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Euclidean Fermi Fields with a Hermitean Feynman-Kac-Nelson Formula. I

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Abstract. We construct free, Euclidean, spin one-half, quantum fields with the following properties: (i) CAR; (ii) Symanzik positivity; (iii) Osterwalder-Schrader positivity; (iv) no doubling of particle or spin states. They admit the recovery of the relativistic Dirac field by the Osterwalder-Schrader technique. We then formally parametrize interacting theories by a natural class of Hermitean, Euclidean actions, and obtain a simple, Hermitean, Feynman-Kac-Nelson formula. The interacting theory formally obeys all the properties (i)–(iv), and admits the reconstruction of a physical Hilbert space, including a Hermitean, contraction semigroup for the Wick rotated time evolution. We propose a system of axioms for the interacting theory.

I. Introduction

Nelson's work on Euclidean field theories for spinless bosons [1, 2], has proved a very significant conceptual and technical stimulus in the program of constructive quantum field theory. It was followed by two important papers of Osterwalder and Schrader [3, 4], who discovered the Osterwalder-Schrader (OS) positivity condition. Modulo a troublesome technicality in their original proof, the OS positivity condition was the key property which allowed the reconstruction of a Wightman field theory from the Euclidean Schwinger functions at unequal arguments, with or without an underlying Euclidean field theory.

They also discussed fermions, and got an algebraically simple, Feynman-Kac-Nelson (FKN) formula, at the expense of doubling the fermion fields, and of a non-Hermitean Euclidean action.

There continues to be an interest in the formulation of the Euclidean fermion problem, and there have been recent technical advances in work by Brydges and Federbush [5], Schrader and Uhlenbrock [6], and Wilde and Perez [7]. We refer particularly to the rather extensive work of Schrader and Uhlenbrock for a more complete list of references.

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We also want to cite explicitly a paper by Hegerfeldt [8], because his axiomatic approach is close in spirit to our own, although he only mentions fermions in a side remark.

In this paper, we choose a different construction of the free, charged, Euclidean-Dirac (ED) field than that of Osterwalder and Schrader. We proceed by first supplying a factor γ_5 in the relativistic two-point function at Schwinger points, to make it Hermitean, and then adding a term with support at the origin of four-dimensional, Euclidean space, to make it positive. The latter step can be done in many ways; but we choose a scheme that not only does not double particles, but also does not double spin states.

The resulting ED fields obey canonical, anticommutation relations (CAR), which turns out to be of some importance in the FKN formula. In particular, the ED field anticommutes with itself, as does its adjoint, while the field and its adjoint anticommute at unequal Euclidean space arguments. We call this situation "local anticommutation relations".

The natural expression of the OS positivity condition turns out to be in terms of an auxiliary, nonlocal field, related to the original ED fields by inverse differential operators. The nonlocal fields are defined at sharp times, and the OS positivity condition, when expressed in terms of them, is of the Nelson type.

In spite of the nonlocality in the OS positivity condition, the abstract proof that one gets a Hermitean, contraction semigroup on the physical space goes through; and one can see by inspection that the entire relativistic reconstruction for the free field goes through.

Next, we consider at the heuristic level local polynomial interactions, and obtain a formal parametrization of a natural class of Euclidean field theories with fermions in terms of Euclidean actions. Our actions are Hermitean, with locally commuting integrands; and our version of OS positivity is preserved. The construction of a physical Hilbert space and a Hermitean, contraction semigroup on it, induced by the Euclidean time evolution, goes through.

Finally, we abstract from our heuristic discussion a proposal for a set of Euclidean axioms for fermions. This proposal remains subject to modification. The recovery of the time evolution goes through, but we are currently studying what might be rigorously sufficient for the existence of the analytic continuation to the relativistic fields in the physical space.

II. Dirac Matrices and the Euclidean Group

Our Minkowski metric is $(+ - - -)$, and we follow the van der Waerden representation for relativistic Dirac matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3, \tag{1}$$

$$\sigma_\mu = \sigma_\mu^* \leftrightarrow (I, \underset{\sim}{\sigma}),$$
$$\tilde{\sigma}_\mu = \sigma^\mu \leftrightarrow (I, -\underset{\sim}{\sigma}). \tag{2}$$

Then $\gamma_0^* = \gamma_0 = \gamma_0^{\mathrm{Tr}}$; $\quad \underset{\sim}{\gamma}^* = -\underset{\sim}{\gamma}$, and

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5^* = \gamma_5^{\mathrm{Tr}}$$
$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{3}$$

Osterwalder and Schrader exploited the fact that $O_+(4, \mathbb{R})$ is isomorphic to the subgroup of $L_+(4, \mathbb{C})$ (identity component of the complex Lorentz group) that leaves the set of Euclidean points $z = (-iy_0, \underset{\sim}{y})$, $y \in \mathbb{R}^4$, invariant. Our notation for the correspondence between $O_+(4, \mathbb{R})$ and its covering group $SU(2) \times SU(2) \subset SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ is the following. We introduce the Euclidean-Pauli four-vector of 2×2 matrices,

$$\tau_\mu = \tau^\mu \leftrightarrow (-iI, \underset{\sim}{\sigma}). \tag{4}$$

Then

$$U_1 \tau_\mu U_2^* = R^\nu{}_\mu \tau_\nu; \quad U_1, U_2 \in SU(2)$$
$$\Rightarrow R \in O_+(4, \mathbb{R}). \tag{5}$$

We define Euclidean-Dirac (ED) matrices:

$$\gamma_{E\mu} \leftrightarrow (-i\gamma_5\gamma_0, \gamma_5\underset{\sim}{\gamma}). \tag{6}$$

(These matrices result from those of Osterwalder and Schrader by multiplying from the left by $-i\gamma_5$.) Then

$$\gamma_E = \begin{pmatrix} 0 & \tau \\ \tau^* & 0 \end{pmatrix} = \gamma_E^*, \tag{7}$$

and

$$\{\gamma_{E\mu}, \gamma_5\} = 0. \tag{8}$$

The ED representation of $SU(2) \times SU(2)$ is

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{9}$$

$$U\gamma_{E\mu}U^* = R^\nu{}_\mu \gamma_{E\nu}. \tag{10}$$

The Euclidean (as well as the Dirac) raising and lowering matrix is

$$K = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \tag{11}$$

$$KUK^{-1} = U^{\mathrm{Tr}-1} = \bar{U}. \tag{12}$$

The last identity is also obeyed by the ED charge conjugation matrix (which plays no role in our theory):

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} = \gamma_5 K = K \gamma_5 \,. \tag{13}$$

These matrices obey the following:

$$K = \bar{K} = -K^{\mathrm{Tr}} = -K^{-1}\,, \tag{14.a}$$

$$C = \bar{C} = -C^{\mathrm{Tr}} = -C^{-1}\,; \tag{14.b}$$

$$K \gamma_E K^{-1} = -\gamma_E^{\mathrm{Tr}}\,, \tag{15.a}$$

$$C \gamma_E C^{-1} = \gamma_E^{\mathrm{Tr}}\,. \tag{15.b}$$

III. The Free Euclidean Two-Point Function

The relativistic two-point functions are

$$\begin{aligned} S_+(x) &= \langle \Omega, \psi(x)\, \bar{\psi}(0)\, \Omega \rangle \\ &= (i\gamma \cdot \partial + m) \frac{1}{(2\pi)^3} \int \frac{d^3 k}{2\omega}\, e^{-ik\cdot x}\,, \end{aligned} \tag{16.a}$$

$$\begin{aligned} S_-(x) &= \langle \Omega, \bar{\psi}(x)\, \psi(0)\, \Omega \rangle \\ &= (i\gamma \cdot \partial - m)^{\mathrm{Tr}} \frac{1}{(2\pi)^3} \int \frac{d^3 k}{2\omega}\, e^{-ik\cdot x}\,, \end{aligned} \tag{16.b}$$

where

$$\omega = (m^2 + \underset{\sim}{k}^2)^{\frac{1}{2}}, \quad \text{and} \quad k = (\omega, \underset{\sim}{k})\,.$$

We define Euclidean two-point functions:

$$\begin{aligned} S_+^E(y) &= \langle \Omega_E, \psi_E(y)\, \psi_E^*(0)\, \Omega_E \rangle \\ &= (2\pi)^{-4} \int d\mu_E (I\lambda + \gamma_E \cdot p + \gamma_5 \mu)\, e^{ip\cdot y}\,, \end{aligned} \tag{17.a}$$

$$\begin{aligned} S_-^E(y) &= \langle \Omega_E, \psi^*(y)\, \psi(0)\, \Omega_E \rangle \\ &= (2\pi)^{-4} \int d\mu_E (I\lambda + \gamma_E \cdot p - \gamma_5 \mu)^{\mathrm{Tr}}\, e^{ip\cdot y}\,, \end{aligned} \tag{17.b}$$

where

$$\begin{aligned} d\mu_E &= d^4 p / (p^2 + m^2)\,, \\ \lambda = (p^2 + m^2)/2m\,, &\qquad \mu = (m^2 - p^2)/2m\,, \end{aligned} \tag{18}$$

and where we follow the convention that dot products involving p rather than k are Euclidean:

$$p \cdot y = p_0 y_0 + \mathbf{p} \cdot \mathbf{y} .$$

The Euclidean two-point functions have the property that for $y_0 > 0$, and $z = (-i y_0, \mathbf{y})$:

$$S_+^E(y) = \gamma_5 S_+(z) = \langle \Omega, \gamma_5 \psi(z) \bar{\psi}(0) \Omega \rangle , \tag{19.a}$$

$$S_-^E(y) = S_-(z) \gamma_5 = \langle \Omega, \bar{\psi}(z) \gamma_5 \psi(0) \Omega \rangle ; \tag{19.b}$$

i.e., we are making the formal correspondence

$$\begin{aligned} \psi_E(y) &\leftrightarrow \gamma_5 \psi(z) , \\ \psi_E^*(y) &\leftrightarrow \psi(z) . \end{aligned} \tag{20}$$

Now $S_\pm^E$ can be regarded as the two-point functions of a field ψ_E and its Hermitean adjoint ψ_E^*, because the matrices in the integrands can be written

$$\begin{aligned} M_\pm(p) &= (\lambda I + \gamma_E \cdot p \pm \mu \gamma_5) \\ &= (\pm \gamma_E \cdot p + m\gamma_5)(I \pm \gamma_5)(\pm \gamma_E \cdot p + m\gamma_5)/2m . \end{aligned} \tag{21}$$

Since $(I \pm \gamma_5)/2$ are orthogonal projections, this displays $M_\pm$ as the squares of Hermitean matrices. It also shows that $M_\pm$ have rank two, so that each particle has only two independent spin states.

The eigenvalues are easy to compute, because $M_\pm$ are unitary equivalent to

$$\begin{aligned} \begin{pmatrix} (\lambda \pm \mu) I & -ipI \\ ipI & (\lambda \mp \mu) I \end{pmatrix} &= (\lambda I \pm \mu \sigma_3 + p \sigma_2) \otimes I \\ &\equiv \eta_\pm \otimes I , \end{aligned} \tag{22}$$

and

$$\det \eta_\pm = \lambda^2 - \mu^2 - p^2 = 0 ,$$

$$\operatorname{Tr} \eta_\pm = 2\lambda ,$$

so the eigenvalues are 0 and 2λ, with two-fold degeneracy.

IV. Euclidean Fock Representation

We define p-space creation and destruction operators on the Euclidean Fock space $\mathscr{F}_E$ in a ED spinor basis, i.e., we do not factor out the analog of the Dirac wave function. The nonvanishing anticommutators are

$$\{b(p), b^*(p')\} = M_+(p)\,\delta(p-p'), \tag{23.a}$$

$$\{c(p), c^*(p')\} = M_-(p)^{\mathrm{Tr}}\,\delta(p-p'). \tag{23.b}$$

We specify an irreducible, Fock representation, with Euclidean vacuum:

$$b(p)\,\Omega_E = c(p)\,\Omega_E = 0. \tag{24}$$

The action of the unitary Fock representation of an element (a, U) of inhomogeneous $SU(2)\times SU(2)$ is:

$$U(a, U)\,\Omega_E = \Omega_E \tag{25.a}$$

$$U(a, U)^*\, b(p)\, U(a, U) = e^{-ip\cdot a}\, U\, b(R^{-1}p), \tag{25.b}$$

$$U(a, U)^*\, c(p)\, U(a, U) = e^{-ip\cdot a}\, \bar{U}\, c(R^{-1}p), \tag{25.c}$$

plus the equations derived from these by Hermitean conjugation. The "bar" notation always means complex conjugation, except on the Minkowski, Dirac field. Thus, b and c transform like ED spinors with lower and upper indices, respectively.

The ED fields are

$$\psi_E(y) = \frac{1}{(2\pi)^2} \int \frac{d^4p}{\sqrt{p^2+m^2}}\,[b(p)\,e^{ip\cdot y} + c^*(p)\,e^{-ip\cdot y}], \tag{26}$$

and the Hermitean conjugate. Note that ψ_E and ψ_E^* transform with lower and upper indices, respectively, and that they obey CAR:

$$\begin{aligned} &\{\psi_E(y), \psi_E(y')\} = \{\psi_E^*(y), \psi_E^*(y')\} = 0, \\ &\{\psi_E(y), \psi_E^*(y')\} = I\,\delta(y-y')/m. \end{aligned} \tag{27}$$

We introduce a complex space of spinor-valued test functions, F, with arguments $y \in \mathbb{R}^4$; e.g., $F = \mathscr{S}(\mathbb{R}^4)\otimes\mathbb{C}^4$, or $F = L_2(\mathbb{R}^4)\otimes\mathbb{C}^4$. The smearing operations

$$f \mapsto \psi_E(f),\ \psi_E^*(f)$$

are linear in f, and we always think of multiplication by f from the right in the spinor space.

Our convention for Fourier transforms is

$$f \mapsto g(p) = (2\pi)^{-2} \int d^4y\, e^{-ip\cdot y} f(y). \tag{28}$$

We shall be especially interested in the differential operators:

$$\begin{aligned} D &= -i\gamma_E \cdot \partial + \gamma_5 m, \\ \bar{D} &= (i\gamma_E \cdot \partial + \gamma_5 m)^{\mathrm{Tr}}, \\ Df &\leftrightarrow (\gamma_E \cdot p + \gamma_5 m)\, g, \\ \bar{D} f &\leftrightarrow (-\gamma_E \cdot p + \gamma_5 m)^{\mathrm{Tr}} g. \end{aligned} \tag{29}$$

The inverses are well-defined by

$$\begin{aligned} D^{-1} f &\leftrightarrow (\gamma_E \cdot p + \gamma_5 m)\, g/(p^2 + m^2), \\ \bar{D}^{-1} f &\leftrightarrow (-\gamma_E \cdot p + \gamma_5 m)^{\mathrm{Tr}} g/(p^2 + m^2). \end{aligned} \tag{30}$$

We define scalar products in the one b and c particle subspaces of $\mathscr{F}_E$, respectively, by

$$\begin{aligned} \langle g, g \rangle_b &= \langle \Omega_E, \psi_E(\bar{f})\, \psi_E^*(f)\, \Omega_E \rangle \\ &= \int d\mu_E g^* M_+ g, \end{aligned} \tag{31.a}$$

$$\begin{aligned} \langle g, g \rangle_c &= \langle \Omega_E, \psi_E^*(\bar{f})\, \psi_E(f)\, \Omega_E \rangle \\ &= \int d\mu_E g^* M_-^{\mathrm{Tr}} g. \end{aligned} \tag{31.b}$$

The corresponding Hilbert spaces are denoted $\mathscr{H}_b$ and $\mathscr{H}_c$, respectively, and the one-particle subspace of $\mathscr{F}_E$ is $\mathscr{F}_1 = \mathscr{H}_b \oplus \mathscr{H}_c$.

Note that $F = L_2(\mathbb{R}^4) \otimes \mathbb{C}^4$ is indeed natural, because the matrices $M_\pm/(p^2 + m^2)$ are uniformly bounded.

For later convenience in describing the appropriate OS positivity condition, we introduce an auxiliary pair of conjugate fields:

$$\begin{aligned} \phi(f) &\equiv (D^{-1}\psi_E)(f) = \psi_E(\bar{D}^{-1} f), \\ \phi^*(f) &\equiv (D^{-1}\psi_E)^*(f) = \psi_E^*(D^{-1} f). \end{aligned} \tag{32}$$

Because of Eq. (21), their two-point functions are

$$\langle \Omega_E, \phi(\bar{f})\, \phi^*(f)\, \Omega_E \rangle \equiv \langle g, g \rangle_{N_+} = \int g^* \left(\frac{I + \gamma_5}{2m} \right) g \, d\mu_E, \tag{33.a}$$

$$\langle \Omega_E, \phi^*(\bar{f})\, \phi(f)\, \Omega_E \rangle \equiv \langle g, g \rangle_{N_-} = \int g^* \left(\frac{I - \gamma_5}{2m} \right) g \, d\mu_E. \tag{33.b}$$

The nonvanishing anticommutation relation is

$$\{\phi(\bar{f}), \phi^*(f')\} = \frac{1}{m} \int g^* g' \, d\mu_E \equiv \frac{1}{m} \langle g, g' \rangle_N. \tag{34}$$

The notation "N" anticipates a Nelson-type application of the Sobolev scalar product. The natural test function space is a Sobolev space, and the anticommutation relation is of course non-local. Covariance under inhomogeneous $SU(2) \times SU(2)$ is preserved.

V. Euclidean Time Reflection and OS Positivity

We define a linear, local, Euclidean time reflection operator Θ as follows:

$$\begin{aligned} \Theta\,\Omega_E &= \Omega_E\,, \\ \Theta\,\psi_E^*(f)\,\Theta^{-1} &= \psi_E(K^{-1}\gamma_{E0}f_\theta)\,, \\ \Theta^{-1*}\,\psi_E(f)\,\Theta^* &= \psi_E^*(K\gamma_{E0}f_\theta)\,, \\ f_\theta(y) \equiv f(\theta y) &= f(-y_0, \mathbf{y})\,. \end{aligned} \tag{35}$$

It is easy to verify that

$$\Theta^2 = I\,, \qquad \Theta^* = \Theta^{-1} = \Theta\,. \tag{36}$$

Note that Θ has the correct action as an automorphism of inhomogeneous $SU(2)\times SU(2)$:

$$\Theta\,U(a, U_1\times U_2)\,\Theta^{-1} = U(\theta a, U_2\times U_1)\,. \tag{37}$$

This is correct, because conjugation of Eq. (5) gives

$$U_2\,\tau_\mu\,U_1^* = (\theta R\theta)^\nu{}_\mu\tau_\nu\,. \tag{38}$$

The action of Θ on ϕ is:

$$\begin{aligned} \Theta\,\phi^*(f)\,\Theta^{-1} &= \phi(K\gamma_{E0}f_\theta)\,, \\ \Theta\,\phi(f)\,\Theta^{-1} &= \phi^*(K^{-1}\gamma_{E0}f_\theta)\,. \end{aligned} \tag{39}$$

The reflection operator that appears in our form of the OS positivity condition is most simply expressed by its action on ϕ. We define a linear, time reflection T by:

$$\begin{aligned} T\,\Omega_E &= \Omega_E\,, \\ T\,\phi^*(f)\,T^{-1} &= \phi^*(f_\theta)\,, \\ T\,\phi(f)\,T^{-1} &= \phi(f_\theta)\,. \end{aligned} \tag{40}$$

Again, one gets

$$T^2 = I\,, \qquad T^{-1} = T^* = T\,. \tag{41}$$

The action of T on ψ_E is nonlocal:

$$\begin{aligned} T\,\psi_E^*(f)\,T^{-1} &= \psi_E^*(D^{-1}D_\theta f_\theta) \\ T\,\psi_E(f)\,T^{-1} &= \psi_E(\bar{D}^{-1}\bar{D}_\theta f_\theta) \end{aligned} \tag{42}$$

where D_θ results from D by sign reflection of the γ_{E0} term.

Let F_+ and $\bar{F}_+$ be the subspaces of test functions f_+ such that Df_+, respectively, $\bar{D}f_+$ have support as distributions in the interior of $y_0 \geqq 0$. That is, $f_+ = D^{-1}h_+$, respectively, $\bar{D}^{-1}h_+$, where h_+ is in N_+, the subspace of the Nelson (Sobolev) space spanned by vectors with strictly positive time support. Recall that

$$\psi_E^*(f_+) = \phi^*(Df_+); \quad \psi_E(f_+) = \phi(\bar{D}f_+). \tag{43}$$

Let $\mathscr{F}_+$ be the linear submanifold of $\mathscr{F}_E$ spanned by polynomials in ψ_E^* and ψ_E smeared, respectively, in F_+ and $\bar{F}_+$, applied to the vacuum. Then $\mathscr{F}_+$ is also the submanifold of $\mathscr{F}_E$ generated from Ω_E by ϕ and ϕ^* smeared in N_+. For the sake of notation, we do not take the completion.

Lemma (OS Positivity). Let $\mathscr{P}_+ \in \mathscr{F}_+$. *Then* $\langle \mathscr{P}_+, T\mathscr{P}_+ \rangle \geqq 0$. (44)

The proof is a simple imitation of an argument due to Osterwalder and Schrader. The statement is true for two-point functions, as one sees most clearly in the ϕ representation of $\mathscr{F}_+$. It then follow easily for Wick ordered monomials. For polynomials, one writes the Wick expansion, making use of the fact that T commutes with the Wick expansion because of its unitarity in the contraction functions, and one then uses orthogonality of the Wick monomials applied to the vacuum.

The reconstruction of the Wightman Fock space follows standard lines. One has the choice of working with the ϕ representation and Nelson's sharp-time method, or of working with either the ϕ or the ψ_E representation and following the OS construction. We prefer the OS construction because of the probable instability of the sharp-time method under interaction in four-dimensional space-time.

The physical Hilbert space $\hat{\mathscr{F}}_+$, then, is obtained from $\mathscr{F}_+$ by dividing out the kernel of the bounded, positive, bilinear form defined on $\mathscr{F}_+$ by T. The unitary, Euclidean time evolution $U(t) = U[(t, \underset{\sim}{0}), I]$ preserves $\mathscr{F}_+$ for $t \geqq 0$, and obeys

$$TU(t)T^{-1} = U(-t). \tag{45}$$

It therefore passes to a Hermitean, contraction semigroup on $\hat{\mathscr{F}}_+$ by an argument due to Osterwalder and Schrader, and refined by Hegerfeldt [8].

Of course, one can verify explicitly, by inspection of the two-point function, that $\hat{\mathscr{F}}_+$ is identified with the Wightman space, and that one recovers the Wick rotated, physical time evolution, and the Wick rotated, free Dirac field operators.

In that regard, we remark that the free Dirac field could already be recovered from the subspace of $\mathscr{F}_+$ generated by ψ_E and ψ_E^* smeared with functions in F having strictly positive time support. After careful consideration of the formal interacting case, we have decided it would be dangerous to try to get away with that in general.

VI. A Heuristic Feynman-Kac-Nelson Formula

Rather than derive in the usual way an expression for the Schwinger functions in terms of the Euclidean action associated with a local relativistic interaction, we proceed by guessing directly how to parametrize Euclidean theories in terms of Euclidean actions. In our heuristic discussion, we constrain ourselves to preserve the following "non-technical" properties:

(i) existence of an invariant vacuum,
(ii) Symanzik positivity,
(iii) Osterwalder-Schrader positivity,
(iv) unitary Euclidean covariance (including reflection invariance),
(v) local anticommutation relations,
(vi) Hermitean, contraction semigroup property.

The kind of Euclidean action V we consider is formally the integral

$$V = \int V(y)\, d^4 y$$

of an invariant, local polynomial, which couples spinless or vector, free, Hermitean, Euclidean boson fields, which of course commute with everything in sight, with powers of local, Hermitean bilinears in the free, ED field of the form

$$(D\psi_E)^*(y)\, \Gamma D \psi_E(y),$$

where Γ is a coupling matrix, and where the reason for the derivative D will emerge. Because our Dirac fields obey CAR, it is easy to write down physically interesting couplings such that the integrand of the Euclidean action is not only formally Hermitean, but formally commutes with itself at different arguments. The action will not, however, commute with the ED field.

We impose the following local conditions on the action integrand:

(i) $V(y)$ is a local function of free fields of the sort described above.
(ii) $V(y) = V^*(y)$.
(iii) $[V(y), V(y')] = 0$, $y \neq y'$.
(iv) $U(a, U)\, V(y)\, U(a, U)^{-1} = V(Ry + a)$.
(v) $\Theta\, V(y)\, \Theta^{-1} = V(\theta y)$.
(vi) $T\, V(y)\, T^{-1} = V(\theta y)$.

A simple example is the Yukawa interaction

$$V(y) = :(D\psi_E)^*(y)\, \gamma_5 D \psi_E(y): \phi_E(y),$$

where ϕ_E is a boson field with even parity under Θ and T. Another is the massive vector interaction

$$V(y) = :(D\psi_E)^*(y)\, \gamma_{E\mu} D \psi_E(y): A_E^{\mu}(y),$$

where

$$\Theta A_E(y)\,\Theta^{-1} = \theta A_E(\theta y), \quad \text{and} \quad T A_E(y)\, T^{-1} = A_E(\theta y).$$

These examples are T covariant because we used $D\psi_E$ rather than ψ_E. The D has no effect on Θ covariance, but it has the virtue that $D\psi_E$ transforms *locally* under T:

$$\begin{aligned} T(D\psi_E)^*(f)\,T^{-1} &= T\psi_E^*(Df)\,T^{-1} \\ &= \psi_E^*(D^{-1}D_\theta D_\theta f_\theta) = \psi_E^*(D D_\theta^{-1} D_\theta f_\theta) \\ &= (D\psi_E)^*(f_\theta). \end{aligned} \tag{46.a}$$

Similarly,

$$T D\psi_E(f)\,T^{-1} = (D\psi_E)(f_\theta). \tag{46.b}$$

Consider the formal expression

$$\mathscr{E} = \exp - V. \tag{47}$$

We think of it heuristically as a Hermitean, positive definite, invertible, invariant operator on the Euclidean Fock space $\mathscr{F}_E$ (which now includes bosons). To imitate its role in the strictly boson theory, we would like to use it as a metric operator to define a new Hilbert space. But the fact that $\mathscr{E}$ fails to commute with the ED field suggests a modification of the field, if we want to preserve the properties mentioned above. This can be done as follows.

Define the Euclidean, pre-Hilbert space $\mathscr{H}_V$ of the interacting theory (for a "suitable" dense set of vectors in $\mathscr{F}_E$, including Ω_E) by the scalar product

$$\langle \psi, \psi \rangle_V = \langle \psi, \mathscr{E}\psi \rangle / \langle \Omega_E, \mathscr{E}\Omega_E \rangle. \tag{48}$$

The denominator normalizes the interacting vacuum state Ω_V, which is identified with Ω_E in the above construction. The Hermitean adjoint operation for operators on $\mathscr{H}_V$ is related to that for operators on $\mathscr{F}_E$ by

$$\mathcal{O}^A = \mathscr{E}^{-1}\mathcal{O}^*\mathscr{E}. \tag{49}$$

Define interacting ED fields in terms of free fields by

$$\begin{aligned} \psi_V &= \mathscr{E}^{-\frac{1}{2}}\psi_E\mathscr{E}^{\frac{1}{2}}, \\ \psi_V^A &= \mathscr{E}^{-\frac{1}{2}}\psi_E^*\mathscr{E}^{\frac{1}{2}}. \end{aligned} \tag{50}$$

The definition of ψ_V^A is consistent with Eq. (49). The interacting fields obey CAR because the free fields do. They should be modified in general by a wave function renormalization constant, which we omit for simplicity. Even then, we would have local anticommutation relations.

The old action of the Euclidean group, including Θ and T, is still unitary, because the invariance of $\mathscr{E}$ implies that

$$
\begin{aligned}
U(a, U)^A &= U(a, U)^* = U(a, U)^{-1}, \\
\Theta^A = \Theta^* = \Theta^{-1}; &\qquad T^A = T^* = T^{-1}.
\end{aligned} \tag{51}
$$

The vacumm $\Omega_V = \Omega_E$ remains invariant. The interacting fields $\psi_V^\#$, where "#" means supply the adjoint or not, appropriate to the subscript, have the same transformation laws under U, Θ, and T as the free fields, again because of the invariance of $\mathscr{E}^{\pm\frac{1}{2}}$.

So far, we clearly have an invariant, normalized vacuum, Euclidean covariance, local anticummutation relations, and Symanzik positivity. It remains to check OS positivity and the semigroup property.

For that purpose, we split V into positive and negative time parts,

$$
\begin{aligned}
V &= V_+ + V_- , \\
V_\pm &= \int_{y_0 \gtrless 0} V(y)\, d^4 y ,
\end{aligned} \tag{52}
$$

and we note the global properties that we actually need:

(i) $V_\pm$ are functions of positive and negative time fields, respectively, with D's acting on the fermi fields;

(ii) $V_\pm = V_\pm^*$;

(iii) $[V_+, V_-] = 0$;

(iv) $\Theta V_\pm \Theta^{-1} = V_\mp$;

(v) $T V_\pm T^{-1} = V_\mp$.

We make the corresponding split:

$$
\begin{aligned}
\mathscr{E} = \mathscr{E}_+ \mathscr{E}_- = \mathscr{E}_- \mathscr{E}_+ ; &\qquad (\mathscr{E}_\pm)^* = \mathscr{E}_\pm ; \\
\Theta \mathscr{E}_\pm \Theta^{-1} = T \mathscr{E}_\pm & T^{-1} = \mathscr{E}_\mp .
\end{aligned} \tag{53}
$$

Now let $\mathscr{H}_{V_+}$ be the submanifold of $\mathscr{H}_V$ generated from the vacuum by ψ_V^A and ψ_V smeared in F_+ and $\overline{F}_+$. Note that the fields $D\psi_E$ have another nice property; they have local anticommutation relations with ϕ: e.g.,

$$
\{D\psi_E(\overline{f}), \phi^*(f')\} = \{\phi(\overline{D^2 f}), \phi^*(f')\} = \langle D^2 f, f' \rangle_N / m . \tag{54}
$$

The last line vanishes if f and f' have disjoint *y-supports,* because $\overline{D}^2 = D^2 \leftrightarrow p^2 + m^2$ kills the nonlocal factor in $d\mu_E$. This has the consequence that $\psi_V^\#(f_+)$ are formally functions only of free fields smeared in the appropriate F_+ and $\bar{F}_+$. For example,

$$\begin{aligned}\psi_V^A(f_+) &= \mathscr{E}^{-\frac{1}{2}}\,\psi_E^*(f_+)\,\mathscr{E}^{\frac{1}{2}} \\ &= \mathscr{E}^{-\frac{1}{2}}\,\phi^*(D f_+)\,\mathscr{E}^{\frac{1}{2}} \\ &= \mathscr{E}_+^{-\frac{1}{2}}\,\phi^*(D f_+)\,\mathscr{E}_+^{\frac{1}{2}} \\ &= \mathscr{E}_+^{-\frac{1}{2}}\,\psi_E^*(f_+)\,\mathscr{E}_+^{\frac{1}{2}}.\end{aligned} \tag{55}$$

In other words, $\mathscr{H}_{V_+}$ is manufactured from $\mathscr{F}_+$.

Theorem A (OS Positivity).

$$\textit{Let}\quad \mathscr{P}_{V_+} \in \mathscr{H}_{V_+}. \quad \textit{Then}\quad \langle \mathscr{P}_{V_+}, T\mathscr{P}_{V_+}\rangle_V \geqq 0. \tag{56}$$

Proof. From the intertwining property of T and the Hermiticity of $\mathscr{E}_\pm$, we have

$$\langle \mathscr{P}_{V_+}, \mathscr{E}\,T\,\mathscr{P}_{V_+}\rangle = \langle \mathscr{E}_+\mathscr{P}_{V_+}, T\,\mathscr{E}_+\mathscr{P}_{V_+}\rangle. \tag{57}$$

The results follows from OS positivity in $\mathscr{F}_+$ and the remarks above.

We can now construct a physical space $\hat{\mathscr{H}}_{V_+}$ from $\mathscr{H}_{V_+}$ by the OS method just as before in the Fock space, and by the same token, a Hermitean, contraction semigroup. (Strong continuity is formally easy.)

To complete the formal reconstruction, we need to construct the Wick rotated, physical field operators. One can give an ultra-heuristic argument, which treats the ϕ field at time zero as an operator and then concludes from the commutation relations with the action that this operator passes to the physical space. That becomes a bit too conjectural for our taste, and so we defer the argument until we can justify it by a more technical, axiomatic study, now in progress. It does seem worth remarking that, even for the formal argument, it seems essential to have the full space $\mathscr{H}_{V_+}$, and not just the subspace of strictly positive times relative to $\psi_V^\#$.

We conclude this section with our version of the Feynman-Kac-Nelson formula:

Theorem B (FKN).

$$\langle \Omega_V, \psi_V^\#(f_1)\dots\psi_V^\#(f_n)\,\Omega_V\rangle_V = \frac{\langle \Omega_E, \mathscr{E}^{\frac{1}{2}}\psi_E^\#(f_1)\dots\psi_E^\#(f_n)\,\mathscr{E}^{\frac{1}{2}}\Omega_E\rangle}{\langle \Omega_E, \mathscr{E}\,\Omega_E\rangle}. \tag{58}$$

VII. Preliminary Axioms for Euclidean Dirac Fields

We propose the following axioms for ED fields ψ and ψ^*. It should be clear to the reader how to include Euclidean bosons.

(i) We are given a separable Hilbert space $\mathscr{H}_E$ and a unitary, strongly continuous representation $U(a, U)$ of inhomogeneous $SU(2) \times SU(2)$, and of the reflections, with a unique, invariant vacuum.

(ii) Irreducible field operators $\psi(f)$ and $\psi^*(f) = \psi(\bar{f})^*$ are densely defined on the usual domain generated by polynomials from the vacuum. The vacuum is cyclic. The fields are operator-valued, tempered distributions in the usual Wightman sense. In particular, the test function space is $F = \mathscr{S}(\mathbb{R}^4) \otimes \mathbb{C}^4$.

(iii) The fields $\psi^{\#}(f)$ transform under the full Euclidean group by the same laws as our ED free fields.

(iv) We have local, anticommutation relations:

$$\{\psi(f), \psi(f')\} = \{\psi^*(f), \psi^*(f')\} = 0\,;$$

$\{\psi(f), \psi^*(f')\} = 0$ if f and f' have disjoint supports.

(v) There is a unitary, Hermitean, time reflection operator T on $\mathscr{H}_E$ which leaves the vacuum invariant and has the same action on $\psi^{\#}(f)$ as described before for the free field.

(vi) Osterwalder-Schrader Positivity: Let $\mathscr{H}_{E_+} \subset \mathscr{H}_E$ be the submanifold of $\mathscr{H}_E$ generated from the vacuum by polynomials in ψ^* and ψ smeared, respectively, in F_+ and $\bar{F}_+$, where

$$F_+ = \{f_+ \in F : D f_+ \in N_+\}$$

$$\bar{F}_+ = \{f_+ \in F : \bar{D} f_+ \in N_+\}$$

$$N_+ = \{h_+ \in F : h_+(y) = 0 \;\text{ for }\; y_0 \leqq 0\}\,.$$

Let $\mathscr{P}_+ \in \mathscr{H}_{V_+}$. Then $\langle \mathscr{P}_+, T\mathscr{P}_+ \rangle \geqq 0$.

(vii) Nelson's Property A'): Let

$$\phi^*(f) = \psi^*(D^{-1} f)\,,$$

$$\phi(f) = \psi(\bar{D}^{-1} f)\,.$$

Then the unsmeared fields $\phi^{\#}(y)$ obey a sufficiently strong version of Nelson's Axiom (A').

We are currently working on the problem of what can be taken as sufficient in the last axiom. As we mentioned in the Introduction, the above scheme is similar in spirit to one proposed by Hegerfeldt [8], for the situation where OS positivity is defined locally.

VIII. Concluding Remarks

(i) The role of the differential operator D in the Euclidean action might be clarified by formally connecting our FKN formula to the relativistic interaction in the physical space. We expect that to be a straightforward exercise.

(ii) We stated the properties of the action that we needed for our heuristic discussion in a global form that is amenable to cutoff; i.e., there is a way to cut off the positive and negative time parts of the action separately, while retaining everything except restricted Euclidean invariance.

(iii) We have of course ignored, at the formal level, any details of renormalization. Superficially at least, the renormalization problem would appear to be made worse in the Euclidean space by the presence of derivatives on the fermi fields in the action. Such difficulties could be ameliorated in the physical space, where the operator D goes over into γ_5 times the Dirac positive energy projection.

(iv) Our free field fits into the functorial scheme of Schrader and Uhlenbrock [6] if one considers the fields $\phi^{\#}$ instead of $\psi_E^{\#}$, in the sense that the second quantization of a minimal Sz.-Nagy semigroup extension is involved.

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Note Added in Proof. We have subsequently learned that the positivity structure of the free ED field is richer than we had thought, and that it obeys another, local form of OS positivity which is preserved under a second, heuristic parametrization of interaction that has the right intuitive properties for reconstruction of relativistic fields. Concluding remark (i) above is over-optimistic because we have not succeeded in even an intuitive reconstruction of relativistic fields (in the interacting case) from the parametrization in this paper. Although we do not think it clearly impossible that the nonlocal method here could be made to work, we now prefer the second, manifestly local structure, which has appeared in preprint as a sequel to this paper.

After submitting the second paper for publication, we received a preprint from J. Fröhlich and K. Osterwalder, which casts doubt on the viability of any scheme for interacting Euclidean fermi fields which attempts to avoid doubling. Their argument is essentially that the extra growth in momentum space which results from extending the Euclidean Green's functions to coinciding arguments makes the renormalization problem for the Euclidean theory unmanageable. This difficulty appears here through the mechanism mentioned in concluding remark (iii), and it appears in our second method as well, as we have indicated in the sequel. While we agree that renormalization could well turn out to be fatal for the utility of either of the methods we propose, we nevertheless think it sensible at this point not to anticipate the results of a more detailed study of specific models of interaction.