{|\tilde{M}}(\partial M) = 0$, then*

$$\frac{1}{d^n} H^n_*([M]) \to c\mu^-, c = \mu^+_{|\tilde{M}}(M).$$

Using this Lemma and the arguments of Lemma 9.1 and Theorem 9.6 in [1], it follows that for some large n, the manifold $H^n(M)$ must intersect the stable manifold of any given periodic saddle point in J^* transversely in any given neighborhood of J^*. Hence the same is true for M. [There might also be some tangential intersections.] So we obtain:

Proposition 1. *Suppose that H is a complex Hénon map. Let $\tilde{M}$ be a one dimensional complex submanifold of $\mathbb{C}^2$ and let $M \subset\subset \tilde{M}$ such that $\mu^+_{|\tilde{M}}(M) > 0$. Let $U \supset J^*$ be some neighborhood. Then for any saddle point p there are arbitrarily large integers so that the manifold $H^n(M)$ intersects $W^s(p)$ transversally in U.*

Proposition 2. *Suppose that H is a complex Hénon map. Let $p \in J \setminus J^*$. Then there are arbitrarily small discs $M \subset \tilde{M}$ through p on which $\mu^+_{|\tilde{M}}(M) > 0$. For every transverse intersection x near p between M and a stable manifold of a saddle point, the function G^- vanishes identically in a neighborhood of the intersection in the stable manifold provided $G^-(x) = 0$, so in particular if G^- vanishes identically on M, which happens if M is an unstable manifold.*

Proof. If not, then the local piece of stable manifold must contain a transverse intersection y with the unstable manifold of some other saddle point. But then the α and ω limit sets of y are periodic saddles. This implies that y is a limit of saddle points, so must be in J^*, a contradiction. □

It is convenient to state the same results for the inverse map.

Proposition 3. *Suppose that H is a complex Hénon map. Let $\tilde{M}$ be a one dimensional complex submanifold of $\mathbb{C}^2$ and let $M \subset\subset \tilde{M}$ such that $\mu^-_{|\tilde{M}}(M) > 0$. Let $U \supset J^*$ be some neighborhood. Then for any saddle point p there are arbitrarily large integers n so that the manifold $H^{-n}(M)$ intersects $W^u(p)$ transversally in U.*

Proposition 4. *Suppose that H is a complex Hénon map. Let $p \in J \setminus J^*$. Then there are arbitrarily small discs $M \subset \tilde{M}$ through p on which $\mu^-_{|\tilde{M}}(M) > 0$. For every transverse intersection y near p between M and an unstable manifold of a saddle point, the function G^+ vanishes identically in a neighborhood of the intersection in the unstable manifold, provided that $G^+(y) = 0$, so in particular if G^+ vanishes identically on M, which happens if M is a stable manifold.*

Theorem 1. *Let H be any complex Hénon map. Then J^* is equal to the closure of the collection of periodic saddle orbits.*

This is in [1].

3. Proof of the Main Theorem

We start by recalling a few standard lemmas which can for example be found in [5].

Lemma 2. *If H is uniformly hyperbolic on J^*, then $W^s_{J^*} = \cup_{p \in J^*} W^s_p$.*

This is [5], Lemma 4.15.

Lemma 3. *Suppose H is uniformly hyperbolic on J^* and that $p \in W^s_{J^*} \cap W^u_{J^*}$. Then $p \in J^*$.*

This is [5], Lemma 4.17.

Lemma 4. *If H is uniformly hyperbolic on J^*, then J^* has local product structure.*

This is [5], Lemma 4.14.

We investigate some consequences of assuming that $J \setminus J^*$ is nonempty. We give first some notation. Let $p \in J^*$ and let $\Phi^s_p : \mathbb{C} \to \mathbb{C}^2$ be a parametrization of the stable manifold W^s_p of p, $\Phi^s_p(0) = p$. This exists a.e. $d\mu$ on J^* (Oseledec regular points) and if H is uniformly hyperbolic on J^*, this exists for all $p \in J^*$. Also, let $J^*_{p,s} := (\Phi^s_p)^{-1}(J^*)$.

For the unstable manifold we use the analogous notation, q, Ψ^u_q, $J^*_{q,u}$.

We next define a notion describing how $J \setminus J^*$ might be attached to J^*. Namely we give a name to points whose stable manifolds or unstable manifolds belong (partly) to $J \setminus J^*$. We will say that a point in J^* is stably exposed (to $J \setminus J^*$) if its stable manifold enters into $J \setminus J^*$ and similarly for unstably exposed. More precisely:

Definition 1. *We say that p is stably exposed if $0 \in \mathbb{C}$ is a boundary point of a simply connected open set $V_{s,p} \subset \mathbb{C}$ such that $\partial V_{s,p} \subset J^*_{p,s}$ and $\Phi^s_p(V_{s,p}) \subset J \setminus J^*$. If we instead have $\Phi^s_p(V_{s,p}) \subset K \setminus J^*$, we say that p is weakly stably exposed.*

We define unstably exposed similarly, using the notation q, $W_{u,q}$.

Remark 1. *When the Jacobian of H has modulus 1, W^s_p, W^u_q cannot intersect the interiors of $K^{\pm}$ so weakly (un)stable is equivalent to (un)stable. If the modulus is $\neq 1$, either K^+ or K^- has empty interior.*

Theorem 2. *Suppose that H is a complex Hénon map which is uniformly hyperbolic on J^*. If $J \setminus J^*$ is nonempty, then there are points $p, q \in J^*$ so that p is weakly stably exposed and q is weakly unstably exposed.*

Definition 2. *Suppose that H is hyperbolic on J^*. Let $p \in J^*$ and let $\Phi^s_p : \mathbb{C} \to W^s_p$, $\Psi^u_p : \mathbb{C} \to W^u_p$ parametrize the stable and unstable manifolds of p, $\Phi^s_p(0) = \Psi^u_p(0) = p$. Suppose that $0 \in \mathbb{C}$ is a boundary point of simply connected components $U \subset \mathbb{C} \setminus J^*_{p,s}$ and $V \subset \mathbb{C} \setminus J^*_{p,u}$. In this case, because of local product structure, J^* is locally contained on one side of each of the laminated local hypersurfaces which are the local unstable set of $\Phi^s_p(\partial U)$ and the local stable set of $\Psi^u_p(\partial V)$ respectively. We say in this case that p is a distinguished boundary point of J^*.*

Remark 2. *Heuristically one can think of the local stable set as $\{(z,w) \in \mathbb{C}^2; |w| = 1\}$ and the local unstable set as $\{(z,w) \in \mathbb{C}^2; |z| = 1\}$ and that J^* contains the distinguished boundary of the bidisc and J^* is contained in the bidisc.*

Theorem 3. *Suppose that H is a complex Hénon map which is uniformly hyperbolic on J^*. If $J \setminus J^*$ is nonempty, then distinguished boundary points are dense in J^*.*

Proposition 5. *Suppose that H is uniformly hyperbolic on J^*. Let $p \in J^*$ and let $\Phi^s_p : \mathbb{C} \to \mathbb{C}^2$ parametrize the stable manifold of p, $\Phi^s_p(0) = p$. Fix a nonempty connected component V of $\mathbb{C} \setminus J^*_{p,s}$. The function $G^+ \circ \Phi^s_p \equiv 0$ on $\mathbb{C}$. On the other hand $(G^- \circ \Phi^s_p)_{|V}$ is either identically zero or strictly positive.*

Proof. The first part is obvious. Suppose there is a point $z \in V$ so that $G^- \circ \Phi^s_p(z) = 0$ but that $G^- \circ \Phi^s_p$ is not identically zero on V. Choosing another z if necessary we can assume that there are discs $z \in \Delta \subset\subset \tilde{\Delta} \subset\subset V$ so that $G^- \circ \Phi^s_p(z) = 0$ but that $G^- \circ \Phi^s_p$ is not vanishing identically on Δ. Then the nonnegative function $G^- \circ \Phi^s_p$ cannot be harmonic on Δ. Let $M := \Phi^s_p(\Delta)$. Then $\mu^-_{|M}$ has mass on M. Let U be a neighborhood of J^* and let $w \in J^*$ be a saddle point. By Proposition 3 there exists an integer $n >> 1$ so that $H^{-n}(M)$ intersects the unstable manifold of w transversally in U. Let x be such an intersection point. Then $x \in W^s_p \cap W^u_w$. By Lemma 3 this implies that $x \in J^*$. Hence $H^n(x) \in J^*$. This contradicts that $(\Phi^s_p)^{-1}(H^n(x)) \in \Delta \subset \mathbb{C} \setminus (\Phi^s_p)^{-1}(J^*)$.

Without the hypothesis of uniform hyperbolicity, the proof still works if we assume that p is a periodic saddle point. Hence:

Proposition 6. *Let H be a complex Hénon map. Let p be a periodic saddle point. Suppose V is a connected component of $\mathbb{C} \setminus J^*_{p,s}$. Then $G^-_{|V}$ either vanishes identically or is strictly positive.*

Proof. Suppose $x \in V$, $\Delta \subset V$, $x \in \Delta$ and $G^-(x) = 0$ while G^- does not vanish identically on Δ. Then there is a $y \in \Delta$ so that y is a transverse intersection with an unstable manifold of a periodic saddle point. But then $y \in J^*$, a contradiction. □

The following result is obvious.

Lemma 5. *Suppose that H is uniformly hyperbolic on J^*. Let $p, q \in J^*$. Suppose that $q \in W^u_p$. Then $W^u_q = W^u_p$. Likewise if $q \in W^s_p$, then $W^s_q = W^s_p$.*

We can now prove Theorem 2.

Proof. Suppose that for every $p \in J^*$, G^- is not vanishing on any open subset of $W^s(p)$. Then there are open subsets $J^* \subset U \subset\subset V$ so that $H(V) \supset\supset U$, $J \setminus \overline{V} \neq \emptyset$ and $G^- > c > 0$ on $W^s_{loc}(J^*) \cap (V \setminus U)$. Pick $x \in J \setminus \overline{V}$. Then $x \in J^+$ and hence, if y is any point in J^*, then there is a sequence $\{y_n\} \subset W^s(y)$, $y_n \to x$. In particular, $G^-(y_n) > c$ for all large n, so $G^-(x) \geq c$, a contradiction since $G^- \equiv 0$ on J.

Hence, by Proposition 6, there exists a $p \in J^*$ and a nonempty connected component $V \subset \mathbb{C}$ of $\mathbb{C} \setminus J^*_{p,s}$ on which $G^- \equiv 0$. Then V is simply connected by Lemma 5: In fact, if V contains a simple closed curve γ whose interior U contains $x \in J^*$, then $G^- \equiv 0$ on U by the maximum principle. Hence by the invariance of G^- it follows that $G^- \equiv 0$ on $H^{-n}(U)$ for all $n \geq 1$. This contradicts that $H^{-n}(U)$ has to be an unbounded sequence clustering all over J^+. It follows that there is a weakly stably exposed point in J^*. Similarly we can find a point $q \in J^*$ which is weakly unstably exposed. □

Next we prove the Theorem 3.

Proof. Pick a weakly stably exposed point $p \in J^*$. Then in a neighborhood of p, the unstable lamination contains a laminated hypersurface with J^* on one side. Let D denote a small unstable disc centered at p. The forward iterates of this disc become dense in J^*. Moreover, for every point $q \in H^n(D) \cap J^*$ there is a connected component $V_{s,q} \subset \mathbb{C} \setminus J^*_{q,s}$ obtained by following $V_{s,p}$ using the local product structure of J^*. Furthermore, by the local product structure each such $V_{s,q}$ is simply connected. Hence, arbitrarily close to any point in J^* there is a laminated hypersurface whis is contained in the local unstable set and with J^* on one side. One can do the same with a weakly unstably exposed point in J^*. Their intersection points give rise to a dense collection of distinguished boundary points. □

We prove the Main Theorem.

Proof. We assume that $J \neq J^*$. The hypothesis that H is not volume preserving is only used at the end. As in the above proof, for every point $z \in H^n(D) \cap J^*$ there is a simply connected component of $W^s_z \setminus J^*$. Using the local product structure of J^* for a complex Hénon map which is hyperbolic on J^*, we get that in fact for every point $z \in J^*$, there is a simply connected component of $W^s_z \setminus J^*$. Also all connected components on which G^- has a zero are simply connected. By [4], if G^- is not identically zero on some simply connected component, then H is stably connected. In particular all connected components of all $W^s_x \setminus J^*$ for all $x \in J^*$ are simply connected and all $W^s_x \setminus J^*$ contain at least one component where $G^+ > 0$ and at least one has a component where $G^- = 0$. The same applies to the unstable set. We prove next two lemmas needed to continue the proof of the Main Theorem. □

Lemma 6. *Suppose that H is a complex Hénon map which is hyperbolic on J^*. Suppose that $J \neq J^*$. If H is not stably connected, then each stable manifold $W^s_z \setminus J^*$ considered as parametrized by $\mathbb{C}$ contains simply connected components in $\mathbb{C}$ on which $G^- \equiv 0$ and all such components are bounded in $\mathbb{C}$.*

Proof. Each $W^s_z \setminus J^*$ contains connected components on which $G^- > 0$. However none of these are simply connected, hence they contain curves γ surrounding a nonempty part of J^*. Any simply connected component of some $W^s_x \setminus J^*$ can be followed to any other $W^s_z \setminus J^*$ using the local product structure and in particular can be followed into the interior of such a γ. However, unbounded components stay unbounded when they are followed in this way. This shows that all simply connected components are bounded. □

Lemma 7. *Let H be a complex Hénon map which is hyperbolic on J^* and suppose that $J \neq J^*$. Then H is stably connected and unstably connected.*

Proof. We show that H is stably connected. Unstable connectedness is proved similarly. Suppose that H is not stably connected. We pick a point $z \in J^*$ and a bounded simply connected component V of $W^s_z \setminus J^*$. After changing z if necessary we can pick a curve $\lambda = \phi(t), t \in [0, 1]$ with $\lambda(t) \in V, t > 0$ and $\lambda(0) = z$. We may assume that $V \subset W^s_{\epsilon,loc}(J^*)$ for some small $\epsilon > 0$ after forward iteration if necessary. Next, consider the backward iterates, $V_n := H^{-n}(V), \gamma_n = H^{-n}(\gamma), z_n = H^{-n}(z)$. For $n \geq n_0$ there is a unique point $w_n = H^{-n}(\phi(t_n)) \in \gamma_n$ so that $w_n \in \partial W^s_{\epsilon,loc}(J^*)$ and the curve γ_n is contained in $W^s_{\epsilon,loc}(J^*)$ for $0 \leq t < t_n$.

Let w_0 be a cluster point of the sequence $\{w_n\}$. Then w_0 is in some W^s_η, $\eta \in J^*$ and belongs to some simply connected component U of $W^s_\eta \setminus J^*$ since $G^-(\eta) = 0$ by continuity. Since this component is bounded, we can after replacing with a forward iterate, assume that U is contained in the local stable manifold of η. Using the local product structure it follows that the diameter of V is arbitrarily small, a contradiction. □

Using [4] Theorem 5.1 and Corollary 7.4 we get:

Theorem 4. *Suppose that H is a complex Hénon map which is hyperbolic on J^*. Moreover assume that $J \setminus J^* \neq \emptyset$. Then H is stably and unstably connected. Morever J is connected and H is volume preserving.*

The Main Theorem is now an immediate consequence. □

Remark 3. *In the case when the complex Hénon map is hyperbolic on J^*, $J \neq J^*$ and H is volume preserving, we can say a little more about the connected components of $W_p^s \setminus J^*$ and similarly for the unstable manifolds for $p \in J^*$. Namely all connected components are simply connected and unbounded in parameter space $\mathbb{C}$.*

Proof. We only need to show that there cannot be a bounded connected component U of some $W_p^s \setminus J^*$. After forward iterations we can assume that U is in the local stable set of p. If there is such a U, consider the local unstable set of ∂U. On this set $G^- \equiv 0$. Hence by slicing with parallel copies of the local stable set of p, we get from the maximum principle that $G^- \equiv 0$ on the inside of the local stable set. But this is an open set in $\mathbb{C}^2$. Hence K^- has nonempty interior. We know that the interior of K^- and K^+ agrees in the volume preserving case. Hence $G^+ \equiv 0$ there also. Hence by continuity G^+ vanishes on a neighborhood of q in the unstable set q for each $q \in \partial U \subset J^*$. This implies that G^+ vanishes identically on forward iterates of this neighborhood. However, these are unbounded and escape out of K^+, a contradiction. □

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