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Laconicity and redundancy of Toeplitz matrices*

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The convergence field of a Toeplitz matrix is a monotonic function of the set of rows that compose the matrix, in the sense that the deletion of some of the rows of the matrix (followed by appropriate renumbering of the rows that remain) can never decrease the convergence field. In the case of certain matrices, the deletion of infinitely many rows always increases the convergence field; but there exist matrices that do not have this property. We shall consider this dichotomy with special reference to the space of bounded sequences and certain classical families of matrices.

1. The concept of laconicity

By a Toeplitz matrix we understand any matrix $A=(a_{nk})$ $(n, k=0, 1, \ldots)$ of complex numbers. For each matrix A, we define the norm of the row with index n as the sum $\sum_k |a_{nk}|$, and the *norm* $\|A\|$ of the matrix as the supremum of the row-norms. A matrix is conservative (German: *konvergenztreu*) if convergence to a finite limit of a sequence $s=\{s_n\}$ implies the existence and convergence of the transform $t=As$ defined formally by the relation $t_n=\sum_{k=0}^{\infty} a_{nk} s_k$. A matrix is *regular* provided it is conservative and $\lim As=\lim s$ whenever the second limit exists and is finite. Except in the present section, we restrict our attention almost exclusively to conservative matrices.

Of two matrices A and B we shall say that B is a *submatrix* of A provided each row of B is a row of A and infinitely many rows of A are not rows of B. If B is a submatrix of A and $s=\{s_n\}$ is a sequence whose transform As exists, then the transform $u=Bs$ is a subsequence of the transform $t=As$; this implies the monotonicity mentioned in the introductory paragraph.

Theorem 1. *If A is a Toeplitz matrix with the property that each sequence of 0's and 1's is the transform As of some sequence s, and if B is a submatrix of A, then the convergence field of A is a proper subset of the convergence field of B.*

Proof. Let A satisfy the hypothesis of the theorem, and let B be obtained from A by the deletion of the rows with indices $n_1, n_2, \ldots$. Let

$$t_n=\begin{cases} 1 & (n\in\{n_i\}), \\ 0 & (n\notin\{n_i\}), \end{cases}$$

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and let s be defined by the condition $s = At$. Then the sequence Bs consists entirely of 0's, while As consists of infinitely many 0's and 1's. This proves the theorem.

Most of the commonly studied classical Toeplitz matrices satisfy the hypothesis in Theorem 1. But it is not generally true of a classical matrix that every sequence of 0's and 1's is the transform As of a *bounded* sequence. Rather, there exist simple, regular matrices A that satisfy the hypothesis in Theorem 1 and some of whose submatrices have the same convergence field as A, *in the space m of bounded sequences*. In other words, some matrices have unnecessarily many rows for the maintenance of their *divergence* fields in m.

Definitions. Corresponding to each Toeplitz matrix A we denote by (A) the set of bounded sequences s whose transform As exists and converges, and we call (A) the *bounded convergence field* of A. We say that A is *redundant* provided it has a submatrix B such that $(A) = (B)$. If A is not redundant, it is *laconic.*

A matrix is redundant, for example, if it has infinitely many pairs of rows that resemble each other sufficiently well. To make this statement precise, we denote by A_{ij} the one-rowed matrix $(a_{i0} - a_{j0},\ a_{i1} - a_{j1}, \ldots)$, and we call the matrix with the rows $A_{10}, A_{20}, A_{21}, A_{30}, A_{31}, A_{32}, A_{40}, \ldots$ the *internal-difference matrix* of A.

Theorem 2. *If A is a matrix of finite norm and its internal-difference matrix has a submatrix whose row-norms tend to* 0, *then A is redundant.*

Proof. The hypothesis of the theorem implies that, for some increasing sequence $\{n_i\}$,

$$\lim_{i \to \infty} \sum_{k=0}^{\infty} |a_{n_{2i},k} - a_{n_{2i+1},k}| = 0.$$

Let B be obtained by the deletion from A of the rows with indices $n_0, n_2, n_4, \ldots$. If s is any bounded sequence, we can obtain Bs from $t = As$ by deleting the elements with indices $n_0, n_2, n_4, \ldots$. The deleted sequence $\{t_{n_0}, t_{n_2}, \ldots\}$ differs by a nullsequence from the sequence $\{t_{n_1}, t_{n_3}, \ldots\}$, which has not been deleted in the passage from As to Bs, and hence Bs converges if and only if As converges. Therefore A is redundant.

2. Laconicity of Hausdorff matrices

The Hausdorff matrices of finite norm are the triangular matrices of the form $A = A(\alpha) = (a_{nk})$ with

$$a_{nk} = \binom{n}{k} \int_0^1 u^k (1-u)^{n-k}\, d\alpha(u), \tag{1}$$

where $\alpha(u)$ is a function of bounded variation on $[0, 1]$, normalized by the rule that $\alpha(0) = 0$ and

$$2\alpha(u) = \alpha(u+0) + \alpha(u-0) \qquad (0 < u < 1)$$

(see [4]).

Theorem 3. *A Hausdorff matrix $A(\alpha)$ is laconic if and only if the function α is discontinuous at $u=1$.*

Proof. Suppose first that α is continuous at $u=1$. The redundancy of A will follow from Theorem 2 when we have shown that

$$\lim_{n\to\infty}\sum_{k=0}^{n+1}|a_{nk}-a_{n+1,k}|=0. \tag{2}$$

Let $\varepsilon>0$, and choose a constant δ $(0<\delta<1)$ such that

$$\int_{\delta^2}^{1}|d\alpha(u)|<\varepsilon.$$

For each n, we consider first separately the indices k less than $n\delta$. Equation (1) implies that

$$a_{nk}-a_{n+1,k}=\int_0^1 f(u,n,k)\,d\alpha(u),$$

where

$$f(u,n,k)=\binom{n}{k}u^k(1-u)^{n-k}\frac{u-k/(n+1)}{1-k/(n+1)} \qquad (0\leqq k\leqq n). \tag{3}$$

Corresponding to each constant η $(0<\eta<1)$ and each pair of integers n and k, we denote by $E(n,k,\eta)$ the part of the interval $[0,1]$ that lies in $[k/n-\eta,\ k/n+\eta]$, and by $F(k,n,\eta)$ the remainder of $[0,1]$. Since

$$\sum_{k=0}^{n}\int_0^1\binom{n}{k}u^k(1-u)^{n-k}|d\alpha(u)|\leqq\int_0^1|d\alpha(u)|,$$

and since the last factor in the right member of (3) is less than ε in $E(n,k,\eta)$ if $k<n\delta$ and $\eta<\eta_\delta$, we can choose η so that the inequality

$$\sum_{k<n\delta}\ \int_{E(n,k,\eta)}|f(u,n,k)|\cdot|d\alpha(u)|<\varepsilon \tag{4}$$

holds for all n. The last factor in the right member of (3) is bounded uniformly with respect to n and k $(k<n\delta)$, and the well-known uniformly rapid convergence to 0 of

$$\binom{n}{k}u^k(1-u)^{n-k}$$

in $F(n,k,\eta)$ implies the analogue of (4) for the range $F(n,k,\eta)$, when n is large enough.

By our choice of η, $\sum_{k\geqq n\delta}|a_{nk}|<2\varepsilon$ when n is large enough. Since the last inequality implies that

$$\sum_{k\geqq n\delta}|a_{nk}-a_{n+1,k}|<4\varepsilon,$$

we now conclude that

$$\sum_{k=0}^{n+1}|a_{nk}-a_{n+1,k}|<6\varepsilon \qquad (n>n_\varepsilon).$$

This proves Theorem 3 for the case where α is continuous at $u=1$.

Suppose next that α is discontinuous at $u=1$, and write $u(1)-u(1-0)=h$. Then $A(\alpha)=A(\beta)+A(\gamma)$, where β is continuous at $u=1$ and γ is constant except for a saltus h at $u=1$. It follows that $A(\alpha)=A(\beta)+hI$, where I is the matrix representing the identity transformation, and where the elements of $A(\beta)$ tend to 0 uniformly as their row-indices become large.

Let B be the matrix obtained from A by deletion of the rows with indices $n_1, n_2, \ldots$. To construct a sequence $s=\{s_n\}$ that lies in (B) but not in (A), we need only choose $s_n=1$ for all n that belong to a sufficiently thin subsequence of $\{n_i\}$, and $s_n=0$ for all other values of n. This concludes the proof of Theorem 3.

Remark. H. G. BARONE [*1*] proved that if a triangular matrix A of finite norm satisfies condition (2), then it transforms every bounded sequence into a sequence whose set of limit points is connected. In particular, he showed that the regular Hölder, Cesàro, and Euler transformations satisfy the conditions [*1*, Theorems 5.3, 6.3, and 9.2].

If a matrix A of finite norm has the property that its diagonal elements are bounded away from 0 while every sequence formed from its remaining elements tends to 0, and if s is a divergent sequence of 0's and 1's (the latter sufficiently scarce), then the origin is an isolated limit point of the sequence As and constitutes a proper subset of the set of all limit points of As.

We can therefore extend BARONE's Theorems 5.3, 6.3, and 9.2 as follows: *In order that a Hausdorff matrix $A(\alpha)$ transform every bounded sequence into a sequence whose set of limit points is connected, it is necessary and sufficient that the function α be continuous at $u=1$.*

3. The little Nörlund transformations

The Nörlund matrices are the triangular matrices of the form

$$N(p)=\begin{pmatrix} p_0/P_0 & & \\ p_1/P_1 & p_0/P_1 & \\ p_2/P_2 & p_1/P_2 & p_0/P_2 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{pmatrix},$$

where $p=\{p_n\}$ is a sequence of complex numbers and $P_n=\sum_0^n p_k \neq 0$ for $n=0, 1, \ldots$ (see NÖRLUND [*5*] and WORONOI [*9*]). In contexts where the transformation $N(p)$ is required to be regular, the sequence $\{P_n\}$ must be bounded away from 0. We shall focus our attention on the cases where

$$\sum |p_n| < \infty. \tag{5}$$

Under the restriction (5), regularity of N is equivalent to the convergence of $\{P_n\}$ to a limit other than zero. In the study of convergence fields of regular Nörlund matrices subject to (5), we shall therefore incur no loss of generality if we replace the element p_r/P_n in the matrix N by p_r. Once the P_n have disappeared, the requirement that $P_n \neq 0$ is superfluous, and we drop it.

With our modified matrix we shall associate the function $f(z)=\sum p_n z^n$. Indeed, we shall use the notation

$$N_f=\begin{pmatrix} p_0 & & \\ p_1 & p_0 & \\ \cdot & \cdot & \cdot \end{pmatrix},$$

and we shall call N_f the *little Nörlund transformation associated with* f (or generated by f). The matrix product of two little Nörlund matrices N_f and N_g is the little Nörlund matrix N_h, where $h(z)=f(z)\,g(z)$. Therefore the little Nörlund matrices form an Abelian semigroup under matrix multiplication.

In the theory of ordinary Nörlund transformations it is customary to use the multiplication

$$N(p)\circ N(q)=N(r), \tag{6}$$

where $r_n=p_0q_n+p_1q_{n-1}+\cdots+p_nq_0$. Now, if for example $\{p_n\}=\{1,1,0,0,0,\ldots\}$ and $\{q_n\}=\{-1,2,0,0,\ldots\}$, then the matrices $N(p)$ and $N(q)$ are regular; but their formal product under (6) is not a Nörlund matrix; for

$$(1+z)(-1+2z)=-1+z+2z^2,$$

and therefore $N(r)$ suffers from the defect that $R_1=r_0+r_1=0$. We see at once that the ordinary Nörlund matrices corresponding to polynomials fail to form a semigroup under the multiplication (6) unless we subject the admissible polynomials to severe restrictions. Hence, under the restriction (5), the use of little instead of ordinary Nörlund matrices has advantages beyond computational and typographical convenience.

Each of the four theorems in the present section is either already in the literature, or it is at least familiar to many specialists in summability theory (see the discussion of *allgemeine Zweierverfahren* by K. Zeller [*10*, p. 126]). We include the material partly to make the paper as nearly self-contained as is feasible, and partly because the inclusion permits us to state the theorems in forms that will be most appropriate in the applications (Section 4).

Theorem 4. *A necessary and sufficient condition for the convergence field of a little Nörlund matrix to contain at least one divergent sequence is that* $f(z)=0$ *for some* z *in* $0<|z|\leqq 1$.

Proof. If $f(z)\equiv 0$, then $N_f s$ converges for all s. If $f(z)=z^h g(z)$, we can obtain $N_f s$ from $N_g s$ by adjoining h elements 0 at the beginning of $N_g s$, and therefore the convergence fields of N_f and N_g are identical. In our proof we may therefore assume that $p_0\neq 0$.

The inverse N_f^{-1} of N_f is the matrix whose n^{th} row is $\{q_n, q_{n-1}, \ldots, q_0, 0, \ldots\}$, where $\sum q_n z^n=1/f(z)$ in the neighborhood of the origin. Since N_f^{-1} can not have finite norm if $1/f(z)$ is unbounded in $|z|\leqq 1$, the sufficiency of the condition in the theorem follows immediately.

To see the necessity, suppose that $\sum|p_n|<\infty$ and $f(z)\neq 0$ in $|z|\leqq 1$. Then $\sum|q_n|<\infty$ (see Wiener [*8*, p. 14] and Zygmund [*11*, middle of p. 246]).

In other words, N_f^{-1} is also a little Nörlund transformation, and therefore it preserves convergence. This completes the proof of Theorem 4.

In view of Theorem 4, we naturally expect the convergence field of a little Nörlund transformation N_f to depend heavily on the position of the zeros of f. For the case where f is a polynomial without zeros on $|z|=1$, a complete description of the convergence field of N_f has been given by G. M. PETERSEN [6, Theorem 2.2] (see A. PEYERIMHOFF [7] for a more general theorem, and D. BORWEIN [2] for related results). Our treatment avoids the restriction that no zeros of f lie on $|z|=1$.

Theorem 5. *Let $\{a_i\}$ be a set of j distinct complex numbers, let $f_i(z)=(z-a_i)^{h_i}$, where h_i is a positive integer, and let $f(z)=\prod_i f_i(z)$. Then the convergence field of N_f is the span of the convergence fields of the N_{f_i}.*

Proof. Again, we may suppose that $f(0)\neq 0$. Let S denote the span of the convergence fields of the transformations N_{f_i}, and suppose first that $s=\{s_n\}\in S$. Then we can write $s_n=\sum_{i=1}^{j} s_n^{(i)}$, where each sequence $s^{(i)}=\{s_n^{(i)}\}$ belongs to the convergence field of the corresponding transformation N_{f_i}. Since the matrix N_f is the commutative matrix product of the matrices N_{f_i}, and since each of the transformations N_{f_i} is conservative, each of the transforms $N_f s^{(i)}$ converges. Therefore s belongs to the convergence field of N_f, and it follows that this convergence field contains S.

To prove that S contains the convergence field of N_f, we again use the fact that, with the notation

$$1/f(z)=\sum q_n z^n \qquad \left(|z|<\min_i |a_i|\right),$$

the n^{th} row of the matrix N_f^{-1} is $\{q_n, q_{n-1}, \ldots, q_0, 0, \ldots\}$. Let $t=N_f s$, that is, let $s=N_f^{-1}t$. There exist polynomials g_i of degree h_i-1 $(i=1,2,\ldots,j)$ such that

$$\sum s_n z^n=\sum q_n z^n \sum t_k z^k=[f(z)]^{-1}\sum t_k z^k$$
$$=\sum_{i=1}^{j} g_i(z)\,(z-a_i)^{-h_i}\sum_{k=0}^{\infty} t_k z^k,$$

in some neighborhood of the origin. Define the j sequences $s^{(i)}=\{s_n^{(i)}\}$ $(i=1,2,\ldots,j)$ by the formulas

$$\sum_{n=0}^{\infty} s_n^{(i)} z^n=g_i(z)\,(z-a_i)^{-h_i}\sum_{k=0}^{\infty} t_k z^k.$$

Then $N_{f_i}s^{(i)}=N_{g_i}t$, and since the transformations N_{g_i} are conservative, the convergence of t implies the convergence of $N_{f_i}s^{(i)}$. Since also $s_n=\sum_{i=1}^{j} s_n^{(i)}$, it follows that if $N_f s$ converges, then s is the sum of j sequences $s^{(i)}$ lying in the convergence fields of the corresponding transformations N_{f_i}. This completes the proof.

Theorem 6 (G. M. PETERSEN). *If $0<|a|<1$ and $f(z)=(z-a)^h$, where h is a positive integer, then $N_f s$ converges if and only if s has the form*

$$s_n = a^{-n}(b_0 + b_1 n + \cdots + b_{h-1} n^{h-1}) + c_n, \tag{7}$$

where the b_i are constants and $\{c_n\}$ is a convergent sequence.

Proof. If s has the form (7) and $N_f s = t$, there exists a polynomial g such that

$$\sum s_n z^n = g(z)\,(z-a)^{-h} + \sum c_n z^n,$$

in other words, such that

$$\sum t_n z^n = (z-a)^h \sum s_n z^n = g(z) + \sum \gamma_n z^n,$$

where $\{\gamma_n\}$ converges. Therefore t converges.

To prove that every sequence in the convergence field of N_f has the form $\{7\}$, we use mathematical induction. Suppose first that $h=1$ and that the sequence $t=N_f s$ converges. Since the row with index n in the matrix $-N_f^{-1}$ is $\{a^{-n-1}, a^{-n}, \ldots, a^{-1}, 0, \ldots\}$, we can write

$$s_n = -a^{-n-1}(t_0 + a t_1 + a^2 t_2 + \cdots + a^n t_n) = -a^{-n-1}\sum_{k=0}^{\infty} a^k t_k - \sum_{k=0}^{\infty} a^k t_{n+1+k}.$$

With the notation $A = -\sum a^{k-1} t_k$, this becomes

$$s_n = A/a^n + \sum_{k=0}^{\infty} a^k t_{n+1+k}.$$

If $\lim t_n = b$, the value of the infinite series on the right is $\varepsilon_n + b/(1-a)$, where $\varepsilon_n \to 0$ as $n \to \infty$. This proves our assertion for the case where $h=1$.

Now write $f(z)=(z-a)^h=(z-a)\,g(z)$ $(h>1)$, and suppose that the convergence field of N_g consists of the sequences

$$\{\tau_n\} = \{a^{-n}(b_0 + b_1 n + \cdots + b_{h-2} n^{h-2}) + c_n\} = \{a^{-n} B_{h-2}(n) + c_n\}, \tag{8}$$

where $\{c_n\}$ represents an arbitrary convergent sequence. If the sequence $N_f s = N_g N_{z-a} s$ converges, then $N_{z-a} s$ has the form (8); we may therefore write $s = N_{z-a}^{-1}\tau$, and it follows that

$$s_n = -a^{-n-1}(\tau_0 + a\tau_1 + \cdots + a^n \tau_n) = -a^{-n-1}\sum_{m=0}^{n} B_{h-2}(m) + \sum_{m=0}^{n} a^m c_m.$$

Since the first sum in the last member is a polynomial in n of degree $h-1$, and since the sequence $\{c_n\}$ converges (so that the contribution $\{-a^{-n-1}\sum a^m c_m\}$ is covered by the first stage of our proof), it follows that $\{s_n\}$ is of the form (8) with B_{h-2} replaced by a polynomial of degree $h-1$. This completes the proof of Theorem 6.

In the following theorem, the symbol Δb_n denotes the difference $b_n - b_{n+1}$, and $\Delta^h b_n$ is defined by the equation $\Delta^h b_n = \Delta(\Delta^{h-1} b_n)$.

Theorem 7. *Let $f(z)=(z-a)^h$, where $|a|=1$ and h is a positive integer. If $a \neq 1$, then the sequence $N_f s$ convergences if and only if s has the form*

$$s_n = a^{-n} b_n + c,$$

where $\{b_n\}$ is a sequence such that $\Delta^h b_n \to 0$, and where c is a constant. If $a=1$, then $N_f s$ converges if and only if $\Delta^h s_n$ tends to a constant.

In the special case where $a=1$, the theorem follows immediately from the fact that for $n \geqq h$ the element t_n of the transform $N_f s$ has the value $t_n = \Delta^h s_{n-h}$.

With regard to the case where $a \neq 1$, we observe first that if $N_f s$ converges to d and $s_n^* = s_n - (1-a)^{-h} d$, then $N_f s^* \to 0$. We may therefore restrict ourselves to the case where $N_f s \to 0$. Now, if $t = N_f s$, then

$$t_{n+h} = a^{-h} \Delta^h (a^n s_n),$$

and since $|a| = 1$, $t_n \to 0$ if and only if $\Delta^h (a^n s_n) \to 0$ as $n \to \infty$. This completes the proof of Theorem 7.

4. Laconicity of little Nörlund transformations

Theorem 8. *If f is a polynomial of degree at most 2 and $f(0) \neq 0$, then N_f is redundant if and only if $f(z) = (z-a)^2$, with $|a| = 1$.*

Proof. If $f(z) \equiv c \neq 0$, then $N_f = cI$, where I is the identity transformation, and N_f is obviously laconic.

For the case where $f(z) = z - a$ $(a \neq 0)$, we shall suppose that M is the matrix obtained by deleting from N_f the rows with indices n_i $(i = 1, 2, \ldots)$, and we shall construct a bounded sequence s such that $Ms \to 0$ while $N_f s$ diverges. It is sufficient to carry out the construction under the assumption that $n_{i+1} - n_i \to \infty$, because the deletion of additional rows would tend to increase the convergence field even further.

Suppose first that $|a| > 1$. For $n < n_1$, we choose $s_n = 0$; for $i = 1, 2, \ldots$, we write

$$s_{n_i + r} = a^{-r} \qquad (0 \leqq r < n_{i+1} - n_i).$$

Inspection shows that the sequence Ms consists exclusively of 0's. Also, the sequence s consists of blocks of elements, the first of which contains only 0's while each of the others consists of an element 1 followed by elements of smaller modulus. Clearly, s is bounded, but $N_f s$ diverges, and therefore N_f is laconic.

If $|a| < 1$, we proceed similarly, except that in order to preserve boundedness of s, we choose

$$s_{n_i - r} = a^{r-1} \qquad (0 < r \leqq n_i - n_{i-1}).$$

That is, we construct s so that it consists of blocks that begin with a small element and end with a 1.

If $|a| = 1$, there is no danger of unboundedness of s; but a sequence constructed according to the pattern used above might accidentally lie in the convergence field of N_f. Should this happen, we multiply all elements of the i^{th} block by $(-1)^i$. This concludes the discussion of the case where f is a first-degree polynomial.

If $f(z)=(z-a)(z-b)$ $(ab\neq 0)$, one of seven more or less different cases arises. We list these cases in order of increasing difficulty:

(i) $|a|=|b|=1, \quad b\neq a;$
(ii) $|a|<1, \quad |b|>1;$
(iii) $|a|=1, \quad |b|>1;$
(iv) $|a|=1, \quad |b|<1;$
(v) $|a|<1, \quad |b|<1;$
(vi) $|a|>1, \quad |b|>1;$
(vii) $|a|=1, \quad b=a.$

Suppose again that M is obtained from N_f by deletion of the rows with indices n_i $(i=1, 2, \ldots;\ n_{i+1}-n_i\to\infty)$. In the first six cases, we shall construct a bounded sequence s such that Ms converges but $N_f s$ diverges. In the last case, we shall show that if Ms converges but $N_f s$ diverges, then s is unbounded.

In case (i), we set $s_n=1$ for $n<n_1$, and for $n_i\leqq n<n_{i+1}$ we write

$$(9) \qquad s_n=\begin{cases} s_{n-1}/a & (i \text{ even}), \\ s_{n-1}/b & (i \text{ odd}). \end{cases}$$

Since $|a|=|b|=1$, s is bounded; and since $a\neq b$, Theorem 7 implies that $N_f s$ diverges. On the other hand, if of three consecutive elements of s the first two are related by one of the formulas in (9), while the second and the third are related by the other formula, then this triplet of elements does not enter the formation of any element of Ms. Hence Ms has only finitely many nonzero elements, and N_f is laconic.

In case (ii) we again use the formulas (9), with a slight modification that is needed to ensure the boundedness of s. We note that in each block $|s_n|$ is an increasing or decreasing function of n, according to whether the first or the second formula is in force. Therefore we choose $s_n=1$ for $n=n_i-1$ $(i=1, 3, 5, \ldots)$, and we let the first formula define s_n for $n=n_i-2, n_i-3, \ldots, n_{i-1}$, while the second determines it for $n=n_i, n_i+1, \ldots, n_{i+1}-1$. Again, $|s_n|\leqq 1$. For even values of i, the element t_{n_i-1} of $N_f s$ involves unrelated "loose ends", and therefore it does not necessarily vanish; but it is small when i is large, because of the condition that $n_{i+1}-n_i\to\infty$.

Case (iii) calls for a further modification, since $|s_n|$ decreases under the reign of the second formula but remains constant under the first. We overcome the difficulty by inserting a harmless growth factor; that is, we retain the second formula, but replace the first with

$$s_n=\frac{n-n_i}{n_{i+1}-n_i}\, s_{n-1}/a\,.$$

Case (iv) is so similar to case (iii) that it needs no further discussion.

In case (v) we can no longer rely on the gradual modification of the elements s_n given by one or the other of the formulas (9). On the other hand, the two formulas produce two blocks of elements in both of which $|s_n|$ increases. If $a \neq b$, we can use a linear combination of the two blocks, with coefficients chosen so that the two contributions cancel each other at the right-hand end of the block that is thus obtained. We write

$$s_n = a^{n_i}/a^{n+1} - b^{n_i}/b^{n+1} \qquad (n_{i-1} \leqq n < n_i). \tag{10}$$

Then $s_n = 0$ for $n = n_i - 1$. Also, s_n is small for $n = n_i$ and for $n = n_i + 1$. Therefore, with the notation $t = N_f s$, we see that $t_n = a - b$ for $n = n_i$, t_n is small for $n = n_i + 1$, and $t_n = 0$ for $n_{i+1} < n < n_i$. Therefore N_f is laconic.

If $a = b$, we replace (10) by the formula

$$s_n = (n_i - 1 - n)\, a^{n_i - n} \qquad (n_{i-1} \leqq n < n_i). \tag{11}$$

Again, s is bounded and $Ms \to 0$; since $t_{n_i} \to a^2 \neq 0$, N_f is laconic.

In case (vi), we replace formulas (10) and (11) by the formulas

$$s_n = a^{n_i - n - 1} - b^{n_i - n - 1} \qquad (n_i - 1 \leqq n \leqq n_{i+1} - 2),$$

$$s_n = (n + 1 - n_i)\, a^{n_i - n} \qquad (n_i - 1 \leqq n \leqq n_{i+1} - 2),$$

respectively.

It remains to treat case (vii), in other words, to show that if $|a| = 1$ and $f(z) = (z - a)^2$, then N_f is redundant.

Suppose again that $n_{i+1} - n_i \to \infty$ and that M is obtained from N_f by deletion of the rows with indices n_i. We shall prove that if Ms converges, then either $N_f s$ converges or else s is unbounded.

Suppose that $Ms \to c$. Then, for $n + 1 \notin \{n_i\}$, the condition $a^2 s_{n+1} - 2a s_n + s_{n-1} \to c$ is satisfied; that is

$$a(a s_{n+1} - s_n) - (a s_n - s_{n-1}) = c + o(1),$$

$$a s_{n+1} - s_n = a^{-1}(a s_n - s_{n-1}) + a^{-1} c + o(1),$$

$$a s_{n+2} - s_{n+1} = a^{-2}(a s_n - s_{n-1}) + (a^{-1} + a^{-2})\, c + o(1),$$

and more generally, for $n_i < n + 1 < n + k < n_{i+1}$,

$$a s_{n+k} - s_{n+k-1} = a^{-k}(a s_n - s_{n-1}) + (a^{-1} + \cdots + a^{-k})\, c + o(k).$$

If $a = 1$, boundedness of s implies that $c = 0$. If $a \neq 1$ and $s_n^* = s_n - c/f(1)$, then $Ms^* \to 0$. We may therefore restrict ourselves to the case where $c = 0$, and our hypothesis on s takes the form

$$\Delta^2(a^n s_n) \to 0 \qquad (n + 2 \notin \{n_i\}). \tag{12}$$

Now, if (12) holds also without its restriction on n, then $N_f s$ converges, by Theorem 7. If (12) does not hold without its restriction, there exists a subsequence of $\{n_i\}$ for whose elements the quantity $|\Delta^2(a^{n_i - 2} s_{n_i - 2})|$ exceeds some positive number 2η. For each of the corresponding indices n_i, at least one of the two quantities $|\Delta(a^{n_i - 2} s_{n_i - 2})|$ and $|\Delta(a^{n_i - 1} s_{n_i - 1})|$ must exceed η. But since the first differences $\Delta(a^n s_n)$ are nearly constant in the two blocks that precede and follow s_{n_i-1}, respectively, this implies that $\{a^n s_n\}$ (and hence $\{s_n\}$) is an unbounded sequence. Therefore N_f and M have the same convergence field in the space of bounded sequences. Hence N_f is redundant in case (vii), and the proof of Theorem 8 is complete.

The question of laconicity and redundancy of Nörlund matrices N_f, where f is a polynomial of degree higher than 2, appears to be difficult. We believe that the following statement holds.

Conjecture. If f is a polynomial and $f(0) \neq 0$, then N_f is laconic if and only if there exists an integer k and a polynomial $g(z)$, of degree at most 2, such that $f(z) = g(z^k)$ and N_g is laconic. (It is easy to prove the sufficiency of the condition.)

5. Laconicity and redundancy of bounded convergence fields

Definitions. We say that the bounded convergence field of a laconic Toeplitz matrix is *laconic*. If a Toeplitz matrix A is redundant, and if moreover *each of its submatrices is either redundant or has a bounded convergence field greater than* (A), then we say that (A) is *redundant.*

Theorem 9. *There exist bounded convergence fields that are both laconic and redundant.*

Proof. Let $f(z) = z+1$ and $g(z) = (z+1)^2$. By Theorem 7, $(N_f) = (N_g)$.

By Theorem 8, (N_f) is laconic, and it remains only to show that (N_g) is redundant.

From the last part of the proof of Theorem 8, we can easily see that if a matrix M is obtained by the deletion from N_g of all rows with indices n_i ($n_i < n_{i+1}$; $i = 1, 2, \ldots$), then $(M) = (N_g)$ if and only if $\lim (n_{i+1} - n_i) = \infty$, and that M is redundant whenever the latter condition is satisfied. Therefore (N_g) is a redundant bounded convergence field, and our theorem is proved. It remains an open question whether every bounded convergence field is both laconic and redundant.

Theorem 10. *The bounded convergence field of the Cesàro-1 transformation is redundant.*

Let the matrix M be obtained by the deletion from C_1 of all rows except those of indices n_r ($n_r < n_{r+1}$, $r = 1, 2, \ldots$). We shall show that $(M) = (C_1)$ if and only if

$$\lim n_{r+1}/n_r = 1 . \tag{13}$$

Suppose first that (13) holds, and let s be a bounded sequence such that $Ms \to 0$. For $n_r \leqq n < n_{r+1}$,

$$(n+1)^{-1} \sum_{k=0}^{n} s_k = (n+1)^{-1} \sum_{k=0}^{n_r} s_k + (n+1)^{-1} \sum_{k=n_r+1}^{n} s_k .$$

The first term on the right tends to 0 by the hypothesis on Ms. Together with the boundedness of s, condition (13) implies that the second term on the right also tends to 0 as $n \to \infty$, and therefore $(M) = (C_1)$.

Suppose next that the deletions have been so extensive that (13) fails. Then there exist a positive number h and a sequence of indices r_j such that $n_{r+1}-n_r>hn_r$ for $r\in\{r_j\}$. Corresponding to each of these indices we choose $s_n=1$ in the first half of the block $n_r<n<n_{r+1}$, and $s_n=-1$ in the second half of the block. All remaining s_n are defined to be 0. Clearly, $Ms\to 0$ and $C_1 s\not\to 0$.

This proves our assertion, and Theorem 10 is established. We point out that, with the notation used in the proof, M is laconic if and only if the sequence $\{n_{r+1}/n_r\}$ is bounded away from 1.

Theorem 11. *If B is a regular Toeplitz matrix, then there exist laconic matrices A and C such that $(A)<(B)<(C)$. In case (B) contains a divergent sequence, the matrix A can be chosen so that (A) also contains a divergent sequence.*

Proof. To construct the required matrix C, we choose an increasing sequence $\{n_i\}$ such that, for some appropriate sequences $\{h_i\}$ and $\{k_i\}$ of integers $(h_i<k_i<h_{i+1})$,

$$\lim_{i\to\infty}\Big(\sum_{k\leqq h_i}+\sum_{k_i\leqq k}\Big)|b_{n_i,k}|=0.$$

We denote by C_i the row of B with index n_i, and we define C to be the matrix whose i^{th} row is C_i. Since C is a submatrix of B, the relation $(C)>(B)$ holds. To see that C is laconic, we note that if D is a submatrix of C, and if $\{s_n\}$ is a sequence consisting of 0's, except for 1's in the blocks $h_i<n<k_i$ corresponding to the rows deleted in the passage from C to D, then $Ds\to 0$ while Cs has the two limit points 0 and 1.

The other half of the theorem is trivial in case (B) contains no divergent sequences. In the case where (B) contains a divergent sequence x, we may suppose that $Bx\to 0$. But because we shall apply the construction in the proof of Theorem 2.2 of [3], we need the hypothesis that our sequence has two limit points other than 0. We therefore replace x by a sequence $y=\{e^{i\vartheta_n}x_n\}$; if $\vartheta_n\to\infty$ slowly enough, then $By\to 0$ and y has two limit points α and β $(\alpha\neq 0\neq\beta)$.

There exists a sequence of integers k_r such that, in the terminology of [3], Bz converges whenever z apes y over $\{k_r\}$, and such that $y(k_{2r})\to\alpha$, $y(k_{2r+1})\to\beta$. With each index n we associate an index $p=p_n$, selected from the greatest two integers k_r less than n in such a way that the sequence $\{(y_n-y_{p_n})^{-1}\}$ is bounded. For each n, the elements a_{nk} of the matrix A are defined by the rule

$$a_{n p_n}=\frac{y_n}{y_n-y_{p_n}},\qquad a_{nn}=\frac{-y_{p_n}}{y_n-y_{p_n}},\qquad a_{nk}=0\qquad (k\neq n,p).$$

The convergence field of A consists of the sequences that ape y over $\{k_r\}$ (see [3, pp. 141–142]), and it is therefore contained in the convergence field of B. To show that A is laconic, suppose that we have obtained D by deleting from A the rows with indices n_i. If $n_i\notin\{p_r\}$, then the n_i^{th} column of A contains only one nonzero element, namely its element on the diagonal. It follows that if infinitely many of the n_i do not belong to $\{p_r\}$, then (D) is larger than (A); for if $z_n=0$ except for $n\in\{n_i\}\setminus\{p_r\}$, then Dz is the sequence $\{0\}$. It remains to deal with the case where all except finitely many of the $\{n_i\}$ belong to $\{p_r\}$. Here we note that there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n=\pm 1$ and $\varepsilon_{n_i}=(-1)^i$, such that for $z_n=\varepsilon_n y_n$ the transform Dz is again the sequence $\{0\}$. Since z does not ape y over $\{k_r\}$, Az does not converge. It follows that A is laconic, and the proof of Theorem 11 is complete.

We do not know whether Theorem 11 can be strengthened so that it asserts the existence, for each divergent sequence x in (B), of a laconic matrix A such that $x\in(A)<(B)$.

Theorem 12. *If B is a regular Toeplitz matrix, there exists a regular matrix E such that (E) is redundant and $(E)>(B)$.*

Proof. We point out that it is not sufficient to construct a redundant matrix E whose bounded convergence field is (B). The matrix E that we seek must have the additional property that each of its laconic submatrices has a larger bounded convergence field than B.

Let the symbols C_i have the same meaning as in the proof of Theorem 11, and let E be the matrix whose n^{th} row is the vector sum $(C_1+C_2+\cdots+C_n)/n$. Clearly, $(E)>(C)>(B)$. If M is obtained by the deletion of all rows of E except those of indices n_r $(n_r<n_{r+1};\ r=1,2,\ldots)$, then M is again laconic if and only if $\{n_{r+1}/n_r\}$ is bounded away from 1, and M is *redundant and equivalent to* E if and only if $n_{r+1}/n_r\to 1$.

Remark. We defined laconicity and redundancy with reference to the space of bounded sequences. Naturally, we could have used a larger or smaller sequence space S. However, the larger the space S used in the definition, the more difficult becomes the construction of a nontrivial redundant matrix. Of course, we can always construct a redundant matrix by overloading a preassigned matrix with superfluous but harmless rows. For the case where S is the space of all sequences, we obtain a more interesting example if to a matrix A for which $As\to 0$ implies that either $s_n\to 0$ or $s_n\to\infty$ we adjoin infinitely many rows of the identity matrix. But we do not know of any matrix whose *convergence field* is redundant relative to the space of all sequences, in the sense analogous to that of our definition at the beginning of this section.

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