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Absolutely λ-Summing Operators, λ a Symmetric Sequence Space

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1. Introduction

Pietsch [5] introduced the concept of absolutely summing operators in Banach spaces and later in [6] extended this concept to absolutely p-summing operators. At the background of these concepts are the sequence spaces l^p and their duality theory. The object of the present paper is to extend the above concept to abstract sequence spaces λ. The sequence spaces λ involved are described in Section 2; the absolutely λ-summing operators are studied in Section 3 while Section 4 discusses the interesting special case $\lambda = n(\phi)$, a sequence space which includes for special ϕ the l^1 and l^∞ spaces and was introduced in the literature by Sargent [10].

2. Notations, Definitions and Preliminary Lemmas

Most of the concepts defined here are well-known and are in Garling [2, 3], Köthe [4], Ruckle [9] and Sargent [11]. For a sequence space λ the α-dual is denoted by λ^α. If $\lambda^{\alpha\alpha} = \lambda$ then λ is said to be a perfect sequence space or a Köthe space. The sequence space λ is said to be symmetric if for each $x \in \lambda$ the sequence x_π which is obtained as a rearrangement of the sequence x corresponding to the permutation π of the positive integers is also in λ for each π. Suppose, in addition, that the topology on λ is generated by a norm p and that $p(x) = p(x_\pi)$ for each x and π, then λ is defined to be a K-symmetric sequence space. The symmetric dual λ^σ of λ is defined as the set $\{y: \sum y_i x_{\pi(i)} < \infty$, for each $x \in \lambda$ and each $\pi\}$. It is known that if λ is symmetric then so is λ^α and that $\lambda^\sigma = \lambda^\alpha$; also, if λ is a symmetric Köthe space then $\lambda = \phi$ or $\lambda = \omega$ or $l^1 \subseteq \lambda \subseteq l^\infty$. If λ is a solid symmetric sequence space then either $\lambda \subseteq c_0$ or $\lambda = \omega$ or $\lambda = l^\infty$.

We now start with the sequence space ω of all scalar sequences and suppose there is given an extended seminorm p on ω. We shall then consider the sequence space $\lambda \subset \omega$ which consists of all $x \in \omega$ for which $p(x) < \infty$. Having constructed this space λ we assume that λ is solid and that it is a K-symmetric Köthe space whose topology is given by the seminorm p which is indeed a norm on λ and that this topology is also the Mackey topology of the dual pair $(\lambda, \lambda^\alpha)$ so that

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$\lambda^\alpha = (\lambda, p)'$. We assume that p is absolutely monotone. One final assumption on (λ, p) is that is has the property AK, viz., for each $x = (x_1, x_2, \ldots, x_n, \ldots) \in \lambda$, the sequences $x^i = (x_1, x_2, \ldots, x_i, 0, 0, \ldots)$, $i = 1, 2, 3, \ldots$ converge, in norm, to x. For further details on this see Zeller [12].

We remark now that the space c of convergent sequences, being not solid, is not included as a special case of our sequence spaces λ; also the same is true of the space c_o with its usual norm topology since that space does not comprise of all sequences $x = (x_n)$ with $\sup |x_n| < \infty$. The spaces l^p, $1 \leqq p \leqq \infty$, are certainly included in the type of spaces λ we discuss as are also the spaces $n(\phi)$ introduced by Sargent [10], brief definitions of which are found in Section 4 of this paper. Also the spaces $\mu_{a,p}$ and $v_{a,p}$ of Garling [3] fit into the set up described above.

Next we start with two normed linear spaces $(E, \| \ \|)$ and $(F, \| \ \|)$. We shall denote by $\lambda(E)$ the vector sequences $x = (x_n)$, $x_n \in E$ which are weakly in λ in the sense that for each $a \in E'$, the sequence $(\langle x_n, a \rangle)$ of scalars is in λ and since λ is solid, the sequence $(|\langle x_n, a \rangle|)$ is also in λ. It is easy to verify that $\lambda(E)$ is a vector sequence space.

Suppose $x = (x_n)$ belongs to $\lambda(E)$. Then from a theorem of Pietsch ([8], Hilfssatz, S. 31) it follows that $\sup_{\|a\| \leqq 1} \sum |\alpha_n \langle x_n, a \rangle| \leqq \varphi$ for all $(\alpha_n) \in B_{\lambda^\alpha}$, the unit ball in λ^α with its topology as the dual of λ. Denoting by α the sequence $(\alpha_n) \in B_{\lambda^\alpha}$ and by $\langle x, a \rangle$ the sequence $(\langle x_n, a \rangle)$ we now get

$$\sup_{\|a\| \leqq 1} |\langle \langle x, a \rangle, \alpha \rangle| \leqq \rho \quad \text{for all } (\alpha) \in B_{\lambda^\alpha}.$$

Thus

$$\sup_{\|a\| \leqq 1} \| |\langle x, a \rangle| \|_\lambda < \infty.$$

We shall denote by ε_λ the functional defined on $\lambda(E)$ by $\varepsilon_\lambda(x) = \sup_{\|a\| \leqq 1} \| |\langle x, a \rangle| \|_\lambda$; ε_λ can easily be verified to be a seminorm, thus giving $\lambda(E)$ a natural topology.

The spaces $\lambda(E)$ corresponding to $\lambda = l^p$, $1 \leqq p < \infty$ have been discussed by Pietsch ([5, 7]).

Next we define the space $\lambda[F]$ as the space of all vector sequences $y = (y_n)$, $y_n \in F$, such that the sequence $(\|y_n\|) \in \lambda$. The space $\lambda[F]$ is topologised in a natural way by the norm $p \cdot \| \ \|$, defined by $(p \cdot \| \ \|)(y) = p(\|y_n\|)$, denoted also by $\|(\|y_n\|)\|_\lambda$ or $\|(y_n)\|_{\lambda[F]}$. These spaces have been discussed in case $\lambda = l^p$ by Pietsch (loc. cit.) and in the general case by Gregory [1].

3. Absolutely λ-Summing Operators

Suppose E and F are normed linear spaces and T is a linear map on E into F. The map T is said to be absolutely λ-summing if for each $x = (x_i) \in \lambda(E)$, the sequence $Tx = (Tx_i) \in \lambda[F]$. If $\lambda = l^p$, these are called absolutely p-summing operators; they are discussed extensively by Pietsch [7].

In the following paragraphs we obtain a characterization of the absolutely λ-summing maps and point out some simple properties of such maps. Most of these results may be looked upon as partial generalizations of Pietsch's work for the l^p-spaces to the setup of abstract sequence spaces.

Theorem 1. *The linear map T is absolutely λ-summing if and only if there exists a $\rho>0$ such that for each finite set of elements $x_1, x_2, \ldots, x_k$ in E the following inequality holds:*

$$\|(Tx_i)\|_{\lambda[F]} \leqq \rho \cdot \sup_{\|a\| \leqq 1} \|(|\langle x_i, a\rangle|)\|_\lambda. \tag{1}$$

Remark. The quantity $\|(Tx_i)\|_{\lambda[F]}$ appearing above is to be interpreted as the norm, in the vector sequence space $\lambda[F]$, of the element $(Tx_1, Tx_2, \ldots, Tx_k, 0, 0, \ldots)$ with a similar interpretation for $\|(|\langle x_i, a\rangle|)\|_\lambda$.

Proof. The sufficiency part is easily proved; suppose $x=(x_i)\in\lambda(E)$. Then for each fixed k, we consider $x^k=(x_1, x_2, \ldots, x_k, 0, 0, \ldots)$ and obtain

$$\|(Tx_1, Tx_2, \ldots, Tx_k, 0, 0, \ldots)\|_{\lambda[F]} \leqq \rho \cdot \sup_{\|a\| \leqq 1} \|(|\langle x^k, a\rangle|)\|_\lambda,$$

and since the space λ is solid and the norm generating it is absolutely monotone, the above expression is $\leqq \rho\cdot\varepsilon_\lambda(x)$. Since λ has AK, it follows that $\|(Tx)\|_{\lambda[F]} \leqq \rho\cdot\varepsilon_\lambda(x)$ and the sufficiency is proven.

Conversely, assume that T is absolutely λ-summing and if possible let for each $\rho>0$ the inequality (1) be not true. Then given $\rho>0$, we can obtain a finite system $x_1^\rho, x_2^\rho, \ldots, x_{n(\rho)}^\rho$ such that

$$\sup_{\|a\| \leqq 1} \|(|\langle x_i^\rho, a\rangle|)\|_\lambda \leqq 1 \quad \text{and} \quad \|(Tx_i^\rho)\|_{\lambda[F]} > \rho.$$

We can do this for $\rho=j\,2^j, j=1,2,3,\ldots$ and obtain correspondingly, finite systems $(x^1), (x^2), \ldots, (x^j), \ldots$. From our assumptions it follows that the sequence x of vectors

$$\frac{x_1^1}{2}, \frac{x_2^1}{2}, \ldots, \frac{x_{n(1)}^1}{2}, \frac{x_1^2}{2^2}, \frac{x_2^2}{2^2}, \ldots, \frac{x_{n(2)}^2}{2^2}, \ldots, \frac{x_1^j}{2^j}, \frac{x_2^j}{2^j}, \ldots, \frac{x_{n(j)}^j}{2^j}, \ldots$$

is in $\lambda(E)$; also since the norm defining the topology of is absolutely monotone it follows that $Tx\notin\lambda[F]$. This completes the proof of the theorem.

Elementary Properties of Absolutely λ-Summing Operators

We shall denote by $\pi_\lambda(E,F)$ the space of all absolutely λ-summing maps on E into F, where both the above spaces are assumed normed. We shall denote by $\pi_\lambda(T)$ the smallest positive ρ satisfying (1) of Theorem 1.

We now make an additional assumption on the sequence space λ. The space λ is said to have the norm preservation property (=n.p.) if $x=(x_i)$ is such that $x_i=0$ for all $i\neq n$, then $\|x\|_\lambda=|x_n|$. The property n.p. along with the basic union property implies that $l^1\subset\lambda$.

Theorem 2. *If $T \in \pi_\lambda(E, F)$ then T is continuous and the operator norm $\|T\| \leqq \pi_\lambda(T)$.*

The proof is trivial and is omitted.

Theorem 3. *The space $\pi_\lambda(E, F)$ is a normed linear space with the norm π_λ and is a Banach space if F is a Banach space.*

Proof. We shall omit the proof of π_λ being a norm and of $\pi_\lambda(E, F)$ being a normed linear space. Assuming that F is a Banach space, we shall prove that $\pi_\lambda(E, F)$ is a Banach space. Let $\{T_n\}$ be a Cauchy sequence in $\pi_\lambda(E, F)$. Then given $\varepsilon > 0$, the inequality $\|T_n - T_m\| \leqq \pi_\lambda(T_n - T_m) < \varepsilon$ holds for $n, m > N$. Thus $\{T_n\}$ is a Cauchy sequence in the space $\mathscr{L}(E, F)$ and therefore there is a $T \in \mathscr{L}(E, F)$ such that $\lim \|T - T_n\| = 0$. Since $\pi_\lambda(T_n - T_m) < \varepsilon$ for $n, m > N$ we get, for $n, m > N$ and for each finite system $x_1, x_2, \ldots, x_k$ in E,

$$\|(T_n x_i - T_m x_i)\|_{\lambda[F]} \leqq \varepsilon \sup_{\|a\| \leqq 1} \|(|\langle x_i, a\rangle|)\|_\lambda.$$

Letting $m \to \infty$ and using continuity of norms we get that $\pi_\lambda(T_n - T) < \varepsilon$ for $n > N$.

Theorem 4. *$\pi_\lambda(E, E)$ is a two sided ideal in $\mathscr{L}(E, E)$ and for $S \in \pi_\lambda(E, E)$ and $T \in \mathscr{L}(E, E)$*

$$\pi_\lambda(ST) \leqq \pi_\lambda(S) \cdot \|T\| \quad \textit{and} \quad \pi_\lambda(TS) \leqq \|T\| \cdot \pi_\lambda(S).$$

The above result is in fact a particular case of the following apparently more general result.

Theorem 4'. (a) *If $S \in \mathscr{L}(E, F)$ and $T \in \pi_\lambda(F, G)$ then $TS \in \pi_\lambda(E, G)$ and $\pi_\lambda(TS) \leqq \pi_\lambda(T) \cdot \|S\|$;*

(b) *If $S \in \pi_\lambda(E, F)$ and $T \in \mathscr{L}(F, G)$ then $TS \in \pi_\lambda(E, G)$ and $\pi_\lambda(TS) \leqq \|T\| \cdot \pi_\lambda(S)$.*

The proofs are simple.

4. Special Case $\lambda = n(\phi)$

Pietsch ([6, 7]) has shown that there exist non-absolutely p-summing operators whose adjoints are absolutely p-summing. In this section we shall prove a similar result for absolutely $n(\phi)$-summing operators and this result, apart from supplementing Pietsch's, will also include his result [6] for the case of absolutely summing (= absolutely 1-summing) operators.

The spaces $n(\phi)$ and $m(\phi)$ described below were introduced by Sargent [10].

For $x = (x_n)$, define the sequence $\Delta x = (x_n - x_{n-1})$, $x_0 = 0$; $S(x)$ denotes the collection of all sequences which are permutations of x. $\mathscr{C}$ is the set of all finite sequences of positive integers. For $\sigma \in \mathscr{C}$ define $c(\sigma) = (c_n(\sigma))$, where $c_n(\sigma) = 1$ if $n \in \sigma$ and $= 0$, otherwise. Let $\mathscr{C}_s = \{\sigma \in \mathscr{C} : \sum c_n(\sigma) \leqq s\}$.

$\phi = (\phi_n)$ is a given (fixed) sequence such that for each n, $0 < \phi_1 \leqq \phi_n \leqq \phi_{n+1}$ and $(n+1)\,\phi_n \geqq n\,\phi_{n+1}$.

The *BK-space*

$$m(\phi)=\left\{x=(x_n):\ \|x\|=\sup_{s\geqq 1}\sup_{\sigma\in\mathscr{C}_s}\left[\frac{1}{\phi_s}\sum_{n\in\sigma}|x_n|\right]<\infty\right\}$$

and the *BK*-space

$$n(\phi)=\{x=(x_n):\ \|x\|=\sup_{u\in s(x)}\sum|u_n|\,\Delta\phi_n<\infty\}.$$

We quote the following lemmas from Sargent [10].

Lemma 1. *The spaces* $n(\phi)$ *and* $m(\phi)$ *are Köthe duals of each other and are solid Köthe spaces. For* $x\in m(\phi)$ *and* $u\in n(\phi)$ *the following inequality holds:*

for each fixed k, $\sum_{i=1}^{k}|u_i x_i|\leqq\|u\|_{n(\phi)}\|x\|_{m(\phi)}$.

Lemma 2. $n(\phi)\supset l^2$ *if and only if* $\Delta\phi\in l^2$.

It is easy to verify that $n(\phi)$ is a K-symmetric sequence space. From earlier statements it also follows that $l^1\subset n(\phi)\subset l^\infty$.

In our discussion we shall assume that $\Delta\phi$ is a decreasing sequence and that $\Delta\phi_1=1$. It then follows from a result of Sargent that $\|\Delta\phi\|_{m(\phi)}=1$. The above assumption also gives that the space $n(\phi)$ has the union property. We assume also that $\sum\Delta\phi_n=\infty$.

We need also the following additional lemmas.

Lemma 3. *The space* $n(\phi)$ *has the sequences* e^i, $e^i=(0,0,\dots,0,1,0,\dots)$, $i=1,2,3,\dots$ *for a basis and the space* $m(\phi)$ *is its topological dual.*

The above lemma is essentially due to Sargent [11].

Lemma 4. *For each* $a\in l^1\subset n(\phi)$, $\|a\|_{n(\phi)}\leqq\|a\|_{l^1}$.

Since

$$\|a\|_{l^1}=\sup_{u\in S(a)}\sum|u_n|\geqq\sup_{u\in S(a)}\sum|u_n|\,\Delta\phi_n=\|a\|_{n(\phi)}$$

the result follows.

Theorem 5. *The identity map* I *of* c_0 *into* l^∞ *is not absolutely* $n(\phi)$*-summing, if* ϕ *is unbounded.*

Proof. If I is absolutely $n(\phi)$-summing then, by Theorem 1, there exists a positive constant ρ such that for each finite system $x^1, x^2, \dots, x^k$ in c_0 we have

$$\|(\|x^i\|_\infty)\|_{n(\phi)}\leqq\rho\sup_{\|a\|_{l^1}\leqq 1}[\|(|\langle x_i,a\rangle|)\|_{n(\phi)}].$$

Choose now $x^i=e^i$, $i=1,2,\dots,k$. Then the above inequality gives

$$\begin{aligned}\|(1,1,1,\dots,1,0,0,0,\dots)\|_{n(\phi)}&\leqq\rho\sup_{\|a\|_{l^1}\leqq 1}\|(a_1,a_2,\dots,a_k,0,0,\dots)\|_{n(\phi)}\\&\leqq\rho\sup_{\|a\|_{l^1}\leqq 1}\|a\|_{n(\phi)}\\&\leqq\rho\sup_{\|a\|_{l^1}\leqq 1}\|a\|_{l^1}=\rho.\end{aligned}$$

But $\|(1, 1, \ldots, 1, 0, 0, \ldots)\|_{n(\phi)} = \phi_k$ and since ϕ is unbounded the result in the theorem follows.

We shall prove now that if ϕ is unbounded and $l^2 \subset n(\phi)$ then the identity map I of l^2 into c_0 is not absolutely $n(\phi)$-summing while its adjoint map, the identity map of l^1 into l^2 is absolutely $n(\phi)$-summing.

Lemma 5. *If $l^2 \subset n(\phi)$ and if $a \in l^2$, then $\|a\|_{n(\phi)} \leqq \|a\|_{l^2} \|\Delta\phi\|_{l^2}$.*

Proof. $\|a\|_{n(\phi)} = \sup\limits_{u \in S(a)} \sum |u_n| \,\Delta\phi_n \leqq \sup\limits_{u \in S(a)} \|u\|_{l^2} \|\Delta\phi\|_{l^2} = \|a\|_{l^2} \|\Delta\phi\|_{l^2}$.

Theorem 6. *The identity map I of l^2 into c_0 is not absolutely $n(\phi)$-summing if ϕ is unbounded and $l^2 \subset n(\phi)$; under the same hypothesis on $n(\phi)$, the identity map I' of l^1 into l^2 is absolutely $n(\phi)$-summing.*

Proof. The proof of the first part is similar to that of Theorem 5 and is omitted. For proving the second part, we consider the orthonormal system $r_n(t)$ of Rademacher functions on the interval $[0, 1]$ defined for $n = 1, 2, 3, \ldots$ by $r_n(t) = (-1)^k$ for $k\,2^{-n} < t < (k+1)\,2^{-n}$ and $=0$ for $t = k\,2^{-n}$.

On l^1 define the linear functional $a(t)$ by the relation

$$\langle x, a(t)\rangle = \sum x_n r_n(t), \qquad x = (x_n).$$

Then $a(t)$ is continuous and $\|a(t)\| \leqq 1$; also,

$$\|I' x\|_{l^2} = \sqrt{\sum |x_n|^2} \leqq \sqrt{3} \int_0^1 |\langle x, a(t)\rangle|\, dt.$$

The above details are in Pietsch ([6], Satz 2.4.2, S. 39). If now $x^1, x^2, \ldots, x^k$ is a finite system in l^1 then, without loss of generality,

$$\begin{aligned}\|(\|I' x^n\|_{l^2})\|_{n(\phi)} &\leqq \sum_{n=1}^{k} \|I' x^n\|_{l^2}\, \Delta\phi_n \leqq \sum_{n=1}^{k} \sqrt{3}\, \Delta\phi_n \int_0^1 |\langle x^n, a(t)\rangle|\, dt \\ &\leqq \sup_{a \in U^0} \sqrt{3} \sum \Delta\phi_n\, |\langle x^n, a\rangle| \\ &\leqq \sup_{a \in U^0} \sqrt{3}\, \|(|\langle x^n, a\rangle|)\|_{n(\phi)},\end{aligned}$$

where U^0 denotes the unit ball in the adjoint space l^∞ of l^1. Thus $I' \in \pi_{n(\phi)}(l^1, l^2)$, with $\pi_{n(\phi)}(I') \leqq \sqrt{3}$.

The above proof is adopted from that of Pietsch [6] who proves the above result for the case $\phi_n = n$.

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