Math. Z. 202, 65–82 (1989)

Mathematische Zeitschrift

T^n-Actions on Holomorphically Separable Complex Manifolds*

David E. Barrett[1], Eric Bedford[2], and Jiri Dadok[2]

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

[2] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

§ 0. Introduction

Here we consider complex manifolds M which are *holomorphically separable*, i.e., the global holomorphic functions $\mathcal{O}(M)$ separate the points of M. We investigate $\mathbf{T}^n$ (n-torus) actions on M, where $\mathbf{T}^n = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \colon \theta \in \mathbb{R}^n\}$, and n is the complex dimension of M. We assume that the action is *effective*, i.e., if $\theta \in \mathbf{T}^n$ is nonzero, then there exists $z \in M$ with $\theta \cdot z \neq z$. Our final assumption on the action is one of smoothness; the map $\mathbf{T}^n \times M \to M$ given by $(\theta, z) \to \theta \cdot z$ is C^1 in both variables and holomorphic in z.

The most obvious examples of such actions are given by Reinhardt domains in $\mathbb{C}^n$, with the usual action $\theta \cdot z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$. The standard Reinhardt action may also be changed as follows: if A is an algebraic automorphism of $\mathbf{T}^n$, then $\theta \cdot_A z = (A\theta) \cdot z$. An obvious question that arises is whether all $\mathbf{T}^n$-actions can arise in this way.

Our main results are contained in the following four theorems.

Theorem 1. *Let M be a connected, separable, complex manifold equipped with an effective, C^1, $\mathbf{T}^n$-action. Then the following conditions are equivalent:*

(i) *M is equivalent by an equivariant biholomorphism to a Reinhardt domain in $\mathbb{C}^n$, after possibly adjusting the $\mathbf{T}^n$-action on M by an automorphism of $\mathbf{T}^n$.*

(ii) *M admits a smooth envelope of holomorphy.*

In particular, if M is Stein, then it is (equivariantly equivalent to) a Reinhardt domain.

The proof of Theorem 1 may also be used to give a simpler proof of the following result of Bialnicki-Birula [1, 2].

Theorem 2. *Suppose that $M = \mathbb{C}^n$ and the $\mathbf{T}^n$-action is algebraic. Then the action is algebraically conjugate to the usual Reinhardt action on $\mathbb{C}^n$.*

It follows from general constructions that our manifold has an envelope Ω of holomorphy of some sort. (See Hayes [7] for a discussion of this subject.)

* Research supported in part by the NSF

In this paper we will use the term *Stein envelope of holomorphy* in the following sense. If M is a holomorphically separable manifold, then the Stein envelope of holomorphy $\tilde{M}$ (if it exists) is a Stein space with $M \subset \tilde{M}$ and such that every $f \in \mathcal{O}(M)$ extends uniquely to a function $\tilde{f} \in \mathcal{O}(\tilde{M})$.

A $\mathbf{T}^n$-action may also be given on $\mathbb{C}^N$ for $N \geqq n$ as follows. If A is an $n \times N$ matrix, then we have $\theta \cdot_A z = e^{iA\theta} z$. If $X \subset \mathbb{C}^N$ is a subvariety invariant under this action, then this $\mathbf{T}^n$-action also operates on X. In this case, we say that the $\mathbf{T}^n$-action on X is *linear*.

Theorem 3. *Let M satisfy the hypotheses of Theorem 1. Them the following are equivalent:*

(i) *there is a finite set of holomorphic functions $\{f_1, \ldots, f_k\} \subset \mathcal{O}(M)$ such that the polynomials in $f_1, \ldots, f_k$ are dense in $\mathcal{O}(M)$ with respect to the topology of uniform convergence on compact sets.*

(ii) *M is equivariantly biholomorphic to an open set of an algebraic variety of $\mathbb{C}^N$ equipped with a linear $\mathbf{T}^n$ action.*

(iii) *M admits a Stein envelope of holomorphy.*

In fact the linearization $f \colon M \to \mathbb{C}^N$ imbeds M as an open subset of an algebraic variety $V \subset \mathbb{C}^N$, and $f(M)$ becomes essentially a piece of a toroidal imbedding. (See [4, 5, 8] for further information on toroidal embeddings.) It is shown in Section 4 that a $\mathbf{T}^n$-invariant domain in such a variety V always has a Stein envelope of holomorphy.

We note that there are separable manifolds for which the conditions i), ii), and iii) do not hold. In Section 5 we give a class of examples of 3-dimensional manifolds without a Stein envelope of holomorphy. These examples are possibly easier to understand than the original example of Grauert [6].

The following result shows that $\mathbf{T}^2$-actions on 2-manifolds are more restricted.

Theorem 4. *Let M satisfy the hypotheses of Theorem 1. If the dimension is $n=2$, then the equivalent conditions* i), ii), *and* iii) *in Theorem 2 all hold.*

§ 1. Preliminaries

Let $\mathbf{T}^n$ be an n-torus. We denote by $\mathfrak{t}$ its Lie algebra and $\exp\colon \mathfrak{t} \to \mathbf{T}^n$ the exponential map. We lat $\Gamma \subset \mathfrak{t}$ be the lattice that is the kernel of the exponential map. For the sake of concreteness, we may identify $\mathfrak{t}$ with $\mathbb{R}^n$, Γ with $(2\pi\mathbb{Z})^n$, and $\mathbf{T}^n = \{g = (e^{i\theta_1}, \ldots, e^{i\theta_n}) = \exp\theta \colon \theta \in \mathbb{R}^n\}$.

We let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and let $\mathcal{L}$ be the lattice dual of Γ:

$$\mathcal{L} = \{\alpha \in \mathfrak{t}^* \colon \alpha(\Gamma) \subset 2\pi\mathbb{Z}\}.$$

Thus each $\alpha \in \mathcal{L}$ gives the character whose value on $\exp\theta$ is $(\exp\theta)^\alpha$ for all $\theta \in \mathfrak{t}$.

Let $v \in \mathfrak{t}$ be given; then $\mathbb{R}v$ generates a closed 1-dimensional subgroup of $\mathbf{T}^n$ if and only if $\mathbb{R}v \cap \Gamma \neq \{0\}$, i.e., if we may take $v \in \Gamma$. Thus we see that there is a correspondence between Γ and the 1-dimensional subgroups of $\mathbf{T}^n$.

There is a further correspondence between the 1-dimensional closed subgroups of $\mathbf{T}^n$. The first is that $v \in \Gamma$ (or the corresponding 1-dimensional subgroups of $\mathbf{T}^n$) corresponds to a hypersurface in $\mathscr{L}$:

$$\mathscr{L}_v = \{\alpha \in \mathscr{L} : \langle \alpha, v \rangle = 0\}.$$

These are the characters that are identically 1 on the torus $\mathbb{R}v$. Thus it follows that these give the lattice of characters on the quotient torus, i.e.,

$$\mathscr{L}_v = \{\chi : \mathbf{T}^n/\mathbb{R}v \to \mathbb{C}^* : \chi(t_1 + t_2) = \chi(t_1)\,\chi(t_2)\}.$$

We record here an elementary result.

Lemma 1.1. *Let $L \subset \mathscr{L}$ be given. The characters in L separate the points of $\mathbf{T}^n$ if and only if the group generated by L equals $\mathscr{L}$.*

Lemma 1.2. *Let $z_0 \in M$ be given, and suppose that $\mathcal{O}(M)$ separates the points of the orbit $\mathbf{T}^n \cdot z_0$. Then $\mathbf{T}^n \cdot z_0$ is a torus of dimension $0 \leqq j \leqq n$. It is a smooth, compact, totally real submanifold of M.*

Proof. The only thing we have to show is that $\mathbf{T}^n \cdot z_0$ is totally real. Let us suppose not. Then there is a complex subspace C inside the tangent space to the orbit $\mathbf{T}^n \cdot z_0$. We then may think of C inside the Lie algebra of $\mathbf{T}^n$. Since $\mathbf{T}^n$ acts by bihomorphisms, exponentiation produces a holomorphic map ϕ: $C \to C \cdot z_0 \subset \mathbf{T}^n \cdot z_0$. If $f \in \mathcal{O}(M)$ then $f \circ \phi$ is bounded and holomorphic on C, therefore a constant. Thus $\mathcal{O}(M)$ cannot separate points on $C \cdot z_0$, which is a contradiction.

For $z_0 \in M$ we define the isotropy subgroup of z_0

$$\mathbf{T}^n_{z_0} = \{\theta \in T^n :\ \theta \cdot z_0 = z_0\}.$$

Since we assume that the action is effective we obtain the following lemma from the Slice Theorem [3].

Lemma 1.3. *For z_0 in an open dense subset of M the map $\theta \to \theta \cdot z_0$ is a diffeomorphism between $\mathbf{T}^n$ and $\mathbf{T}^n \cdot z_0$.*

For $\alpha \in (\mathbf{T}^n)^*$, we say that $f_\alpha \in \mathcal{O}(M)$ is a *holomorphic character* associated to α if $f_\alpha(\theta \cdot z) = e^{i\alpha \cdot \theta} f_\alpha(z)$ holds for all $z \in M$, $\theta \in \mathbf{T}^n$. Let f_α and g_α be holomorphic characters. If z_0 is as in Lemma 1.3, then f_α/g_α is constant on $\mathbf{T}^n \cdot z_0$ and so $f_\alpha = c g_\alpha$ on M. Thus there is (up to constant multiple) at most one character for each α. Let

$$\mathscr{L}_{\mathcal{O}}(M) = \{\alpha \in (\mathbf{T}^n)^* : \text{there is a nonzero holomorphic character } f_\alpha\}.$$

Any $f \in \mathcal{O}(M)$ has a Fourier expansion $f = \Sigma f_\alpha$, where

$$f_\alpha(z) = (2\pi)^{-n} \int_{\mathbf{T}^n} f(z \cdot \theta)\, e^{-i\alpha \cdot \theta}\, d\theta$$

is a holomorphic character associated to α. The Fourier series converges to f uniformly on compact subsets of M.

Lemma 1.4. *$\mathcal{L}_{\mathcal{O}}(M)$ generates $\mathcal{L}$ as a group.*

Proof. Let z_0 be a point of M as in Lemma 1.3. If $f \in \mathcal{O}(M)$ is written in a Fourier expansion $f = \Sigma f_\alpha$, then f_α restricted to $\mathbf{T}^n \cdot z_0$ is just a character on $\mathbf{T}^n$. Since the f_α's must separate points, the result follows from Lemma 1.1.

Lemma 1.5. *For $z_0 \in M$, $\mathbf{T}^n_{z_0} \neq 0$ if and only if there exists a nontrivial holomorphic character f_α with $f_\alpha(z_0) = 0$.*

Proof. Suppose $\theta_0 \in \mathbf{T}^n_{z_0}$, $\theta_0 \neq 0$. By Lemma 1.1, there exists $\alpha \in \mathcal{L}_{\mathcal{O}}(M)$ such that $e^{i\alpha \cdot \theta_0} \neq 1$. Thus

$$f_\alpha(\theta_0 \cdot z_0) = e^{i\alpha \cdot \theta_0} f_\alpha(z_0)$$

so $f_\alpha(z_0) = 0$.

Suppose that $\mathbf{T}^n_{z_0} = 0$. Then $\mathbf{T}^n \cdot z_0$ is totally real n-torus. If $f_\alpha(z_0) = 0$ for some nontrivial character, then $f_\alpha(\mathbf{T}^n \cdot z_0) = 0$. Thus f_α must vanish identically, which is a contradiction.

Now let us define the set $S = \{z \in M \colon \mathbf{T}^n_z \neq 0\}$. In light of Lemma 1.5, we see that S is given by

$$S = \cup \{f_\alpha = 0\}$$

with the union taken over all holomorphic characters on M.

Lemma 1.6. *$\mathcal{L}_{\mathcal{O}}(M)$ is given as an intersection of half-spaces, i.e., there is a subset $\{\beta_j\} \subset \mathcal{L}^*$ such that*

$$\mathcal{L}_{\mathcal{O}}(M) = \{\alpha \in \mathcal{L} \colon \langle \alpha, \beta_j \rangle \geqq 0 \text{ for all } j\}.$$

Proof. Let us write $S = S_1 \cup S_2 \cup \ldots$ as a union of irreducible components. By Lemma 1.4 there are $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_{\mathcal{O}}(M)$ which generate $\mathcal{L}$ as a group. For $1 \leqq j \leqq n$, we let $\mu_{j,k}$ be the order of vanishing of f_{α_j} on S_k.

Any $\alpha \in \mathcal{L}$ may be written as $\alpha = v_1 \alpha_1 + \ldots + v_n \alpha_n$ for $v_1, \ldots, v_n \in \mathbf{Z}$, and thus there is a meromorphic character

$$f_\alpha = (f_{\alpha_1})^{v_1} \ldots (f_{\alpha_n})^{v_n}.$$

This character will be holomorphic if and only if f_α vanishes to nonnegative order on S_k for each k. Thus we have the condition $\alpha \in \mathcal{L}_{\mathcal{O}}(M)$ if and only if

$$\sum_{j=1}^{n} \mu_{j,k} v_j \geqq 0$$

for $k = 1, 2, 3, \ldots$. Thus we have the statement of the Lemma if we take

$$\beta_k = \sum_{j=1}^{n} \mu_{j,k} \alpha_j^*$$

where $\{\alpha_1^*, \ldots, \alpha_n^*\}$ is the dual basis to $\{\alpha_1, \ldots, \alpha_n\}$.

Inspecting of proof of Lemma 1.6, we see that we also have the following.

Corollary 1.7. *To each irreducible component S_j of S there corresponds an element $\beta_j \in \mathscr{L}^*$ with the following two properties:*

(i) *f_α is identically zero on S_j if and only if $\langle a, b_j \rangle > 0$.*

(ii) *the set $\{a \in \mathscr{L}_{\mathcal{O}}(M): \langle \alpha, \beta_j \rangle = 0\}$ contains $(n-1)$ linearly independent elements.*

Proof. The property (i) follows from the condition that f_α vanishes to positive order on S_j. Property (ii) follows because $n-1$ functions are required to separate points on S_j.

§ 2. Local Linearization

Here we discuss the local linearization of a $\mathbf{T}^n$-action: everything can be made locally equivalent to a domain with an A-Reinhardt action. Throughout this section, we will assume that $\mathbf{T}^n$-orbits are totally real, but M is not required to be holomorphically separable (see Lemma 1.2). At the end of this Section we give an example of a separable manifold which is locally but not globally Reinhardt.

Lemma 2.1. *Let $\mathbf{T}^n \cdot z_0 \subset M^n$ be a fixed $\mathbf{T}^n$ orbit. Then it has a $\mathbf{T}^n$-invariant Stein neighborhood U in M.*

Proof. By averaging we can produce real $\mathbf{T}^n$-invariant functions $r_1, \ldots, r_n, \ldots, r_{n+k}$ defined near $Z = \mathbf{T}^n \cdot z_0$ such that:

(i) $Z = \{x: r_j(x) = 0, \text{all } j\}$

(ii) $dr_1 \wedge \ldots \wedge dr_n \neq 0$ on Z.

Since Z is totally real, we may assume $\partial r_1 \wedge \ldots \wedge \partial r_n \neq 0$ on Z. A standard computation then shows that $\sigma = \sum_{i=1}^{n} r_i^2$ is strongly plurisubharmonic on a sufficiently small neighborhood of Z. We then set $U = \{x: \sigma(x) < \varepsilon\}$ for $\varepsilon > 0$ small.

Proposition 2.2. *Given any $z_0 \in M$, there is a $\mathbf{T}^n$-invariant neighborhood V of z_0 and a biholomorphism $\varphi: V \to \varphi(V) \subset \mathbb{C}^n$ intertwining the $\mathbf{T}^n$-action on V and an effective linear action of $\mathbf{T}^n$ on $\mathbb{C}^n$.*

Proof. Let V be a Stein $\mathbf{T}^n$-invariant neighborhood of z_0. Let $k = \dim \mathbf{T}^n \cdot z_0$. Let Λ be the lattice generated (over $\mathbb{Z}$) by

$$\{a \in \mathscr{L}(V): f_\alpha(z_0) \neq 0\}.$$

Let $\beta_1, \beta_2, \ldots, \beta_l$ be a basis of Λ. By Corollary 1.7, $l \leq k$. The corresponding characters $f_{\beta_1}, \ldots, f_{\beta_l}$ are meromorphic functions on V, nonsingular on $T^n \cdot z_0$. We choose V small enough so $f_{\beta_j} \in \mathcal{O}(V)$. Since $\mathcal{O}(V)$ separates points, $f_{\beta_1}, \ldots, f_{\beta_l}$ must separate points on $T^n \cdot z_0$ and therefore $l = k$ and

$$df_{\beta_1} \wedge \ldots \wedge df_{\beta_k}(z_0) \neq 0.$$

Now let $g_1, g_2, \ldots, g_n \in \mathcal{O}(V)$ give local coordinates at z_0. Expanding g_i into Fourier series we find characters $f_{\sigma_1}, f_{\sigma_2}, \ldots, f_{\sigma_n}$ such that

$$df_{\sigma_1} \wedge \ldots \wedge df_{\sigma_n}(z_0) \neq 0.$$

We now let $\alpha_i = \beta_i$, $1 \leqq i \leqq k$, and choose $\alpha_{k+1} \ldots \alpha_n$ from $\{\sigma_1, \ldots, \sigma_n\}$ so

$$df_{\alpha_1} \wedge \ldots \wedge df_{\alpha_n}(z_0) \neq 0.$$

The map φ: $V \to \mathbb{C}^n$ given by

$$\varphi = (f_{\alpha_1}, \ldots, f_{\alpha_n})$$

is a local biholomorphism near $T^n \cdot z_0$ and is one-to-one on $T^n \cdot z_0$. Shrinking V if necessary, we achieve that φ is one-to-one and thus is the desired biholomorphism.

Corollary 2.3. *If the isotropy subgroup $T^n_{z_0}$ is trivial, then the map φ in Proposition 2.2 may be chosen so that φ: $V \to \varphi(V) \subset \mathbb{C}^n$ is equivariant with the usual (Reinhardt) $\mathbf{T}^n$-action on $\mathbb{C}^n$, i.e., $\varphi(V)$ is a Reinhardt domain. Further, in this case, φ is uniquely determined by the value $\varphi(z_0)$.*

Corollary 2.4. *For $z_0 \in S$, there exists $1 \leqq d < n$ such that the local linearizing map φ takes a neighborhood of z_0 in S to $\{z \in \mathbb{C}^n : |z - \varphi(z_0)| < \varepsilon, z_1 \ldots z_d = 0\}$.*

Now we discuss the possibility of continuing φ analytically to all of M. Let $\pi_1(\mathbf{T}^n) \subset \pi_1(M)$ denote the subgroup generated by a $\mathbf{T}^n$-orbit. This is a normal subgroup, and $\pi_1(M)/\pi_1(\mathbf{T}^n)$ represents the fundamental group of the orbit space $M/\mathbf{T}^n$.

Theorem 2.5. *If $S = \phi$ and if $\pi_1(M)/\pi_1(\mathbf{T}^n) = 0$ then M is equivariantly equivalent to a Reinhardt Riemann domain over $(\mathbb{C}^*)^n$.*

Proof. Let us start with $z_0 \in M$. By Lemma 2.2 there is a $\mathbf{T}^n$-invariant neighborhood V of z_0 and characters $f_{\alpha_j} \in \mathcal{O}(V)$, $1 \leqq j \leqq n$ such that $\varphi = (f_{\alpha_1}, \ldots, f_{\alpha_n})$ is an equivariant imbedding of V into $\mathbb{C}^n$. Since $S = \phi$, this imbeds V as a Reinhardt domain in $(\mathbb{C}^*)^n$. If γ is any path in M starting at z_0 we may continue f_{α_j}, and thus φ, along γ to analytic function which we denote by $\tilde{f}_{\alpha_j}$, and $\tilde{\varphi}$.

If γ lies in $\mathbf{T}^n \cdot z_0$, then it is clear that the continuation is trivial and $\varphi = \tilde{\varphi}$. The hypothesis that $\pi_1(M)/\pi_1(\mathbf{T}^n) = 0$ means that any loop based at z_0 may be deformed to a closed curve lying in $\mathbf{T}^n \cdot z_0$. Thus $\varphi(z_0) = \tilde{\varphi}(z_0)$ for all closed curves γ, and thus φ is globally defined on M.

Since φ is a local imbedding and equivariant, γ: $M \to (\mathbb{C}^*)^n$ gives M the structure of a Reinhardt Riemann domain.

It is possible to show, by a power series argument, that if M is holomorphically separable, then φ is one to one and thus in this case M is a Reinhardt domain. We do not do this here, however, since we will prove this same result in Section 3 by other means.

Example 2.6. Let M be as in Theorem 2.5 except that we do not assume $\pi_1(M) = \pi_1(\mathbf{T}^n \cdot z_0)$. Then M is covered by a Reinhardt Riemann domain.

To see this, let X be the universal cover of the manifold $M/\mathbf{T}^n$. Let $\tilde{M}$ be the pullback to X of the principal $\mathbf{T}^n$ bundle M over $M/\mathbf{T}^n$:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{p} & M \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & M/\mathbf{T}^n \end{array}.$$

Then $p\colon \tilde{M}\to M$ is a $\mathbf{T}^n$-equivariant covering map, and we can apply Theorem 2.5 to $\tilde{M}$.

A concrete example of such an M may be obtained as follows: Let $N\subset(\mathbb{C}^*)^2$ be a Reinhardt domain for which the base $N/\mathbf{T}^2$ is an annulus. We cut N equivariantly to annihilate $\pi_1(N/\mathbf{T}^2)$ and then reglue it with a fixed twist $(z_1, z_2)\simeq(e^{2\pi i\kappa_1}z_1, e^{2\pi i\kappa_2}z_2)$ along the $\mathbf{T}^2$-orbits. If $\kappa_1, \kappa_2\notin\mathbb{Z}$, then the resulting manifold $\hat{N}$ is not holomorphically separable. And if κ_1, $\kappa_2\notin\mathbb{Q}$, then $\hat{N}$ has no holomorphic characters (and thus no nonconstant holomorphic functions) at all.

Example 2.7. We will construct a holomorphically separable 2-manifold M with $\mathbf{T}^2$-action as a union of two coordinate charts. M will not be biholomorphic to a Reinhardt domain. We let

$$\Omega_1=\{(z_1,z_2)\in\mathbb{C}^2\colon\ \tfrac{1}{4}<|z_1|<\tfrac{5}{4}, |z_2|<\tfrac{5}{4}\}$$
$$\Omega_2=\{(\tilde{z}_1,\tilde{z}_2)\in\mathbb{C}^2\colon\ |\tilde{z}_1|<\tfrac{5}{4}, \tfrac{3}{4}<|\tilde{z}_2|<\tfrac{5}{4}\}.$$

We give Ω_1 the ordinary Reinhardt action $z\cdot\theta=(e^{i\theta_1}z_1, e^{i\theta_2}z_2)$. For Ω_2 we set $A=\begin{pmatrix}2&3\\1&2\end{pmatrix}$, and we use the action

$$\theta\cdot_A\tilde{z}_1=A\theta\cdot\tilde{z}=(e^{i\tilde{\theta}_1}\tilde{z}_1, e^{i\tilde{\theta}_2}\tilde{z}_2)$$

where $\tilde{\theta}=A\theta$.

Let us define the map

$$f(z)=z^A=\begin{pmatrix}z_1^2 & z_2^3\\ z_1 & z_2^2\end{pmatrix},$$

and $U=\{z\in\Omega_1\colon f(z)\in\Omega_2\}$. Thus $f\colon U\to f(U)$ is an equivariant map from the $\mathbf{T}^2$ action on Ω_1 to the $\mathbf{T}^2$ action on Ω_2, and we may define

$$M=(\Omega_1\cup\Omega_2)/\sim$$

where we identify $z\sim\tilde{z}$ for $z\in\Omega_1\cap U$, $\tilde{z}\in\Omega_2\cap f(U)$, and $f(z)=\tilde{z}$.

It follows, then, that

$$S=\{z\in\Omega_1\colon\ z_2=0\}\cup\{\tilde{z}\in\Omega_2\colon\ \tilde{z}=0\}=S_1\cup S_2.$$

Further, the isotropy subgroups are

$$\mathbf{T}^2_{S_1}=\left\{\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\colon\ \theta_1=0\right\}\quad\text{and}\quad\mathbf{T}^2_{S_2}=\left\{\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\colon\ (A\theta)_2=0\right\}=\{\theta_1+2\theta_2=0\}.$$

On the basis of the isotropy subgroups we will see that the action on M is not equivalent to a linear action on $\mathbb{C}^2$. For it it were, it would have to have a $\mathbf{T}^2$-action for which the isotropy subgroups were $\{\theta_1=0\}$ and $\{\theta_2=0\}$. But the only changes of $\mathbf{T}^2$ action on M are given by changes of basis on $\mathbf{T}^2$. These would be given by taking the existing action on M and pre-multiplying it by some matrix $\mathbf{T}\in GL(2,\mathbb{Z})$. If the isotropy group $\mathbf{T}^2_{S_1}=\{\theta_1=0\}$ is to preserved, then $\mathbf{T}$ must have the form $\mathbf{T}=\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. It follows, then, that $\mathbf{T}=\begin{pmatrix} \pm 1 & 0 \\ k & \pm 1 \end{pmatrix}$ for some $k\in\mathbb{Z}$. But now the isotropy subgroup of S_2 becomes

$$\mathbf{T}^2_{S_2}=\left\{\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}:\ (AT\theta)_2=0\right\}=\{(\pm 1+2k)\,\theta_1\pm 2\theta_2=0\}.$$

Since $\pm 1+2k$ is always odd, this can never be changed to $\{\theta_2=0\}$, which would be necessary for M to be Reinhardt.

Next we see what the characters on M are and that M is holomorphically separable. If $f\in\mathcal{O}(M)$, then it has a Laurent series on Ω_1

$$\begin{aligned} f(z)&=\sum a_{j_1 j_2}\, z_1^{j_1}\, z_2^{j_2}, \qquad j_2\geqq 0,\\ &=\sum a_J\, z^J \end{aligned}$$

and it also has a Laurent series on Ω_2

$$f(z)=\sum \tilde{a}_{k_1 k_2}\, \tilde{z}_1^{k_1}\, \tilde{z}_2^{k_2}, \qquad k_1\geqq 0.$$

To see the relation between a_J and $\tilde{a}_K$, we write the first series using the identification $\tilde{z}=z^A$, so that

$$f(z)=\sum a_J(\tilde{z}^{A^{-1}})^J=\sum a_J\, \tilde{z}^{(JA^{-1})}$$

so that $a_J=\tilde{a}_K$ with $K=JA^{-1}$. It follows that the sum is taken over

$$\mathscr{L}=\{J\in\mathbb{Z}^2:\ J_{(2)}\geqq 0, (JA^{-1})_{(1)}\geqq 0\}=\{(j_1,j_2):\ j_2\geqq 0, 2j_1-j_2\geqq 0\},$$

which is easily seen to be the lattice of characters.

At this stage we also see that M is not Reinhardt since $\mathscr{L}=\mathscr{L}_{\mathcal{O}}(M)$ is not equivalent as a semigroup to $(\mathbb{Z}_+)^2$. This is because $\mathscr{L}$ has minimal elements $e_1=(1,0)$, $e_2=(1,1)$, and $e_3=(1,2)$ which are needed for any set of generators of $\mathscr{L}$. The corresponding characters give an imbedding of M into $\mathbb{C}^3$ via

$$F(z)=(f_{e_1}, f_{e_2}, f_{e_3})=(z_1, z_1 z_2, z_1 z_2^2)=(\zeta_1,\zeta_2,\zeta_3).$$

By the relation $e_1+e_3=2e_2$, it follows that $f(M)$ is an open subset of the variety $V=\{\zeta_1\zeta_3=\zeta_2^2\}$. The $\mathbf{T}^2$ action on $M\simeq f(M)\subset V$ is given by

$$\theta\cdot\zeta=(e^{i\theta\cdot e_1}\zeta_1, e^{i\theta\cdot e_2}\zeta_2, e^{i\theta\cdot e_3}\zeta_3).$$

§ 3. Holomorphically Separable Manifolds

In this Section we give first the proof of Theorem 1. Then we give most of the proof of Theorem 3, with only the part concerning envelopes of holomorphy being deferred to Section 4.

Proposition 3.1. *Let M be a connected Stein manifold with $S \neq \phi$. Then there are at most n distinct S_j's, and $\bigcap S_j \neq \phi$.*

Proof. Let $S_1, S_2, \ldots$ denote the irreducible components of S. Since S is $\mathbf{T}^n$-invariant, so is each S_j. We need to show that $\bigcap_{j=1}^{p} S_j \neq \phi$ for all p, and we proceed by induction. Let us assume that $\bigcap_{j=1}^{k} S_j \neq \phi$ and $\bigcap_{j=1}^{k+1} S_j = \phi$. Then since M is Stein, there exists an analytic function f which is $=0$ on $\bigcap_{j=1}^{k} S_j$ and $=1$ on S_{k+1}. If we average f over $\mathbf{T}^n$, then we obtain a function which is constant on orbits and thus globally constant. On the other hand, the function remains the same on $\bigcap_{j=1}^{k} S_j$ and S_{k+1}, which is a contradiction.

We see that there are at most n S_j's because if there are $(n+1)$ of them, then there is a point $z_0 \in S_1 \cap \ldots \cap S_{n+1}$. But this is a contradiction to Corollary 2.4.

Proof of Theorem 1. We need to show that statements (i) and (ii) are equivalent.

(i)$\Rightarrow$(ii). This is clear, since every Reinhardt domain has an envelope of holomorphy (its logarithmically convex hull).

(ii)$\Rightarrow$(i). Let $\tilde{M}$ be the Stein manifold that is the envelope of holomorphy of M. Then every $\phi \in \operatorname{Aut}(M)$ extends to $\tilde{\phi} \in \operatorname{Aut}(\tilde{M})$, and so the $\mathbf{T}^n$-action extends to $\tilde{M}$. To see that the action is smooth on $\tilde{M}$, we note that the Lie algebra of $\mathbf{T}^n$ is given by holomorphic vector fields Z on M. Since $\tilde{M}$ Stein, we may imbed it in $\mathbb{C}^m$, $m=2n+1$, and so Z may be written on M as $Z = \sum a_j \frac{\partial}{\partial z_j}$. Since we may extend each holomorphic function a_j to $\tilde{M}$, we may extend the vector field Z, too, and so the action is smooth.

We let $\tilde{S}$ denote the subset of $\tilde{M}$ where the $\mathbf{T}^n$-action in singular. By Proposition 3.1, there exists a point $z_0 \in \cap \tilde{S}_j$. Let U be a $\mathbf{T}^n$-invariant neighborhood of z_0. Then $\mathscr{L}_{\mathcal{O}}(U) = \mathscr{L}_{\mathcal{O}}(\tilde{M})$ by the criterion of Section 1. That is, $\alpha \in \mathscr{L}_{\mathcal{O}}(\tilde{M})$ if and only if the set or linear inequalities in Lemma 1.6 are satisfied. Since these involve conditions on the $\tilde{S}_j$, the inequalities will hold if and only if they hold in a neighborhood of z_0, and thus they are the same for U or $\tilde{M}$.

Finally, by Proposition 2.2, we may linearize the T^n-action in a neighborhood U of $\mathbf{T}^n \cdot z_0$, and the mapping is given as $f = (f_{\alpha_1}, \ldots, f_{\alpha_n})$ with α_j generating $\mathscr{L}_{\mathcal{O}}(U)$. Since $\mathscr{L}_{\mathcal{O}}(U) = \mathscr{L}_{\mathcal{O}}(\tilde{M})$, it follows that f is holomorphic on $\tilde{M}$. Also, f is one-one, since f_{α_j} generate $\mathcal{O}(\tilde{M})$ and $\tilde{M}$ is holomorphically separable. Thus f is a biholomorphism.

Proof of Theorem 2. Here we use the letter M to denote our copy of $\mathbb{C}^n$ with the given $\mathbf{T}^n$-action. Carrying through the proof of Theorem 1 in this case, we obtain a biholomorphic mapping $f=(f_{\alpha_1}, \dots, f_{\alpha_n})$: $M \to \mathbb{C}^n$ that linearizes the action. Since a character f_α may be obtained by averaging a polynomial over $\mathbf{T}^n$,

$$f_\alpha(z) = \int_{\mathbf{T}^n} p(\theta \cdot z)\, e^{-i\alpha \cdot \theta} d\theta,$$

it follows that f_α is itself a polynomial. Finally, it is known that if f is a polynomial mapping which is biholomorphic, then the inverse is a polynomial mapping ([9, 10]) which completes the proof.

Theorem 3.2. *The equivalent conditions* (i) *and* (ii) *of Theorem 1 are also equivalent to the condition that* $\mathscr{L}_{\mathscr{O}}(M) \cong \mathbb{Z}^k \times (\mathbb{Z}_+)^{n-k}$.

Proof. Let us first assume that (i) and (ii) are satisfied. Without loss of generality, we may renumber the coordinates such that $\{z_j=0\} \cap M = \emptyset$ if and only if $k+1 \leq j \leq n$. Thus $z_1, z_1^{-1}, \dots, z_k, z_k^{-1}, z_{k+1}, \dots, z_n$ generate $\mathscr{O}(M)$.

Conversely, we suppose that $\mathscr{L}_{\mathscr{O}}(M) \cong \mathbb{Z}^k \times (\mathbb{Z}_+)^{n-k}$, and we let $e_j = (0, \dots, 1, \dots 0)$ where the 1 is in the j-th slot. If f_{e_j} denotes the corresponding holomorphic character, then $(f_{e_1}, \dots, f_{e_n})$: $M \to \mathbb{C}^n$ maps M to a Reinhardt domain inside $\{z_1 \dots z_k \neq 0\}$.

Corollary 3.3. *If M is as in Theorem 1, and if there is at most one nontrivial subgroup of $\mathbf{T}^n$ that appears as an isotropy subgroup then the action on M is equivariantly equivalent to a linear action on $\mathbb{C}^n$.*

Proof. Suppose there is only one isotropy subgroup, which must be given as $\mathbb{R}\beta$ for some $\beta \in \mathscr{L}^*$. Then, by Lemma 1.6 and Corollary 1.7, it follows that $\mathscr{L}_{\mathscr{O}}(M) = \{\alpha \in \mathscr{L}: \langle \alpha, \beta \rangle > 0\}$, where we replace β by $-\beta$ if necessary. Thus $\mathscr{L}_{\mathscr{O}}(M)$ is equivalent to $\mathbb{Z}_+ \times \mathbb{Z}^{n-1}$, and the result follows from Theorem 3.2.

Now let us discuss the condition that $\mathscr{L}_{\mathscr{O}}(M)$ is finitely generated as a semigroup. Let

$$\mathscr{L}_0 = \mathscr{L}_{\mathscr{O}}(M) \cap -\mathscr{L}_{\mathscr{O}}(M)$$

denote the largest subgroup of $\mathscr{L}_{\mathscr{O}}(M)$. Since $\mathscr{L}_0 \subset \mathbb{Z}^n$, it follows that $\mathscr{L}_0$ has rank $\leq n$ and thus is generated by $\leq n$ elements. Thus $\mathscr{L}_{\mathscr{O}}(M)$ is finitely generated if and only if $\mathscr{L}_{\mathscr{O}}(M)/\mathscr{L}_0$ is.

We say that an element $\alpha \in \mathscr{L}_{\mathscr{O}}(M)$ is *minimal* if whenever $\beta_1, \beta_2 \in \mathscr{L}_{\mathscr{O}}(M)$, $\beta_2 + \beta_2 = \alpha$, we have $\beta_1 \in \mathscr{L}_0$ or $\beta_2 \in \mathscr{L}_0$. We see that if $\mathscr{L}_{\mathscr{O}}/\mathscr{L}_0 \neq 0$, then $\mathscr{L}_{\mathscr{O}}$ has minimal elements as follows.

Let $\hat{\mathscr{L}}_{\mathscr{O}}$ and $\hat{\mathscr{L}}_0$ denote the convex hulls of $\mathscr{L}_{\mathscr{O}}$ and $\mathscr{L}_0$. Thus $\hat{\mathscr{L}}_{\mathscr{O}}/\hat{\mathscr{L}}_0$ is a subcone of $\mathbb{R}^n/\hat{\mathscr{L}}_0$ which contains no linear subspaces. It follows, then, that $\hat{\mathscr{L}}_{\mathscr{O}}/\hat{\mathscr{L}}_0$ is the convex hull of its extreme points. Now since $\mathscr{L}_{\mathscr{O}}(M)$ is a polyhedral cone generated by linear inequalities over $\mathbb{Z}^n$, it contains a point corresponding to every extreme ray. For each such extreme ray, $\mathscr{L}_{\mathscr{O}}(M)$ contains a minimal element. In general, $\mathscr{L}_{\mathscr{O}}(M)$ also contains other minimal elements.

Finally, we make the observation that $\mathscr{L}_{\mathscr{O}}(M)$ *is finitely generated if and only if* $\hat{\mathscr{L}}_{\mathscr{O}}(M)/\hat{\mathscr{L}}_0$ *has a finite number of extreme rays*, generated by $\alpha_1, \dots, \alpha_N$. The only thing nontrivial about this assertion is that if there is a finite set $\alpha_1, \ldots, \alpha_N$, then $\mathscr{L}_{\mathcal{O}}(M)/\mathscr{L}_0$ is finitely generated. But it is geometrically evident that a set of generators is given by a fundamental polyhedron in this lattice, i.e.,

$$\mathscr{L} \cap \text{convex hull} \{\alpha_{i_1} + \ldots + \alpha_{i_j}\colon 1 \leqq i_1 \leqq \ldots \leqq i_j \leqq N\}.$$

We also see that $\mathscr{L}_{\mathcal{O}}$ *is finitely generated if and only if* $\mathscr{L}_{\mathcal{O}}/\mathscr{L}_0$ *has a finite number of minimal elements.*

Let us make the definition that $\mathcal{O}(M)$ is *generated by k functions* if there are $\varphi_1, \ldots, \varphi_k \in \mathcal{O}(M)$ such that the set of polynomials in $\varphi_1, \ldots, \varphi_k$ is dense in $\mathcal{O}(M)$ in the sense of uniform convergence on compact subsets of M.

Proposition 3.4. *The semigroup $\mathscr{L}_{\mathcal{O}}(M)$ is finitely generated if and only if $\mathcal{O}(M)$ is. In particular, if $\mathcal{O}(M)$ has a finite set of generators, it is generated by a finite number of characters.*

Proof. If $\mathscr{L}_{\mathcal{O}}M)$ is generated by $\alpha_1, \ldots, \alpha_N$, then $f_{\alpha_1}, \ldots, f_{\alpha_N}$ generate f_α for all $\alpha \in \mathscr{L}_{\mathcal{O}}(M)$. Thus polynomials in the $f_{\alpha_1}, \ldots, f_{\alpha_N}$ are dense in $\mathcal{O}(M)$.

Conversely, let $\{\varphi_1, \ldots, \varphi_N\} \subset \mathcal{O}(M)$ be a set of generators. Without loss of generality we may assume that $\varphi_1 = 1$ is constant, and $\varphi_2, \ldots, \varphi_N$ contain no constant terms in their Fourier expansions. Let $\mathscr{L}_0(M)/\mathscr{L}_0$ have distinct minimal elements represented by $\{\alpha_1, \alpha_2, \ldots\}$, and for $j \geqq 2$ let us write

$$\varphi_j = \sum_k c_j^k f_{\alpha_k} + \varphi'_j$$

where φ'_j is the part of the Fourier series that does not involve any minimal elements, and Fourier coefficients of c_j^k lie in $\mathscr{L}_0$. We note that any product $\varphi_{i_1}, \ldots, \varphi_{i_p}$ of generators cannot involve any minimal elements. The reason is that φ_j contains no constant term (corresponding to $\alpha = 0$) and so all the terms $f_{\alpha_r} f_{\alpha_s} = f_{\alpha_r + \alpha_s}$ are compound. Thus all of the minimal terms f_{α_k} that can be generated by polynomials in $\{\varphi_1, \ldots, \varphi_N)$ can be generated already by linear combinations. Thus there can be at most N minimal elements, and thus $\mathscr{L}_{\mathcal{O}}(M)$ is finitely generated.

Corrolary 3.5. *The minimal number of generators for $\mathcal{O}(M)$ is $2k+m$, where k is the rank of $\mathscr{L}_0$ and m is the number of minimal elements of $\mathscr{L}_{\mathcal{O}}(M)/\mathscr{L}_0$.*

Proof of Theorem 3. Our proof will proceed as follows.

(ii)$\Rightarrow$(iii). This is a direct consequence of Theorem 4.1 which is proved in Section 4.

(i)$\Rightarrow$(ii). By Proposition 3.4, $\mathscr{L}_{\mathcal{O}}(M)$ is finitely generated with generators $\gamma_1, \ldots, \gamma_N$. Let $F = (f_{\gamma_1}, \ldots, f_{\gamma_N})$ denote the mapping given by the corresponding characters. Thus the mapping $F\colon M \to \mathbb{C}^N$ is an holomorphic mapping which is equivariant to the $\mathbf{T}^n$ action $\theta \cdot \zeta = (\zeta_1 e^{i\gamma_1 \cdot \theta}, \ldots, \zeta_N e^{i\gamma_N \cdot \theta})$. Since the f_{γ_j} generate $\mathcal{O}(M)$, it follows that F is one-to-one.

To show that F is biholomorphic it suffices to show that dF is injective at any point $z_0 \in M$. By Proposition 2.2, we may choose local coordinates so that $z_0 = (0, \ldots, 0, 1, \ldots, 1)$ with d 0's and $n-d$ 1's, and the $\mathbf{T}^n$-action is equivalent to the standard Reinhardt action. Identifying $\mathscr{L}$ with $\mathbb{Z}^n$ via this equivalence, we conclude that $\mathscr{L}_{\mathcal{O}}(M)$ is contained in $\mathbb{Z}_+^d \times \mathbb{Z}^{n-d}$. Since $\mathcal{O}(M)$ separates points of $\mathbf{T}^n \cdot z_0$ it follows that $\mathscr{L}_{\mathcal{O}}(M) \cap (\{0\}^d \times \mathbb{Z}^{n-d})$ is not contained in any proper linear subspace of $\{0\}^d \times \mathbb{Z}^{n-d}$. Thus we may choose linearly independent elements $\alpha_{n-d+1}, \ldots, \alpha_n$ in $\mathscr{L}_{\mathcal{O}}(M) \cap (\{0\}^d \times \mathbb{Z}^{n-d})$. Further, we may choose γ in the interior of $\mathscr{L}_{\mathcal{O}}(M) \cap (\{0\}^d \times \mathbb{Z}^{n-d})$, i.e., γ does not lie in any of the hyperplanes that define the boundary.

Now for M sufficiently large, $\alpha_j = M\gamma + e_j$ lies in $\mathscr{L}_{\mathcal{O}}(M)$ for $1 \leqq j \leqq d$, where e_j is a standard basis element. We see that the characters $f_{\alpha_1}, \ldots, f_{\alpha_n}$ give local coordinates at z_0; this follows from the identity

$$d f_{\alpha_1} \wedge \ldots \wedge d f_{\alpha_n} = \det A \cdot z^{\alpha_1 + \cdots + \alpha_n} (z_1 \ldots z_n)^{-1} \, d z_1 \wedge \ldots \wedge d z_n$$

where A is the matrix with column vectors $\alpha_1, \ldots, \alpha_n$. Finally, since $\gamma_1, \ldots, \gamma_N$ are generators of $\mathscr{L}_{\mathcal{O}}(M)$, each f_{α_j} is a monomial in the functions $f_{\gamma_1}, \ldots, f_{\gamma_N}$. Thus dF is injective at z_0.

Finally $F(M)$ is an open subset of the variety $V = \cap \{\zeta_1^{m_1} \ldots \zeta_N^{m_N} = C\}$ where the intersection is taken over all $m = (m_1, \ldots, m_N) \in \mathbb{Z}^N$ such that $m_1 \gamma_1 + \ldots + m_N \gamma_N = 0$, and $C = f_{\gamma_1}^{m_1}(z_0) \ldots f_{\gamma_N}^{m_N}(z_0)$ for some $z_0 \in M$. It is evident that $F(M) \subset V$; what we must show is that $F(M)$ does not have strictly smaller dimension. To see that this cannot happen we consider the mapping η: $\mathbb{Z}^N \to \mathscr{L}$ given by $\eta(k_1, \ldots, k_N) = k_1 \gamma_1 + \ldots + k_N \gamma_N$, and we let $\mathscr{M} = \mathbb{Z}^N / \ker \eta$, so that $\mathscr{M}$ is a lattice isomorphic via η to $\mathscr{L} \cong \mathbb{Z}^n$.

Now if $F(M)$ is not open in V, there is locally a holomorphic function h which is nonconstant on V, and such that $\{h=0\} \supset F(M)$. Without loss of generality, we may assume that

$$h(\zeta) = \sum_{\beta \in \mathscr{M}} \alpha_\beta \, \zeta^\beta,$$

i.e. the only powers come from $\mathscr{M}$. But now

$$h(F(\theta \cdot z)) = \sum_{\beta \in \mathscr{M}} a_\beta \, e^{i\theta \cdot \eta(\beta)} F(z)^\beta$$

vanishes for all $\theta \in \mathbf{T}^n$. Since η: $\mathscr{M} \to \mathscr{L}$ is an isomorphism, we conclude that $a_\beta = 0$ for all β. Thus $h = 0$ on V, which completes the proof.

(iii)$\Rightarrow$(i). Let us suppose that there is a Stein space $\tilde{M}$ which is the envelope of holomorphy of M. Since $\mathcal{O}(\tilde{M})$ is a Stein algebra, it is finitely generated, and the same set of generators also generates $\mathcal{O}(M)$.

The Case of Dimension Two

Proof of Theorem 4. If $\mathscr{L}_{\mathcal{O}}(M) \neq \mathbb{Z}^2$ then it is the intersection of rational half-spaces in $\mathbb{Z}^2$, and by Corollary 1.7, there are only two possibilities: $\mathscr{L}_{\mathcal{O}}(M)$ is a rational half-space, or it is the intersection of two of them. In either case, it is immediate that $\mathscr{L}_{\mathcal{O}}(M)$ if finitely generated, which completes the proof.

A more precise version of this result is as follows.

Theorem 3.5. *Let M be a connected, holomorphically separable 2-manifold with smooth, effective $\mathbf{T}^2$-action. Then $M = M_1 \cup M_2$ where M_j is equivariantly equivalent to a Reinhardt domain in $\mathbb{C}^2$ with an A_j-action. In particular, M can have at most two distinct, nonzero isotropy subgroups $\mathbf{T}^2_z$.*

Proof. As was noted in the proof of Theorem 4, $\mathscr{L}_{\mathcal{O}}(M)$ is either a rational half-space in $\mathbb{Z}^2$ or the intersection of two of them. It suffices to consider the second case, in which

$$\mathscr{L}_{\mathcal{O}}(M) = \{\alpha \in \mathscr{L} : \langle \alpha, \beta_j \rangle \geqq 0, j = 1, 2\}.$$

The set $S = S_1 \cup S_2$ where the action is singular is given by $S_j = \{f_{\alpha_j} = 0\}$, where we let α_j be any $\alpha \in \mathscr{L}_{\mathcal{O}}(M)$ such that $\langle \alpha_j, \beta_j \rangle = 0$. Thus the manifolds $M_1 = M - S_2$ and $M_2 = M - S_1$ have the desired properties, and the only nontrivial isotropy subgroups that occur are the 1-tori generated by $\mathbb{R}\beta_j$.

An analogous result holds for $n \geqq 2$, but in this case we write $M = M_1 \cup \ldots \cup M_J$, where M_j corresponds to the j-th face in the boundary of $\mathscr{L}_{\mathcal{O}}(M)$, orthogonal to β_j. Of course if $n > 2$, then J can be arbitrarily large.

§ 4. Envelopes of Holomorphy

We let $X \subset \mathbb{C}^N$ denote a variety of the form

$$X = \bigcap_{j=1}^{\infty} \{\zeta \in \mathbb{C}^N : \zeta^{\alpha_j} = c_j\}$$

for some sequence $\{\alpha_j\} \subset \mathbb{Z}^N$ and $\{c_j\} \subset \mathbb{C}$. We will assume that the set $\{\alpha_j\}$ has the property: if $m\alpha \in \{\alpha_j\}$ for some positive integer m, then $\alpha \in \{\alpha_j\}$. It follows that X is a normal variety (see [8], p. 5). Our main result is the following.

Theorem 4.1. *Let X be as above, and let $\Omega \subset X$ be a connected, open set. Then there exists an open set $\hat{\Omega} \subset X$ containing Ω such that*

(i) *$\hat{\Omega}$ is Stein.*

(ii) *every $f \in \mathcal{O}(\Omega)$ extends uniquely to an element $\hat{f} \in \mathcal{O}(\hat{\Omega})$.*

Let us define $\tilde{\mathscr{L}} \subset \mathbb{Z}^N$ by

$$\tilde{\mathscr{L}} = \{\alpha \in \mathbb{Z}^N : \zeta^\alpha \in \mathcal{O}(\Omega)\}.$$

Since X is a normal complex space, a function is holomorphic if it is weakly holomorphic (i.e., if it is holomorphic on the regular points and locally bounded at the singular points). Thus $\tilde{\mathscr{L}} = \left\{\alpha \in \mathbb{Z}^N : \zeta^\alpha \text{ is locally bounded near } \Omega \cap \left(\bigcup_{j=1}^{N} \{\zeta_j = 0\}\right)\right\}$.

Next we define $\mathscr{a}$ to be the closure of $\mathbb{Q}^+ \tilde{\mathscr{L}}$ in $\mathbb{R}^N$.

Lemma 4.2. *If $a \in \mathscr{a}$ then*

$$\mathscr{u}(\zeta, a) = \sum a_i \log |\zeta_i|$$

is locally bounded above on $\Omega \cap \left\{ \bigcup_{j=1}^{N} \{\zeta_j = 0\} \right\}$.

Proof. Let $\{a^j\} \subset \mathbb{Q}^+ \tilde{\mathscr{L}}$ be a sequence converging to $a^0 \in \mathbb{R}^N$, and let $\zeta^0 \in \Omega \cap \left(\bigcup_{j=1}^{N} \{\zeta_j = 0\} \right)$. Since $\Omega \cap \left(\bigcup_{j=1}^{N} \{\zeta_j = 0\} \right)$ is a subvariety, we may find a compact set K inside $\Omega - \left(\bigcup_{j=1}^{N} \{\zeta_j = 0\} \right)$ such that

$$\varphi(\zeta_0) \leqq \sup_K \varphi$$

holds for all φ psh on Ω. Thus we have

$$u(\zeta_0, a^j) \leqq \sup_{\zeta \in K} u(\zeta, a^j).$$

Now the right hand side is bounded above independently of j so we obtain

$$u(\zeta_0, a^0) \leqq C.$$

For points ζ_1 near ζ_0, we may have a similar maximum principle with K_1 near K, and this completes the proof.

Now we set

$$\mathscr{a}_1 = \mathscr{a} \cap \{x \in \mathbb{R}^N : |x| = 1\},$$

so that $\mathscr{a}_1$ is compact. For $K \subset \Omega$ a fixed compact set, we define

$$c_K(a) = \sup_{\zeta \in K} |\zeta|^a$$

for $a \in \mathscr{a}_1$. We note that $c_K(a)$ is continuous on $\mathscr{a}_1$.

Lemma 4.3.

$$K^\varepsilon := \{\zeta \in X : |\zeta|^a \leqq c_K(a) + \varepsilon \text{ for all } a \in \mathscr{a}_1\}$$

contains a neighborhood of K in X.

Proof. This follows from the compactness of $\mathscr{a}_1$. We must show that if $x \in K$, then there exists $\delta_x > 0$ such that $B(x, \delta_x) \subset K^\varepsilon$. For each $a \in \mathscr{a}_1$ let δ_x^a be the largest positive number such that $B(x, \delta_x^a) \subset \{|\zeta|^a \leqq c_K(a) + \varepsilon\}$. Now $\delta_x^a > 0$ and δ_x^a varies continuously with a, so

$$\delta_x := \min_{\mathscr{a}_1} d_x^a > 0.$$

One final notation:

$$\tilde{K} = \{\zeta \in X : |\zeta|^a \leqq c_K(a) \text{ for all } a \in \mathscr{a}_1\}.$$

Proof of Theorem 4.1. Let us define $\hat{\Omega} = \cup \tilde{K}$, where we take the union over all compact $K \subset \Omega$. First we show that $\tilde{\Omega}$ is open. Let $L \subset \Omega$ be a compact set with $K \subset \operatorname{int} L$. It is easily seen, then, that $c_K(a) < c_L(a)$ for all $a \in \mathscr{a}_1$. Thus there exists $\varepsilon > 0$ such that $c_K + \varepsilon < c_L$. We conclude, then, by Lemma 4.3, that

$$\tilde{K} \subset \operatorname{int} \tilde{L} \subset \Omega.$$

Thus $\hat{\Omega}$ is open.

If $K \subset \hat{\Omega}$ is compact, then the holomorphic hull

$$\hat{K} = \{z \in \Omega : |f(z)| \leqq \sup_K |f| \text{ for all } f \in \mathcal{O}(\hat{\Omega})\}$$

satisfies $\hat{K} \subset \tilde{K}$ since $\mathbb{Q}^+ \mathscr{L}$ is dense in $\mathscr{a}$. Since also $K \subset \tilde{K}_1$ for some $K_1 \subset \Omega$ it follows that $\hat{K} \subset \tilde{K} \subset \tilde{K}_1$ is a closed subset of the compact set $\tilde{K}_1$ and hence compact. This concludes the proof that $\hat{\Omega}$ is Stein.

For the proof of (ii) we let $f \in \mathcal{O}(\Omega)$ be given and expand f as a series of characters:

$$f = \sum c_\alpha f_\alpha.$$

On $\Omega \subset X$ this takes the form $f(\zeta) = \sum c_\alpha \zeta^\alpha$. Since the series converges uniformly and absolutely on compact subsets of Ω, it follows immediately that the series converges on $\hat{\Omega}$. This gives us our analytic continuation.

§ 5. Examples

In this section we give the construction of a holomorphically separable 3-dimensional complex manifold with a $\mathbf{T}^3$-action such that the equivalent properties (i), (ii), and (iii) of Theorem 3 do not hold. We will construct a separable manifold M such that $\mathscr{L}_{\mathcal{O}}(M)$ is the cone on a rational polygon with infinitely many sides. It will be shown that M cannot have a Stein envelope of holomorphy, since there would be an infinite number of irreducible components passing though a common point. Our construction, as summarized in Theorem 5.1, is a rather general procedure for making examples.

We will define our manifold M as a union of manifolds $M_0, M_1, M_2, \ldots$, modulo an identification, i.e. $M = (\cup M_k)/\sim$. For $j \geqq 1$, M_j is given as: $M_j \subset \mathbb{C}^n$ is a small neighborhood of the compact disk $\{(1, 1, \zeta) : \zeta \in \mathbb{C}, |\zeta| \leqq 1\}$, and M_j has the action

$$\theta \cdot_{A_j} z = e^{iA_j\theta} z,$$

for some $A_j \in GL(3, \mathbb{Z})$. We will later specify A_j by requiring that the 3rd column of A_j^1 is a given vector of integers $c^j = (c_1^j, c_2^j, c_3^j)$ with $\gcd(c^j) = 1$.

The set $M_0 \subset (\mathbb{C}^*)^3$ will be a Reinhardt domain with the usual Reinhardt action, given by $A_0 = I$. We will choose M_0 so that its logarithmic image is an open set $\omega \subset \mathbb{R}^3$ containing $\bigcup_{j=1}^{\infty} \{tc^j : t \in \mathbb{R}, t < 0\}$. We let ω_j denote the connected component of $\omega \cap \{x \in \mathbb{R}^3 : |x| \geqq 1\}$ which intersects $\{tc^j : t \in \mathbb{R}, t < 0\}$, and we call ω_j the j-th *leg*. We will choose the $c^1, c^2, c^3, \ldots$ and the $M_0, M_1, M_2, \ldots$ such that:

(i) the legs $\omega_1, \omega_2, \ldots$ are disjoint.

(ii) $A_j \omega \supset \log M_j$, where $\log M_j$ denotes the logarithmic image of $M_j \cap (\mathbb{C}^*)^3$ *in* $\mathbb{R}^3$.

(iii) $A_j \omega_j \cap \{\xi_3 < 2\} = \log(M_j) \cap \{\xi_3 < -2\}$.

We define M via the obvious identification: $z^{(0)} \sim z^{(j)}$ if $z^{(0)} \in M_0$, $z^{(j)} \in M^{(j)}$, and $(z^{(0)})^{A_j} = z^{(j)}$. Thus the $\mathbf{T}^3$ actions on each of the pieces match up under this identification, and it follows from (ii) that the manifold $M = (UM_j)/\sim$ is a Hausdorff, complex manifold.

The singular set for the $\mathbf{T}^3$-action consists of $S = \cup S_j$, where

$$S_j = \{(z_1, z_2, z_3) \in M_j\colon\ z_3 = 0\}.$$

The corresponding isotropy subgroup is

$$\begin{aligned} G_j &= \left\{ \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \in \mathbf{T}^3\colon\ A_j \theta = \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} \text{ for some } \psi \in \mathbf{T}^1 \right\} \\ &= \left\{ \theta = A_j^{-1} \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} \right\} = \{\xi^j\colon\ \psi \in \mathbf{T}^1\}. \end{aligned}$$

We conclude, then, that $G_i \neq G_j$ for $i \neq j$. It follows that M *cannot have a Stein envelope of holomorphy* $\hat{M}$. For if $\hat{M}$ existed, the intersection $\cap \hat{S}_j$ would have to be nonempty by Proposition 3.1. But since the isotropy subgroups are all different, $\hat{S}$ would have infinitely many irreducible components at this point, which is a contradiction.

We conclude this section by showing: *the vectors c_j can be chosen so that M is holomorphically separable.*

Repeating the calculation given at the end of Section 2, we see that the holomorphic characters are given on M_0 as $(z^{(0)})^J$, where $J \in \mathscr{L}$, and

$$\begin{aligned} \mathscr{L} &= \{J \in \mathbb{Z}^3 : (J^t A_k^{-1})_{(3)} \geqq 0, k = 1, 2, \ldots\} \\ &= \{J \in \mathbb{Z}^3 : J \cdot c^k \geqq 0, k = 1, 2, \ldots\}. \end{aligned}$$

The set of holomorphic characters which do not vanish identically on S_k is given by

$$\mathscr{L}_k = \{J \in \mathscr{L}\colon\ J \cdot c^k = 0\}.$$

Thus, for a set of vectors $\{c^j\} \subset \mathbb{Z}^3$, $\mathscr{L}$ is given as the dual cone, and $\mathscr{L}_k$ is the "face" of $\partial \mathscr{L}$ which is orthogonal to c^k.

Since S_k is a small, $\mathbf{T}^3$-invariant neighborhood of (1, 1, 0) in $\{z_3 = 0\}$, the characters of $\mathscr{L}_k$ vanish nowhere on S_k. It follows that holomorphic characters on M will separate points on each S_k if

(iv) $\mathscr{L}_k$ contains 2 linear independent points for each k.

The geometric interpretation of (iv) is that $\mathscr{L}$ is a polyhedral cone whose boundary consists of "true" 2-dimensional faces $\mathscr{L}_k$.

Finally, we observe that we may choose $\{c^j\}\subset(\mathbb{Z}_+)^3$, (or equivalently, $\{c^j\}\subset(\mathbb{Q}_+)^3$) such that (i), (ii), (iii), and (iv) hold. Since $\{c^j\}\subset(\mathbb{Z}_+)^3$, it follows that $\mathscr{L}\supset(\mathbb{Z}_+)^3$, and so holomorphic characters separate any two points $z\in M_0$ and $w\in M$. I.e., the coordinate functions z_1, z_2, z_3 on M_0 extend holomorphically to M and vanish on $\bigcup_{j=1}^{\infty} S_j$. By (iv), the characters in $\mathscr{L}_k$ separate points in S_k. Finally, by (iv), we may take linearly independent vectors $\alpha_k, \beta_k\in\mathscr{L}_k-\bigcup_{j\neq k}\mathscr{L}_j$. It follows that the characters $(z^{(k)})^{\alpha_k}$ and $(z^{(k)})^{\beta_k}$ separate points and do not vanish on S_k, but they vanish on S_j for all $j\neq k$. This completes the proof that M is holomorphically separable.

We have used this Section to give a construction of 3-dimensional examples. The same ideas may be used to give n-dimensional examples. We note here a general result that may be proved along these lines.

First if M is holomorphically separable, then $\mathscr{L}$ has the structure:

$$\mathscr{L}=\{\alpha\in(\mathbf{T}^n)^*:\ \alpha\cdot\gamma_j\geqq 0,\quad j=1,2,\ldots\},$$

for some set $\{\gamma_j\}\subset\mathscr{L}^*$. The connection between the γ_j and the S_j is as follows. By Corollary 2.4, the isotropy subgroup $\mathbf{T}^n_{S_j}$ is 1-dimensional for each j. Thus $\mathbf{T}^n_{S_j}$ is generated by an element $\gamma_j\in\mathscr{L}^*$. By Corollary 1.7, the characters that do not vanish identically on S_j are given by $\mathscr{L}_j=\{\alpha\in\mathscr{L}:\ \alpha\cdot\gamma_j=0\}$. Replacing γ_j possibly by $-\gamma_j$, we see that $\mathscr{L}$ has the representation obtained at the end of Section 1:

a) $\mathscr{L}_{\mathcal{O}}(M)=\{\alpha\in\mathscr{L}:\ \alpha\cdot\gamma_j\geqq 0$ for $j=1,2,\ldots\}$.

And for each j, $\mathcal{O}(S_j)$ is generated by the characters $\{f_\alpha:\ \alpha\in\mathscr{L}_j\}$. If M is holomorphically separable, it follows that

b) for each j, $\mathscr{L}_j$ contains n-1 linearly independent elements.

Theorem 5.1. *If $\mathscr{L}_{\mathcal{O}}\subset(\mathbf{T}^n)^*$ is of the form a) and satisfies b) above, then there is a holomorphically separable n-manifold M with a holomorphic $\mathbf{T}^n$-action such that $\mathscr{L}_{\mathcal{O}}(M)=\mathscr{L}_{\mathcal{O}}$.*

References

1. Bialnicki-Birula, A.: Remarks on the action of an algebraic torus on k^n. Bull. Pol. Acad. Sci., Math. **14**, 177–181 (1966)
2. Bialnicki-Birula, A.: Remarks on the action of an algebraic torus on k^n II. Bull. Pol. Acad. Sci. Math. **15**, 123–125 (1967)
3. Bredon, G.: Introduction to compact transformation groups. New York: Academic Press 1972
4. Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33**, 87–154 (1978)
5. Ehlers, F.: Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierten Singularitäten. Math. Ann. **218**, 127–156 (1975)
6. Grauert, H.: Bemerkenswerte pseudokonvexe Mannigfaltigkeiten. Math. Z. **81**, 377–391 (1963)
7. Hayes, S.: Maximal simultaneous continuation in complex analysis. Expo. Math. **3**, 261–273 (1983)

8. Kempf, G., Knudsen, F., Mumford, D, Saint-Donat, B.: Toroidal Embeddings I. (Lect. Notes Math., vol. 339), Berlin Heidelberg New York: Springer 1972
9. Rudin, W.: A geometric criterion for algebraic varieties. J. Math. Mech. **7** (1963)
10. Winiarski, T.: Inverse of polynomial automorphisms of $\mathbb{C}^n$. Bull Pol. Acad. Sci. **XXVII**, 673–674 (1979)

Received December 18, 1987; in final form June 16, 1988