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Bounded Analytic Functions in the Dirichlet Space

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In this paper we study the Hilbert space of analytic functions with finite Dirichlet integral in a connected open set Ω in the complex plane. We show that every such function can be represented as a quotient of two bounded analytic functions, each of which has a finite Dirichlet integral. This has several consequences for the structure of invariant subspaces of the algebra of multiplication operators on the Dirichlet space, in case Ω is simply connected. Namely, we show that every nontrivial invariant subspace contains a nontrivial bounded function, that each two nontrivial invariant subspaces have a nontrivial intersection (that is, the algebra is "cellular indecomposable"), and that each nontrivial invariant subspace has the "codimension one" property with respect to certain multiplication operators.

1. Introduction

Let Ω be a connected open set in the complex plane, and let $H^\infty(\Omega)$ denote the algebra of bounded analytic functions in Ω, with the supremum norm. By the *Bergman space* of Ω, denoted $B(\Omega)$ or $L^2_a(\Omega)$, we mean the set of analytic functions in Ω that are square integrable with respect to area measure. With the L^2 norm this is a Hilbert space.

The *Dirichlet space*, denoted $D(\Omega)$, is the set of (single-valued) analytic functions in Ω whose first derivative is in $B(\Omega)$; equivalently, these are the functions that map Ω onto a region of finite area (counting multiplicity). This space becomes a Hilbert space with the following norm. Select some point z_0 in Ω and let

(1) $$\|f\|^2_D = |f(z_0)|^2 + \|f'\|^2_B.$$

If dA denotes area measure, then

(2) $$\|f'\|^2_B = \int_\Omega |f'|^2 \, dA$$

is called the *Dirichlet integral* of f in Ω.

Finally, we introduce the space $M(D(\Omega))$ of *multipliers* on $D(\Omega)$, that is, the space of those functions φ in Ω such that $\varphi D(\Omega) \subset D(\Omega)$. Using the closed graph theorem one shows that if φ is a multiplier then multiplication by φ is a bounded operator on $D(\Omega)$. With the operator norm $M(D(\Omega))$ is a Banach algebra. Since point evaluations are multiplicative linear functionals on this algebra, the multipliers are bounded functions (in a commutative Banach algebra all nonzero multiplicative linear functionals have norm one). Also, since the constant function 1 is in $D(\Omega)$ we see that the multipliers are contained in $D(\Omega)$. Thus

$$M(D(\Omega)) \subset H^\infty(\Omega) \cap D(\Omega). \tag{3}$$

If g is a conformal map (that is, an analytic homeomorphism) of Ω onto some domain G, then composition with g induces an isometric isomorphism of the Banach algebra $H^\infty(G)$ onto $H^\infty(\Omega)$. Also, the space $D(G)$ is mapped onto $D(\Omega)$ and the Dirichlet integral is preserved. In addition, $M(D(G))$ is mapped onto $M(D(\Omega))$, preserving its action on the Dirichlet space. Finally, if g maps the distinguished point z_0 of Ω to the distinguished point w_0 in G, then g induces a unitary map between the Dirichlet spaces, and an isometric isomorphism between the multiplier spaces.

In the special case when Ω is the unit disk, $\mathbb{D}$, if f has the power series $f(z) = \Sigma \hat{f}(n) z^n$, then a calculation shows that

$$\|f'\|_B^2 = \int_{\mathbb{D}} |f'|^2 \, dA = \pi \sum_1^\infty n |\hat{f}(n)|^2. \tag{4}$$

Thus if $z_0 = 0$ then $\|f\|_D^2 = |f(0)|^2 + \pi \Sigma \, n |\hat{f}(n)|^2$.

The Dirichlet space in $\mathbb{D}$ lies in a family $\{D_\alpha\}_{\alpha \in \mathbb{R}}$ of Hilbert spaces; the norm is given in terms of the power series coefficients:

$$\|f\|_\alpha^2 = \Sigma (n+1)^\alpha |\hat{f}(n)|^2. \tag{5}$$

For $\alpha = 1$ one has the Dirichlet space (with an equivalent norm). Two other special cases are $\alpha = 0$ (the Hardy space H^2), and $\alpha = -1$ (the Bergman space B). A brief survey of the spaces D_α is given in [9]. We note that D_α is an algebra, contained in the disc algebra, when $\alpha > 1$.

In his dissertation [10] Carleson considered spaces T_α related to the spaces D_α, $0 < \alpha \leq 1$. The space T_0 is the Nevanlinna class of meromorphic functions of bounded characteristic, and the space T_1 consists of those meromorphic functions that map $\mathbb{D}$ onto a region of finite *spherical* area, counting multiplicity (that is, $(1+|f|^2)^{-1} f'$ is in $L^2(\mathbb{D})$); the spaces $\{T_\alpha\}$ decrease with increasing α. Also, D_α is contained within T_α (this can be proved using the formulae on page 19 of [10]).

The Nevanlinna class contains H^∞, the space of bounded analytic functions in $\mathbb{D}$, and every Nevanlinna function is the quotient of two bounded functions. Carleson found a partial analogue of this for the spaces T_α: each function in T_α is a quotient of two bounded functions, each of which is in T_β for all $\beta < \alpha$ (see [10], Theorem 3.6). He conjectured that one could not take $\beta = \alpha$ here, however, since there seemed to be an essential difference between bounded functions in T_α and general functions in T_α with regard to the exceptional set where the radial limit may fail to exist. Indeed, he shows (Theorem 3.3, p. 34; also p. 28) that this set has outer capacity zero of order $(1-\alpha)$ for general functions in T_α, whereas, at least for inner functions in T_α, it has Hausdorff measure 0 of order $1-\alpha$. (For $\alpha=1$ one has logarithmic capacity and logarithmic measure, see p. 14 of [10]).

We show in Theorem 1 below that for $\alpha=1$ one can take $\beta=1$, at least for functions in the Dirichlet space D. Thus for $\alpha=1$ the question is reduced to asking whether each function in T_1 is the quotient of two functions in D? Curiously this sort of result is also of interest in the theory of transitive operator algebras initiated by Arveson [3]; see the remarks at the end of this paper.

2. Dirichlet Functions as Quotients of Bounded Functions

We now show that every function in $D(\Omega)$ is a quotient of two bounded functions in $D(\Omega)$; this is also valid when Ω is a Riemann surface. The proof works for more general classes of functions (for the Dirichlet space it can be simplified somewhat); we require two lemmas.

Let E be a set of analytic functions in a region Ω. We say that the set E is *solid*, if $h\in E$, g holomorphic in Ω, and $|g(z)|\leqq c|h(z)|$ in Ω, imply that $g\in E$. For example, the space of p^{th} power integrable analytic functions (with respect to area measure in Ω) is solid. If E is a set of analytic functions in Ω, then by E_1 we denote the set of all those analytic functions in Ω for which $f'\in E$. We note that $H^\infty(\Omega)\cap E_1$ is an algebra whenever E is a solid linear space of analytic functions on Ω.

Lemma 1. *Let Ω be a connected open set in the complex plane, let E be a solid set of analytic functions in Ω,* and let *$f\in E_1$. If there is a function $F\in H^\infty(\Omega)$, not identically zero, and a positive constant c, such that, for all $z\in\Omega$,*

a) $|f(z)|^\delta |F(z)|\leqq c$ *for some* δ, $0<\delta\leqq 1$,
b) $|F'(z)|\leqq c|f'(z)|$,

then $f=\varphi/\psi$, *where* $\varphi,\psi\in H^\infty(\Omega)\cap E_1$.

Proof. By choosing the constant c larger, if necessary, we may assume that $|F|\leqq c$ in Ω. Select a positive integer n such that $n\delta\geqq 1$, and let $\varphi=fF^{n+1}$, $\psi=F^{n+1}$. We first show that $\psi\in H^\infty(\Omega)\cap E_1$. Indeed, $\psi\in H^\infty(\Omega)$ since $|\psi|\leqq c^{n+1}$ in Ω. Also,

$$|\psi'|\leqq(n+1)|F^n||F'|\leqq(n+1)c^{n+1}|f'|.$$

Thus $\psi'\in E$, and so $\psi\in E_1$, as desired.

Next we show that $|fF^n|\leqq c^n$ in Ω. Indeed, if $|f(z)|\leqq 1$, then $|fF^n|\leqq|F|^n\leqq c^n$. On the other hand, if $|f(z)|>1$, then

$$|f(z)\,F^n(z)|\leqq(|f(z)|^\delta|F(z)|)^n\leqq c^n.$$

We now show that $\varphi\in H^\infty(\Omega)\cap E_1$. Indeed, $|\varphi|=|fF^n||F|\leqq c^{n+1}$. Also,

$$\begin{aligned}|\varphi'|&\leqq|f'F^{n+1}|+(n+1)|fF^nF'|\\&\leqq c^{n+1}|f'|+(n+1)c^{n+1}|f'|\\&\leqq(n+2)c^{n+1}|f'|.\end{aligned}$$

Thus $\varphi\in E_1$ which completes the proof. □

Lemma 2. *Let Ω be a connected open set in the complex plane, let E be a solid set of analytic functions in Ω, and let $f \in E_1$. If there is a compact set K having positive two dimensional Lebesgue measure, such that K and $f(\Omega)$ are disjoint, then $f = \varphi/\psi$, where $\varphi, \psi \in H^\infty(\Omega) \cap E_1$.*

Proof. We may assume that the complement of K is connected (otherwise we replace K by a totally disconnected compact subset, still of positive measure; the complement of this new set will be connected). By a result of Nguyen Xuan Uy [18] (see Hruščëv [16] for another proof), there exists g, holomorphic and not constant in the complement of K (with respect to the extended plane), such that both g and g' are bounded there. By subtracting a constant we may assume that $g(\infty) = 0$. Let $F(z) = g(f(z))$. To complete the proof we show that F satisfies the conditions of Lemma 1.

First, F is bounded since g is bounded. Now let

$$g(w) = a_1 w^{-1} + a_2 w^{-2} + \dots$$

be the power series expansion of g about infinity. Then $wg(w) \to a_1$ as $|w| \to \infty$. It follows that $f(z)F(z) \to a_1$ as $|f(z)| \to \infty$, and thus hypothesis a) of Lemma 1 is satisfied, with $\delta = 1$. Since $F' = g'(f)f'$, and g' is bounded, we see that hypothesis b) is satisfied also. □

Remark. Nguyen [18] produces a bounded holomorphic function g that satisfies a Lipschitz condition of order one in the complement of K; the boundedness of g' follows from this. E.P. Dolženko [12] proved that if K has 2-dimensional measure zero, if U is an open set containing K, if g is holomorphic in $U \setminus K$ and satisfies a Lipschitz condition of order one there, then g extends to be holomorphic on K (see also Garnett [14], Chap. 3, Theorem 2.3, p. 66). One says that K is a removable set for the class Lip 1. Dolženko also showed that for functions of class Lip α, $0 < \alpha < 1$, the necessary and sufficient condition for removability is that the set have Hausdorff measure zero of order $1 + \alpha$.

Theorem 1. *If $f \in D(\Omega)$, then $f = \varphi/\psi$, where $\varphi, \psi \in D(\Omega) \cap H^\infty(\Omega)$.*

Proof. Since $f(\Omega)$ has finite area, counting multiplicity, the complement has infinite 2-dimensional measure, and the result follows from Lemma 2. □

Remarks 1. The theorem is valid also when Ω is a Riemann surface; the proof is the same, though some details have to be reinterpreted in the language of differential forms. (An analytic function of Ω is said to have a finite Dirichlet integral if the 2-form $df \wedge d\bar{f}$ is integrable.)

2. In case Ω is simply connected one can avoid Nguyen's theorem, and base the proof on Lemma 1. Indeed, for $f \in D(\Omega)$ one can choose a compact set K, of positive 2-dimensional measure, disjoint from $f(\Omega)$. One can define the square root so that $k(z, w) = (w - f(z))^{-1/2}$ is continuous in w, and analytic in z, for $w \in K$, $z \in \Omega$. Let $F(z) = \int k(z, w)\, dA(w)$. One shows that the hypotheses of Lemma 1 are satisfied, with $\delta = 1/2$.

3. A different proof for the simply connected case can be given, using a formula of Carleson. Indeed, by conformal invariance it is sufficient to consider the unit

disk, $\mathbb{D}$. If $f \in D(\mathbb{D})$, then f has boundary values almost everywhere. Let $\psi_f(e^{it}) = 1$, whenever $|f(e^{it})| \leqq 1$, and $\psi_f = |f|^{-1}$ otherwise; let $\psi(z)$ be the outer function determined by ψ_f. Then $|\psi(z)| \leqq 1$ in $\mathbb{D}$. Using Carleson's formula for the Dirichlet integral, [11], it is shown in [9] (Lemma 7 (a), (c), (d), p. 284) that in this situation one has ψ, $\psi f \in D(\mathbb{D}) \cap H^\infty$. (The discussion in [9] is incomplete: when considering the contribution of the inner factor to the formula in [11] one needs to remark that $|\psi f| \leqq |f|$ a.e. on $\partial \mathbb{D}$.) An advantage of this proof is that the denominator, ψ, has no zeros in Ω; it is not clear how to achieve this in the generality of Theorem 1.

See the remarks at the end of the paper for further comments and questions related to Theorem 1.

3. Invariant Subspaces

In this section we assume that Ω is simply connected and we consider the algebra $M(D(\Omega))$ of multipliers on $D(\Omega)$ (see the Introduction). By the conformal invariance of the Dirichlet space and of the space of multipliers, we may restrict attention to the unit disk $\mathbb{D}$. Throughout this section D will denote the Dirichlet space in $\mathbb{D}$. Here we have the special operator M_z of multiplication by z. It is known that every multiplier is the limit, in the strong operator topology, of a sequence of polynomials in this special operator (see [25], Theorem 12, p. 90). Warning: the operator M_z does not always operate on $D(\Omega)$. It never does when Ω is unbounded, since multipliers are always bounded. But even for rectifiable Jordan domains M_z may fail to operate (see Theorems 1 and 10 of [4]).

We recall the family $\{D_\alpha\}$ of Hilbert spaces in $\mathbb{D}$, defined in the Introduction. For all $\alpha \in \mathbb{R}$ the map $M_z, f \mapsto zf$ defines a bounded linear transformation on D_α. We shall denote this operator by M_z or by (M_z, D_α). A subspace $\mathcal{M}$ of D_α is called *invariant*, if (M_z, D_α) maps $\mathcal{M}$ into itself.

By subspace we always mean closed subspace.

If $f \in D$ then $[f]$ denotes the closure of the polynomial multiples of f, that is, the smallest invariant subspace containing f.

If g is an analytic function on $\mathbb{D}$ then g_r denotes the function defined by $g_r(z) = g(rz)$, $0 \leqq r < 1$. Since g_r is analytic on the closed disc $\mathbb{D}^-$ one shows that $g_r f \in [f]$ for all f in D.

Lemma 3. *If $f \in D$, $\varphi \in D \cap H^\infty$, and $\varphi f \in D$, then $\varphi_r f \to \varphi f$ and $\varphi f \in [f]$.*

Proof. As noted above, $\varphi_r f \in [f]$, and so it will be sufficient to show that $\varphi_r f \to \varphi f$. If $h \in D$ with $h(0) = 0$, then it follows from the definition of the Dirichlet norm given in the Introduction that

$$\| h \|_D^2 = \pi \Sigma\, n |\hat{h}(n)|^2 = \| h' \|_B^2,$$

where B denotes the Bergman space. Thus we have

$$\begin{aligned} \| \varphi_r f - \varphi f \|_D &\leqq \| \varphi_r (f - f_r) \|_D + \| (\varphi f)_r - \varphi f \|_D \\ &= \| \{ \varphi_r (f - f_r) \}' \|_B + \| (\varphi f)_r - \varphi f \|_D \\ &\leqq \| (\varphi_r)' (f - f_r) \|_B + \| \varphi_r \|_\infty \| (f - f_r)' \|_B + \| (\varphi f)_r - \varphi f \|_D. \end{aligned}$$

As $r\to 1$ the first term in the last expression approaches 0 by Lemma 3, page 278 of [9]. It is trivial that the second and third terms tend to zero, and so $\varphi_r f\to\varphi f$ as $r\to 1$. □

Theorem 2. *Let $\mathcal{M}\neq\{0\}$, $\mathcal{N}\neq\{0\}$ be invariant subspaces for (M_z, D). Then*

a) $\mathcal{M}\cap H^\infty\neq\{0\}$,
b) $\mathcal{M}\cap\mathcal{N}\neq\{0\}$,
c) $z\mathcal{M}$ *is a closed subspace of* $\mathcal{M}$ *of codimension* 1.

Proof. a) Let $f\in\mathcal{M}, f\neq 0$. By Theorem 1 there are functions $\varphi, \psi\in H^\infty\cap D$ such that $f=\varphi/\psi$. By the Lemma, $\varphi=\psi f\in[f]\subset\mathcal{M}$, as required.

b) By a) there are bounded functions $\varphi\in\mathcal{M}$ and $\psi\in\mathcal{N}$. By the Lemma we have $\varphi\psi\in[\varphi]\cap[\psi]\subset\mathcal{M}\cap\mathcal{N}$, as required.

c) A calculation shows that $\|zf\|_D\geqq\|f\|_D$ for all $f\in D$; thus $z\mathcal{M}$ is a closed set. The codimension one property stated in c) follows from b) and a theorem of Bourdon [8]. □

4. Remarks and Questions

As operator is called *cellular indecomposable* if every two nontrivial invariant subspaces have a nontrivial intersection (see [19]). Thus Theorem 2 b) states that (M_z, D_α) is cellular indecomposable when $\alpha=1$. What about other values of α?

As mentioned earlier the spaces D_α are algebras for $\alpha>1$. Thus if $f, g\in D_\alpha$ then $fg\in[f]\cap[g]$ (see Proposition 7, p. 275 of [9]). Thus (M_z, D_α) is cellular indecomposable when $\alpha>1$.

Furthermore, (M_z, H^2) is cellular-indecomposable for a similar reason. In fact, every invariant subspace contains an H^∞ function, and H^∞ coincides with the set of multipliers on H^2.

For $\alpha<0$ the operators (M_z, D_α) are not cellular indecomposable. This was first established by Horowitz [15]; it is also a consequence of the results in Chapter 10 of [6].

Conjecture 1. (M_z, D_α) is cellular indecomposable for $0<\alpha<1$.

We now make some comments about part c) of Theorem 2. One verifies that the operator $M_z-\lambda$ is bounded below on D_α for all $\lambda\in\mathbb{D}$ and all real α. Hence if $\mathcal{M}$ is an invariant subspace for (M_z, D_α), then $(M_z-\lambda)|\mathcal{M}$ is bounded below and thus is a semi-Fredholm operator. Therefore the number

$$\dim(\mathcal{M}\ominus(M_z-\lambda)\mathcal{M})=\dim\ker((M_z-\lambda)|\mathcal{M})^*=-\operatorname{index}((M_z-\lambda)|\mathcal{M})$$

does not depend on $\lambda\in\mathbb{D}$.

We say that an invariant subspace $\mathcal{M}$ of (M_z, D_α) has the *codimension one property*, if $\dim\mathcal{M}\ominus zM=1$ or if $\mathcal{M}=\{0\}$.

Thus Theorem 2 c) states that every invariant subspace of (M_z, D) has the codimension one property.

Invariant subspaces with the codimension one property are of a simpler structure than general invariant subspaces. In the few cases where a complete function theoretic description of all invariant subspaces is known (H^2: Beurling [7]; D_2: Korenblum [17]) all invariant subspaces have the codimension one property. (See Šamoyan [23] for an extension of Korenblum's work to other algebras of analytic functions; see also Aleksandrov [2]). On the other hand, it follows from results of Apostol, Bercovici, Foias, and Pearcy (see Chap. 10 of [6]) that (M_z, D_α) has invariant subspaces which do not have the codimension one property, for each $\alpha < 0$.

For more information on the significance of the codimension one property see [22].

Conjecture 2. Every invariant subspace of (M_z, D_α) has the codimension one property, $0 < \alpha < 1$.

Invariant subspaces of (M_z, D_α) having the form $[f]$ are said to be *cyclic* (or *singly generated*). Since $[f]$ is the span of $\{f, zf, z^2f, \dots\}$ it is easy to verify that $[f]$ has the codimension one property.

Conjecture 3. Every invariant subspace of (M_z, D_α) is singly generated, for $0 < \alpha \leqq 1$.

The first and third conjectures each imply the second. Note that the third conjecture is open even for $\alpha = 1$.

Next we say a few words about a possible extension of Theorem 1 suggested by the theory of operator algebras. An algebra $\mathscr{A}$ of operators (on a Banach space E) is said to be *transitive* if no proper subspace is invariant for all the operators in $\mathscr{A}$. The transitive algebra problem asks whether a transitive algebra must be dense (in the strong operator topology) in the algebra of all operators on E. An affirmative answer would imply that every operator on E has a nontrivial invariant subspace (and much more). Some Banach spaces admit operators with no nontrivial invariant subspaces (see Beauzamy [5], Enflo [13], Read [20], [21]), thus the transitive algebra problem has a negative answer for such spaces. The problem is still open for reflexive Banach spaces.

One usually makes some additional assumption about the algebra $\mathscr{A}$. For example, Arveson [3] showed that if $\mathscr{A}$ contains the unilateral shift operator on Hilbert space, then $\mathscr{A}$ must be strongly dense. Ever since it has been an open problem whether his result remains true if $\mathscr{A}$ contains a weighted shift operator. Thus far this is only known for strictly cyclic shifts, which includes (M_z, D_α), $\alpha > 1$. In particular it is unknown for (M_z, D_α), $0 < \alpha \leqq 1$, (and for $\alpha < 0$). The unilateral shift may be identified with M_z on H^2, and Arveson's proof used the fact that each function in H^2 is the quotient of two bounded functions, that is, of two multipliers on H^2. The proof would work for (M_z, D_α), $0 < \alpha \leqq 1$, if the following conjecture were correct.

Conjecture 4. Each $f \in D_\alpha$ is the quotient of two multipliers on D_α.

Recall that the multipliers on D_α are contained in $D_\alpha \cap H^\infty$. Thus this conjecture represents a generalization of Theorem 1 (for $\alpha = 1$).

The multipliers on D_α have been characterized by Stegenga [26]. For $0<\alpha<1$ even the direct analogue of Theorem 1 is open.

Conjecture 5. Each $f\in D_\alpha$ is the quotient of two bounded functions in D_α.

Conjecture 6. Each invariant subspace of (M_z, D_α) contains a nontrivial multiplier $(0<\alpha\leq 1)$.

This is correct for $\alpha>1$ and for $\alpha=0$. It is false for $\alpha<0$ (this follows from Theorem 6 of [24]; see also Horowitz [14]).

Conjecture 7. If Ω is a plane region (or Riemann surface) that has nonconstant analytic functions with finite Dirichlet integral, then it has nonconstant multipliers of the Dirichlet space.

It follows from Theorem 1 that Ω admits nonconstant H^∞-functions with finite Dirichlet integral, but such functions are not necessarily multipliers. Ahlfors and Beurling [1] (Theorem 4) have characterized the plane domains Ω for which $D(\Omega)$ is nontrivial: there must be another domain conformally equivalent to Ω, whose complement has positive two dimensional measure.

Bibliography

1. Ahlfors, L.V., Beurling, A.: Conformal invariants and function-theoretic null sets. Acta Math. **83**, 101–129 (1950). MR 12, p. 171
2. Aleksandrov, A.B.: Invariant subspaces of shift operators. An axiomatic approach. Zap. Naučn. Sem. Leningrad Otdel. Math. Inst. Steklov (LOMI) **113**, 7–26, 264 (1981) (Series title: Investigations on linear operators and the theory of functions, vol. XI.) Russian with English summary. MR 83g:47031
3. Arveson, W.B.: A density theorem for operator algebras. Duke Math. J. **34**, 635–647 (1967). MR 36 #4345
4. Axler, S., Shields, A.L.: Univalent multipliers of the Dirichlet space. Mich. Math. J. **32**, 65–80 (1985). MR 86c:30043
5. Beauzamy, B.: Un operateur sans sous-espace invariant: simplification de l'example de P. Enflo. Integral Equations Oper. Theory. **8**, 314–384 (1985)
6. Bercovici, H., Foias, C., Pearcy, C.M.: Dual algebras with applications to invariant subspaces and dilation theory, CBMS Regional Conf. Ser. Math, vol. 56 (1985), Am. Math. Soc., Providence, Rhode Island
7. Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math. **81**, 239–255 (1949) MR 10, p. 381
8. Bourdon, P.: Cellular-indecomposable shifts and Beurling's theorem. Mich. Math. J. **33**, 187–193 (1986)
9. Brown, L., Shields, A.: Cyclic vectors in the Dirichlet space. Trans. Am. Math. Soc. **285**, 269–304 (1984) MR 86d: 30079
10. Carleson, L.: On a class of meromorphic functions and its associated exceptional sets. Appelbergs Boktryckeri, Uppsala, 1950. MR 11, p. 427
11. Carleson, L.: A representation formula for the Dirichlet integral. Math. Z. **73**, 190–196 (1960). MR 22 #3803
12. Dolženko, E.P.: On the removal of singularities of analytic functions. Uspehi Mat. Nauk **18**, 135–142 (1963); English translation: Am. Math. Soc. Transl., New Series **97**, 33–41 (1970). MR 27 #5898
13. Enflo, P.: On the invariant subspace problem in Banach spaces. Acta Mathematica. To appear
14. Garnett, J.: Analytic capacity and measure. Lecture Notes Math. 297, Berlin-Heidelberg-New York: Springer 1972. MR 56 # 12257
15. Horowitz, C.: Zeros of functions in Bergman spaces. Duke Math. J. **41**, 693–710 (1974). MR 50 #10215

16. Hruščëv, S.V.: A simple proof of the theorem on removable singularities for analytic functions satisfying a Lipschitz condition. Akad. Nauk SSSR, Leningrad Otdel., Zapiski nauč. sem. LOMI 113, Issled. lin. op. teor. funk. **XI**, 199–203 (1981). (Russian, with English summary). MR 84d: 30082
17. Korenblum, B.: Invariant subspaces of the shift operator in a weighted Hilbert space. Mat. Sbor. **89**(131), 110–137 (1972); English translation: Math USSR Sb. **18**, 111–138 (1972). MR 47 #5621
18. Uy, N.X.: Removable sets of analytic functions satisfying a Lipschitz condition. Arkiv Mat. **17**, 19–27 (1979). MR 81i: 30076
19. Olin, R.F., Thomson, J.E.: Cellular-indecomposable subnormal operators. Integral Equations Oper. Theory. **7**, 392–430 (1984). MR 85m: 47016
20. Read, C.J.: A solution to the invariant subspace problem on the space l^1. Bull. Lond. Math. Soc. **17**, 305–317 (1985)
21. Read, C.J.: A short proof concerning the invariant subspace problem. J. Lond. Math. Soc. (2)**34**, 335–348 (1986)
22. Richter, S.: On invariant subspaces of multiplication operators on Banach spaces of analytic functions. Ph.D. dissertation, University of Michigan, 1986
23. Šamoyan, F.A.: Closed ideals in algebras of analytic functions that are smooth on the boundary. Izv. Akad. Nauk Arm. SSR Ser., Mat. **16**, 173–191 (1981). MR 83b: 46074
24. Shapiro, H.S., Shields, A.L.: On the zeros of functions with finite Dirichlet integral and some related function spaces. Math. Z. **80**, 217–229 (1962). MR 26 #2617
25. Shields, A.L.: Weighted shift operators and analytic function theory. Topics in operator theory, 49–128. Math. Surveys **13**, (1974) Am. Math. Soc., Providence, R.I. MR50 #14341
26. Stegenga, D.A.: Multipliers of the Dirichlet space. Ill. J. Math. **24**, 113–139 (1980). MR 81a: 30027

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