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A Phase Cell Approach to Yang-Mills Theory

III. Local Stability, Modified Renormalization Group Transformation

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Abstract. In this paper the basic local stability result is obtained, in a form valid in both small field and large field regions. To achieve this, some modifications are made in both the action and the renormalization group transformation. Though there is some sacrifice of elegance in these modifications, the establishment of this local stability estimate yields the most basic ingredient of the phase cell cluster expansion, good estimates for all the actions.

Incidental to the estimates of this paper we establish some results on "lattice geometry," interesting in their own right. A bound on the "minimum area" of a loop of length l, in d dimensions, is obtained as $\frac{l^2}{8}\left(1-\frac{1}{d}\right)$. This, a best possible bound, was obtained for us by A. Blass. We also construct a "radial" maximal tree for the lattice in d dimensions. We hope to stimulate someone to find a better construction of "radial" trees.

Introduction

It is not far amiss to say that each machine in Constructive Quantum Field Theory has two essential ingredients, a perturbative aspect (to handle renormalization cancellations) and a positivity or stability aspect (the non-perturbative feature). This latter occurs under different names in different programs: positivity of the vacuum energy in the traditional cluster expansion; the bounds on partition functions in the method of exact renormalization transformations; and α-positivity and α-stability in the phase cell cluster expansion approach to boson models. In this paper we establish essential stability results for our phase cell attack on Yang-Mills theories. (This may have been the most difficult problem we have to face.) The ideas in this paper may also be useful in other approaches to the study of four dimensional gauge theories.

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As we have argued in [3], we are not dealing with lattice approximations to a Yang-Mills field. Rather we are dealing with a continuum Yang-Mills field. The group elements we have assigned to bonds of our lattices (of different scales) are variables associated to the continuum field, as the Fourier coefficients of a function are associated to the function. This association is non-linear – but fiendishly clever – to provide ready access to gauge invariance features of the theory.

To achieve the stability results of the present paper we make a number of modifications in the formalism as presented in [3]. In particular we change the block size from 2^4 to N^4, we change the averaging procedure, and we change the form of the plaquette action. The two later changes have no effect in the small field region. We have thus changed our choice of non-linear variables and our action. We may alternatively say that we have changed our block spin renormalization transform and our action. These changes are detailed in Sect. 3 and the Appendix.

We have started with the small field stability result of [3] and patterned our treatment of the general situation on this limiting case. Section 2 presents the small field result again, for notational reasons. Two of our modifications, of the block size and of the averaging, have a very physical motivation. We want the non-linearly associated variables assigned to large scale bonds to be minimally effected by small scale lattice excitations beneath them. As the N^4 lattice bonds are averaged to get the non-linear variable at the next scale, bonds corresponding to large excitations are suppressed in the averaging. We are not just speaking about the $1/N^4$ factor occurring in the averaging, but we have added additional suppression in how the averaging is performed. These ideas should be useful in other approaches, where also treating the effects of large fields at small scales coupling to the scales above them, is a basic difficulty of the four dimensional theory.

We have one reservation in our satisfaction with the present treatment. Perhaps the modifications we make are more drastic than necessary, only because of our ignorance; much simpler modifications may yield similar stability bounds. The averaging procedure, C) of Sect. 3 and the Appendix, is not complete, but sufficiently specified to yield our present result.

In Sect. 1 we present some aesthetic properties of our average of group elements (either before modification, or in small field regions where the modification does not matter). Section 4 contains the basic stability results. These are proved in Sect. 5. Section 5.1 is particularly basic and of general interest. The results on "lattice geometry" are included in Sect. 5.4.

1. Pure Averages of Group Elements

In this section we discuss an averaging procedure different from the one studied in [2]. We let G be a compact Lie Group and $d(\cdot, \cdot)$ an invariant distance constructed from an invariant metric on G. ε will denote the identity, and we consider elements $g = e^A$ that are sufficiently close to the identity, with $|A|$ sufficiently small. For a collection of such elements $\{g_i = e^{A_i}\}_{i=1,\ldots,n}$ we define an average $\bar{g}$

$$\bar{g} = e^x = \overline{e^{A_i}} = \overline{g_i} \tag{1.1}$$

as the element minimizing the expression

$$\sum_{i=1}^{n} d^2(\bar{g}, g_i). \tag{1.2}$$

We here collect some easy results.

We may normalize d to satisfy

Lemma 1.0.

$$d^2(\varepsilon, e^A) = A^2 \equiv -\mathrm{Tr}(A^2) \equiv |A|^2. \tag{1.3}$$

(Note the definition of A^2, for A in the Lie Algebra.)

Lemma 1.1.

$$d^2(e^A, e^B) = (B-A)^2 + F, \tag{1.4}$$

where

$$|F| \leqq c(|B|^4 + |A|^4). \tag{1.5}$$

Sketch of proof of Lemma 1.1. By invariance

$$d^2(e^A, e^B) = d^2(\varepsilon, e^{-A} e^B) = C^2, \tag{1.6}$$

where

$$e^{-A} e^B = e^C. \tag{1.7}$$

By the Baker-Campbell-Hausdorff formula there is a *convergent* power series for

$$C = B - A + \tfrac{1}{2}[B, A] + \ldots. \tag{1.8}$$

In the expression for $C^2 = -\mathrm{Tr}(CC)$, the third order term vanishes since

$$\mathrm{Tr}(M[M, N]) = 0 \tag{1.9}$$

for any M and N.

Lemma 1.2. *With the notation above*

$$x = \frac{1}{n}\Sigma A_i + \frac{1}{n}\Sigma T_i \tag{1.10}$$

with

$$|T_i| \leqq c(|x|^3 + |A_i|^3).$$

This result is obtained by differentiating (1.2) with respect to (components of) x. The derivative of F in (1.4) may be estimated as (with y a component of A)

$$\left|\frac{dF}{dy}\right| \leqq c(|B|^3 + |A|^3) \tag{1.11}$$

by the same arguments as in the proof of Lemma 1.1.

Lemma 1.3. $x = 0 \Leftrightarrow \Sigma A_i = 0$.

This surprising result follows by noting that

$$\sum_i d^2(e^x, e^{A_i}) = \sum_i (x - A_i)^2 + E, \tag{1.12}$$

where E is a power series with non-zero powers of x all $\geqq 2$. Again the Baker-Campbell-Hausdorff formula and Eq. (1.9) yield this property.

2. Pure Small Field Stability

In this section we consider a "pure small field" situation. All bond assignments may be chosen simultaneously to be elements "very close" to the identity, and we may consider G to be an abelian group. Basically we recapitulate the discussion in Sect. 1 of [3], with minor modification.

We work with a block size N^4 instead of 2^4 as in [3]. We consider two levels r and $r+1$, with plaquettes $\{P_i\}$ at level r and $\{p_i\}$ at level $r+1$. There are non-negative numbers $\alpha(i)_j$ such that (with $g_{\partial P} = e^{A_{\partial P}}$)

$$A_{\partial P_i} = \sum_j \alpha(i)_j A_{\partial p_j}, \tag{2.1}$$

with

$$\sum_i \alpha(i)_j = 1/N^2, \tag{2.2}$$

and

$$\sum_j \alpha(i)_j = N^2. \tag{2.3}$$

By *convexity* or by the *Schwartz inequality* it follows from (2.1) and (2.3) that

$$(A_{\partial P_i})^2 \leqq N^2 \sum_j \alpha(i)_j (A_{\partial p_j})^2, \tag{2.4}$$

and from (2.2) that

$$\sum_i (A_{\partial P_i})^2 \leqq \sum_j (A_{\partial p_j})^2. \tag{2.5}$$

In d dimensions $1/N^2$ in (2.2) would be replaced by $1/N^{d-2}$, and in (2.5) there would be a factor N^{4-d} on the right side of the inequality.

In this paper we understand a relation that holds in the "small field" region to mean a relation that is true to linear order in deviations of Lie Algebra elements from zero.

3. Modifications

We here describe the three modifications in the formalism, as differing from the presentation in [3] and elsewhere.

A) *Block size.* As stated in Sect. 2 we will consider a block of size N^4 instead of 2^4. (N will be large.)

B) *Action.* The Wilson action of a plaquette, P, as a function of $g_{\partial P}$ is

$$\sim \mathrm{Re}(\mathrm{Tr}(I - U(g_{\partial P}))). \tag{3.1}$$

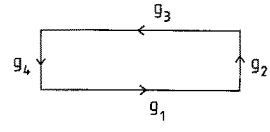


Fig. 1

$g_{\partial P}$ may be understood as $g_1g_2g_3g_4$, $g_2g_3g_4g_1$, $g_3g_4g_1g_2$, or $g_4g_1g_2g_3$, with the g_i the group elements associated to the bonds in the plaquette. For many cases any of the four choices give the same answer, as in Eq. (3.1) and e.g. (3.4). In other situations a convenient choice is made. If we write

$$g_{\partial P} = e^{A_{\partial P}}, \tag{3.2}$$

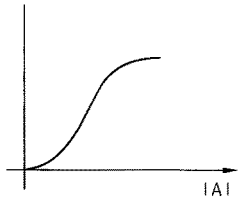
where we choose $A_{\partial P}$ small if $g_{\partial P}$ is close to ε, we will choose the action to be

$$S_M(P) = f_S(A_{\partial P}), \tag{3.3}$$

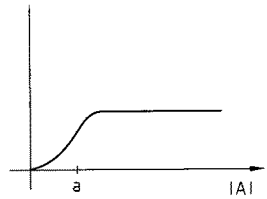
where $f_S(A) = f_S(|A|)$ is C^∞ and monotonic in $|A|$, and satisfies

$$f_S(A) = \begin{cases} |A|^2 & \text{if } |A| \leqq a \\ 3/2a^2 & \text{if } |A| \geqq 2a. \end{cases} \tag{3.4}$$

(a will be small.) Graphically we sketch the Wilson action and the modified action S_M as a function of $|A|$.



Wilson Action
Fig. 2



Modified Action
Fig. 3

For $|A| \leqq a$ the two actions essentially agree.

C) *Averaging*. We modify $d(e^A, e^B)$ to $d_M(e^A, e^B)$ also an invariant distance with

$$d_M(\varepsilon, e^A) = f_d(|A|), \tag{3.5}$$

where $|A|$ is chosen small for elements close to ε. We choose

$$f_d(|A|) = \begin{pmatrix} |A| & |A| < c_d N^2 a \\ (3/2)(c_d N^2 a) & |A| > 2c_d N^2 a \end{pmatrix} \tag{3.6}$$

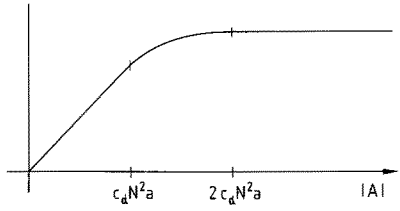


Fig. 4

and require f_d to be C^∞ with bounds on its derivatives

$$|f_d^{(r)}| < c_r (c_d N^2 a)^{-r}, \tag{3.7}$$

$\bar{g}$ will be chosen by minimizing the expression

$$\sum_1^n d_M^2(\bar{g}, g_i) \tag{3.8}$$

as a variation of (1.2). Notice (3.8) and (1.2) yield the same result when the g_i are sufficiently close together. In this paper we will not need to define $\bar{g}$ when the minimization does not yield a unique result. We will obtain results independent of the definition of $\bar{g}$ in these cases.

We will later give conditions on N, a, c_d.

In C) we have so far modified the averaging by changing how the average of elements, $\overline{g_j}$, is defined. The g_i to be averaged arise as group elements associated to contours [see the discussion before and after Eq. (5.11) for example]. We will modify the procedure of [1] for defining the contours to be picked. In particular the choice of contours connecting two points on the lattice (and whose associated group elements are to be averaged) will vary with the field in the large field region. We leave details to the Appendix. As unattractive as modifications in C) may be, they preserve gauge invariance; they have no effect in the purely small field region; and they have the physically desirable effect of suppressing the effects of small scale large field excitations on the large scale fields above them. It is best to read the Appendix after reading Sect. 4.

4. Local Stability Statements

A plaquette p is s.f. if $|A_{\partial p}| < a$ (more properly, $|g_{\partial p}| < a$, with definition at beginning of Sect. 5) and l.f. if $|A_{\partial p}| \geqq a$. Each vertex of a plaquette p at level s is contained in one N^4 size block, a vertex in the level $s-1$ lattice; it is said to *hit* this vertex. A plaquette p at level s is said to *hit* a bond or plaquette of the $s-1$ lattice if it hits a vertex of this bond or plaquette. A plaquette p at level s is S.F. if it is not hit by any l.f. plaquettes (of level $s+1$), and otherwise is L.F. If one half the action, $\frac{1}{2}a^2$, of a l.f. plaquette is distributed equally among all the plaquettes it hits, $A_{\text{l.f.}}(p)$ is the action associated to p by this process.

We now specialize to two levels r and $r+1$, and consider a single plaquette P in the r level. We also consider exactly those plaquettes $\{p_i\}_{i \in P}$ in the level $r+1$ all of whose vertices lie in the four N^4 size blocks, vertices of the plaquette P. This is a smaller set of p_i than hit P. We distinguish three regimes:

Regime 1. All the p_i (p_i with $i \in P$) are s.f..
Regime 2. Some of the p_i are l.f., but $A_{\text{l.f.}}(P) < 3a^2$.
Regime 3. $A_{\text{l.f.}}(P) \geqq 3a^2$.

We note there is an absolute number r_1 such that if more than r_1 p_i are l.f., we are in Regime 3.

We now use the definitions of Sect. 2 for $\alpha_j(P_i = P)$, and Sect. 3 for S_M. In studying stability it is natural to seek lower bounds for $\Delta S(P)$ defined by

$$\Delta S(P) = N^2 \sum_j \alpha_j S_M(p_j) - S_M(P). \tag{4.1}$$

(See [3, Sect. 2.4].). With the redistribution of the actions of l.f. plaquettes as above it is appropriate for us to study the slightly different quantity $\tilde{\Delta} S(P)$,

$$\tilde{\Delta} S(P) = A_{\text{l.f.}}(P) + N^2 \sum_{j \neq \text{l.f.}} \alpha_j S_M(p_j) - S_M(P). \tag{4.2}$$

The sum in (4.2) is over plaquettes p_j that are s.f.

We now present the stability theorem, divided into results for the three regimes.

Local Stability Theorem 4.1. *In Regime 3,*

$$\tilde{\Delta}S(P) \geqq a^2 . \tag{4.3}$$

This theorem is immediate from the definitions of S_M, α_j, and Regime 3; it requires no proof.

Local Stability Theorem 4.2. *There is a number $f_1 > 0$ such that in Regime 2,*

$$\tilde{\Delta}S(P) \geqq f_1 a^2 . \tag{4.4}$$

Local Stability Theorem 4.3. *There is a number f_2 such that in Regime 1,*

$$\tilde{\Delta}S(P) \geqq -f_2(\Sigma |A_{\partial p_j}|^2)^{3/2} . \tag{4.5}$$

All these results will hold for suitable fixed N, c_d, f_1, f_2, and a_0, where a must satisfy $a < a_0$. Equation (5.25) below is actually a stronger form of Local Stability Theorem 4.3 that is needed for applications.

5. Proofs

5.1. Preliminaries

We define $|g| = d(\varepsilon, g)$.

Lemma 5.0.

$$|g_1 g_2^{-1}| = d(g_1, g_2) \tag{5.1}$$

by invariance of the metric.

Lemma 5.1.

$$|g_1 g_2| \leqq |g_1| + |g_2| . \tag{5.2}$$

Proof.

$$d(\varepsilon, g_1 g_2) \leqq d(\varepsilon, g_1) + d(g_1, g_1 g_2) \leqq d(\varepsilon, g_1) + d(\varepsilon, g_2) , \tag{5.3}$$

since

$$d(g_1, g_1 g_2) = d(\varepsilon, g_2) \tag{5.4}$$

by invariance of the metric.

Lemma 5.2.

$$|e^A| \leqq |A| . \tag{5.5}$$

Proof. This follows from Lemma 5.1 and Lemma 1.0.

Lemma 5.3.

$$|g| = |g^{-1}| . \tag{5.6}$$

Proof. This follows since $g \to g^{-1}$ is an isometric mapping.

We now let R be a rectangle made up of $M \times N$ plaquettes $\{p_i\}_{i \in R}$ in a lattice, to whose oriented bonds we have group elements assigned. There is naturally defined a $g_{\partial R}$.

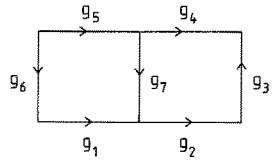


Fig. 5

For the figure picturing a 1×2 rectangle R,

$$g_{\partial R} = g_1 g_2 g_3 g_4^{-1} g_5^{-1} g_6 .$$

Lemma 5.4.

If

$$R = \sum_{i \in R} p_i , \tag{5.7}$$

then

$$|g_{\partial R}| \leqq \sum |g_{\partial p_i}| . \tag{5.8}$$

Proof. We consider this gauge invariant problem in a gauge where bonds on the bottom edge of the rectangle and all vertical bonds are assigned ε. Consider a single plaquette p as figured below. We have

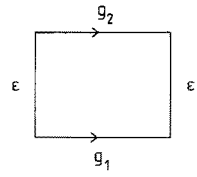


Fig. 6

$$|g_2| \leqq |g_1| + |g_{\partial p}| . \tag{5.9}$$

This follows from

$$g_2 = g_{\partial p}^{-1} g_1 \tag{5.10}$$

and Lemmas 5.1 and 5.3. Using Lemma 5.1 and (5.9) the result, Lemma 5.4, easily follows.

We may consider a more general situation. To a contour, an oriented curve on the lattice, made up of course of oriented line segments, we may associate a group element.

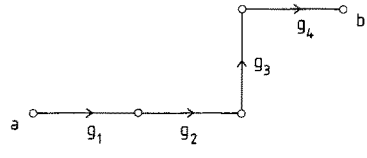


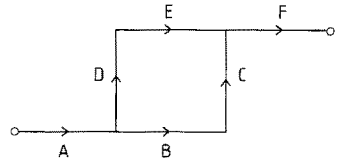
Fig. 7

To the sketched contour Γ from a to b we have a group element associated

$$g_\Gamma = g_1 g_2 g_3 g_4 . \tag{5.11}$$

We consider two contours Γ_1, Γ_2 differing by a single plaquette p. The plaquette provides an *elementary homotopy* between Γ_1 and Γ_2. In the figure

Fig. 8



The contours $ABCF$ and $ADEF$ differ by the plaquette $BCED$. In the figure each contour contains two sides of the plaquette. In general Γ_1 may contain m sides, and Γ_2 n sides, of the plaquette realizing the elementary homotopy, with $m, n \in \{0,1,2,3,4\}$, $m+n=4$.

Lemma 5.5. *If Γ_1 and Γ_2 are connected by a sequence of elementary homotopies provided by plaquettes $p_1, p_2, \ldots, p_n$, then*

$$d(g_{\Gamma_1}, g_{\Gamma_2}) \leqq \sum_{i=1}^{n} |g_{\partial p_i}| . \tag{5.12}$$

The proof is direct. Note that Lemma 5.5 includes Lemma 5.4.

5.2. Regime 1

The group element assigned to a bond in P is an average of N^4 group elements associated with N^4 contours as enter (in the Abelian case) in (1.8) of [1]. $\Gamma_{c_-,x}$ of (1.8) contains $\leqq 4(N-1)$ bonds. If we consider x' so that $\Gamma_{c_-,x'} \subset \Gamma_{c_-,x}$ with one less bond, then the contours associated to x and x' are related by $\leqq N$ elementary homotopies. Thus any contour is related to the straight line contour $\langle c_-, c_+ \rangle$ by $\leqq 4(N-1)N$ elementary homotopies; and any two contours are related by $\leqq 8(N-1)N$ elementary homotopies.

Lemma 5.6. *Let Γ_1 and Γ_2 be two contours whose associated group elements are averaged (along with N^4-2 others) to yield a group assignment to a bond of P. Assume all p_i are s.f. Then*

$$d(g_{\Gamma_1}, g_{\Gamma_2}) \leqq 8(N-1)Na . \tag{5.13}$$

We now specialize to the situation where all p_i are s.f.

Parameter Conditions 5.7.

$$c_d = 600 . \tag{5.14}$$

We here and elsewhere are vastly over-generous in choice of constants, to accomodate worst case scenarios arising in estimates from Sects. 5.3 and 5.4.

Parameter Condition 5.8.

$$4000(N-1)Na < \varepsilon_1 , \tag{5.15}$$

where ε_1 is sufficiently small so that the estimates in Sect. 1 hold and so that one has the following proposition (using the notation of Sect. 1).

Proposition 5.9. *Let g_i be n elements of the group G, then*

$$d(g_i, g_j) < \varepsilon_1 \Rightarrow d(\bar{g}, g_i) < \varepsilon_1 . \tag{5.16}$$

It is an easy geometric exercise to show this holds for ε_1 small enough.

We now work in a Balaban axial gauge. Three of the bonds in the plaquette P are assigned the identity element. Each contour $\Gamma_{c,x}$, $\Gamma_{x(c),c+}$ in (1.8) of [1] is assigned the identity element.

Our approach to stability has been of course to make the small field situation of Sect. 2 dominate the picture. We now divide the small field proof into a sequence of steps and then discuss the size of corrections to yield the stability theorem in Regime 1. We let b_1, b_2, b_3, b_4 be the four bonds in P and as noted b_1, b_2, b_3 are assigned ε and b_4 is assigned g

$$g = e^{\overline{A_{\Gamma_{4i}}}}, \quad g_{\Gamma_{4i}} = e^{A_{\Gamma_{4i}}}, \tag{5.17}$$

where the $g_{\Gamma_{4i}}$ are the group elements assigned to contours Γ_{4i} and averaged to yield g.

Step 1.

$$g_{\partial P} = \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot g, \quad \varepsilon = e^{\frac{1}{N^4}\Sigma A_{\Gamma_{ki}}}, \quad k = 1,2,3, \tag{5.18}$$

$$g = e^{\frac{1}{N^4}\Sigma A_{\Gamma_{4i}}}. \tag{5.19}$$

Step 2.

$$g_{\partial P} = e^{\frac{1}{N^4}\Sigma(A_{\Gamma_{1i}} + A_{\Gamma_{2i}} + A_{\Gamma_{3i}} + A_{\Gamma_{4i}})}. \tag{5.20}$$

Step 3.

$$g_{\partial P} = e^{A_{\partial P}}, \quad A_{\partial P} = \sum \alpha_j A_{\partial p_j}. \tag{5.21}$$

Step 4. The Local Stability Theorem.

We proceed to discuss corrections to these steps. We begin with the most basic ingredient. We note at the small field limit, the relation between A_{b_α} and $A_{\partial p_i}$, where $g_{b_\alpha} = e^{A_{b_\alpha}}$ is group element assigned to bond b_α of the $r+1$ lattice (the portion we are considering)

$$A_{\partial p_i} = \sum_\alpha M_{i\alpha} A_{b_\alpha}. \tag{5.22}$$

The right side is the first term in a power series convergent for small fields. Since the Balaban gauge provides a complete gauge specification we have in the small field region (with $A_{\partial p_i}$, $i \in J$, a linearly independent set)

$$A_{b_\alpha} = \sum_{i \in J} N_{\alpha i} A_{\partial p_i} \tag{5.23}$$

(for bonds b_α not assigned ε by the gauge). By the inverse function theorem we have for $A_{\partial p_i}$ small enough (depending on N),

$$\left| A_{b_\alpha} - \sum_i N_{\alpha i} A_{\partial p_i} \right| < c \sum |A_{\partial p_i}|^2. \tag{5.24}$$

This estimate will enable us to control all corrections to steps in the small field proof. We make a number of comments:

a) Equation (5.18) is generally correct by Lemma 1.3.

b) Equation (5.19) has a correction to be added in the exponent as dominated by estimate in Lemma 1.2.

c) The exponent in (5.20) must have the same correction mentioned in b) added to it, and is otherwise correct.

d) For a small enough (depending on N) we find from Baker-Campbell-Hausdorff estimates the basic result:

$$|A_{\partial P}|^2 \leqq N^2 \sum_j \alpha_j |A_{\partial p_j}|^2 + c(\sum |A_{\partial p_j}|^2)^{3/2}. \tag{5.25}$$

Parameter Condition 5.10. a_0 is picked small enough (as a function of N) such that (5.25) holds.

Equation (5.25) is Local Stability Theorem 4.3 with f_2 determined (as a function of N).

5.3. Regime 2

The proof of Theorem 4.2 is complicated and tricky, but mushy; it involves no hard estimates or precise non-trivial inequalities. All the modifications introduced in Sect. 3 and the Appendix are needed and used. A number of constants arise that in general will depend on the positions of the l.f. plaquettes. Since there are only a finite number of positions for the l.f. plaquettes, we may choose the constants independent of l.f. plaquette positions by maximizing or minimizing over a finite set, and we do so. Some of the constants will depend on N, which we will note. Our results are obtained for N large enough, and a_0 small enough depending on N.

1) We include the l.f. plaquettes (among the p_i) inside $\leqq r_1$ balls of radius $\leqq c_2$. (We are considering a total lattice of $4N^4$ vertices, in the four N^4 blocks, vertices of P.) Inside each N^4 block, the vertices and bonds not in any of the balls includes a large connected set of bonds and vertices, the *major block sublattice.* The set of vertices in the block, not included in the major block sublattice contains $\leqq c_3$ elements. The union of the four major block sublattices naturally combine to form the *major sublattice,* a connected lattice, with only s.f. plaquettes.

2) We will say a lattice is *simply connected* if every closed contour can be modified to a trivial contour (a point) by a sequence of elementary homotopies. (See the discussion before Lemma 5.5.) By removing $\leqq c_4$ vertices and bonds each major block sublattice and the major sublattice may be made simply connected and connected. (This could not be done in two dimensions.) We thus arrive at the *nice block sublattices* and *nice sublattice.*

Observation. In a simply connected lattice, if the bonds in a maximal tree are assigned the identity as a choice of gauge, then the bond variables are uniquely determined by the plaquette variables.

3) The bonds assigned ε inside a block in the Balaban axial gauge will also be assigned ε in the nice block sublattice. This in general will not define a gauge inside the nice block. We assign ε to a sufficient number of additional bonds to define an axial gauge in each nice block.

4) For each of the three bonds of P assigned ε in Sect. 5.2, we will set a certain average to be ε. We first note, if in a given block as illustrated in the Appendix, contours radiate out from the center of a ball, B in the figure, instead of C; then we may in this section let B be the base point, instead of C. This does not effect our estimate, it is a gauge change. With this convention, we average to ε, for the three bonds, those group elements assigned to the subset of the relevant contours

contained in the nice lattice. We are averaging as in Sect. 5.2, possibly with a different base point, but not over all the contours. We are not assigning three of the bonds in $P\,\varepsilon$, but approximately ε; the average we are setting to be ε is not precisely the same average that assigns group elements to the bonds of P. (We no longer mention the shift in base point.)

5), 3), and 4) specify a gauge. In this gauge we deduce an inequality like (5.24), with both sums over only s.f. plaquettes. We also have, for N large enough, (5.13) with 8 replaced by $\frac{17}{16}(22)^2$ for the contours kept in the average in 4). (This is a gauge independent statement.) See Sect. 5.4.

6) We write, as in Step 1 of Sect. 5.2,

$$g_{\partial P} = g_1 \cdot g_2 \cdot g_3 \cdot g_4\,, \tag{5.26}$$

$$g_k = e^{\frac{1}{N_k}\sum A^n_{\Gamma_{k_i}} + E_k}, \qquad k = 1, \ldots, 4\,. \tag{5.27}$$

The n superscript indicates all contours lie in the nice lattice,

$$\sum A^n_{\Gamma_{k_i}} = 0\,, \qquad k = 1, 2, 3\,. \tag{5.28}$$

We also have N_k as the number of these "nice contours" entering the bond k average. We can require

$$N_k = N^4 - N_k^E\,, \tag{5.29}$$

with

$$N_k^E < c_5 N\,. \tag{5.30}$$

This requires a reasonable choice of the balls in 1) and a reasonable choice of the additional bonds and vertices removed in 2). We may require all bonds and vertices within distance $c_1 N - c_6$ of the base point to be in the nice lattice. Note that the inequality (5.30) depends on the modification of the Appendix; and also indicates how close parts of the contours have to be to the radial straight line paths (see the discussion near the end of the Appendix).

7) It follows from the modification introduced in Eq. (3.5) and (3.6) and from (5.30) that

$$|E_k| < \frac{c_7}{N} a_0\,, \qquad k = 1, 2, 3\,, \tag{5.31}$$

$$|E_4| < \frac{c_7}{N} a_0 + c_8 (N^2 a_0)^2\,. \tag{5.32}$$

For fixed N large enough, a_0 small enough, these E_k are small enough not to require careful treatment in deriving our estimates.

8) We write thus

$$g_{\partial P} = e^{\frac{1}{N^4}\sum_k \sum_i A^n_{\Gamma_{k_i}} + E'} \tag{5.33}$$

with E' very small [with estimate like (5.32)]. We consider $N \times N$ squares in the nice lattice, parallel to P, such as were averaged in Appendix A of [4]. We may rewrite the exponent in (5.33),

$$\frac{1}{N^4} \sum_s A^n_s + E''\,, \tag{5.34}$$

where the sum is over such squares in the nice lattice. The number of such s, N_s, satisfies

$$N^4 - N_s \leqq c_9 N . \tag{5.35}$$

E'' is small [with an estimate like (5.32)] and differs from E' by some $A^n_{\Gamma_{k_i}}$ not in any A^n_s.

9) With (5.34) we may feel we are almost there, that the remaining proof is just as in the last subsection. But there is a major difficulty yet to be overcome, the trickiest aspect of the present proof. Although the square contours summed over in (5.34) lie in the nice lattice, not all their "interiors" do. (The interior of a square contour is the $N \times N$ union of plaquettes of which the square is an outer boundary.) The number of such "bad squares" is easily estimated to be $\leqq cN^2$ in number, but this is too many to be moved over to the error term E''. And the proof in the last subsection will not go over to the bad squares. Thus we have a real problem.

10) Given any $\varepsilon > 0$, we will show in the next step that we may rewrite (5.34) as

$$\frac{1}{N^4} \sum_{\mathrm{GS}} \alpha(\mathrm{GS}) A^n_{\mathrm{GS}} + E''' \tag{5.36}$$

with

$$1 \leqq \alpha(\mathrm{GS}) \leqq 1 + \varepsilon , \tag{5.37}$$

where the sum over GS is the sum over only "good squares" (ones with interiors in the nice lattice) and E''' satisfying an estimate as (5.32) with constants a function of ε. As in the last subsection we deduce from (5.36) for N large enough, a_0 small enough (depending on N)

$$(1+\varepsilon)^2 N^2 \sum_{\substack{j \\ p_j \text{ s.f.}}} \alpha_j S_M(p_j) - |A_{\partial P}|^2 \geqq -c(N,\varepsilon) a_0^3 - \varepsilon a_0^2 \tag{5.38}$$

with the sum in (5.38) as indicated over s.f. plaquettes only. From (5.38) it is straightforward to deduce Theorem 4.2, for ε small enough, N (as a function of ε) large enough, and a_0 (as a function of N) small enough.

11) We pick one of the two lattice directions, say $\mathbf{n}$, perpendicular to the plane of P. To each bad square, say BS, we find c_{10} good squares GS_α that are each parallel displacements of BS in the direction of $\mathbf{n}$. We can also require that the distance between BS and each GS_α is $\leqq c_{11}$ lattice spacings. We then can write

$$A^n_{\mathrm{BS}} = \frac{1}{c_{10}} \sum_\alpha A^n_{\mathrm{GS}_\alpha} + e(\mathrm{BS}) \tag{5.39}$$

with

$$|e(\mathrm{BS})| \leqq c_{12} N a . \tag{5.40}$$

(c_{12} will depend on c_{10}.)

Equations (5.39) and (5.40) are derived by relating the contour BS to each of the BS_α by a sequence of elementary homotopies, and studying the change in assigned group elements as in the analysis of Lemmas 5.4 and 5.5. If c_{10} is picked large enough (as a function of ε) we can achieve (5.36) and (5.37).

12) It is clear many of our estimates are rather wasteful, such as (5.40), and it is a very worthwhile problem to try and find a sharper stability theorem than we have stated.

5.4. Simple Lattice Geometry

A) Minimum Area of a Loop

We first consider the problem of bounding the number of elementary homotopies it takes to contract a closed contour of l bonds in d dimensions in a complete unit lattice. We will not count as a step removing a portion of the contour consisting of the same bond traversed in both directions in sequence. We note the folk theorem on the *isoperimetric inequality for square packing* which states that a square on the $2-d$ lattice maximizes the area to perimeter ratio. From this it is easy to deduce that in two dimensions a contour of length l can be contracted to a point by $\leqq \left(\frac{l}{4}\right)^2$ elementary homotopies. The minimum number of elementary homotopies it requires to contract a loop to a point is sometimes called the *minimum area* of the loop.

Theorem on Minimum Area (due to A. Blass). *In d dimensions the minimum area of a loop of length l is*

$$\leqq \frac{l^2}{8}\left(1-\frac{1}{d}\right).$$

Proof. We convert the problem to one of the study of words in a free group. We consider the free group generated by $g_1, \ldots, g_d$ (and $g_1^{-1}, \ldots, g_d^{-1}$). Associating g_i to any edge in the i^{th} direction with positive orientation, and g_i^{-1} to these edges with negative orientation; there is a natural assignment to a loop of length l, of a word of length l (which we note contains equal numbers of elements g_i and g_i^{-1}). We wish to convert this word to the identity element (or zero word). In this conversion we allow three kinds of steps:

a) removal of adjacent elements of form $g_i g_i^{-1}$ or $g_i^{-1} g_i$,
b) interchanging adjacent elements,
c) replacing word $w_1 w_2$ by $w_2 w_1$.

We note that steps of type a) correspond to the removal, in a loop, of a subpath traversing an edge in one direction and then immediately the opposite direction. Step b) corresponds to an elementary homotopy. In a sequence of steps converting our word to the identity, we count the number of steps of type b). The minimum number of such steps in a sequence "contracting" the word to the identity is the "minimum area" of the associated loop.

We leave to the reader the elementary (but non-trivial) argument required to show that the worst cases, configurations that maximize the minimum area, are of the form

$$g_1^r g_2^r \ldots g_d^r (g_1^{-1})^r (g_2^{-1})^r \ldots (g_d^{-1})^r .$$

This, a word of length $(2dr)=l$, is easily seen to require

$$r[(d-1)r+(d-2)r+\ldots+r]=\frac{l^2}{8}\left(1-\frac{1}{d}\right)$$

steps of type b) to contract. For a bound of the form $c(d)l^2$, we have found the best constant $c(d)=\frac{1}{8}\left(1-\frac{1}{d}\right)$.

B) Radial Maximal Trees

We now consider constructing radial trees on a lattice as required in the Appendix and Sect. 5.3. For its intrinsic interest we do the construction in any number of dimensions. We first state what properties our tree will have – this will define in what sense the tree is radial. We then construct a maximal tree on the lattice Z^d, radial about the origin, i.e. satisfying all the properties listed.

Properties of Radial Maximal Trees in Z^d. For a point p in Z^d, we write $|p|$ for the distance to the origin, and $d_T(p)$ for the distance to the origin along T, a maximal tree. The following properties define the term "radial" applied to T.

Radial Property 1. There is a c_a such that for all p, $d_T(p) \leqq c_a|p|$.

Radial Property 2. There is a c_b such that for each point p, the portion of the tree joining p to the origin (a path from the origin to p, lying in T) lies within a ball about the origin of radius, $|p|+c_b$.

Radial Property 3. For each $c_0>1$ there is a $m(c_0)$ such that for each $r>0$ one has that: if T is cut at any point p with $|p| \geqq r$, then the number of vertices lying in a ball of radius $c_0 r$ about the origin and disconnected from the origin is $\leqq mr$.

Construction of a Radial Maximal Tree. We construct our tree by an inductive process. We have trees $T_1, T_2, \ldots$ such that $T_i \subset T_{i+1}$ and $T_i \subset T$. T_i is a maximal tree on the vertices lying within the ball B_i of radius $r_i = 2^i$ about the origin. We assume T_i has been constructed, and specify the construction of T_{i+1}.

Construction of T_{i+1}. We choose constants c_1, c_2, c_3 (independent of i) satisfying

$$c_1 > 2\sqrt{d}, \quad c_2 > 3c_1, \quad c_3 > 3c_2.$$

We select a set of lattice points, p_α, satisfying

1) $2^i < |p_\alpha| < 2^i + \sqrt{d}$.
2) The points p_α have mutual separations $\geqq c_2$.
3) Each point on the boundary of B_i (these need not be lattice points) is within distance c_3 of some p_α.

We now for each p_α draw a line through the origin and p_α, call this l_α. We pick a shortest path on the lattice $t_{0\alpha}$ satisfying

1) $t_{0\alpha}$ connects p_α to a point within distance $\sqrt{d}$ of the boundary of B_{i+1}.
2) $t_{0\alpha}$ lies in $(B_{i+1} - B_i)$.
3) The maximum distance between a point in $t_{0\alpha}$ and l_α is $\leqq c_1$.

We note that $t_{0\alpha}$ is of length $\leqq 2^i c_4$, for a constant c_4 independent of i (and α).

We arrange our $t_{0\alpha}$ in a (finite) sequence, and expand these inductively. With an ordering $\alpha_1, \ldots, \alpha_s$ on the α's, we will first expand $t_{0\alpha_1}$ to become $t_{1\alpha_1}$, then $t_{0\alpha_2}$ to become $t_{1\alpha_2}$. In general after expanding $t_{r\alpha_j}$ to become $t_{(r+1)\alpha_j}$, we next expand $t_{r\alpha_{j+1}}$ to become $t_{(r+1)\alpha_{j+1}}$ if $j<s$, $t_{(r+1)\alpha_1}$ to become $t_{(r+2)\alpha_1}$ if $j=s$. Each $t_{r\alpha_j}$ will be a tree in $(B_{i+1} - B_i)$. At a given expansion step, when $t_{r\alpha_j}$ is expanded to become $t_{(r+1)\alpha_j}$, one must have

1) $t_{r\alpha_j} \subset t_{(r+1)\alpha_j}$, i.e. it is an expansion.
2) $t_{(r+1)\alpha_j}$ is disjoint from all other trees as they have developed so far in the inductive process.

3) Each bond in $t_{(r+1)\alpha_j}$ touches a bond in $t_{r\alpha_j}$.
4) $t_{(r+1)\alpha_j}$ is a maximal element satisfying 1), 2), 3).

At the end of this expansion process. We have disjoint trees $t_{f\alpha}$ hitting all vertices in $(B_{i+1} - B_i)$. T_{i+1} is obtained from T_i and the $t_{f\alpha}$ by adding one bond for each $t_{f\alpha}$ to join it to T_i.

This has been what at first seems a rather complicated construction. But living with it for a while one may see the resulting tree satisfies the three Radial Properties.

Appendix. Field Dependent Contour Selection

We begin with a brief discussion of the problem motivating the modification introduced in this Appendix. We consider a $2-d$ 5^2 block with root vertex C, and bonds assigned ε by a typical Balaban axial gauge drawn. We prefer

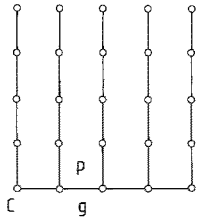


Fig. 9

a centrally located root vertex, so we draw an alternative gauge selection

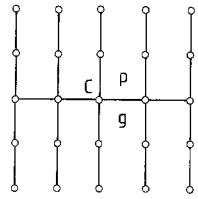


Fig. 10

p, g label plaquettes. Now imagine these two figures with 5 replaced by a very large number N, and p, g similarly located, near midway along the bottom row, adjacent to C, respectively. Now imagine a very localized field excitation; all plaquettes p_i in the whole lattice have small $g_{\partial p_i}$, except p and g which have large $g_{\partial p} = g_{\partial g}^{-1}$. It is easy to see that this very localized excitation, buried in the N^2 block wreaks havoc on the group elements assigned to contours, that have to be averaged. The group elements will be divided into two sets, a large number of elements in each set, and the group elements within each set nearly equal, but very different between the sets. We wish to establish an averaging procedure that minimizes the effects of a local excitation such as this – it appears to be impossible to do with the Balaban averaging procedure. More immediately we want a procedure that will enable us to establish stability estimates.

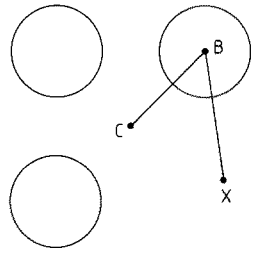


Fig. 11

We put the root vertex C at the center of the N^4 block. We put r_1+1 balls of radius $c_1 N$ in the block, the balls are separated from each other, from C, and from the boundary of the block by $c_1 N$. For a field configuration in Regime 2 there will be less than r_1+1 l.f. plaquettes in the block. We may thus find a ball, say the one illustrated with center B, with no l.f. plaquettes inside it. Having selected (by some decision process we do not now detail) the ball centered at B, we define the Balaban axial gauge. Better, we define contours connecting C to each vertex in the block, in such a way that no closed loop can be made of portions of the union of contours. These contours are selected to be made of the path from C to B followed by a path from B to the point. In the figure, C is joined to X by $\mathbf{CB} \cup \mathbf{BX}$. On the lattice, of course, one cannot choose straight line paths, but one chooses "good approximations" to the straight line paths. The proofs in Sect. 5.3 specify how good an approximation is here required, not one demanding a precision algorithm. We will require that the number of bonds in the path from the center of a ball (B in the figure) to any point (X in the figure) be less than $5(N-1)$. Likewise for the path from C to B. These points are discussed further in Sect. 5.4.

When there are no l.f. plaquettes in the block, or more than r_1 l.f. plaquettes in the block any axial gauge suffices.

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