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A Note on a Problem of D. Pompeiu

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1. Introduction

As Pompeiu's problem – named after the late Rumanian mathematician Dimitrie Pompeiu – we shall denote the following question. Let B be a bounded region of the xy-plane and let T be the set of the Euclidean transformations of the plane. Suppose that $f(x, y)$ is a function of the real variables x and y, continuous in the whole xy-plane and satisfying

$$\iint_{\tau(B)} f(x,y)\,dx\,dy=0, \qquad \tau\in T_1 \tag{1}$$

for some subset $T_1 \subset T$. Does this imply that $f(x, y)\equiv 0$? Pompeiu thought that if B is a disc and if $T_1 = T$, then $f(x, y)\equiv 0$ [6]. It was noted later that the result is not correct in this form. The function $f(x, y)=\sin ax$, for a suitable choice of a, provides a counter example. This example shows that even if one assumes that f is bounded, the conclusion $f=0$ need not hold.

Pompeiu proved in [7] the following result: if B denotes a square and if $f(x, y)$ is a function of the real variables x and y, continuous in the whole xy-plane and having a limit as $x^2+y^2\to\infty$, then (1) holds for all $\tau\in T$ if, and only if, $f(x, y)\equiv 0$. The Bulgarian mathematician Christo Christov established in [1] and [2] Pompeiu's result (and similar results if B is a triangle or a parallelogram) without the condition of the existence of $\lim f(x, y)$. See [3] for further references.

It is the purpose of this note to present a result which can be considered as a contribution to the study of Pompeiu's problem. Our result is more special than Christov's in that we assume that $f(x, y)$ has a limit as $x^2+y^2\to\infty$. On the other hand our result is more general in that we use only the translations rather than all Euclidean transformations. Also, we replace the Lebesque area measure $dx\,dy$ on a square by any product measure $d\mu(x)\,d\nu(y)$ where μ, ν are arbitrary complex-valued measures of compact support. The method applied uses a few facts from the theory of mean periodic functions in one variable as, for example, presented in the lecture notes by J. P. Kahane [4].

2. Background Material

A complex-valued continuous function f defined on the real line is said to be *mean periodic* if there is a complex-valued measure μ, not identically zero, of

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compact support, such that

$$\int f(x+y)\,d\mu(y)=0 \qquad (-\infty<x<\infty).$$

Next, we shall define *almost periodic* functions (in the sense of BOHR). We say, given any $\varepsilon>0$, that the number $\tau>0$ is an almost period corresponding to ε for the complex-valued continuous function $f(x)$ if $|f_\tau(x)-f(x)|<\varepsilon$ for all x, where f_τ is the translate of f: $f_\tau(x)=f(x+\tau)$. A set M of reals will be called relatively dense if there exists an $L>0$ such that in any interval of length L there lies at least one point of M. HARALD BOHR then defined [5]: A function $f(x)$ complex-valued and continuous, is almost periodic if, for every $\varepsilon>0$, the almost periods of f form a relatively dense set.

For the proof of our theorem we need the following facts about mean periodic and almost periodic functions.

Theorem A (KAHANE [4]). *A uniformly continuous, bounded mean periodic function is almost periodic* (*in the sense of* BOHR).

Lemma A. *An almost periodic function $f(x)$ for which $\lim_{x\to\infty} f(x)$ exists is necessarily a constant.*

Theorem A lies quite deep and is difficult to prove; on the other hand, Lemma A is trivial.

3. Main Result

Theorem. *Let μ and ν be arbitrary complex-valued measures of compact support on the real line (with neither of them being the zero measure). Let $f(x,y)$ be continuous in the plane and satisfy*

(1) $$\lim f(x,y) \quad \textit{exists} \quad (x^2+y^2\to\infty),$$

(2) $$\iint f(x+s,y+t)\,d\mu(x)\,d\nu(y)=0, \qquad -\infty<s,t<\infty.$$

Then $f=$ constant. If further

(3) $$\int d\mu\neq 0 \quad \textit{and} \quad \int d\nu\neq 0$$

then $f=0$.

Proof. Let $c=\lim f(x,y)$ $(x^2+y^2\to\infty)$. First note that if (3) does not hold then $f-c$ satisfies (1) and (2). On the other hand, if (3) does hold then $c=0$, since $\iint c\,d\mu\,d\nu=0$ by (2).

Thus in any case we may assume without loss of generality that $c=0$.

For fixed s consider the function $\phi(y)=\int f(s+x,y)\,d\mu(x)$. We have:

$$\int \phi(y+t)\,d\nu(y)=0 \qquad (-\infty<t<\infty)$$

and so $\phi(y)$ is mean periodic. Furthermore, ϕ is continuous and

$$\lim \phi(y)=0 \qquad (|y|\to\infty),$$

and so ϕ is bounded and uniformly continuous. By Theorem A and Lemma A, $\phi=0$.

Thus we have shown that for each fixed s and for all y we have

$$\int f(s+x, y)\, d\mu(x)=0.$$

For fixed y, let $g(x)=f(x, y)$. The above equation tells us that g is mean periodic. Also, $\lim g(x)=0$ ($|x| \to \infty$) and so, just as in the previous paragraph, $g=0$. But this says that for each y the function $f(x, y)=0$ for all x, i.e., $f=0$, which completes the proof.

The fact that we had a product measure enabled us to make use of the theory of mean periodic function in one variable for solving our variant of POMPEIU's problem. The problem of a general measure with compact support in the plane remains open.

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