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Orlicz capacities and Hausdorff measures on metric spaces

Jana Björn[1], **Jani Onninen**[2]

[1] Department of Mathematics, Linköping University, SE-581 83, Linköping, Sweden (e-mail: jabjo@mai.liu.se)

[2] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA (e-mail: jonninen@umich.edu)

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Abstract. In the setting of doubling metric measure spaces with a 1-Poincaré inequality, we show that sets of Orlicz Φ-capacity zero have generalized Hausdorff h-measure zero provided that

$$\int_0^1 \Theta^{-1}(t^{1-s}h(t))\,dt < \infty,$$

where Θ^{-1} is the inverse of the function $\Theta(t) = \Phi(t)/t$, and s is the "upper dimension" of the metric measure space. This condition is a generalization of a well known condition in $\mathbf{R}^n$. For spaces satisfying the weaker q-Poincaré inequality, we obtain a similar but slightly more restrictive condition. Several examples are also provided.

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1. Introduction

In $\mathbf{R}^n$, it is known that sets of p-capacity zero have generalized Hausdorff h-measure zero provided that

$$\int_0^1 (t^{p-n}h(t))^{1/(p-1)}\,\frac{dt}{t} < \infty,$$

see Frostman [7] ($p = 2$) and Theorem 7.1 in Havin–Maz′ya [10], or Theorem 5.1.13 in Adams–Hedberg [2]. In particular, the Hausdorff dimension of such sets is at most $n - p$. Similar results for weighted capacities and Hausdorff measures in $\mathbf{R}^n$ can be found e.g. in Heinonen–Kilpeläinen–Martio [11], Theorem 2.32. For capacities associated with potentials generated by various nonnegative kernels, a generalization of the above condition was proved by Aikawa in [1], Theorem 2. Among Hausdorff measures and capacities generated by the powers t^p, all these results are sharp.

In this paper, we study the relation between Orlicz capacities cap_Φ and generalized Hausdorff measures Λ_h on metric measure spaces. This problem has been motivated by the recent development in the theory of Sobolev spaces and calculus on metric measure spaces without a differentiable structure (see e.g Hajłasz [8], Heinonen–Koskela [13], Cheeger [6] and Shanmugalingam [18]), and by applications in the theory of mappings with finite distortion, which appear e.g. in nonlinear elasticity and are a generalization of mappings with bounded distortion. The fact that sets of p-capacity zero have Hausdorff dimension at most $n - p$ has been used to show that certain mappings with L^p-integrable distortion are open and discrete, see e.g. Reshetnyak [17], Heinonen–Koskela [12] and Villamor–Manfredi [19]. Our results about Orlicz capacities are used in Björn [4] to obtain openness and discreteness for some mappings with distortion in Orlicz spaces. Orlicz–Sobolev capacities on metric spaces have also been treated by e.g. Aïssaoui [3].

Under the assumption that the metric space is doubling and supports a 1-Poincaré inequality, we show that $\mathrm{cap}_\Phi(K) = 0$ implies $\Lambda_h(K) = 0$, provided that

$$\int_0^1 \Theta^{-1}(t^{1-s}h(t))\,dt < \infty, \tag{1}$$

where Θ^{-1} is the inverse of the function $\Theta(t) = \Phi(t)/t$, and s is the "upper dimension" of the metric measure space, see Theorem 3 and Proposition 2 for the exact formulation.

In particular, the implication is true in the following two cases (Examples 2 and 3):

$$\Phi(t) = t^p,\ p > 1 \text{ and } \int_0^1 (t^{p-s}h(t))^{1/(p-1)}\,\frac{dt}{t} < \infty;$$

$$h(t) = t^\alpha,\ 0 < \alpha < s - 1 \text{ and } \int_1^\infty \left(\frac{\Phi(r)}{r}\right)^{1/(1-s+\alpha)} dr < \infty.$$

The former condition is the same as the condition valid in $\mathbf{R}^n$, while the latter is satisfied e.g. for $\Phi(t) = t^{s-\alpha+\varepsilon}$ and $\Phi(t) = t^{s-\alpha}\log^{s-1-\alpha+\varepsilon}(e+t)$ with $\varepsilon > 0$.

Under the weaker assumption that the metric measure space is doubling and supports a q-Poincaré inequality for some $q > 1$, we show that $\mathrm{cap}_\Phi(K) = 0$ implies $\Lambda_h(K) = 0$, provided that

$$\int_0^1 \Phi^{-1}(t^{-s}h(t))\,dt < \infty,$$

where Φ^{-1} is the inverse of the Young function Φ. This condition is somewhat stronger than (1), as shown by Proposition 3 and Examples 4 and 5.

2. Notation and preliminaries

Throughout the paper, $X = (X, d, \mu)$ will be a metric space equipped with a Borel regular measure μ satisfying $0 < \mu(B) < \infty$ for all balls $B = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ in X with $0 < r < \infty$. We shall also assume that the measure μ is doubling and that X supports a Poincaré inequality, see Definitions 1 and 3 below.

Definition 1. *If $\sigma > 0$ and $B = B(x_0, r)$ is a ball, we let σB denote the ball $B(x_0, \sigma r)$. We say that the measure μ is* doubling, *if there exists $C > 0$ such that*

$$\mu(2B) < C\mu(B)$$

for all balls B in X.

In [13], Heinonen and Koskela introduced upper gradients as a substitute for the modulus of the usual gradient. The advantage of this notion is that it can be defined without the notion of partial derivatives and can easily be used in the metric space setting.

Definition 2. *A Borel function g on X is an* upper gradient *of a real-valued function u on X if for all rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by the arc length ds,*

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g\, ds.$$

Definition 3. *We say that the space X supports a* weak q-Poincaré inequality *with $q \geq 1$, if there exist $C > 0$ and $\sigma \geq 1$ such that*

$$\frac{1}{\mu(B)} \int_B |u - u_B|\, d\mu \leq Cr \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g^q\, d\mu \right)^{1/q}$$

holds for all balls $B = B(x_0, r)$ in X and all pairs (u, g), where u is a Lipschitz function on X and g is an upper gradient of u. Here and in what follows, $u_B = \mu(B)^{-1} \int_B u\, d\mu$.

Definition 4. *A convex function $\Phi : (0, \infty) \to (0, \infty)$ satisfying*

$$\lim_{t \to 0+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$$

is called a Young function. *If $\Phi(2t) \leq C\Phi(t)$ for some constant C and all $t \in (0, \infty)$, then Φ is said to be* doubling (*or satisfying the Δ_2-condition*).

Definition 5. *The* Orlicz space $L^\Phi(X)$ *is the set of all measurable functions with the* Luxemburg norm

$$\|f\|_{L^\Phi(X)} = \inf\left\{ \lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty,$$

where we interpret $\Phi(0) = 0$.

The estimates

$$\begin{aligned} \|f\|_{L^\Phi(X)} &\le \int_X \Phi(f)\,d\mu, \quad \text{if } \|f\|_{L^\Phi(X)} \ge 1, \\ \|f\|_{L^\Phi(X)} &\ge \int_X \Phi(f)\,d\mu, \quad \text{if } \|f\|_{L^\Phi(X)} \le 1 \end{aligned} \tag{2}$$

for the Luxemburg norm will be useful. For the proofs see e.g. Lemma 3.8.4 in Kufner–John–Fučík [14] or Theorem III.13 in Rao–Ren [16].

Definition 6. *Let Φ be a Young function and let*

$$\psi(\tau) := \inf\{t : \varphi(t) > \tau\},$$

where φ is the left derivative of Φ (it exists everywhere and is nondecreasing). Then the function

$$\Psi(t) := \int_0^t \psi(\tau)\,d\tau$$

is called the complementary Young function *to Φ.*

Note that Φ is the complementary function to Ψ. By Theorem 3.4.7 in Kufner–John–Fučík [14] or Corollary II.4 in Rao–Ren [16], the complementary Young function to Φ is doubling if and only if there exist $k_0 > 1$ and $T_0 > 0$ such that for all $t \ge T_0$,

$$\Phi(t) \le \frac{\Phi(k_0 t)}{2k_0}. \tag{3}$$

The following generalized Hölder inequality for Orlicz spaces is proved e.g. in Theorem 3.7.5 in Kufner–John–Fučík [14] or in Rao–Ren [16].

Theorem 1. *For a pair Φ, Ψ of complementary Young functions and for $f \in L^\Phi(X)$ and $g \in L^\Psi(X)$,*

$$\int_X |fg|\,d\mu \le 2\|f\|_{L^\Phi(X)}\|g\|_{L^\Psi(X)}.$$

Definition 7. *Let Φ be a Young function and $K \subset B(x_0, R)$ be compact. The Φ-capacity of K with respect to the ball $B(x_0, 2R)$ is*

$$\operatorname{cap}_\Phi(K, B(x_0, 2R)) = \inf \|g\|_{L^\Phi(X)},$$

where the infimum is taken over all upper gradients g of all Lipschitz continuous functions φ with compact support in $B(x_0, 2R)$ and $\varphi \ge 1$ on K.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant and depends only on the fixed parameters, such as the doubling constant of μ and the constants in the Poincaré inequality. When needed, we will point out the dependence on other parameters.

3. Generalized Hausdorff measures

Definition 8. *Let $h : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that* $\lim_{t\to 0+} h(t) = h(0) = 0$. *For $0 < \delta \le \infty$, and $E \subset X$, we define*

$$\Lambda_h^\delta(E) = \inf \sum_{j=1}^{\infty} h(\operatorname{diam} E_j),$$

where the infimum is taken over all collections $\{E_j\}_{j=1}^\infty$ such that $E_j \subset X$, $\operatorname{diam} E_j \le \delta$ *and $E \subset \bigcup_{j=1}^\infty E_j$. The* Hausdorff h-measure *is then*

$$\Lambda_h(E) = \lim_{\delta\to 0+} \Lambda_h^\delta(E).$$

Let also

$$\lambda_h^\delta(E) = \inf \sum_{j=1}^{\infty} c_j h(\operatorname{diam} E_j),$$

where the infimum is taken over all collections $\{(E_j, c_j)\}_{j=1}^\infty$ such that $E_j \subset X$, $\operatorname{diam} E_j \le \delta$, $0 < c_j \le 1$ *and* $\chi_E \le \sum_{j=1}^\infty c_j \chi_{E_j}$.

In the proof of our main result, Theorem 3, we shall need the following Frostman lemma. For $h(t) = t^s$, it is proved in Mattila [15], Theorem 8.17, and the proof is the same for general h.

Theorem 2. *Let $K \subset X$ be compact and $0 < \delta \le \infty$. Then there exists a Radon measure ω supported on K such that $\omega(K) = \lambda_h^\delta(K)$ and $\omega(E) \le h(\operatorname{diam} E)$ for all $E \subset X$ with* $\operatorname{diam} E \le \delta$.

Theorem 2 is formulated in terms of the weighted Hausdorff content λ_h^δ, rather than Λ_h^δ. Clearly, $\lambda_h^\delta(E) \le \Lambda_h^\delta(E)$ for all $E \subset X$ and $0 < \delta \le \infty$. The following lemma gives a partial converse to this inequality. As we have not been able to find it in this generality in the literature, we show how the proof of Lemma 8.16 in Mattila [15] can be modified to obtain the result we need.

Lemma 1. *There exists a constant C depending only on the doubling constant of μ such that for all compact sets $K \subset X$ and $0 < \delta \le \infty$,*

$$\Lambda_h^\delta(K) \le C\lambda_h^\delta(K).$$

Proof. Let $0 < \delta \le \infty$ and fix $0 < t < 1$. Let c_j and E_j be as in the definition of $\lambda_h^\delta(K)$ and find open balls $B_j \supset E_j$ with radii $r_j \le 2 \operatorname{diam} E_j$. Cover each ball B_j by N_j balls B_{ij} with radii $r_j/20 \le \operatorname{diam} E_j/10$. This can be done so that $N_j \le N$ for all j, where N depends only on the doubling constant of the measure μ.

The sets $\left\{x : \sum_{j=1}^k c_j \chi_{B_j}(x) > t\right\}$, $k = 1, 2 \dots$, form an open cover of the compact set K and hence there exists k such that

$$K \subset \left\{x : \sum_{j=1}^{k} c_j \chi_{B_j}(x) > t\right\}.$$

Now, apply the argument from the proof of Lemma 8.16 in Mattila [15] to $\mathcal{B} = \{B_{ij} : j = 1, 2, \dots, k; i = 1, 2, \dots, N_j\}$. It shows that

$$\Lambda_h^\delta(K) \le \Lambda_h^\delta\Biggl(\Biggl\{x : \sum_{j=1}^{k}\sum_{i=1}^{N_j} c_j \chi_{B_{ij}}(x) > t\Biggr\}\Biggr) \le \frac{1}{t}\sum_{j=1}^{k}\sum_{i=1}^{N_j} c_j h(\operatorname{diam} 5B_{ij})$$
$$\le \frac{N}{t}\sum_{j=1}^{k} c_j h(\operatorname{diam} E_j).$$

Taking infimum over all admissible collections $\{(E_j, c_j)\}_{j=1}^\infty$ finishes the proof. $\square$

4. Measures and maximal functions

In this section we prove an integral estimate for maximal functions, which will be used in the proof of our main result. First, we make the following definition.

Definition 9. *For a Borel measure ω on X, we let*

$$W(x,\rho) = \int_0^\rho \frac{\omega(B(x,t))}{\mu(B(x,t))}\,dt \quad \textit{and} \quad M(x,\rho) = \sup_{0<r<\rho} \frac{r\omega(B(x,r))}{\mu(B(x,r))}.$$

The following two estimates are metric space analogues of Theorem 3.6.1 in Adams–Hedberg [2]. The proofs given here are similar to those in [2].

Proposition 1. *Let X be a metric space equipped with a doubling measure μ. Let ω be a Borel measure on X with* $\operatorname{supp}\omega \subset B(x_0, R)$. *Let Ψ be a doubling Young function. Then*

$$\int_X \Psi(W(x,R))\,d\mu(x) \le C\int_{B(x_0,2R)} \Psi(M(x,3R))\,d\mu(x),$$

where C depends only on the doubling constants of μ and Ψ.

Lemma 2. *Let X be a metric space equipped with a doubling measure μ. Let ω be a Borel measure on X with* $\operatorname{supp}\omega \subset B(x_0, R)$. *Then there exist constants $a > 1$ and $C > 0$, depending only on the doubling constant of μ, such that for all $\lambda > 0$ and all* $0 < \varepsilon \le 1$, *the following* "good λ inequality" *holds,*

$$\mu(\{x : W(x,R) > a\lambda\}) \le C\varepsilon\mu(\{x : W(x,R) > \lambda\})$$
$$+\mu(\{x \in B(x_0,2R) : M(x,3R) > \varepsilon\lambda\}).$$

Proof of Proposition 1. Multiply the "good λ inequality" from Lemma 2 by $\Psi'(\lambda)$ (which exists for a.e. λ by the convexity of Ψ) and integrate with respect to λ from 0 to Λ,

$$\int_0^\Lambda \mu(\{x : W(x,R) > a\lambda\})\Psi'(\lambda)\,d\lambda$$
$$\le C\varepsilon\int_0^\Lambda \mu(\{x : W(x,R) > \lambda\})\Psi'(\lambda)\,d\lambda$$
$$+\int_0^\Lambda \mu(\{x \in B(x_0,2R) : M(x,3R) > \varepsilon\lambda\})\Psi'(\lambda)\,d\lambda.$$

As ω has compact support, so does W and the integrals are finite. The monotonicity of Ψ' and the doubling condition on Ψ yield for a.e. $\lambda > 0$,

$$\begin{aligned}\Psi'(a\lambda) &\le \frac{1}{a\lambda}\int_{a\lambda}^{2a\lambda}\Psi'(\tau)\,d\tau \le \frac{\Psi(2a\lambda)}{a\lambda}\\ &\le \frac{C\Psi(\lambda)}{\lambda} \le \frac{C}{\lambda}\int_0^{\lambda}\Psi'(\tau)\,d\tau \le C\Psi'(\lambda),\end{aligned}$$

where C depends only on a and the doubling constant of Ψ. Similarly, $\Psi'(\lambda) \le C\Psi'(\varepsilon\lambda)$. Changing variables we now obtain

$$\begin{aligned}&\int_0^{a\Lambda}\mu(\{x : W(x,R) > \lambda\})\Psi'(\lambda)\,d\lambda\\ &\quad\le C\varepsilon\int_0^{\Lambda}\mu(\{x : W(x,R) > \lambda\})\Psi'(\lambda)\,d\lambda\\ &\qquad+C'\int_0^{\varepsilon\Lambda}\mu(\{x\in B(x_0,2R) : M(x,3R) > \lambda\})\Psi'(\lambda)\,d\lambda,\end{aligned}$$

where C depends on a and the doubling constants of Ψ and μ, and C' depends on a, ε and the doubling constant of Ψ. Choosing $\varepsilon > 0$ small enough so that $C\varepsilon < \frac{1}{2}$, and letting $\Lambda \to \infty$ then finishes the proof. □

Proof of Lemma 2. As in Lemma 4.2 in Björn–MacManus–Shanmugalingam [5], it can be shown that the function $x \mapsto W(x,R)$ is lower semicontinuous. Hence, the set

$$\Omega = \{x : W(x,R) > \lambda\} \subset B(x_0,2R)$$

is open. Assume that Ω^c is nonempty (the case $\Omega = X$ is easier). Then, for every $x\in\Omega$ there exists $r_x > 0$ such that $B(x,r_x)\subset\Omega$ and $B(x,2r_x)\cap\Omega^c$ is nonempty. The collection $\{B(x,r_x/5)\}_{x\in\Omega}$ covers Ω and by the Vitali type covering theorem (Theorem 2.1 in Mattila [15]), there exist $x_j\in\Omega$ and $r_j = r_{x_j}$, $j = 1,2,\dots$, such that the balls $B(x_j,r_j/5)$ are pairwise disjoint and $\Omega = \bigcup_{j=1}^{\infty} B(x_j,r_j)$.

Let $B = B(x_j,r_j)$, $a > 1$ and $0 < \varepsilon \le 1$ be fixed for a while but arbitrary. Then either $B \subset \{x : M(x,3R) > \varepsilon\lambda\}$ or there exists $x'\in B$ such that $M(x',3R)\le\varepsilon\lambda$. Assume that the latter occurs and let $x\in B$ be arbitrary.

If $r_j < R/4$, then find $x''\in\Omega^c$ so that $d(x,x'')\le 3r_j$. Then for $t\ge r_j$, we have $B(x,t)\subset B(x'',4t)\subset B(x,7t)$ and $B(x,t)\subset B(x',3t)\subset B(x,5t)$. The doubling property of μ then yields,

$$\int_{r_j}^{R/4}\frac{\omega(B(x,t))}{\mu(B(x,t))}\,dt \le C\int_{r_j}^{R/4}\frac{\omega(B(x'',4t))}{\mu(B(x'',4t))}\,dt \le CW(x'',R)\le C\lambda$$

and

$$\int_{R/4}^{R}\frac{\omega(B(x,t))}{\mu(B(x,t))}\,dt \le C\int_{R/4}^{R}\frac{\omega(B(x',3t))}{\mu(B(x',3t))}\,dt \le CM(x',3R)\int_{R/4}^{R}\frac{dt}{3t}\le C\varepsilon\lambda.$$

These two estimates together give

$$\int_{r_j}^{R} \frac{\omega(B(x,t))}{\mu(B(x,t))}\,dt \le C\lambda,$$

where C depends only on the doubling constant of μ. Hence, if $W(x,R) > a\lambda$ and a is sufficiently large, then $W(x,r_j) > a\lambda - C\lambda \ge a\lambda/2$. It follows that for $r_j < R/4$,

$$\begin{aligned}
\mu(\{x \in B : W(x,R) > a\lambda\}) &\le \mu(\{x \in B : W(x,r_j) > a\lambda/2\}) \\
&\le \frac{C}{\lambda}\int_B W(x,r_j)\,d\mu(x) \\
&= \frac{C}{\lambda}\int_0^{r_j}\int_X \frac{\chi_B(x)}{\mu(B(x,t))}\int_X \chi_{B(x,t)}(y)\,d\omega(y)\,d\mu(x)\,dt. \qquad (4)
\end{aligned}$$

Note that $\chi_B(x)\chi_{B(x,t)}(y)$ is nonzero only if $x \in B(y,t) \cap B$. In this case, we have $\mu(B(y,t)) \le \mu(B(x,2t)) \le C\mu(B(x,t))$ and $y \in B(x',3r_j)$. The Fubini theorem then implies

$$\begin{aligned}
\frac{C}{\lambda}\int_B W(x,r_j)\,d\mu(x) &\le \frac{C}{\lambda}\int_0^{r_j}\int_{B(x',3r_j)} \frac{\mu(B\cap B(y,t))}{\mu(B(y,t))}\,d\omega(y)\,dt \\
&\le \frac{C}{\lambda} M(x',3r_j)\mu(B(x',3r_j)).
\end{aligned}$$

Inserting this into (4) we have (still for $r_j < R/4$)

$$\begin{aligned}
\mu(\{x \in B : W(x,R) > a\lambda\}) &\le \frac{C}{\lambda} M(x',3R)\mu(B(x_j,4r_j) \\
&\le C\varepsilon\mu(B(x_j,r_j/5). \qquad (5)
\end{aligned}$$

If $r_j \ge R/4$, then we have as above,

$$\begin{aligned}
&\mu(\{x \in B : W(x,R) > a\lambda\}) \\
&\quad \le \frac{C}{\lambda}\int_0^R\int_X \frac{\chi_B(x)}{\mu(B(x,t))}\int_X \chi_{B(x,t)}(y)\,d\omega(y)\,d\mu(x)\,dt \\
&\quad \le \frac{C}{\lambda}\int_0^R\int_{B(x_0,R)} \frac{\mu(B\cap B(y,t))}{\mu(B(y,t))}\,d\omega(y)\,dt \\
&\quad \le \frac{C}{\lambda} M(x',3R)\mu(B(x',3R)) \le C\varepsilon\mu(x_j,r_j/5),
\end{aligned}$$

i.e. (5) holds also for $r_j \ge R/4$. Finally,

$$\begin{aligned}
\{x : W(x,R) > a\lambda\} \subset{} & \{x \in B(x_0,2R) : M(x,3R) > \varepsilon\lambda\} \\
& \cup \bigcup \{x \in B(x_j,r_j) : W(x,R) > a\lambda\},
\end{aligned}$$

where the last union is taken over all balls $B(x_j, r_j)$ such that $B(x_j, r_j) \not\subset \{x : M(x, 3R) > \varepsilon\lambda\}$. It then follows using (5) that

$$\begin{aligned}
&\mu(\{x : W(x, R) > a\lambda\}) \\
&\quad \le \mu(\{x \in B(x_0, 2R) : M(x, 3R) > \varepsilon\lambda\}) + C\varepsilon \sum_{j=1}^{\infty} \mu(B(x_j, r_j/5)) \\
&\quad \le \mu(\{x \in B(x_0, 2R) : M(x, 3R) > \varepsilon\lambda\}) + C\varepsilon\mu(\Omega),
\end{aligned}$$

which finishes the proof. □

5. The main result

Theorem 3. *Let X be a metric space equipped with a doubling measure μ and supporting a weak 1-Poincaré inequality. Let $s > 0$ be such that for every ball $B \subset X$, there is $C > 0$ such that for all balls $B(y, t) \subset B$,*

$$\mu(B(y, t)) \ge Ct^s. \tag{6}$$

Let Φ be a Young function with a doubling complementary function Ψ. Let $h : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\lim_{t\to 0+} h(t) = h(0) = 0$. If

$$\int_0^1 \frac{\Psi(t^{1-s}h(t))}{t^{1-s}h(t)}\, dt < \infty, \tag{7}$$

then $\Lambda_h(K) = 0$ for every compact $K \subset X$ with $\operatorname{cap}_\Phi(K) = 0$.

Remark 1. A simple iteration of the doubling condition for μ shows that (6) holds with $s = \log_2 C_d$, where C_d is the doubling constant of μ.

Proof. Assume that $K \subset B(x_0, R/4)$. Let φ be a Lipschitz function with support in $B(x_0, R/2)$ such that $\varphi \ge 1$ on K. Let $x \in B(x_0, R/2)$ be arbitrary and let $r_j = 2^{-j}R$ and $B_j = B(x, r_j)$, $j = 0, 1, 2, \dots$. Then by the Lebesgue differentiation theorem,

$$\begin{aligned}
\varphi(x) &= \lim_{j\to\infty} \frac{1}{\mu(B_j)} \int_{B_j} \varphi\, d\mu \\
&= \frac{1}{\mu(B_0)} \int_{B_0} \varphi\, d\mu + \sum_{j=0}^{\infty} \frac{1}{\mu(B_{j+1})} \int_{B_{j+1}} (\varphi - \varphi_{B_j})\, d\mu,
\end{aligned}$$

where $\varphi_{B_j} = \mu(B_j)^{-1} \int_{B_j} \varphi\, d\mu$. Let g be an upper gradient of φ. Then the first term on the right-hand side can be estimated using the Sobolev inequality (see e.g. the proof of Theorem 13.1 in Hajłasz–Koskela [9]) and the second term is estimated by the weak 1-Poincaré inequality as follows

$$\begin{aligned}\varphi(x) &\le \frac{Cr_0}{\mu(B_0)}\int_{B_0} g\,d\mu + C\sum_{j=0}^{\infty}\frac{r_j}{\mu(5\sigma B_j)}\int_{5\sigma B_j} g\,d\mu \\ &\le C\int_0^R \frac{1}{\mu(B(x,t))}\int_{B(x,t)} g\,d\mu\,dt.\end{aligned}$$

Let ω be the measure provided by the Frostman lemma (Theorem 2), i.e. $\operatorname{supp}\omega \subset K$, $\omega(K) = \lambda_h^\infty(K)$ and $\omega(B(x,r)) \le h(2r)$ for all $x \in X$ and $r > 0$. As $\varphi \ge 1$ on K, we get integrating the last estimate with respect to ω,

$$\omega(K) \le C\int_X\int_0^R \frac{1}{\mu(B(x,t))}\int_{B(x,t)} g\,d\mu\,dt\,d\omega(x).$$

Write $\int_{B(x,t)} g\,d\mu$ as $\int_X \chi_{B(x,t)}(y)g(y)\,d\mu(y)$. The Fubini theorem and the fact that $\chi_{B(x,t)}(y) = \chi_{B(y,t)}(x)$ then yield

$$\omega(K) \le C\int_X g(y)\int_0^R\int_X \frac{\chi_{B(y,t)}(x)}{\mu(B(x,t))}\,d\omega(x)\,dt\,d\mu(y). \tag{8}$$

Note that $\chi_{B(y,t)}(x) \ne 0$ only if $d(x,y) < t$ and in this case the doubling condition implies $\mu(B(x,t)) \ge C\mu(B(y,t))$. Inserting this into (8) and using the generalized Hölder inequality (Lemma 1) we get

$$\begin{aligned}\omega(K) &\le C\int_X g(y)\int_0^R \frac{\omega(B(y,t))}{\mu(B(y,t))}\,dt\,d\mu(y) \\ &\le C\|g\|_{L^\Phi(X)}\,\|W(\,\cdot\,,R)\|_{L^\Psi(X)},\end{aligned} \tag{9}$$

where $W(y,R)$ is as in Definition 9 and Ψ is the Young function complementary to Φ. We shall now show that the last norm in (9) is finite. By (2) and Proposition 1 we have

$$\begin{aligned}\|W(\,\cdot\,,R)\|_{L^\Psi(X)} &\le 1 + \int_X \Psi(W(y,R))\,d\mu(y) \\ &\le 1 + \int_X \Psi(M(y,3R))\,d\mu(y).\end{aligned} \tag{10}$$

To estimate the last integral, we first note that by the doubling property and monotonicity of Ψ, we have for all $r > 0$,

$$\int_r^{2r} \Psi\left(\frac{t\omega(B(y,t))}{\mu(B(y,t))}\right)\frac{dt}{t} \ge C\Psi\left(\frac{r\omega(B(y,r))}{\mu(B(y,r))}\right).$$

Hence,

$$\int_X \Psi(M(y,3R))\,d\mu(y) \le C\int_0^{6R}\int_X \Psi\left(\frac{t\omega(B(y,t))}{\mu(B(y,t))}\right)d\mu(y)\,\frac{dt}{t}. \tag{11}$$

Next, let us write

$$\Psi\left(\frac{t\omega(B(y,t))}{\mu(B(y,t))}\right)=\frac{\Psi\big(t\omega(B(y,t))/\mu(B(y,t))\big)}{t\omega(B(y,t))/\mu(B(y,t))}\int_X\frac{\chi_{B(y,t)}(x)}{\mu(B(y,t))}\,t\,d\omega(x). \quad (12)$$

The integrand is nonzero only if $d(x,y)<t$ and in that case we have $B(x,t)\subset B(y,2t)$ and $\chi_{B(y,t)}(x)=\chi_{B(x,t)}(y)$. Moreover, by (6),

$$\frac{t\omega(B(y,t))}{\mu(B(y,t))}\le Ct^{1-s}h(2t).$$

Inserting this into (12), together with the monotonicity of the function $t\mapsto\Psi(t)/t$ and the Fubini theorem, then yields

$$\begin{aligned}&\int_X\Psi\left(\frac{t\omega(B(y,t))}{\mu(B(y,t))}\right)d\mu(y)\\&\quad\le C't\frac{\Psi(Ct^{1-s}h(2t))}{Ct^{1-s}h(2t)}\int_X\int_X\frac{\chi_{B(x,t)}(y)}{\mu(B(x,t))}\,d\mu(y)\,d\omega(x)\\&\quad\le C'\omega(K)t\frac{\Psi(Ct^{1-s}h(2t))}{t^{1-s}h(2t)}.\end{aligned}$$

As Ψ is doubling, inserting this estimate into (11) and (10) gives

$$\|W(\,\cdot\,,R)\|_{L^\Psi(X)}\le 1+C\omega(K)\int_0^{12R}\frac{\Psi(t^{1-s}h(t))}{t^{1-s}h(t)}\,dt,$$

which together with (9) and the assumption (7) yields

$$\omega(K)\le C(1+\omega(K))\|g\|_{L^\Phi(X)},$$

where C depends only on Ψ, h, R, s and μ, but not on g. Taking infimum over all functions φ and g admissible in the definition of $\operatorname{cap}_\Phi(K)$ finishes the proof. $\square$

6. Equivalent conditions and special cases

Proposition 2. *If Φ is a Young function with a doubling complementary function Ψ, then the condition* (7) *is equivalent to each of the conditions*

$$\int_0^1\psi(t^{1-s}h(t))\,dt<\infty, \quad (13)$$

$$\int_0^1\Theta^{-1}(t^{1-s}h(t))\,dt<\infty, \quad (14)$$

where ψ is the left derivative of Ψ and Θ^{-1} is the inverse of the function $\Theta(t)=\Phi(t)/t$.

Proof. As Ψ is doubling, we have by the monotonicity of ψ,

$$\frac{\Psi(t)}{t} = \frac{1}{t}\int_0^t \psi(\tau)\,d\tau \le \psi(t) \le \frac{1}{t}\int_t^{2t} \psi(\tau)\,d\tau \le \frac{\Psi(2t)}{t} \le \frac{C\Psi(t)}{t},$$

which shows (7) $\Leftrightarrow$ (13).

Similarly, we have $\Theta(\sigma) \le \varphi(\sigma) \le 2\Theta(2\sigma)$ for all $\sigma > 0$, where φ is the left derivative of φ. Hence, for all $\tau > 0$,

$$\psi(\tau) = \inf\{\sigma : \varphi(\sigma) > \tau\} \le \inf\{\sigma : \Theta(\sigma) > \tau\} = \Theta^{-1}(\tau),$$

which yields (14) $\Rightarrow$ (13). Similarly, we get $\Theta^{-1}(\tau) \le 2\psi(2\tau) \le C\Psi(\tau)/\tau$ and hence (7) $\Rightarrow$ (14). $\square$

Example 1. If $h(t) \le Ct^{s-1}$, then the value of the integral in (7) is at most $\Psi(C)/C < \infty$, i.e. the Hausdorff h-measure of K is zero whenever $\operatorname{cap}_\Phi(K) = 0$, independently of Φ. In particular, this is true for the $(s-1)$-dimensional Hausdorff measure.

Example 2. If $\Phi(t) = t^p$, $p > 1$, (i.e. cap_Φ is the p-capacity), then the condition (14) can be written as

$$\int_0^1 (t^{p-s}h(t))^{1/(p-1)}\,\frac{dt}{t} < \infty.$$

Theorem 7.1 in Havin–Maz′ya [10] (or Theorem 5.1.13 in Adams–Hedberg [2]) states that every set in $\mathbf{R}^n$ with zero $C_{\alpha,p}$-capacity, $0 < \alpha p < n$, has Hausdorff h-measure zero provided that

$$\int_0^1 (t^{\alpha p-n}h(t))^{1/(p-1)}\,\frac{dt}{t} < \infty.$$

So, for $\alpha = 1$, our condition is a generalization of the condition in $\mathbf{R}^n$.

Example 3. If $h(t) = t^\alpha$, $0 < \alpha < s-1$, (i.e. Λ_h is the α-dimensional Hausdorff measure), then (14) (and thus (7)) holds if and only if

$$\int_1^\infty \left(\frac{\Phi(r)}{r}\right)^{1/(1-s+\alpha)} dr < \infty. \tag{15}$$

Indeed, the change of variables $t = \Theta(r)^{-\beta}$, with $-\beta = 1/(1-s+\alpha) < 0$, and integration by parts show that the integral in (14) is equal to the limit, as $R \to \infty$, of

$$\begin{aligned}
&\Big[-r\Theta(r)^{-\beta}\Big]_{\Theta^{-1}(1)}^R + \int_{\Theta^{-1}(1)}^R \Theta(r)^{-\beta}\,dr \\
&\quad = \Theta^{-1}(1)\big(1-\Theta(R)^{-\beta}\big) + \int_{\Theta^{-1}(1)}^R \big(\Theta(r)^{-\beta} - \Theta(R)^{-\beta}\big)\,dr \\
&\quad \le C + \int_1^R \Theta(r)^{-\beta}\,dr.
\end{aligned}$$

Hence, (15) implies (14). Conversely, (3) yields $\Theta(R) \geq 2\Theta(R/k_0)$. Thus, for sufficiently large R,

$$\int_{\Theta^{-1}(1)}^{R/k_0} \left(\Theta(r)^{-\beta} - \Theta(R)^{-\beta}\right) dr \geq \left(1 - 2^{-\beta}\right) \int_{\Theta^{-1}(1)}^{R/k_0} \Theta(r)^{-\beta}\, dr,$$

which shows that (14) implies (15).

7. A weaker Poincaré inequality

In this section we weaken the assumption of 1-Poincaré inequality from Theorem 3. Instead, we assume a q-Poincaré inequality for some $q \geq 1$ and obtain a condition sufficient for the validity of the implication

$$\operatorname{cap}_\Phi(K) = 0 \quad \Longrightarrow \quad \Lambda_h(K) = 0,$$

which is somewhat stronger than the condition in Theorem 3, see Proposition 3 and Example 4.

Note that in the following theorem we do not assume that the complementary function to Φ is doubling. Also, the proof is simpler than that of Theorem 3.

Theorem 4. *Assume that X supports a weak q-Poincaré inequality for some $q \geq 1$ and that μ is doubling. Let Φ be a Young function such that the function $t \mapsto \Phi(t^{1/q})$ is convex and let $h : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that* $\lim_{t\to 0+} h(t) = h(0) = 0$. *If*

$$\int_0^1 \Phi^{-1}(t^{-s}h(t))\, dt < \infty, \tag{16}$$

then $\Lambda_h(K) = 0$ for all compact $K \subset X$ with $\operatorname{cap}_\Phi(K) = 0$.

Proof. Assume that $K \subset B(x_0, R/4)$ and $\Lambda_h(K) > 0$. As in the proof of Theorem 3 with the 1-Poincaré inequality replaced by the q-Poincaré inequality, we have

$$0 < \omega(K) \leq C \int_0^R \int_X \left(\frac{1}{\mu(B(x,t))} \int_{B(x,t)} g^q\, d\mu\right)^{1/q} d\omega(x)\, dt,$$

where ω is the Frostman measure from Theorem 2, g is an upper gradient of a Lipschitz function φ with support in $B(x_0, R/2)$ and $\varphi \geq 1$ on K. As the function $t \mapsto \Phi(t^{1/q})$ is convex, two applications of the Jensen inequality imply

$$\begin{aligned} 1 &\leq C \int_0^R \int_X \Phi^{-1}\left(\frac{1}{\mu(B(x,t))} \int_{B(x,t)} \Phi(g)\, d\mu\right) \frac{d\omega(x)}{\omega(K)}\, dt \\ &\leq C \int_0^R \Phi^{-1}\left(\int_X \int_X \frac{\chi_{B(x,t)}(y)}{\mu(B(x,t))}\, \Phi(g(y))\, d\mu(y)\, \frac{d\omega(x)}{\omega(K)}\right) dt. \end{aligned}$$

The integrand is nonzero only if $x \in B(y,t)$ and in this case we have $\mu(B(x,t)) \ge C\mu(B(y,t))$. The Fubini theorem and the fact that $\chi_{B(x,t)}(y) = \chi_{B(y,t)}(x)$ then imply

$$1 \le C' \int_0^R \Phi^{-1}\biggl(\int_X \frac{\omega(B(y,t))\,\Phi(g(y))}{C\mu(B(y,t))\,\omega(K)}\,d\mu(y)\biggr)\,dt.$$

As $\omega(B(y,t)) \le h(2t)$ and $\mu(B(y,t)) \ge Ct^s$, we have

$$1 \le C' \int_0^R \Phi^{-1}\biggl(\frac{Ct^{-s}h(2t)}{\omega(K)} \int_X \Phi(g)\,d\mu\biggr)\,dt. \tag{17}$$

If $\operatorname{cap}_\Phi(K, B(x_0, R/2)) = 0$, we can find Lipschitz functions φ_j with support in $B(x_0, R/2)$ and upper gradients g_j, such that $\varphi_j \ge 1$ on K and $\|g_j\|_{L^\Phi(X)} \to 0$, as $j \to 0$. By (2) we then have for all $t \in (0, R]$,

$$\Phi^{-1}\biggl(\frac{Ct^{-s}h(2t)}{\omega(K)} \int_X \Phi(g_j)\,d\mu\biggr) \to 0, \quad \text{as } j \to 0.$$

Using (16), the dominated convergence theorem then shows that the right-hand side in (17) can be made arbitrarily small, which is a contradiction. Thus, we must have $\operatorname{cap}_\Phi(K, B(x_0, R/2)) > 0$. □

Proposition 3. *If the complementary function to* Φ *is doubling, then the condition* (16) *implies* (14).

Proof. Assume that (16) holds, i.e. that

$$\int_t^{2t} \Phi^{-1}(\tau^{-s}h(\tau))\,d\tau < \int_0^{2t} \Phi^{-1}(\tau^{-s}h(\tau))\,d\tau \le 1$$

for sufficiently small t. It follows that $\Phi^{-1}((2t)^{-s}h(t)) \le 1/t$ and hence, using (3),

$$t^{-s}h(t) \le 2^s\Phi(1/t) \le \Phi(C/t),$$

or equivalently,

$$\Phi^{-1}(t^{-s}h(t)) \le C/t$$

for some $C > 1$ and sufficiently small t. Next, with $\sigma = \Theta^{-1}(t^{1-s}h(t))$, we have

$$\Theta(\Phi^{-1}(t^{-s}h(t))) = \frac{t^{-s}h(t)}{\Phi^{-1}(t^{-s}h(t))} \ge \frac{t^{-s}h(t)}{C/t} = \frac{t^{1-s}h(t)}{C} = \frac{\Theta(\sigma)}{C}.$$

Iterating the inequality $\Theta(\sigma) \ge 2\Theta(\sigma/k_0)$ (which follows from (3)) and inserting it into the last estimate yields for some $C > 1$,

$$\Theta(\Phi^{-1}(t^{-s}h(t))) \ge \Theta(\sigma/C).$$

The monotonicity of Θ then gives

$$\Phi^{-1}(t^{-s}h(t)) \ge \frac{\sigma}{C} = \frac{\Theta^{-1}(t^{1-s}h(t))}{C}$$

and the condition (14) follows. □

Example 4. Let $h(t) = t^\alpha$, $0 < \alpha < s$, (i.e. Λ_h is the α-dimensional Hausdorff measure). Then it can be shown as in Example 3 that (16) holds if and only if

$$\int_1^\infty \Phi(r)^{1/(\alpha-s)}\,dr < \infty.$$

In particular, if $\Phi(r) = r^p \log^d(e+r)$, then (16) is satisfied if and only if $p > s-\alpha$ or $p = s-\alpha$ and $d > s-\alpha$.

At the same time, again by Example 3, the condition (14) holds if and only if $p > s-\alpha$ or $p = s-\alpha$ and $d > s-\alpha-1$, which is sharp in $\mathbf{R}^n$.

This shows that the condition (16) is more restrictive than (14).

Example 5. Let $\Phi(t) = t^p$, $p \geq q$, and $h(t) = t^\alpha$. Then the conditions (14) and (16) are equivalent and hold if and only if $p > s-\alpha$, which is sharp in $\mathbf{R}^n$.

Problem 1. Is the condition (14) sufficient also under the assumption of q-Poincaré inequality with $q > 1$? Is it necessary?

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