On Finite Affine Planes of Rank 3

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1. Introduction. LÜNEBURG has proposed the problem of determining the finite affine planes admitting rank 3 groups of collineations — i.e., groups of collineations which act as rank 3 permutation groups on the points — observing that the members of his new family of planes [3], Section 12, have this property. In the present note we take the first step towards a solution of this problem by proving the following

Theorem. If a finite affine plane $\Pi$ admits a rank 3 group $G$ of collineations then it is a translation plane and $G$ contains the group $T$ of all translations.

As a consequence we prove

Corollary. If $\Pi$ admits a rank 3 group $G$ of collineations such that $G$ induces a regular group of permutations of the points at infinity then $\Pi$ is a desarguesian plane of odd order.

The proof of the Theorem is reduced to an application of a theorem of WAGNER [4]. We conjecture that if a rank 3 group induces a doubly transitive group on the points at infinity then the plane is a Lüneburg plane.

Throughout the paper $\Pi$ denotes a finite affine plane of order $n$ and $G$ denotes a rank 3 group of collineations of $\Pi$. When convenient we regard $\Pi$ as being obtained from a projective plane $\Pi'$ by specializing a line $l_\infty$ to be the line at infinity, and then we regard $G$ as a group of collineations of $\Pi'$. But the terms “point” and “line” will always refer to the points and lines of $\Pi$, the points of $l_\infty$ being referred to as “points at infinity”.

2. Proof of the Theorem. We use the notation of [1] for rank 3 permutation groups, denoting by $\Delta(P)$ and $\Gamma(P)$ the two $G_p$-orbits $\neq \{P\}$ of points, for a given point $P$, with $\Delta(P)^g=\Delta(P^g)$ and $\Gamma(P)^g=\Gamma(P^g)$ for all $g\in G$. By the theorem of WAGNER [4] it suffices to prove that $G$ is flag-transitive, which we now do.

If $G$ has odd order then for $P$ a point, $|\Delta(P)|=|\Gamma(P)|=(n^2-1)/2$, and this number divides the order of $G$ and so must be an odd integer, which is impossible. Hence the order of $G$ is even and therefore $\Delta$ and $\Gamma$ are self-paired.

Now suppose that no line through $P$ meets both $\Delta(P)$ and $\Gamma(P)$. Then the lines through $P$ fall into two classes, $\Delta_P$ and $\Gamma_P$, such that

$$X\in \Delta(P) \iff XP\in \Delta_P,$$

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and

\[ Y \in \Gamma(P) \iff YP \in \Gamma_P. \]

If \(|A_p| = t\) then \(|\Delta(P)| = t(n-1), \quad |\Gamma_P| = n-t+1\) and \(|\Gamma(P)| = (n-t+1)(n-1)\). Now

\[ XP \in A_P \iff X \in A(\Delta(P)) \iff P \in A(X) \iff XP \in A_X \]

and hence

\[ \lambda = |\Delta(P) \cap \Delta(X)| = (t-1)^2 + n-2. \]

Similarly, if \(Y \in \Gamma(P)\) then \(YP\) is in neither \(A_P\) nor \(A_Y\) so \(|\mu| = |\Delta(P) \cap \Delta(Y)| = t^2\).

By [1], Lemma 5, we have

\[ (t-1)(n-t) = t^2(n-t+1)(n-1) \]

giving \(n = 0\), which is impossible. Hence there is a line \(l\) through \(P\) which meets both \(\Delta(P)\) and \(\Gamma(P)\), and it follows that \(G_P\) is transitive on the set of lines through \(P\). Hence \(G\) is flag-transitive, completing the proof of the Theorem.

3. Proof of the Corollary. Some preparation is needed before we introduce the assumption about the action of \(G\) on the points at infinity. By the Theorem, \(\Pi\) is a translation plane and \(G\) contains the group \(T\) of all translations of \(\Pi\). So we must have \(n = q\), a prime power, and \(G = TG_P\) for any point \(P\).

Let \(P, l\) be an incident point-line pair in \(\Pi\) and let \(P_\infty\) be the point on \(l_\infty\) incident with \(l\). Then \(G_{P_{\infty}} \geq T\) so \(G_{P_{\infty}} = TG_{P_{\infty}}, T\), and \(G_t = T_{l}G_{P_{\infty}, t}\).

Now we determine the \(G_l\)-orbits of lines in \(\Pi\). Since \(G_P\) has four point orbits in \(\Pi'\) it has four line orbits in \(\Pi'\) and therefore three line orbits in \(\Pi\). Hence \(G_t\) has three point orbits in \(\Pi\). Let \(\rho\) be the number of \(G_l\)-orbits of points in \(l_\infty\). Then \(G_t\) has \(3 + \rho\) point orbits in \(\Pi'\) and hence \(2 + \rho\) line orbits in \(\Pi\). On the other hand, \(G_t\) stabilizes the following \(1 + \rho\) sets of lines in \(\Pi\):

\[ \{l\}, \text{ the set of lines } \not\parallel l \text{ parallel to } l, \text{ and the } \rho-1 \text{ sets of lines meeting } l_\infty \text{ in the } \rho-1 \text{ point orbits } \{P_{\infty}\} \text{ of } G \text{ in } l_\infty. \]

Each \(G_l\)-orbit of lines \(\not\parallel l\) through \(P_{\infty}\) is a \(G_{P_{\infty}, l}\)-orbit since \(G_t = T_{l}G_{P_{\infty}, l}\) and \(T_{l}\) fixes every line through \(P_{\infty}\). Suppose that there is just one such orbit. Then every line \(\not\parallel l\) parallel to \(l\) carries the same number, say \(a\), of points of \(\Delta(P)\). Hence

\[ |\Delta(P)| = (q-1)a + b \]

where \(b = |l \cap \Delta(P)|\). On the other hand, since each line through \(P\) meets \(\Delta(P)\) in \(b\) points,

\[ |\Delta(P)| = (q+1)b \]

so \(b = q-1\), which is impossible. It follows that the \(G_l\)-orbits of lines in \(\Pi\) are the listed sets, except that the set of lines \(\not\parallel l\) parallel to \(l\) is the union of two orbits \(A(l)\) and \(B(l)\). As we remarked above, \(A(l)\) and \(B(l)\) are \(G_{P_{\infty}, l}\)-orbits.

Now assume that \(G\) induces a regular group of permutations on \(l_\infty\). Then the same is true of \(G_P\), and the elements \(\not\parallel l\) of \(G_{P_{\infty}, l}\) are \((P, l_\infty)\)-homologies. Hence for \(m \in A(l)\) and \(n \in B(l)\),

\[ G_{P_{\infty}, l, m} = G_{P_{\infty}, l, n} = 1, \quad so \quad |A(l)| = |B(l)| = G_{P_{\infty}, l}, \]
and therefore
\[ |G_{P, l}| = (q - 1)/2. \]

The existence of a group of \((P, l, o):\) homologies of this order implies that \(\Pi\)

is desarguesian (cf. [3], Section 11), proving the Corollary.

**4. Remarks.** (1) In the case of the Corollary it can be seen that \(G_P\) is cyclic

of order \((q^2 - 1)/2.\)

(2) Since \(G\) is flag-transitive by the Theorem, it is primitive on the points

of \(\Pi\) by [2], Proposition 3, and \(T\) is its unique minimal normal subgroup.

(3) It is easily seen that a group of collineations of a finite affine plane \(\Pi\)

has rank 3 on the lines of \(\Pi\) if and only if it is doubly transitive on the points

\(\Pi\) and on the points at infinity.

**References**

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