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Prime ideals and the ideal-radical of a distributively generated near-ring

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The concepts of a prime ideal of a distributively generated (d.g.) near-ring R, a prime d.g. near-ring and an irreducible R-group are introduced[1]). The annihilating ideal of an irreducible R-group with an R-generator is a prime ideal. Consequently we define a prime ideal to be primitively prime if it is the annihilating ideal of such an R-group, and a d.g. near-ring to be a primitively prime near-ring if it acts faithfully on such a group. The intersection of all the primitively prime ideals of a d.g. near-ring is called the ideal-radical; this ideal contains all the nilpotent ideals of the near-ring and a relationship between it and the quasi-radical of the near-ring is established.

In section 2 we consider d.g. near-rings R which satisfy the descending chain condition for left R-modules. In this case, the ideal-radical is nilpotent. Any non-zero prime d.g. near-ring is a primitively prime d.g. near-ring. All irreducible R-groups with an R-generator of a non-zero prime d.g. near-ring R are shown to be isomorphic to the finite number of direct summands of the group $R^+ - N$, where N is the quasi-radical of R. If R has finite order, then it has, to within an isomorphism, but one faithful representation on an irreducible R-group with an R-generator and all its irreducible R-groups with R-generators are homomorphic images of R-subgroups of this group.

In section 3, a number of equivalent conditions is given for a d.g. near-ring to have a nilpotent radical. One of them is that all its proper prime ideals are maximal ideals. In section 4, we construct an example of a finite d.g. near-ring whose radical is not nilpotent and whose quasi-radical is not an ideal.

1. Definitions and general properties

A *near-ring* R is a system with two binary operations, addition and multiplication, such that:

(i) The elements of R form a group R^+ under addition.

(ii) The elements of R form a semi-group under multiplication.

(iii) $(x+y)z = xz + yz$, for all $x, y, z \in R$.

[1]) The notation in this paper is different from that adopted in the two previous papers [7] and [8]. What was previously called an irreducible R-group is now called a minimal R-group. The term irreducible R-group is given another and more general meaning in this paper (see section 1).

If S is a multiplicative semi-group contained in R, whose elements generate R^+ and satisfy

(iv) $s(x+y)=sx+sy$,

we say that R is generated by the distributive semi-group S. A near-ring R which is generated by some distributive semi-group is said to be *distributively generated*.

Throughout this article, we shall mean by a d.g. near-ring a d. g. near-ring with an identity element which will be denoted by e and R will always denote such a near-ring. The symbol S stands for some distributive semi-group generating R. It will always be assumed that S contains e; this imposes no further structural restriction on R.

An (R, S)-*group* Ω is given by an additive group, the additive notation not to imply commutativity, together with a mapping $(x, w)\to xw$ of $R\times\Omega$ into Ω such that

(i) For all $x, y\in R$ and all $w\in\Omega$, $(x+y)w=xw+yw$.
(ii) For all $x, y\in R$ and all $w\in\Omega$, $(xy)w=x(yw)$.
(iii) For all $s\in S$ and $w_1, w_2\in\Omega$, $s(w_1+w_2)=sw_1+sw_2$.
(iv) For all $w\in\Omega$, $ew=w$.

If the particular semi-group S occurring in this definition does not need to be specified, we simply speak of an R-*group* (see [5]).

The zero 0 of R^+ is a two-sided annihilator of R ([2], 1.1, 1.5). The zero of an additive group Ω will be denoted by 0_Ω, or, if no confusion is possible, simply by 0. If Ω is an R-group, then $x0_\Omega=0_\Omega$, for all $x\in R$ ([3], 1.4).

A *faithful* R-group Ω is an R-group such that if $x\in R$ and $xw=0$ for all $w\in\Omega$, then $x=0$. A *minimal* R-group is a non-zero R-group which contains no R-groups as proper, non-zero sub-groups. A *primitive* d.g. near-ring R is a d.g. near-ring which has a faithful representation on a minimal R-group (see [7]).

A homomorphism Φ of an R-group Ω into another R-group is called an R-*homomorphism* if $\Phi(xw)=x(\Phi w)$, for all $x\in R$ and $w\in\Omega$.

A sub-group $\mathfrak{a}$ of R^+ is a *left (right) R-module* of R if $xy\in\mathfrak{a}$ $(yx\in\mathfrak{a})$ for all $x\in R$ and $y\in\mathfrak{a}$. A left R-module that is also a right R-module is a two-sided R-module. A *left (right, two-sided) ideal* is a left (right, two-sided) R-module that is also normal in R^+. A two-sided ideal is simply called an *ideal*. Left R-modules are the R-subgroups of R^+. Left ideals are precisely the kernels of R-homomorphisms of R^+, and ideals are precisely the kernels of the near-ring homomorphisms of R ([2], 1.3.3, [3], 2.1.4).

Definition 1. An ideal $\mathfrak{p}$ in a d.g. near-ring R is called a *prime ideal* if and only if whenever $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R and $\mathfrak{a}\,\mathfrak{b}\subseteqq\mathfrak{p}$, then either $\mathfrak{a}$ or $\mathfrak{b}$ is contained in $\mathfrak{p}$[2]). A d.g. near-ring whose zero ideal is prime is called a *prime d.g. near-ring*.

[2]) If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are subsets of R, then $\mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_r$ denotes the additive group generated by all elements of the form $a_1a_2\ldots a_r$, where $a_i\in\mathfrak{a}_i$, $i=1, \ldots, r$. If $\mathfrak{a}$ is a subset of R, we put $\mathfrak{a}^n=\mathfrak{a}\,\mathfrak{a}\ldots\mathfrak{a}$ (n times). We say that the set $\mathfrak{a}$ of R is *nilpotent* if $\mathfrak{a}^n=(0)$ for some positive integer n.

Lemma. *If* $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ *are ideals and* $\mathfrak{p}$ *is a prime ideal of* R, *then* $\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_r \subseteqq \mathfrak{p}$ *implies that* $\mathfrak{a}_i \subseteqq \mathfrak{p}$ *for some* i.

Proof. If a set $\mathfrak{a}$ is contained in the ideal $\mathfrak{p}$, then so is the least ideal of R containing $\mathfrak{a}$. We denote this ideal by $\bar{\mathfrak{a}}$. Hence if $\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_n$ is contained in $\mathfrak{p}$ so is the ideal $\overline{\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_n}$. Now $(\overline{\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1}})\, \mathfrak{a}_n$ is a product of two ideals and $(\overline{\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1}})\, \mathfrak{a}_n \subseteqq \overline{\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1} \mathfrak{a}_n} \subseteqq \mathfrak{p}$. Therefore if $\mathfrak{a}_n \nsubseteqq \mathfrak{p}$, $\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1} \subseteqq \overline{\mathfrak{a}_1 \ldots \mathfrak{a}_{n-1}} \subseteqq \mathfrak{p}$. Repeating this argument gives the required result.

Definition 2. A non-zero R-group is called an *irreducible* R-group if it possesses no proper, non-zero, normal R-subgroups[3]).

In the following, we shall be concerned with irreducible R-groups Ω which possess an R-generator, that is, an element w in Ω such that $Rw = \Omega$. Such groups will be called *cyclic irreducible R-groups*. It is clear that a minimal R-group is a cyclic irreducible R-group.

Proposition 1. *If* Ω *is a cyclic irreducible R-group, then the annihilating ideal* $(0:\Omega)$ *is a prime ideal of* R[4]).

Proof. The ideal $(0:\Omega)$ consists of all those $x \in R$ such that $xw = 0$ for all $w \in \Omega$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \nsubseteqq (0:\Omega)$ and $\mathfrak{b} \nsubseteqq (0:\Omega)$, and w an R-generator of Ω. Then the R-group $\mathfrak{b}\Omega = \mathfrak{b}(Rw) = (\mathfrak{b}R)\, w = \mathfrak{b} w$ is not the zero sub-group of Ω. If $w' \in \Omega$, then there exists an element $z \in R$ such that $w' = zw$ and so $w' + bw - w' = zw + bw - zw = (z + b - z)\, w \in \mathfrak{b} w$, for all $b \in \mathfrak{b}$. Thus $\mathfrak{b} w$ is a non-zero normal R-subgroup of Ω; hence $\mathfrak{b} w = \Omega$. Now the group $(\mathfrak{a}\, \mathfrak{b})\Omega \supseteqq (\mathfrak{a}\, \mathfrak{b})\, w = \mathfrak{a}(\mathfrak{b}\, w) = \mathfrak{a}\Omega$ is a non-zero subgroup of Ω. Therefore $(\mathfrak{a}\, \mathfrak{b}) \nsubseteqq (0:\Omega)$ and this proves that the ideal $(0:\Omega)$ is prime.

Definition 3. A d.g. near-ring R which has a faithful representation on a cyclic irreducible R-group is called a *primitively prime d.g. near-ring*. An ideal $\mathfrak{p}$ of a d.g. near-ring R is called a *primitively prime ideal* if $R/\mathfrak{p}$ is a primitively prime d.g. near-ring.

The above proposition shows that a primitively prime d.g. near-ring is a prime d.g. near-ring and a primitively prime ideal is a prime ideal. A primitively prime ideal is a proper ideal.

Proposition 2. *An ideal* $\mathfrak{p}$ *in* R *is primitively prime if and only if* $\mathfrak{p} = (\mathfrak{l}:R)$, *where* $\mathfrak{l}$ *is a maximal left ideal of* R.

Proof. Let $\mathfrak{p} = (\mathfrak{l}:R)$; then $R/\mathfrak{p}$ acts faithfully on the irreducible $R/\mathfrak{p}$-group $R^+ - \mathfrak{l}$. The image of the identity element e of R under the homomorphism of R^+ onto $R^+ - \mathfrak{l}$ is an $R/\mathfrak{p}$-generator of $R^+ - \mathfrak{l}$. Conversely, if $\mathfrak{p}$ is a primitively prime ideal, let Ω be a faithful, irreducible $R/\mathfrak{p}$-group with an $R/\mathfrak{p}$-generator w, say. Then Ω is an R-group, $Rw = \Omega$ and the mapping of R^+ onto Ω given by $x \to xw$, for all $x \in R$, is an R-homomorphism. Hence $R^+ - (0:w) \cong \Omega$; $(0:w)$ is a maximal left ideal of R and $\mathfrak{p} = ((0:w):R)$.

If $\mathfrak{a}$ is a subset of R and Δ is a subset of an R-group Ω, then $\mathfrak{a}\Delta$ denotes the group generated by all elements of the form aw, where $a \in \mathfrak{a}$ and $w \in \Delta$.

[3]) See footnote [1]).

[4]) If Δ_1 and Δ_2 are two subsets of an R-group, then $(\Delta_1 : \Delta_2)$ denotes the set of elements of R which map Δ_2 into Δ_1. We note that if $\mathfrak{l}$ is a left ideal of R, then $(\mathfrak{l}:R)$ is an ideal of R contained in $\mathfrak{l}$ which contains all ideals of R in $\mathfrak{l}$ ([2], 3.7.1).

The primitive ideals of R are by definition the annihilating ideals of minimal R-groups (see [8]). Hence:

Proposition 3. *The primitive ideals of a d.g. near-ring are primitively prime ideals and primitive d.g. near-rings are primitively prime d.g. near-rings.*

Proposition 4. *A prime ideal $\mathfrak{p}$ of a d.g. near-ring R contains all the nilpotent ideals of R. The factor near-ring $R/\mathfrak{p}$ has no non-zero nilpotent ideals.*

Proof. If $\mathfrak{a}$ is an ideal and $\mathfrak{a}\,\mathfrak{a} \ldots \mathfrak{a} \leqq \mathfrak{p}$, then $\mathfrak{a} \leqq \mathfrak{p}$ by the lemma.

The intersection of all the primitive ideals of a d.g. near-ring R is called the *radical* (if the near-ring has no primitive ideals, then the radical is taken to be the whole near-ring) and is equal to the intersection of all the maximal left ideals which are maximal left R-modules of R. The intersection of all the maximal left ideals of a d.g. near-ring is called the *quasi-radical* (see [8]). Any d.g. near-ring with an identity element contains maximal left ideals (which are proper left ideals). This is proved by an application of ZORN's lemma. Therefore, by proposition 2, any d.g. near-ring with an identity element possesses primitively prime ideals.

Definition 4. The intersection of all the primitively prime ideals of a d.g. near-ring is called the *ideal-radical* of the near-ring.

Theorem 1. *Let N denote the quasi-radical of a d.g. near-ring R. The ideal-radical is the ideal $(N:R)$ which is contained in N. The ideal-radical contains all the nilpotent ideals of the near-ring.*

Proof. Let P denote the ideal-radical of R. P is contained in the ideal $(\mathfrak{l}:R)$ for each maximal left ideal $\mathfrak{l}$ of R by proposition 2. Since $(\mathfrak{l}:R) \leqq \mathfrak{l}$, P is contained in each maximal left ideal of R and so in the intersection of all the maximal left ideals of R, i.e., the quasi-radical N of R. But P is an ideal and therefore $P \leqq (N:R)$. Also N is contained in each maximal left ideal $\mathfrak{l}$ of R and hence $(N:R) \leqq (\mathfrak{l}:R)$, for each maximal left ideal $\mathfrak{l}$ of R. Therefore $(N:R)$ is contained in the intersection of the ideals $(\mathfrak{l}:R)$ where $\mathfrak{l}$ is a maximal left ideal of R, i.e., $(N:R) \leqq P$ by proposition 2. Therefore the ideal $(N:R)$ is the ideal-radical of R.

The last part of the theorem follows immediately from proposition 4.

We have the following situation for a d.g. near-ring R: the radical contains the quasi-radical which contains the ideal-radical. The radical is an ideal (which may or may not be proper) containing all the nilpotent left R-modules of R. The quasi-radical is a proper left ideal containing all the nilpotent left ideals of R. The ideal-radical is an ideal containing all the nilpotent ideals of R. Further properties of these three "radicals" are obtained in the next section for any d.g. near-ring R which satisfies the descending chain condition for left R-modules (see [8]).

2. Further properties for d.g. near-rings which satisfy the descending chain condition

Any d.g. near-ring R which appears in this and the following section satisfies the descending chain condition (d.c.c.) for left R-modules. In this case, the primitive ideals of R are maximal ideals and primitive d.g. near-rings are simple near-rings, that is, they possess no proper, non-zero ideals (see [7]). The restatement of proposition 3 is

Proposition 5. *The maximal ideals of a d.g. near-ring R which satisfies the d.c.c. for left R-modules are primitively prime ideals. If R is a simple d.g. near-ring, then it is a primitively prime near-ring.*

Theorem 2. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. The ideal-radical is a nilpotent ideal containing all the nilpotent ideals of R.*

Proof. The quasi-radical of R is nilpotent (see [8]) and so the ideal-radical must be nilpotent.

Corollary. *R has a non-zero nilpotent ideal if and only if its ideal-radical P is non-zero. The near-ring R/P has no non-zero, nilpotent ideals.*

We consider the R-group $R^+ - N$, where N is the quasi-radical. The intersection of all the maximal left ideals of R is N. Since R satisfies the d.c.c. for left R-modules, there exists a finite number of distinct maximal left ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ such that $\bigcap_{i=1}^{n} \mathfrak{l}_i = N$ and $\mathfrak{L}_k = \bigcap_{\substack{i=1\\ i\neq k}}^{n} \mathfrak{l}_i \neq N$ for any $k=1, \ldots, n$. In the canonical R-homomorphism of R^+ onto $R^+ - N$ let $\mathfrak{l}_k$ be mapped onto Ω_k and $\mathfrak{L}_k$ onto Δ_k for all $k=1, \ldots, n$. Since $\bigcap_{i=1}^{n} \mathfrak{l}_i = N$ it follows that $\bigcap_{i=1}^{n} \Omega_i = (0)$. Since the $\mathfrak{l}_k$ are maximal left ideals $R = \mathfrak{l}_k + \mathfrak{L}_k$ and so $R^+ - N = \Omega_k \oplus \Delta_k$ for $k=1, \ldots, n$. Also $\mathfrak{l}_k \supseteq \mathfrak{L}_j$ for all $j \neq k$ and $\mathfrak{l}_k \cap \mathfrak{L}_k = N$ for all $k=1, \ldots, n$, therefore $\Omega_k \supseteq \Delta_j$ for all $j \neq k$ and $\Omega_k \cap \Delta_k = (0)$ for all $k=1, \ldots, n$.

If $m \neq k$, $\Omega_m = \Omega_m \cap (R^+ - N) = \Omega_m \cap (\Delta_k \oplus \Omega_k) = \Delta_k \oplus \Omega_k \cap \Omega_m$ by the modular law which holds for the lattice of normal subgroups of $R^+ - N$. Hence $R^+ - N = \Omega_m \oplus \Delta_m = (\Delta_k \oplus \Omega_k \cap \Omega_m) \oplus \Delta_m = \Delta_k \oplus \Delta_m \oplus \Omega_k \cap \Omega_m$.

If $n \neq m$, k, $\Omega_k \cap \Omega_m = \Omega_k \cap \Omega_m \cap (\Delta_n \oplus \Omega_n) = \Delta_n \oplus \Omega_k \cap \Omega_m \cap \Omega_n$ and hence $R^+ - N = \Delta_k \oplus \Delta_m \oplus \Delta_n \oplus \Omega_k \cap \Omega_m \cap \Omega_n$. Since $\bigcap_{i=1}^{n} \Omega_i = (0)$ it follows that we obtain finally

$$R^+ - N = \Delta_1 \oplus \cdots \oplus \Delta_n, \tag{1}$$

where $\Delta_k \cong (R^+ - N) - \Omega_k \cong R^+ - \mathfrak{l}_k$ for $k=1, \ldots, n$. From these isomorphisms it follows that the Δ_k are irreducible R-groups. If e is mapped onto $\bar{e}$ under the canonical homomorphism of R^+ onto $R^+ - N$ and $\bar{e} = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n$, $\bar{e}_k \in \Delta_k$, then $R\bar{e}_k = \Delta_k$ for $k=1, \ldots, n$[5]). Thus the Δ_k are cyclic irreducible R-groups.

Let Ω be a cyclic irreducible R-group with an R-generator w, say. $R^+ - (0:w) \cong \Omega$ and $(0:w)$ is a maximal left ideal. Thus $(0:w)$ contains N

[5]) We note that if R is a d.g. near-ring and the (R, S)-group $\Omega = \Omega_1 \oplus \cdots \oplus \Omega_n$ is a direct sum of R-groups, then $x(w_1 + \cdots + w_n) = x w_1 + \cdots + x w_n$ for all $x \in R$, where $w_k \in \Omega_k$ for $k=1, \ldots, n$.

and from (1) $R^+ - (0:w)$ must be R-isomorphic to one of the R-groups Δ_k[6]). Hence

Theorem 3. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. Any cyclic irreducible R-group is R-isomorphic to one of the finite number of irreducible R-groups which appear in the direct sum decomposition* (1) *of $R^+ - N$.*

Now let $\mathfrak{p}$ be a proper prime ideal in R; put $\overline{R} = R/\mathfrak{p}$ and denote by $\overline{N}$ the quasi-radical of $\overline{R}$. $\overline{R}$ is a prime d.g. near-ring satisfying the d.c.c. for left $\overline{R}$-modules. The zero ideal $(\overline{0})$ of $\overline{R}$ is prime and so $\overline{R}$ contains no non-zero nilpotent ideals, in particular, $(\overline{N}:\overline{R}) = (\overline{0})$. Hence $\overline{R}$ acts faithfully on the group $\overline{R}^+ - \overline{N}$. By (1), $\overline{R}^+ - \overline{N} = \overline{\Delta}_1 + \cdots + \overline{\Delta}_r$, where the $\overline{\Delta}_k$ are cyclic irreducible $\overline{R}$-groups. The annihilating ideals $(\overline{0}:\overline{\Delta}_k)$ are primitively prime ideals and $\bigcap_{i=1}^{r} (\overline{0}:\overline{\Delta}_k) = (\overline{0})$. Therefore

$$(\overline{0}:\overline{\Delta}_1)(\overline{0}:\overline{\Delta}_2)\ldots(\overline{0}:\overline{\Delta}_r) \subseteqq \bigcap_{i=1}^{r} (\overline{0}:\overline{\Delta}_i) = (\overline{0})$$

and, since $(\overline{0})$ is a prime ideal, $(\overline{0}:\overline{\Delta}_k) = (\overline{0})$ for some k. In other words, $\overline{R}$ acts faithfully on some $\overline{\Delta}_k$. Hence

Theorem 4. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. Any proper prime ideal of R is a primitively prime ideal. If R is a non-zero prime d.g. near-ring, then it is a primitively prime d.g. near-ring.*

We end this section with a structure theorem for finite prime d.g. near-rings which gives a relationship between all the cyclic irreducible R-groups. To a certain extent it generalizes the theorem for simple d.g. near-rings which states that all the minimal R-groups of a simple d.g. near-ring R are R-isomorphic (see [7]).

Theorem 5. *Let R be a non-zero finite, prime d.g. near-ring. All cyclic irreducible R-groups are R-homomorphic images of R-subgroups of a faithful, cyclic irreducible R-group. To within an isomorphism, R has one, and only one, faithful, cyclic irreducible R-group.*

Proof. R is not the zero near-ring. Let N be the quasi-radical of R. Then $R^+ - N = \Delta_1 \oplus \cdots \oplus \Delta_n$, where the Δ_k are irreducible R-groups with R-generators $\overline{e}_k$, $\overline{e} = \overline{e}_1 + \cdots + \overline{e}_n$. From the previous theorem, R acts faithfully on some Δ_k, say Δ_n. Denote by I the set of all groups which are R-homomorphic images of R-subgroups of Δ_n and, by I', those groups of I which are cyclic

[6]) We are using here the fact that an R-group is an (R, S)-group for some set S of distributive elements of R. Thus an R-group is an operator group with S as a set of operators. Since S generated R^+ a subgroup of an (R, S)-group is an R-subgroup if and only if it is an admissible subgroup for the set of operators S and two (R, S)-groups are R-isomorphic if they are isomorphic as operator groups for the set of operators S. It follows that in the present case the decomposition (1) of $R^+ - N$ provides us with the composition series $\Delta_1 \oplus \cdots \oplus \Delta_n > \Delta_2 \oplus \cdots \oplus \Delta_n > \cdots > \Delta_n > (0)$ for $R^+ - N$ as an operator group for any distributively generating set S of R. Hence we can use the Jordan-Hölder theorem for operator groups to show that any two composition series of R-subgroups of $R^+ - N$ are of the same length and the factor groups of one of the series are R-isomorphic to the factor groups of the other series in some order.

irreducible R-groups. We define a left ideal N' as follows:

$$N' = \cap \{\mathfrak{l} : \mathfrak{l} \text{ is a left ideal and } R^+ - \mathfrak{l} \in I'\}.$$

Clearly such left ideals $\mathfrak{l}$ are maximal left ideals and so $N' \supseteqq N$. It will be shown that some power of N' annihilates Δ_n but, since Δ_n is faithful, this means that N' is a nilpotent left ideal. Hence $N' \subseteqq N$ and consequently $N' = N$. Thus N is the intersection $\bigcap_{i=1}^{m} \mathfrak{l}_i$ of maximal left ideals $\mathfrak{l}_i$ of R for which $R^+ - \mathfrak{l}_i \in I'$. We may assume that $\bigcap_{\substack{i=1 \\ i \neq j}}^{m} \mathfrak{l}_i \neq N$ for all $j = 1, \ldots, m$. As indicated for the decomposition (1) of $R^+ - N$, this leads to the decomposition $R^+ - N = \Delta'_1 \oplus \cdots \oplus \Delta'_m$, where $\Delta'_k \cong R^+ - \mathfrak{l}_k \in I'$ for $k = 1, \ldots, m$. These Δ'_k are cyclic irreducible R-groups. Thus both this decomposition and decomposition (1) provide us with composition series of $R^+ - N$ and so $n = m$ and the Δ_k are R-isomorphic to the Δ'_j in some order [7]). The theorem now follows from theorem 3.

We shall prove that every group in I is annihilated by some power of N' by induction on the order of the groups in I.

If Ω has least order among the non-zero groups in I, it is a minimal R-group and so is in I'. Let $w \in \Omega$, $w \neq 0$; then $Rw = \Omega$ and hence $R^+ - (0:w) \cong \Omega$ so that the left ideal $(0:w)$ contains N'. This is true for every element of Ω and consequently N' annihilates Ω itself.

Now let $\Omega \in I$ be an R-group of order $m > 0$ and assume that all groups in I of order less than m are annihilated by some power of N'. Let $\Omega = \Omega_0 > \Omega_1 > \cdots > \Omega_l = (0)$ be a strictly descending chain of R-groups in Ω such that each Ω_i is normal in Ω_{i-1} and $\Omega_{i-1} - \Omega_i$ is an irreducible R-group (not necessarily cyclic). Each $\Omega_{i-1} - \Omega_i$ is contained in I. If $l > 1$, then these factor groups have order less than m and so, by hypothesis, are annihilated by some power of N'. Therefore Ω itself is annihilated by some power of N'. We are left with the case $l = 1$, i.e., when Ω is an irreducible R-group. If $w \in \Omega$ and $Rw = \Omega$, then $\Omega \in I'$ and $R^+ - (0:w) \cong \Omega$. Therefore $(0:w)$ contains N' and hence N' annihilates w. If $w \in \Omega$ and $Rw \subsetneqq \Omega$, then $Rw \in I$ has order less than m and by hypothesis Rw, and so w, is annihilated by some power of N'. Thus every element of Ω and therefore Ω itself is annihilated by some power of N'. It follows, therefore, that every group in I, in particular Δ_n itself, is annihilated by some power of N'. This proves the first part of the theorem.

Now let Ω and Ω' be two faithful, cyclic irreducible R-groups. They are finite groups. We have shown that Ω is an R-homomorphic image of an R-subgroup of Ω' and, conversely, Ω' is an R-homomorphic image of an R-subgroup of Ω. Thus Ω and Ω' are isomorphic.

Besides the properties of a d.g. near-ring described at the end of section 1 we have the following additional properties for a d.g. near-ring R which satisfies the d.c.c. for left R-modules:

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[7]) See footnote [5]).

The radical M of R is an ideal containing all the nilpotent left R-modules of R and the factor d.g. near-ring R/M contains no non-zero nilpotent left R/M-modules, i.e., R/M is semi-simple. The quasi-radical is a nilpotent left ideal containing all the nilpotent left ideals of R (see [8]). The ideal-radical P is a nilpotent ideal containing all the nilpotent ideals of R. Furthermore, the factor d.g. near-ring R/P contains no non-zero nilpotent ideals.

3. The nilpotency of the radical

In [8] we gave necessary and sufficient conditions for the radical of a d.g. near-ring to be nilpotent. We restate these conditions below and add new conditions in terms of prime ideals.

Theorem 6. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. The following conditions are equivalent:*

(a) The radical is nilpotent.

(b) The radical is the quasi-radical.

(c) Every maximal left ideal is a maximal left R-module.

(d) The radical is the ideal-radical.

(e) Every proper prime ideal is maximal.

(f) Every cyclic irreducible R-group is a minimal R-group.

Proof. (a) implies (b). The radical contains quasi-radical and the quasi-radical contains all the nilpotent left ideals of R. Therefore the radical is the quasi-radical if it is nilpotent.

(b) implies (c). If the radical M is the quasi-radical, then every maximal left ideal of R contains M. But R/M is a semi-simple d.g. near-ring and so is a direct sum of left ideals of R/M which are minimal left R/M-modules (see [8]). It follows that every maximal left ideal is a maximal left R-module.

(c) implies (d). If every maximal left ideal is a maximal left R-module, then the radical is the quasi-radical and so is a nilpotent ideal. But the radical contains the ideal-radical and the ideal-radical contains every nilpotent ideal of R. Therefore the radical is the ideal-radical if it is nilpotent.

(d) implies (e). If the radical M is the ideal-radical, then every prime ideal of R contains M. Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{m}_1 \mathfrak{m}_2 \ldots \mathfrak{m}_r \subseteqq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_r = M \subseteqq \mathfrak{p}$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are maximal ideals of R (if R satisfies the d.c.c. for left R-modules, then the radical is, in fact, an intersection of a finite number of maximal ideals). Hence $\mathfrak{m}_i \subseteqq \mathfrak{p}$ for some i and therefore $\mathfrak{m}_i = \mathfrak{p}$.

(e) implies (f). Let Ω be a cyclic irreducible R-group. Then the annihilating ideal $\mathfrak{p} = (0:\Omega)$ is prime and is a maximal ideal if condition (e) is satisfied. In this case $R/\mathfrak{p}$ is a simple d.g. near-ring and so is a direct sum of isomorphic minimal $R/\mathfrak{p}$-groups (see [7]). Since Ω is an $R/\mathfrak{p}$-group and has an $R/\mathfrak{p}$-generator it follows that it is a direct sum of minimal $R/\mathfrak{p}$-groups. But Ω is irreducible and so it must be a minimal $R/\mathfrak{p}$-group. Hence Ω is a minimal R-group.

(f) implies (a). Let $\mathfrak{l}$ be a maximal left ideal of R. Then $R^+ - \mathfrak{l}$ is a cyclic irreducible R-group and hence it is a minimal R-group if condition (f) is satisfied. Thus $\mathfrak{l}$ is a maximal left R-module. Therefore the radical and the quasi-radical must be equal in this case and therefore the radical is nilpotent.

4. An example of a d.g. near-ring with a non-nilpotent radical

We construct an example of a finite d.g. near-ring with a non-nilpotent radical and a quasi-radical which is not an ideal.

Let Ω be a finite, non-abelian, simple group and R the near-ring generated by all the inner-automorphisms of Ω. It has been shown in [4] and [7] that R is a finite, simple d.g. near-ring with an identity element. If Δ is a subgroup of Ω, $(\Delta:\Omega)$ is a right R-module and any right R-module $\mathfrak{r}=(\Delta:\Omega)$ for some subgroup Δ. There is a one-to-one lattice correspondence given by $\mathfrak{r}=(\Delta:\Omega)\leftrightarrow \mathfrak{r}\Omega=\Delta$ between the right R-modules of R and the subgroups of Ω. Furthermore, right ideals and normal subgroups correspond to each other. Consequently, R has proper, non-zero right R-modules but no proper, non-zero right ideals. Finally, we note that each right R-module $\mathfrak{r}$ has an element $e_{\mathfrak{r}}$ such that $\mathfrak{r}=e_{\mathfrak{r}}R$.

Now consider the near-ring T generated by all the endomorphisms Φ_x of R^+, for all $x\in R$, where $\Phi_x(y)=yx$ for all $y\in R^+$ [8]). It is clear that T is a finite d.g. near-ring with an identity element; R^+ is a faithful T-group and the identity element of R is a T-generator of R^+. But the T-subgroups of R^+ are precisely the right R-modules of R and so R^+ is an irreducible T-group with a T-generator which is not a minimal T-group. It follows from theorem 5 that T does not have a nilpotent radical.

Since R^+ is faithful, the ideal radical of T is the zero ideal (T is a finite, prime d.g. near-ring which is not simple). If the quasi-radical were an ideal, it would be the zero ideal (since the quasi-radical is nilpotent). But then T is a direct sum of irreducible T-groups by the equality (1) after the corollary to theorem 2. Every T-subgroup $\mathfrak{r}$ of R^+ has a T-generator $e_{\mathfrak{r}}$ and consequently every T-subgroup is a direct sum of irreducible T-subgroups. By the lattice-isomorphism, this means that every subgroup of Ω is a direct sum of simple groups. If we take Ω to be the alternating group on nine symbols, then it has a cyclic subgroup of order nine and this subgroup is not a direct sum of simple subgroups, i.e., two subgroups of order three. It follows that the quasi-radical of T corresponding to the alternating group on nine symbols is not an ideal.

It is an open question whether or not the quasi-radical of a d.g. near-ring can be an ideal and yet not be the radical also. Or, to put it another way, whether a d.g. near-ring can possess non-zero nilpotent left R-modules and yet possess no non-zero nilpotent left ideals.

[8]) These mappings of R^+ into itself are endomorphisms because R satisfies the right distributive law.

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