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Unrefined minimal K-types for p-adic groups

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Dedicated to Armand Borel

1. Introduction

Let G be a reductive group over a p-adic field k. The study of admissible representations of $G(k)$ via their restriction to compact open subgroups was begun by Mautner, Shalika and Tanaka for groups of type A_1. In contrast to real reductive groups where the representation theory of a maximal compact subgroup K is given in terms of slight modifications to Cartan's theory of the highest weight, the representation theory of a (compact) parahoric subgroup $\mathscr{P}$ is quite complicated. There is still no comprehensive theory for classifying the irreducible representations. In the case of $GL_n(k)$, Howe, in [8], defined the notion of an essential character of a filtration subgroup K_i of $\mathrm{GL}_n(\mathfrak{o})$ based on realizing characters of $\mathsf{K}_i/\mathsf{K}_j$ $(2i \geqq j)$ as cosets in $M_n(k)$. In particular, Howe proved that any admissible representation of $\mathrm{GL}_n(k)$ has only finitely many essential characters.

In [12], based on work of Howe and the first author in regard to Hecke algebra isomorphisms, the first author gave a more precise formulation of Howe's ideas and defined the notion of an unrefined minimal K-type as certain representations of parahoric filtrations subgroups $\mathscr{P}_i$ in terms of semisimple and nilpotent elements in $M_n(k)$. It was conjectured in [12] that every irreducible admissible representation of $\mathrm{GL}_n(k)$ contained an unrefined minimal K-type. This conjecture was proved by Howe and the first author via a combinatorial argument and it was also shown that any two unrefined minimal K-types contained in an irreducible representation must be closely related, namely they must be *associates* of one another.

The term "minimal K-type" was used by the first author in analogy with a similar notion in the case of real groups. In both cases there is a measure of the depth of a representation. In the real case Vogan [16] defines the depth of an irreducible representation of K as the length of the highest weight with a rho shift. A minimal K-type of an admissible irreducible representation π is then defined as a K-type whose depth is minimal among all K-types occuring in π. In particular, in the real case, a representation of K can be a minimal K-type in one representation and not in another. In the p-adic case an unrefined minimal K-type is intrinsically defined (see 5.1), i.e. its definition depends only on the representation and not on how it sits in the restriction of an admissible representation relative to other representations of filtration subgroups. Because the definition is intrinsic, one must now prove that there is an unrefined minimal K-type in any irreducible admissible representation π. In both the real and p-adic setting a minimal or unrefined minimal K-type is a very important constituent of a representation. In the real case Vogan has proved that a minimal K-type occurs with multiplicity one and used minimal K-types as an anchor in classifying the irreducible admissible representations. Unrefined minimal K-types and their refinements to "refined" minimal K-types should hopefully play a similar role in the classification of irreducible admissible representations of a p-adic group.

Based on insight obtained from the combinatorial proof with Howe, the first author discovered that the existence of unrefined minimal K-types would follow quite easily if certain cosets, which realize characters of filtrations subgroups, satisfy a descent property whenever a coset contains a nilpotent element. The descent property for $M_n(k)$ was established in [5] and [9] and for Lie algebras of certain classical groups in [11]. The formulation of the descent property in terms of the Lie algebra leads to certain unnecessary restrictions. In the case of $\mathrm{GL}_n(k)$, the vector space $M_n(k)$ is both naturally the Lie algebra and the dual of the Lie algebra. This double role of $M_n(k)$ obscures the distinction between the Lie algebra and its dual.

In this paper, the existence of unrefined minimal K-types is established for all reductive p-adic groups (Theorem 5.2). We also prove that any two unrefined minimal K-types occuring in an irreducible admissible representation are associates of each other. Depth is defined in terms of the *congruence level* of a filtration subgroup and it is proved that an unrefined minimal K-type minimizes depth. To each point x in the Bruhat–Tits building of G/k there is a naturally attached filtration $\{\mathscr{P}_{x,r_i}\}$ of the parahoric subgroup $\mathscr{P}_x$ which is the isotropy subgroup at x. These filtrations include as special cases the filtrations introduced in [14]. The point x also defines filtrations $\{\mathfrak{g}_{x,r_i}\}$ (resp. $\{\mathfrak{g}^*_{x,-r_i}\}$) of the Lie algebra $\mathfrak{g}$ (resp. its dual $\mathfrak{g}^*$). In particular, cosets in the dual $\mathfrak{g}^*$ naturally parametrize characters of the abelian group $\mathscr{P}_{x,r_i}/\mathscr{P}_{x,r_{i+1}}$, $r_i > 0$. The descent property can be formulated in the proper setting of the nilpotent, i.e. unstable, elements in the dual, and its truth implies the existence of unrefined minimal K-types.

The proof of existence of unrefined minimal K-types presented in this paper is conceptually different from the approach followed by earlier authors to prove the existence in particular cases. Our proofs are uniform and do not require explicit realizations of the reductive groups over local fields.

The existence and associativity properties of unrefined minimal K-type allows one to attach a nonnegative rational number $\varrho(\pi)$ – the depth of an unrefined minimal K-type contained in π – to any irreducible admissible representation π. This number should be important for certain aspects of the representation π. Thus, it is quite natural to conjecture that Harish–Chandra's local character expansion of the character Θ_π of π is valid for all regular $g \in \mathscr{P}_{x,\varrho(\pi)^+}$ for any point x in the Bruhat–Tits building of G/k.

2. Parahoric subgroups and their natural filtrations

The goal of this section is to introduce natural filtrations of any parahoric subgroup in terms of the (absolute) affine root system.

2.1. Let k be a nonarchimedean local field and K be a fixed maximal unramified extension of k. Let $\mathfrak{o}$ (resp. $\mathfrak{O}$) be the ring of integers in k (resp. K) and $\mathfrak{f}$ (resp. $\mathfrak{F}$) be the residue field of k (resp. K). Note that $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$. Let $\Gamma = \mathrm{Gal}(K/k)$ be the Galois group of K/k; Γ has a natural identification with the Galois group of $\mathfrak{F}/\mathfrak{f}$. We fix a uniformizing element ϖ in $\mathfrak{o}$. Let ω be the discrete valuation of K such that $\omega(K^{\times}) = \mathbf{Z}$.

Let G be an absolutely quasi-simple, simply connected algebraic group defined over k. Let $\mathfrak{g}$ be the Lie algebra of G/k and $\mathbf{g} = \mathfrak{g} \otimes_k K$. Recall that the Bruhat–Tits building of G/K is a contractible simplicial complex on which $G(K)$ and the Galois group Γ operate by simplicial automorphisms. The Bruhat–Tits building of G/k is the set of points in the building (of G/K) fixed under Γ. For a point x of the Bruhat–Tits building of G/K, P_x will denote the isotropy subgroup at x in the natural action of $G(K)$ on the building; P_x is a parahoric subgroup of $G(K)$ and all the parahoric subgroups of $G(K)$ arise this way. If x is fixed under Γ, then P_x is defined over k (i.e. it is Γ-stable) and we shall denote $P_x \cap G(k)$ by $\mathscr{P}_x$.

In the sequel when we say that a point lies in a particular simplex of the building, we shall mean that it lies in the interior of the simplex.

2.2. Let S be a maximal k-split torus of G and let T be a maximal K-split torus of G defined over k and containing S. According to the Bruhat–Tits theory such a torus T exists [4, 5.1.12]. Since the residue field of K is algebraically closed, according to a well known result of Steinberg, G is quasi-split over K i.e., it contains a Borel subgroup defined over K. Equivalently, the centralizer Z of T in G is a (maximal) torus; it is defined over k since T is. Let N be the normalizer of T in G.

Let $X^*(T)$ be the group of characters of T, and $X_*(T)$ be the group of 1-parameter subgroups of T (recall that a 1-parameter subgroup of T is a rational homomorphism $\lambda: \mathrm{GL}_1 \to T$). There is a nondegenerate pairing

$$\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbf{Z},$$

defined as follows: For $\chi \in X^*(T)$ and $\lambda \in X_*(T)$, $\chi \circ \lambda$ is an endomorphism of GL_1. Now $\mathrm{End}(\mathrm{GL}_1) = \mathbf{Z}$ and $\langle \chi, \lambda \rangle$ is set to be equal to the integer corresponding to $\chi \circ \lambda$. Let $\mathscr{V} = X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$, and $\mathscr{V}^* = X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Then $\mathscr{V}^*$ is the vector space dual of $\mathscr{V}$.

The apartment A associated with the torus T in the Bruhat–Tits building of G/K is an affine space under $\mathscr{V}$. As T is defined over k, A is stable under the action of the Galois group Γ on the building of G/K. There is a natural transitive action of $\mathscr{V}$ on A by translations; for $x \in A$ and $v \in \mathscr{V}$, we shall denote the translate of x by v by $x + v$. In particular, if λ is a 1-parameter subgroup of T and r is a real number, then the translate of $x \in A$ under $r\lambda (\in \mathscr{V})$ is denoted $x + r\lambda$. For a given $x \in A$, the subset $\{x + r\lambda \mid r \in \mathbf{R} \text{ and } \lambda \in X_*(T)\}$ is clearly dense in A.

Let F be the space of $\mathbf{R}$-valued affine-linear functions on A. The function which takes the value 1 at all the points of A will be denoted by δ in the sequel.

2.3. Let $\Phi (\subset X^*(T))$ be the set of roots of G with respect to T and Π be the basis of the root system determined by a Borel subgroup containing T and defined over the splitting field of T. For a root $b \in \Phi$, let U_b be the corresponding root subgroup. It is a connected unipotent subgroup of G defined over K, normalized by Z, and of dimension $\leqq 3$. The Lie algebra of U_b consists of root spaces corresponding to the roots which are positive integral multiples of b. For $b \in \Phi$, let G_b be the subgroup generated by U_b and U_{-b}; G_b is a simply connected semi-simple subgroup of G defined over K. Set $Z_b = G_b \cap Z$. Then Z_b is a maximal torus of G_b and Z is a direct product of the subtori Z_a, $a \in \Pi$. Let Z_0^b be the maximal bounded subgroup of $Z_b(K)$ and for any positive integer s, let $Z_{s\delta}^b$ be the congruence subgroup described in [14: 2.6]. Let Z_0 be the maximal bounded subgroup of $Z(K)$; then Z_0 is a direct product of the subgroups Z_0^a, $a \in \Pi$.

2.4. *The subgroup* U_ψ. To every affine function $\psi \in F$, with gradient b belonging to Φ, one associates a subgroup U_ψ of $U_b(K)$. (This subgroup has been denoted by X_ψ in [15].) As we shall need to make extensive use of this subgroup, for the reader's convenience we reproduce its description from [15: 1.4] below. An equivalent description of the subgroup, which uses a "Chevalley basis" over the splitting field of the torus Z, is given in [14: §2].

The group $X_K^*(Z)$ of K-rational characters of Z can be identified with a subgroup of finite index of the character group $X^*(T)$. Let $v: Z(K) \to \mathscr{V}$ be the homomorphism defined by

$$\chi(v(z)) = -\omega(\chi(z)),$$

for $z \in Z(K)$ and $\chi \in X_K^*(Z)$. Then the maximal bounded subgroup Z_0 of $Z(K)$ is precisely the kernel of v, and $\Lambda := Z(K)/Z_0$ is a free abelian group whose rank is the K-rank of G (which is equal to $\dim \mathscr{V}$). The *affine* Weyl group $N(K)/Z_0$ is an extension of the K-Weyl group $N(K)/Z(K)$ by Λ. There is an extension of v to a homomorphism of $N(K)$ in the group of affine transformations of A. This extension will be denoted again by v.

Let $b \in \Phi$, and $u \in U_b(K) - \{1\}$. It is known ([2: §5]) that the intersection $U_{-b}uU_{-b} \cap N(K)$ consists of a single element $m(u)$ whose image in the K-Weyl group is the reflection r_b associated with b. Hence $r(u) := v(m(u))$ is an affine reflection whose gradient is r_b. Let $\alpha(b, u)$ $(\in F)$ denote the affine function on A whose gradient is b and which vanishes on the hyperplane of points fixed by $r(u)$. Now for any affine function $\psi \in F$ with gradient $b \in \Phi$, let

$$U_\psi = \{u \in U_b(K) \mid u = 1 \text{ or } \alpha(b, u) \geqq \psi\}.$$

Then U_ψ is a subgroup and it is normalized by Z_0. If b is a multipliable root (i.e. $2b$ is also a root), then $U_{2\psi} \subset U_\psi$.

2.5. An affine function ψ with gradient $b \in \Phi$ is called an *affine root of* G (relative to T and K) if U_ψ is not contained in $U_{\psi+\varepsilon} \cdot U_{2b}(K)$ $(= U_{\psi+\varepsilon}$, if $2b$ is not a root) for any $\varepsilon > 0$. Let $\Psi (\subset F)$ be the set of affine-roots of G relative to T and K. As T is defined over k, there is a natural action of the Galois group Γ on Ψ. Now let x be a point of the apartment A, then the parahoric subgroup P_x is generated by Z_0 and the subgroups U_ψ, $\psi \in \Psi$ such that $\psi(x) \geqq 0$.

2.6. *Filtrations of parahoric subgroups.* Let $x \in A$. There is a natural filtration of P_x defined as follows: For any nonnegative real number r, let $P_{x,r}$ be the subgroup of P_x generated by $Z_{n\delta}^a$, $a \in \Pi$ and $n \in \mathbf{Z}$, $n \geqq r$, and the U_ψ, for $\psi \in \Psi$ such that $\psi(x) \geqq r$. It is obvious that $P_{x,0} = P_x$, and if $s \geqq r$, then $P_{x,s} \subset P_{x,r}$. For $r \geqq 0$, let $P_{x,r^+} = \bigcup_{s>r} P_{x,s}$. It follows easily from Lemmas 2.4 and 2.7 of [14] that for all $r \geqq 0$, $P_{x,r}$ is a normal subgroup of P_x. In fact, for any nonnegative real numbers r and s, the commutator subgroup $[P_{x,r}, P_{x,s}]$ is contained in $P_{x,r+s}$. This implies in particular that for all $r > 0$, $P_{x,r}/P_{x,r^+}$ is abelian; moreover as

$[P_{x,0^+}, P_{x,r}] \subset P_{x,r^+}$, the conjugation action of $P_x = P_{x,0}$ on $P_{x,r}$ induces an action of the group $P_{x,0}/P_{x,0^+}$ on $P_{x,r}/P_{x,r^+}$. If x is fixed under Γ, i.e. if it is a point of the Bruhat–Tits building of G/k, then for all $r \geqq 0$, $P_{x,r}$ is Γ-stable.

The apartment A' corresponding to a maximal K-split torus T' of G contains x if, and only if, $T'(K) \cap P_x$ is the maximal bounded subgroup of $T'(K)$ (see [15: 3.6.1]), and if this is the case, then T' (resp. A') is conjugate to T (resp. A) under an element of P_x. This at once implies that the filtration introduced above is independent of the choice of the apartment containing x.

Now let $y = g \cdot x$, $g \in G(K)$, be a point of the Bruhat–Tits building of G/K. Then $P_y = gP_xg^{-1}$. For $r \geqq 0$, we set $P_{y,r} = gP_{x,r}g^{-1}$. Then $P_{y,r}$ depends only on r and y and not on the choice of the conjugating element g. Note that if x and y are points contained in the same simplex, then $P_x = P_y$ and $P_{x,0^+} = P_{y,0^+}$; however, for $r > 0$, $P_{x,r}$ may not in general be equal to $P_{y,r}$.

For a parahoric subgroup $P = P_x$ defined over k, the group $P_x/P_{x,0^+}$ has a natural identification with the group of $\mathfrak{F}$-rational points of a connected reductive $\mathfrak{f}$-group (see 3.2 below). The *pro-nil radical* $R_n(P)$ of P is by definition the subgroup $P_{x,0^+}$. The *pro-nil radical* $R_n(\mathscr{P})$ of the parahoric subgroup $\mathscr{P} = P \cap G(k)$ of $G(k)$ is $R_n(P) \cap G(k)$.

2.7. *Standard parahoric subgroups.* We fix a chamber (i.e., a simplex of maximal dimension) C lying in A, which is stable under Γ. Then given a point y of the Bruhat–Tits building of G/K, there is a unique point x in the closure $\bar{C}$ of C ($\bar{C}$ consists of points lying in C and all its faces) and an element $g \in G(K)$ such that $y = g \cdot x$. If y is fixed under Γ, then so is x and the element g can be chosen to be k-rational. The isotropy I of C in $G(K)$ is an Iwahori subgroup defined over k. I determines a basis Δ of the affine-root system Ψ. An affine root $\psi \in \Psi$ is said to be *positive* ($\psi > 0$) if it is a nonnegative integral linear combination of roots in Δ. Equivalently, $\psi > 0$ if, and only if, $U_\psi \subset I$. An affine root ψ is said to be *negative* ($\psi < 0$) if $-\psi$ is positive. As I is defined over k, Δ is stable under the action of the Galois group Γ on the affine root system Ψ. For a subset Θ of Δ, let P_Θ be the subgroup of $G(K)$ generated by I and the $U_{-\theta}, \theta \in \Theta$. Then P_Θ is a parahoric subgroup of $G(K)$ and any parahoric subgroup (of $G(K)$) containing I equals P_Θ for a unique $\Theta \subset \Delta$. The parahoric subgroup P_Θ is defined over k if, and only if, Θ is stable under Γ. Any parahoric subgroup P of $G(K)$ is conjugate (in $G(K)$) to a unique P_Θ, moreover if P is defined over k, then the unique P_Θ to which it can be conjugated is also defined over k, and then P and P_Θ are in fact conjugate to each other under an element of $G(k)$. The parahoric subgroups P_Θ, for $\Theta \subset \Delta$, will be referred to as the standard parahoric subgroups. These are the isotropy subgroups of points lying in $\bar{C}$.

Let x be a point of $\bar{C}$ and let $\Theta = \{\theta \in \Delta \mid \theta(x) = 0\}$, then $P_x = P_\Theta$. Since $x \in \bar{C}$, if for an affine root ψ, $\psi(x) > 0$, then ψ is positive. This implies that $P_{x,r}$ is contained in the Iwahori subgroup I for all $r > 0$.

It is obvious that for a suitable point x in the simplex fixed by a parahoric subgroup P, the filtration subgroups $P_{x,r}$ of $P = P_x$, defined above, coincide with the filtration subgroups introduced in [14: 2.14].

3. The associated filtrations of the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$

Every point of the Bruhat–Tits building of G/k determines a natural filtration of the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The purpose of this section is to describe these filtrations.

3.1. If G/K is not a triality form of type ${}^6D_4$, let L be the smallest Galois extension of K over which G splits. If G/K is a triality form of type ${}^6D_4$, let L be a fixed extension of K of degree 3 contained in the Galois extension of K, of degree 6, over which G splits. There are three such extensions, all isomorphic to each other over K. We fix a uniformizing element ϖ_L of L. Let $\ell = [L:K]$. Then ϖ_L^ℓ/ϖ is a unit.

For a root $b \in \Phi$, let L_b equal K or L according as b is a long or a short root. (If all the roots of the root system Φ are of equal length, then by convention all roots are long: if Φ is nonreduced, then all divisible roots are long and all nondivisible roots are short.) Let $\varpi_b = \varpi$ if $L_b = K$ and $\varpi_b = \varpi_L$ if $L_b = L$. We shall denote the ring of integers of L_b by $\mathfrak{o}(L_b)$ and let $\ell_b = [L:L_b]$. If ψ is an affine root whose gradient is b, set $L_\psi = L_b$, $\varpi_\psi = \varpi_b$ and $\ell_\psi = \ell_b$; then for an integer n, $\psi + n\delta$ is an affine root if, and only if, n is a multiple of ℓ_ψ(cf. [14: §2]). We shall denote the affine root $\psi + \ell_\psi\delta$ by ψ^+.

If $b \in \Phi$ is a nonmultipliable root, then $U_b(K)$ is isomorphic to L_b. Also, if b is a multipliable root, then $U_b(K)/U_{2b}(K)$ is isomorphic to L_b as well. For any affine root ψ, U_ψ/U_{ψ^+} is isomorphic to $\mathfrak{o}(L_\psi)/\varpi_\psi\mathfrak{o}(L_\psi)$; see [14: §2].

3.2. Let x be a Γ-invariant point of $\bar{C}$. We shall now describe the filtration of the Lie algebra $\mathfrak{g}$ associated with x.

The Bruhat–Tits theory associates a smooth affine $\mathfrak{o}$-group scheme $\mathscr{I}$ (resp. $\mathscr{G}_x$) to I (resp. P_x) whose generic fiber $\mathscr{I} \otimes_{\mathfrak{o}} k$ (resp. $\mathscr{G}_x \otimes_{\mathfrak{o}} k$) is G and whose group of $\mathfrak{O}$-rational points is P_x (resp. I). Let $L(\mathscr{I})$ and $L(\mathscr{G}_x)$ be the Lie algebras of $\mathscr{I}$ and $\mathscr{G}_x$ respectively. Since the generic fibers of both $\mathscr{I}$ and $\mathscr{G}_x$ are equal to G, we have $L(\mathscr{I}) \otimes_{\mathfrak{o}} k = \mathfrak{g} = L(\mathscr{G}_x) \otimes_{\mathfrak{o}} k$. In particular, both $L(\mathscr{I})$ and $L(\mathscr{G}_x)$ are lattices in $\mathfrak{g}$. We denote them by $\mathfrak{i}$ and $\mathfrak{g}_x$ respectively. Set

$$\mathbf{i} = \mathfrak{i} \otimes_{\mathfrak{o}} \mathfrak{O}$$

$$\mathbf{g}_x = \mathfrak{g}_x \otimes_{\mathfrak{o}} \mathfrak{O}.$$

The inclusion of I into P_x induces an $\mathfrak{o}$-group scheme homomorphism of $\mathscr{I}$ into $\mathscr{G}_x$; we use it to identify $\mathfrak{i}$ with a $\mathfrak{o}$-Lie subalgebra of $\mathfrak{g}_x$ and $\mathbf{i}$ with a $\mathfrak{O}$-Lie subalgebra of $\mathbf{g}_x$. Note that $\mathbf{g}_x$ is itself a lattice in the Lie algebra $\mathbf{g} = \mathfrak{g} \otimes_k K$.

The special fiber $\mathscr{G}_x \otimes_{\mathfrak{o}} \mathfrak{f}$ of $\mathscr{G}_x$ is a connected algebraic group defined over the residue field $\mathfrak{f}$; it admits a Levi decomposition over $\mathfrak{f}$. Let M_x be the quotient of $\mathscr{G}_x \otimes_{\mathfrak{o}} \mathfrak{f}$ by its unipotent radical. Then $P_{x,0}/P_{x,0+}$ has a natural identification with the group of $\mathfrak{F}$-rational points of the reductive group M_x.

Let $\mathfrak{z} \subset \mathfrak{g}$ be the Lie algebra of Z, and let

$$\mathbf{z} = \mathfrak{z} \otimes_k K.$$

For any $b \in \Phi$, let $\mathbf{z}_b \subset \mathbf{z}$ be the subalgebra corresponding to the subgroup Z_b and let $\mathbf{g}_b$ be the root subspace in $\mathbf{g}$ corresponding to b.

It is known that for each $b \in \Phi$, both

$$\mathbf{i}^b := \mathbf{i} \cap \mathbf{g}_b \quad \text{and} \quad \mathbf{z}_0^b := \mathbf{i} \cap \mathbf{z}_b$$

are "canonically" isomorphic to $\mathfrak{o}(L_b)$ as $\mathfrak{O}$-modules. For any nonnegative integral multiple n of ℓ_b, denote the $\mathfrak{O}$-submodule of $\mathbf{i}^b$ (resp. $\mathbf{z}_0^b$) corresponding to the submodule $\varpi_b^{n/\ell_b}\mathfrak{o}(L_b)$ of $\mathfrak{o}(L_b)$ by $\mathbf{i}_n^b$ (resp. $\mathbf{z}_n^b$). Let ψ be an affine root, b be its gradient, and let ψ_b be the smallest positive affine root with gradient b.

Then $\psi - \psi_b = n\delta$; where n is a multiple of ℓ_b. Define the $\mathfrak{O}$-submodule

$$\mathfrak{u}_\psi = \begin{cases} \mathfrak{i}_n^b & \text{if } n \geqq 0 \\ \varpi^n \mathfrak{i}^b_{-n(\ell-1)} & \text{if } n < 0. \end{cases}$$

If n is a negative integral multiple of ℓ_b, set $\mathfrak{z}_n^b = \varpi^n \mathfrak{z}^b_{-n(\ell-1)}$. It is clear that the $\mathfrak{u}_\psi$'s and the $\mathfrak{z}_n^b$'s do not depend on the choice of the uniformizing elements ϖ of K and ϖ_L of L.

Now for any real number r, let $\mathfrak{z}_r$ be the $\mathfrak{O}$-submodule of $\mathfrak{z}$ spanned by the $\mathfrak{z}_n^a$'s, $a \in \Pi$, and $n \in \ell_a \mathbf{Z}$ with $n \geqq r$. Let $\mathfrak{g}_{x,r}$ be the $\mathfrak{O}$-submodule of $\mathfrak{g}$ spanned by $\mathfrak{z}_r$ and the $\mathfrak{u}_\psi$'s for affine roots ψ such that $\psi(x) \geqq r$. It is obvious that $\mathfrak{g}_{x,0} = \mathfrak{g}_x$, and for $r \geqq 0$, $\mathfrak{g}_{x,r}$ is an ideal in $\mathfrak{g}_x$. For any r, $\mathfrak{g}_{x,r}$ is stable under the action of the Galois group Γ and the adjoint action of P_x on $\mathfrak{g}$.

For $r \leqq s$, we have $\mathfrak{g}_{x,r} \supset \mathfrak{g}_{x,s}$. Set $\mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$. The induced action of $P_{x,0^+}$ on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ is trivial, so there is a natural action of $\mathsf{M}_x(\mathfrak{F}) = P_{x,0}/P_{x,0^+}$ on it. Note that for all r, $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+\ell}$ and so $\mathfrak{g}_{x,r^+} \supset \varpi\mathfrak{g}_{x,r}$.

For all $r > 0$, there is a natural isomorphism of $P_{x,r}/P_{x,r^+}$ with $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ which is $\Gamma \ltimes \mathsf{M}_x(\mathfrak{F})$ equivariant.

The $\mathfrak{o}$-submodule of $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}_{x,r^+}$) consisting of the elements fixed under Γ will be denoted by $\mathrm{g}_{x,r}$ (resp. g_{x,r^+}) in the sequel.

3.3. Now let y be a point of the Bruhat–Tits building of G/k which is conjugate to x and choose $g \in G(k)$ such that $y = g \cdot x$. We define

$$\mathfrak{g}_{y,r} = \operatorname{Ad} g(\mathfrak{g}_{x,r}) \quad \text{and} \quad \mathrm{g}_{y,r} = \operatorname{Ad} g(\mathrm{g}_{x,r}).$$

Then since P_x keeps $\mathfrak{g}_{x,r}$ stable, it is clear that $\mathfrak{g}_{y,r}$ and $\mathrm{g}_{y,r}$ depend on y and r and not on the choice of $g(\in G(k))$. If $y \in A$, it can be conjugated in $\bar{C}$ by an element of $N(k)$, and hence $\mathfrak{g}_{y,r}$ is the $\mathfrak{O}$-submodule of $\mathfrak{g}$ spanned by $\mathfrak{z}_r$ and the $\mathfrak{u}_\psi$'s for affine roots ψ such that $\psi(y) \geqq r$.

3.4. To every point x of the Bruhat–Tits building of G/K, we associate a monotone increasing sequence of nonnegative numbers as follows: If x is in $\bar{C}$, let $\{r_i | i \in \mathbf{N} \cup \{0\}\}$ be the set of nonnegative values assumed by the affine functions in $\Psi \cup (n\delta | n \in \mathbf{N}\}$ at x arranged in a monotone increasing sequence. Now to an arbitrary point x of the building of G/K, we associate the sequence associated to the (unique) point of $\bar{C}$ conjugate to x. Then for $r_{i-1} < s \leqq r_i$, we have $P_{x,s} = P_{x,r_i}$ and $\mathfrak{g}_{x,s} = \mathfrak{g}_{x,r_i}$. In particular, $P_{x,r_i^+} = P_{x,r_{i+1}}$ and $\mathfrak{g}_{x,r_i^+} = \mathfrak{g}_{x,r_{i+1}}$.

3.5. Filtration of the dual. Let $\mathrm{g}^* = \operatorname{Hom}_k(\mathrm{g}, k)$ and $\mathfrak{g}^* = \operatorname{Hom}_K(\mathfrak{g}, K)$. There is a natural action of the Galois group Γ on $\mathfrak{g}^*$ by semi-linear automorphisms, and g^* is the k-subspace of $\mathfrak{g}^*$ of elements fixed under Γ. In this paper we shall always view $\mathfrak{g}^*$ as a G-module under the coadjoint action of G. An element X of $\mathfrak{g}^*$ will be called *nilpotent* (resp. *semi-simple*), if in the coadjoint action, the G-orbit through X contains zero in its closure (resp. is closed) in the Zariski-topology. As confusion is unlikely, we shall denote the coadjoint action also by Ad. Let

$$\mathfrak{g}^* = \mathfrak{g}_0^* \oplus \bigoplus_{b \in \Phi} \mathfrak{g}_b^*$$

be the weight space decomposition of $\mathfrak{g}^*$ with respect to T. Here,

$$\mathfrak{g}_b^* = \{X \in \mathfrak{g}^* | \mathrm{Ad}(t)X = b(t)X \text{ for all } t \in T\}.$$

Note that for $X \in \mathfrak{g}_b^*$ and $Y \in \mathfrak{g}_c$, if $X(Y) \neq 0$, then $b = -c$. Thus, we can identify $\mathfrak{g}_0^*$ with the dual $\mathfrak{z}^* = \mathrm{Hom}(\mathfrak{z}, K)$ of $\mathfrak{z}$, and for every $b \in \Phi$, $\mathfrak{g}_b^*$ with the dual of $\mathfrak{g}_{-b}$. For an affine root ψ, with gradient b, let

$$\mathfrak{u}_\psi^* = \{X \in \mathfrak{g}_b^* | X(Y) \in \mathfrak{O} \quad \text{for all} \quad Y \in \mathfrak{u}_{-\psi}\}.$$

Let x be a Γ-invariant point of the apartment A. For $r \in \mathbf{R}$, let

$$\mathfrak{g}_{x,r+}^\bullet = \{X \in \mathfrak{g}^* | X(Y) \in \mathfrak{O} \quad \text{for all} \quad Y \in \mathfrak{g}_{x,r^+}\}.$$

For $t \in \mathbf{R}$, let

$$\mathfrak{z}_t^* = \{X \in \mathfrak{z}^* | X(Y) \in \mathfrak{O} \quad \text{for all } Y \in \mathfrak{z}_s \quad \text{and} \quad s > -t - \ell\},$$

and let $\mathfrak{g}_{x,t}^*$ be the $\mathfrak{O}$-submodule of $\mathfrak{g}^*$ spanned by $\mathfrak{z}_t^*$ and $\mathfrak{u}_\psi^*$'s for affine roots ψ such that $\psi(x) \geqq t + (\ell - \ell_\psi)$. Then

$$\mathfrak{g}_{x,r^+}^\bullet = \mathfrak{g}_{x,-r-\ell}^*,$$

and

$$(*) \qquad \mathfrak{g}_{x,-r}^* = \varpi \mathfrak{g}_{x,-r-\ell}^* = \{X \in \mathfrak{g}^* | X(Y) \in \varpi\mathfrak{O} \quad \text{for all } Y \in \mathfrak{g}_{x,r^+}\}.$$

From the above description it is clear that for all r, $\mathfrak{g}_{x,-r}^*$ is stable under the action of the Galois group Γ and also under the coadjoint action of the parahoric subgroup P_x on $\mathfrak{g}^*$. For $r \geqq s$, $\mathfrak{g}_{x,-r}^* \supset \mathfrak{g}_{x,-s}^*$. Set $\mathfrak{g}_{x,-r+}^* = \bigcup_{s<r} \mathfrak{g}_{x,-s}^*$. Then for all r, $\mathfrak{g}_{x,-r+}^* \supset \varpi \mathfrak{g}_{x,-r}^*$. The induced action of $P_{x,0^+}$ on $\mathfrak{g}_{x,-r}^*/\mathfrak{g}_{x,-r^+}^*$ is trivial and hence there is a natural action of $\mathsf{M}_x(\mathfrak{F}) = P_{x,0}/P_{x,0^+}$ on it.

The $\mathfrak{o}$-submodule of $\mathfrak{g}_{x,-r}^*$ (resp. $\mathfrak{g}_{x,-r+}^*$) consisting of the elements fixed under Γ will be denoted by $g_{x,-r}^*$ (resp. $g_{x,-r+}^*$).

3.6. If y is a point of the Bruhat–Tits building of G/k which is conjugate to x under an element g of $G(k)$, we set

$$\mathfrak{g}_{y,-r}^* = \mathrm{Ad}\, g(\mathfrak{g}_{x,-r}^*) \quad \text{and} \quad g_{y,-r}^* = \mathrm{Ad}\, g(g_{x,-r}^*).$$

Then $\mathfrak{g}_{y,-r}^*$ and $g_{y,-r}^*$ are well defined i.e., they depend only on y and r and not on the choice of the conjugating element g.

3.7. Now let x be a point of the Bruhat–Tits building of G/k and let $\{r_i\}$ be the sequence associated to it in 3.4. Then $\mathfrak{g}_{x,r_j^+} = \mathfrak{g}_{x,r_{j+1}}$ for all j. For $i \geqq 1$, the $\mathfrak{f}$-bilinear map

$$g_{x,-r_i}^*/g_{x,-r_{i-1}}^* \times g_{x,r_i}/g_{x,r_{i+1}} \to \mathfrak{f}$$
$$(X, Y) \mapsto X(Y) \bmod \varpi\mathfrak{o}$$

is a nondegenerate $\mathsf{M}_x(\mathfrak{f})$-invariant pairing. This nondegenerate pairing composed with a fixed nontrivial character of the prime field of $\mathfrak{f}$ provides an $\mathsf{M}_x(\mathfrak{f})$-equivariant isomorphism of the Pontrjagin dual of $g_{x,r_i}/g_{x,r_{i+1}}$ with $g_{x,-r_i}^*/g_{x,-r_{i-1}}^*$.

3.8. Let $\mathscr{P}_x = P_x \cap G(k)$ and for $r \geqq 0$, $\mathscr{P}_{x,r} = P_{x,r} \cap G(k)$, $\mathscr{P}_{x,r+} = P_{x,r^+} \cap G(k)$. Then for $r > 0$, the natural isomorphism of $P_{x,r}/P_{x,r^+}$ with $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ gives an isomorphism of $\mathscr{P}_{x,r}/\mathscr{P}_{x,r^+}$ onto $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ which is $\mathsf{M}_x(\mathfrak{f})$-equivariant (cf. [14: 2.24]). Thus the above also provides an identification of the Pontrjagin dual of $\mathscr{P}_{x,r_i}/\mathscr{P}_{x,r_{i+1}}$ with $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$. The character of $\mathscr{P}_{x,r_i}/\mathscr{P}_{x,r_{i+1}}$ corresponding to the coset $\mathfrak{X} = X + \mathfrak{g}^*_{x,-r_{i-1}}$ will be denoted by $\chi_{\mathfrak{X}}$. We shall say that this character is *nondegenerate* if the coset $X + \mathfrak{g}^*_{x,-r_{i-1}}$ does not contain any nilpotent elements.

4. Unstable elements in integral representations

4.1. Let P be a parahoric subgroup of $G(K)$ and T be a maximal K-split torus such that the apartment corresponding to T in the Bruhat–Tits building of G/K contains the simplex whose isotropy subgroup is P, or, equivalently, P contains the maximal bounded subgroup T_0 of $T(K)$. We assume that both P and T are defined over k and further that T contains a maximal k-split torus S of G.

Let $\mathscr{T}$ be the smooth affine $\mathfrak{o}$-group scheme with generic fiber T and which is diagonalizable over $\mathfrak{O}$. (The ring of regular function of $\mathscr{T}$ is $(\mathfrak{O}[X^*(T)])^\Gamma$; where $\mathfrak{O}[X^*(T)]$ is the group ring of the character group $X^*(T)$ with coefficients in $\mathfrak{O}$.) The group of $\mathfrak{O}$-rational points of $\mathscr{T}$ is T_0. Let $\mathscr{S}$ be the closed $\mathfrak{o}$-subgroup scheme of $\mathscr{T}$ corresponding to the subtorus S of T. Let $\mathsf{S} = \mathscr{S} \otimes_{\mathfrak{o}} \mathfrak{f}$ and $\mathsf{T} = \mathscr{T} \otimes_{\mathfrak{o}} \mathfrak{f}$. Then S is the maximal $\mathfrak{f}$-split subtorus of T. There is a canonical identification of the character group $X^*(\mathsf{S})$ (resp. $X^*(\mathsf{T})$) of S (resp. T) with the character group $X^*(S)$ (resp. $X^*(T)$) of S (resp. T), and of the group $X_*(\mathsf{S})$ (resp. $X_*(\mathsf{T})$) of 1-parameter subgroups of S (resp. T) with the group $X_*(S)$ (resp. $X_*(T)$) of 1-parameter subgroups of S (resp. T).

Let V be a free $\mathfrak{o}$-module of finite rank and $\rho: \mathscr{T} \to \mathrm{GL}(V)$ be a rational representation of $\mathscr{T}$ defined over $\mathfrak{o}$. Then ρ is completely reducible over $\mathfrak{O}$ i.e., $V \otimes_{\mathfrak{o}} \mathfrak{O}$ is a direct sum of the weight submodules (see [6: Proposition on p. 177]). ρ induces a rational representation ρ_k of $T = \mathscr{T} \otimes_{\mathfrak{o}} k$ on $V \otimes_{\mathfrak{o}} k$ and a representation $\bar{\rho}$ of $\mathsf{T} = \mathscr{T} \otimes_{\mathfrak{o}} \mathfrak{f}$ on $V \otimes_{\mathfrak{o}} \mathfrak{f}$. The weights of ρ_k and $\bar{\rho}$ are the same if we use the canonical identification of $X^*(T)$ with $X^*(\mathsf{T})$.

Now let $\mathscr{G}$ be the smooth affine $\mathfrak{o}$-group scheme associated to P by the Bruhat–Tits theory. The generic fiber $\mathscr{G} \otimes_{\mathfrak{o}} k$ of $\mathscr{G}$ is G/k, the group of its $\mathfrak{O}$-rational points is P and the special fiber $\bar{\mathscr{G}} = \mathscr{G} \otimes_{\mathfrak{o}} \mathfrak{f}$ is a connected algebraic $\mathfrak{f}$-group which admits a Levi decomposition defined over $\mathfrak{f}$. Let M be the quotient of $\bar{\mathscr{G}}$ by its unipotent radical. Then M is a reductive $\mathfrak{f}$-group. There is a natural embedding of T in M, defined over $\mathfrak{f}$, which corresponds to the inclusion of T_0 in P. In this embedding, T is a maximal torus of M and $\mathsf{S}(\subset \mathsf{T})$ is a maximal $\mathfrak{f}$-split torus.

4.2. In the rest of this section we shall use the notation introduced in §§2, 3. Thus x is a Γ-invariant element of $\bar{C}$, $P_x = P_\Theta$ is the associated parahoric subgroup, and $\mathscr{G}_x$ is the $\mathfrak{o}$-group scheme associated with the parahoric subgroup P_x. Denote by M_x the quotient of the special fiber $\mathscr{G}_x \otimes_{\mathfrak{o}} \mathfrak{f}$ (of $\mathscr{G}_x$) by its unipotent radical; M_x is a connected reductive group defined over $\mathfrak{f}$. The filtration lattices $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}^*_{x,s}$), are free $\mathfrak{o}$-modules. The adjoint (resp. coadjoint) action of $\mathscr{G}_x$ on $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}^*_{x,s}$) can be placed in the context of a rational representation $\rho: \mathscr{G}_x \to \mathrm{GL}(V)$ of $\mathscr{G}_x$ on a finite rank free $\mathfrak{o}$-module V. We also denote by ρ, the extended representation of $G = \mathscr{G}_x \otimes_{\mathfrak{o}} k$ on $V \otimes_{\mathfrak{o}} k$. In what follows, we shorten $\rho(g)X$ to either $g \cdot X$ or gX. As usual, we shall say that an element $X \in V$ is *unstable* if the Zariski closure of the orbit $G \cdot X$ contains zero. If k is of characteristic zero and $X \in V$ is an unstable element then, according to a result of Kempf [10] and Rousseau, there exists a 1-parameter subgroup $\lambda: \mathrm{GL}_1 \to G$, defined over k, such that $\mathrm{Lim}_{t \to 0} \lambda(t)X = 0$.

Nilpotent elements contained in $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}^*_{x,s}$) are unstable for the adjoint (resp. coadjoint) action of $\mathscr{G}_x$.

4.3. Proposition. *Suppose that $\rho: \mathscr{G}_x \to \mathrm{GL}(V)$ is a rational representation of $\mathscr{G}_x$ on a free finite rank $\mathfrak{o}$-module V and W is a $\mathscr{G}_x(\mathfrak{o})$-submodule containing ϖV. Assume that the induced action of $\mathscr{G}_x \otimes_{\mathfrak{o}} \mathfrak{f}$ on V/W has the unipotent radical acting trivially so that ρ gives rise to a representation of M_x on V/W. Let $X \in V$ be such that there is a 1-parameter subgroup $\lambda: \mathrm{GL}_1 \to G$, defined over K, so that*

$$\operatorname*{Lim}_{t \to 0} \lambda(t)X = 0.$$

Then the M_x-orbit through the image $\bar{X}$ of X in V/W contains zero in its closure.

Proof. Let Q be a special maximal parahoric subgroup of $G(K)$ containing the Iwahori subgroup I. Let

$$\mathfrak{P}_\lambda = \{g \in G | \operatorname*{Lim}_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\}.$$

Then $\mathfrak{P}_\lambda$ is a parabolic subgroup of G defined over K. Since by Iwasawa decomposition ([15: 3.3.2]), $G(K) = Q \cdot \mathfrak{P}_\lambda(K)$, and for any $p \in \mathfrak{P}_\lambda(K)$, $\mathrm{Lim}_{t \to 0} p\lambda(t)p^{-1}X = 0$, after replacing the 1-parameter subgroup λ by a conjugate under an element of $\mathfrak{P}_\lambda(K)$, we may, and we shall, assume that it is contained in the maximal K-split torus $T' := qTq^{-1}$; where q is an element of Q. Then $T' \subset \mathfrak{P}_\lambda$.

Let $\mathscr{Q}$ be the $\mathfrak{O}$-group scheme associated to the parahoric subgroup Q by the Bruhat–Tits theory. The generic fiber $\mathscr{Q} \otimes_{\mathfrak{O}} K$ of $\mathscr{Q}$ is G/K and the special fiber $\mathsf{Q} := \mathscr{Q} \otimes_{\mathfrak{O}} \mathfrak{F}$ is a connected algebraic $\mathfrak{F}$-group. The group of $\mathfrak{O}$-rational points of $\mathscr{Q}$ is Q and the reduction "mod $\mathfrak{p}$" map $Q \to \mathsf{Q}(\mathfrak{F})$ is known to be surjective. Let $R_u(\mathsf{Q})$ be the unipotent radical of Q and $\mathsf{M} = \mathsf{Q}/R_u(\mathsf{Q})$. Now observe that the image of $I \subset Q$ under the composite of the reduction map $Q \to \mathsf{Q}(\mathfrak{F})$ and the natural projection $\mathsf{Q}(\mathfrak{F}) \to \mathsf{M}(\mathfrak{F})$ is the group of $\mathfrak{F}$-rational points of a Borel subgroup B of M; the image of $\mathfrak{P}_\lambda(K) \cap Q$ is the group of $\mathfrak{F}$-rational points of a parabolic subgroup P (of M) and the image of $T'(K) \cap Q$ is the group of $\mathfrak{F}$-rational points of a maximal torus T' contained in P (recall that $T' \subset \mathfrak{P}_\lambda$ and as $T' = qTq^{-1}$ with q in Q, $T'(K) \cap Q$ is the maximal bounded subgroup of $T'(K)$). By the Bruhat decomposition, the intersection $\mathsf{B} \cap \mathsf{P}$ contains a maximal torus T'' ([1:14.13]). Then T'' is conjugate to T' in P. Fix a $p \in \mathfrak{P}_\lambda(K) \cap Q$ such that the image of p in $\mathsf{M}(\mathfrak{F})$ conjugates T' to T'' and let $T'' = pT'p^{-1}$. Then as $\mathsf{T}'' \subset \mathsf{B}$, the maximal bounded subgroup of $T''(K)$ is contained in I which in turn is contained in P_x. This implies that T'' embedds in a natural way in M_x as a maximal $\mathfrak{F}$-torus. Let λ'' be the conjugate of the 1-parameter subgroup λ under p. Then as p is in $\mathfrak{P}_\lambda(K)$, $\mathrm{Lim}_{t \to 0} \lambda''(t)X = 0$. On reduction λ'' gives a 1-parameter subgroup $\overline{\lambda''}$ of T'' such that $\mathrm{Lim}_{t \to 0} \overline{\lambda''}(t)\bar{X} = 0$. Hence $\mathsf{T}'' \cdot \bar{X}$, and so *a fortiori* $\mathsf{M}_x \cdot \bar{X}$, contains zero in its closure.

Notation. For an affine root φ, we let $\dot{\varphi}$ denote its gradient, then $\dot{\varphi} \in \Phi(\subset X^*(T))$. For $\lambda \in X_*(T)$, we set $\langle \varphi, \lambda \rangle = \langle \dot{\varphi}, \lambda \rangle$.

4.4. Proposition. *Let V, W, ρ and $X \in V$ be as in the preceding proposition. Then there is a $p \in \mathscr{P}_x = P_x \cap G(k)$ and a 1-parameter subgroup $\mu : \mathrm{GL}_1 \to S(\subset G)$, defined over k, such that*

(1) $\operatorname{Lim}_{t \to 0} \mu(t)(pX) \equiv 0 \pmod{W}$,

(2) *for all $\theta \in \Theta$, $\langle \theta, \mu \rangle \geqq 0$.*

Proof. Let $\bar{X}$ denote the projection of X on V/W. Then according to the preceding proposition, the closure of $\mathsf{M}_x \cdot \bar{X}$ contains zero. Now in view of a result of Kempf [10] and Rousseau, there is a 1-parameter subgroup $\lambda : \mathrm{GL}_1 \to \mathsf{M}_x$ defined over the finite field $\mathfrak{f}$ such that $\operatorname{Lim}_{t \to 0} \lambda(t)\bar{X} = 0$. Let T be the maximal $\mathfrak{f}$-torus of M_x corresponding to T; it contains a maximal $\mathfrak{f}$-split torus S of M_x. Now we recall that the set consisting of the restrictions to S of the gradients of the affine roots belonging to Θ is a basis of the root system of M_x with respect to S ($X^*(\mathsf{S})$ identified with $X^*(S)$ here), cf. [15: 3.5.1]. By conjugacy of maximal $\mathfrak{f}$-split tori under $\mathsf{M}_x(\mathfrak{f})$ and the fact that the $\mathfrak{f}$-Weyl group of M_x acts transitively on the set of Weyl chambers in $X_*(\mathsf{S})$, we conclude now that there is a conjugate λ' of λ, under an element of $\mathsf{M}_x(\mathfrak{f})$, such that λ' is contained in $X_*(\mathsf{S})(\subset X_*(\mathsf{M}_x))$ and $\langle \theta, \lambda' \rangle \geqq 0$ for all $\theta \in \Theta$. Let μ be the lift of λ' to S. Then μ has the desired properties. To determine $p \in \mathscr{P}_x$, use the fact that the natural map $\mathscr{P}_x \to \mathsf{M}_x(\mathfrak{f})$ is surjective.

5. Unrefined minimal K-types and the main theorem

Given a point x of the Bruhat–Tits building of G/k, let P_x be the associated parahoric subgroup. Let $\mathscr{P}_x = P_x \cap G(k)$ and for $r \geqq 0$, let $\mathscr{P}_{x,r} = P_{x,r} \cap G(k)$. Let the sequence $\{r_i\}$ be as in 3.4. For $i \geqq 1$, realize the characters of $\mathscr{P}_{x,r_i}/\mathscr{P}_{x,r_{i+1}}$ as in 3.8.

5.1. Definition. An *unrefined minimal* K-*type* is a pair $(\mathscr{P}_{x,r}, \chi)$, where x is a point of the Bruhat–Tits building of G/k, r is a nonnegative real number, χ is a representation of $\mathscr{P}_{x,r}$ trivial on $\mathscr{P}_{x,r+}$ and

i) if $r = 0$, then χ is a cuspidal representation of $\mathscr{P}_{x,0}/\mathscr{P}_{x,0^+}$ inflated to $\mathscr{P}_x\ (= \mathscr{P}_{x,0})$

ii) if $r > 0$, then $\mathscr{P}_{x,r} \neq \mathscr{P}_{x,r^+}$ and χ is a nondegenerate character of $\mathscr{P}_{x,r}/\mathscr{P}_{x,r^+}$.

In the remainder of the paper, we shall drop the adjective "unrefined". The nonnegative number r is called the *depth* of the minimal K-type. Let y be another point of the Bruhat–Tits building of G/k, and let $\{s_j\}$ be the monotone increasing sequence of nonnegative real numbers associated to it as in 3.4. Then two minimal K-types $(\mathscr{P}_{x,r_i}, \chi)$ and $(\mathscr{P}_{y,s_j}, \xi)$, are said to be *associates* if they have the same depth (i.e. $r_i = s_j$), and

i) in the case of zero depth, there is an element $g \in G(k)$ so that $\mathscr{P}_x \cap \mathscr{P}_{gy}$ surjects onto both $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$ and χ is isomorphic to $\mathrm{Ad}(g)\xi$.

ii) in the case of positive depth, the $G(k)$ orbit of the coset $X + \mathfrak{g}^*_{x,-r_{i-1}}$, which realizes χ, intersects the coset $Y + \mathfrak{g}^*_{y,-s_{j-1}}$, which realizes ξ.

5.2. Theorem. *Assume that k is of characteristic zero. Given an irreducible admissible complex representation (π, V_π) of $G(k)$, there is a nonnegative rational number $\varrho(\pi)$ with the following properties.*

(1) *For some x in the Bruhat–Tits building of G/k, the space* $V_\pi^{\mathscr{P}_{x,\varrho(\pi)^+}}$ *of* $\mathscr{P}_{x,\varrho(\pi)^+}$*-fixed vectors is nonzero and* $\varrho(\pi)$ *is the smallest number with this property.*

(2) *For any y in the Bruhat–Tits building, if* $W = V_\pi^{\mathscr{P}_{y,\varrho(\pi)^+}} \neq \{0\}$*, then*

i) *if* $\varrho(\pi) = 0$*, any irreducible* $\mathscr{P}_{y,\varrho(\pi)}$*-submodule of W contains a minimal* K*-type of depth zero of a parahoric* $\mathscr{Q} \subset \mathscr{P}_y$*;*

ii) *if* $\varrho(\pi) > 0$*, any irreducible* $\mathscr{P}_{y,\varrho(\pi)}$*-submodule of W is a minimal* K*-type*

Moreover, any two minimal K*-types contained in* π *are associates of each other.*

6. Three key propositions

6.1. Optimal points. Let

$$\Sigma = \{\psi \in \Psi | \psi > 0 \quad \text{and} \quad \psi - \ell\delta < 0\}.$$

Then Σ is a finite set. For any nonempty subset $\mathfrak{S}$ of Σ, let $f_{\mathfrak{S}}$ be the real valued function on $\bar{C}$ defined as follows:

$$f_{\mathfrak{S}}(x) = \min\{\psi(x) - (\ell - \ell_\psi) | \psi \in \mathfrak{S}\} \quad \text{for } x \in \bar{C}.$$

Now for each nonempty Γ-stable subset $\mathfrak{S}$ of Σ, we fix an element $x_{\mathfrak{S}}$ of $\bar{C}$ such that

i) the function $f_{\mathfrak{S}}$ takes its maximum value at $x_{\mathfrak{S}}$,

ii) $\psi(x_{\mathfrak{S}})$ is a rational number for all affine roots ψ,

iii) $x_{\mathfrak{S}}$ is fixed under Γ.

Note that the set of points where $f_{\mathfrak{S}}$ takes its maximum value is the intersection of $\bar{C}$ with a Γ-stable hyperplane of A which is defined over **Q**. Hence there exist elements in $\bar{C}$ satisfying the above conditions. We shall call $x_{\mathfrak{S}}$ an *optimal point* for the subset $\mathfrak{S}$ of Σ. Let $\mathcal{O}$ be the (finite) set consisting of the optimal points $x_{\mathfrak{S}}$'s, for all Γ-stable nonempty subsets $\mathfrak{S}$ of Σ.

Finding optimal points explicitly is a problem of linear programming. For the basic results and techniques of this theory see [13]. That there exists a point $x_{\mathfrak{S}}$ such that the function $f_{\mathfrak{S}}$ takes its maximum value at $x_{\mathfrak{S}}$, and for all affine roots ψ, $\psi(x_{\mathfrak{S}})$ is rational, also follows from an observation on page 33 of [13].

Remark. It can be shown that if $G = \mathrm{SL}_n$, then given any nonempty subset $\mathfrak{S}$ of Σ, the barycenter of a suitable face of C is an optimal point for $\mathfrak{S}$.

6.2. We say that a subset Ξ of the affine root system Ψ is *bounded below* if there exists an integer n such that for all $\psi \in \Xi$, $\psi + \ell n\delta > 0$. Now let Ξ be a nonempty Γ-stable subset of Ψ which is bounded below and let n be the smallest integer such that every root in the set $\Xi^+ := \{\psi + \ell n\delta | \psi \in \Xi\}$ is positive. To Ξ we associate the subset $\mathfrak{S}(\Xi) := \Sigma \cap \Xi^+$ of Σ, and set $x_\Xi = x_{\mathfrak{S}(\Xi)}$; where Σ, and for a Γ-stable nonempty subset $\mathfrak{S}$ of Σ, the optimal point $x_{\mathfrak{S}} (\in \bar{C})$, are as in 6.1.

6.3. Proposition. *Assume that k is of characteristic zero. Let x be a* Γ*-invariant point of* $\bar{C}$*. Let r be a real number such that* $\mathfrak{g}^*_{x,r}$ *properly contains* $\mathfrak{g}^*_{x,r+}$ *and let* $X \in \mathfrak{g}^*_{x,r}$ *be a nilpotent element. Then there is a* $p \in \mathscr{P}_x = P_x \cap G(k)$ *and a* $y \in \mathcal{O}$ *such that for some* $s > r$

$$pX + \mathfrak{g}^*_{x,r+} \subset \mathfrak{g}^*_{y,s},$$

or, equivalently, $X + \mathfrak{g}^*_{x,r^+} \subset \mathfrak{g}^*_{p^{-1}y,s}$.

Proof. From Proposition 4.4 applied to the case where $V = \mathfrak{g}^*_{x,r}$ and $W = \mathfrak{g}^*_{x,r^+}$, we conclude that there is a $p \in \mathscr{P}_x$ and a 1-parameter subgroup $\mu: GL_1 \to S$ defined over k such that $\mathrm{Lim}_{t \to 0} \mu(t) pX \equiv 0 \pmod{\mathfrak{g}^*_{x,r^+}}$ and $\langle \theta, \mu \rangle \geqq 0$ for all $\theta \in \Theta$. Denote pX by Y. Then $Y \equiv Y_0 + \Sigma Y_\psi \pmod{\mathfrak{g}^*_{x,r^+}}$; where $Y_0 \in \mathfrak{z}^*_r$, $Y_\psi \in \mathfrak{u}^*_\psi$, and the summation is over the set of affine roots ψ such that $\psi(x) = r + (\ell - \ell_\psi)$. Now let $\mathfrak{S}$ be the set of affine roots ψ such that $\psi(x) = r + (\ell - \ell_\psi)$ and $Y_\psi \not\equiv 0 \pmod{\mathfrak{g}^*_{x,r^+}}$. As X is k-rational, $\mathfrak{S}$ is Γ-stable.

Since μ is a 1-parameter subgroup contained in S, the coadjoint action of μ on $\mathfrak{z}^*$ is trivial. From this and the fact that $\mathrm{Lim}_{t \to 0} \mu(t) Y \equiv 0 \pmod{\mathfrak{g}^*_{x,r^+}}$, we conclude that

(1) $Y_0 \equiv 0 \pmod{\mathfrak{g}^*_{x,r^+}}$,

(2) $\langle \psi, \mu \rangle > 0$ for $\psi \in \mathfrak{S}$.

Now, for $\varepsilon \geqq 0$, consider the element $x + \varepsilon\mu \in A$. As x is Γ-invariant and μ is defined over k, the point $x + \varepsilon\mu$ is Γ-invariant for all ε. Recall that for any affine root θ, $\theta(x + \varepsilon\mu) = \theta(x) + \varepsilon\langle \theta, \mu \rangle$. Now,

(1) if $\theta \in \Theta$, then $\theta(x) = 0$ and $\langle \theta, \mu \rangle \geqq 0$ so that $\theta(x + \varepsilon\mu) \geqq 0$ for all $\varepsilon \geqq 0$,

(2) if $\theta \in \Delta - \Theta$, then $\theta(x) > 0$ and hence for all sufficiently small $\varepsilon \geqq 0$, we have $\theta(x + \varepsilon\mu) \geqq 0$.

Therefore, for all sufficiently small $\varepsilon \geqq 0$, $x + \varepsilon\mu \in \bar{C}$.

For $\psi \in \mathfrak{S}$ and $\varepsilon > 0$ we have $\psi(x + \varepsilon\mu) = \psi(x) + \varepsilon\langle \psi, \mu \rangle > \psi(x)$ $(= r + (\ell - \ell_\psi))$ since $\langle \psi, \mu \rangle > 0$. Also, for the affine roots ψ such that $\psi(x) > r + (\ell - \ell_\psi)$, it is clear that $\psi(x + \varepsilon\mu) > r + (\ell - \ell_\psi)$ for all sufficiently small positive ε. Hence, if $z = x + \varepsilon\mu$, where ε is a sufficiently small positive number, we can find a real number $u > r$ such that $\mathfrak{g}^*_{z,u} \supset Y + \mathfrak{g}^*_{x,r^+}$.

Let $\Xi = \mathfrak{S} \cup \{\psi | \mathfrak{u}^*_\psi \subset \mathfrak{g}^*_{x,r^+}\}$. Then Ξ is clearly Γ-stable and bounded below. Take $y = x_\Xi$; where x_Ξ is in 6.2. Then as y is an optimal point for the subset Ξ, there exists a s, $s \geqq u > r$ such that $\mathfrak{g}^*_{y,s} \supset Y + \mathfrak{g}^*_{x,r^+}$. This proves the proposition.

The following is a converse to the above proposition.

6.4. Proposition. *Let x be an element of the Bruhat–Tits building of G/k, and r be a real number such that $\mathfrak{g}^*_{x,r}$ contains $\mathfrak{g}^*_{x,r^+}$ properly. Suppose $X \in \mathfrak{g}^*_{x,r}$ is such that the coset $X + \mathfrak{g}^*_{x,r^+}$ does not contain any nilpotent elements. Then for all y in the building, $\mathfrak{g}^*_{y,s} \cap (X + \mathfrak{g}^*_{x,r^+}) = \varnothing$ for $s > r$.*

Proof. We argue by contradiction. Assume there is a y in the building and a real number $s > r$ such that $\mathfrak{g}^*_{y,s} \cap (X + \mathfrak{g}^*_{x,r^+}) \neq \varnothing$. After replacing x, y and X by their conjugates under an element of $G(k)$, we assume that both x and y lie on A. Replacing X by $X + Z$ for suitable $Z \in \mathfrak{g}^*_{x,r^+}$, we further assume that $X \in \mathfrak{g}^*_{y,s}$. Let $X = X_0 + \sum X_\psi$; where $X_0 \in \mathfrak{z}^*_s$, $X_\psi \in \mathfrak{u}^*_\psi$ and the summation is over a (finite) set of affine roots ψ with distinct gradients and such that $\psi(x) \geqq r + (\ell - \ell_\psi)$. Let $\mathfrak{S}$ be the set of affine roots ψ such that $\psi(x) = r + (\ell - \ell_\psi)$ and $X_\psi \not\equiv 0 \pmod{\mathfrak{g}^*_{x,r^+}}$. Then, as $\mathfrak{g}^*_{y,s} \cap \mathfrak{z}^* = \mathfrak{z}^*_s = \mathfrak{g}^*_{x,s} \cap \mathfrak{z}^*$, and $s > r$, we conclude that $X_0 \in \mathfrak{g}^*_{x,r^+}$. Hence $X \equiv \sum_{\psi \in \mathfrak{S}} X_\psi \pmod{\mathfrak{g}^*_{x,r^+}}$. Since X, and so also X_ψ, is in $\mathfrak{g}^*_{y,s}$, $\psi(y) \geqq s + (\ell - \ell_\psi) > r + (\ell - \ell_\psi)$ for every affine root ψ such that $X_\psi \neq 0$. Now as the set of elements in A of the form $x + u\lambda$, with $u \in \mathbf{R}$ and $\lambda \in X_*(T)$, is dense, we can find an $u > 0$ and a 1-parameter subgroup λ defined over k such that $\psi(x + u\lambda) > r + (\ell - \ell_\psi)$ for all $\psi \in \mathfrak{S}$. But as $\psi(x + u\lambda) = \psi(x) + u\langle \psi, \lambda \rangle = r + (\ell - \ell_\psi) + u\langle \psi, \lambda \rangle$, it follows that $\langle \psi, \lambda \rangle > 0$ for all $\psi \in \mathfrak{S}$. This implies that $\mathrm{Lim}_{t \to 0} \lambda(t) \sum_{\psi \in \mathfrak{S}} X_\psi = 0$. Thus $\sum_{\psi \in \mathfrak{S}} X_\psi$ is nilpotent and we have shown that X is congruent to a nilpotent element modulo $\mathfrak{g}^*_{x,r^+}$. This contradiction completes the proof of the proposition.

6.5. Proposition. *Let x and y be two points of the Bruhat–Tits building of G/k and let r and s be real numbers so that $\mathfrak{g}_{x,r} \supset \mathfrak{g}_{y,s}$. Then $P_{x,r} \supset P_{y,s}$ and hence, $\mathscr{P}_{x,r} \supset \mathscr{P}_{y,s}$.*

Proof. After replacing x and y by their conjugates under a suitable element of $G(k)$, we assume that both x and y lie in the apartment A. Recall that $P_{x,r}$ (resp. $P_{y,s}$) is generated by the subgroups U_ψ, $\psi \in \Psi$ such that $\psi(x) \geqq r$, and the subgroups $Z^a_{n\delta}$, $a \in \Pi$, $n \geqq r$ (resp. by the subgroups U_ψ, $\psi \in \Psi$ such that $\psi(y) \geqq s$, and the subgroups $Z^a_{n\delta}$, $a \in \Pi$, $n \geqq s$). As $\mathfrak{g}_{x,r}$ contains $\mathfrak{g}_{y,s}$, $\mathbf{g}_{x,r}$ contains $\mathbf{g}_{y,s}$. Now since $\mathbf{u}_\psi \subset \mathbf{g}_{x,r}$ (resp. $\mathbf{u}_\psi \subset \mathbf{g}_{y,s}$) if, and only if, $\psi(x) \geqq r$ (resp. $\psi(x) \geqq s$) and $\mathfrak{z}^a_n \subset \mathbf{g}_{x,r}$ (resp. $\mathfrak{z}^a_n \subset \mathbf{g}_{y,s}$) if, and only if, $n \geqq r$ (resp. $n \geqq s$), the assertion of the proposition is obvious.

7. Proof of Theorem 5.2

In this section the local field k is assumed to be of characteristic zero.

Suppose (π, V_π) is an irreducible admissible complex representation of $G(k)$ and $\mathcal{O}$ be as in 6.1.

Given a point x of the Bruhat–Tits building of G/k, we shall let $\{r_i\}$ with $r_0 = 0$, denote the monotone increasing sequence associated to it in 3.4. If y is another point of the building, we shall denote the monotone increasing sequence associated to it by $\{s_j\}$ below.

7.1. Existence of minimal K*-types.* We claim (π, V_π) contains a minimal K-type. Let r be the smallest nonnegative rational number such that there is a point $x \in \mathcal{O}$ so that the subspace W of elements of V_π fixed under $\mathscr{P}_{x,r^+}$ is nontrivial. (The existence of r is assured since $\mathcal{O}$ is finite.) Then $r = r_i$ for some i. As $\mathscr{P}_{x,r^+}$ is a normal subgroup of $\mathscr{P}_{x,r}$, there is an induced representation of $\mathscr{P}_{x,r}/\mathscr{P}_{x,r^+}$ on W. If $r = 0$, then $\mathscr{P}_{x,r}/\mathscr{P}_{x,r^+} = \mathsf{M}_x(\mathfrak{f})$. By Harish-Chandra's theory of Eisenstein series for reductive groups over finite fields, [7: Vol IV], there is a parahoric subgroup $\mathscr{Q} \subset \mathscr{P}_x$ and a cuspidal representation χ of $\mathscr{Q}/R_u(\mathscr{Q})$ whose inflation to $\mathscr{Q}$ is contained in $W_{|\mathscr{Q}}$. In particular, π contains a minimal K-type of depth zero in this case. Therefore, we can assume that $i > 0$ and thus the group $\mathscr{P}_{x,r_i}/\mathscr{P}_{x,r_{i+1}}$ is abelian. Realize its characters as the cosets of $\mathfrak{g}^*_{x,-r_{i-1}}$ in $\mathfrak{g}^*_{x,-r_i}$ (see 3.8). Let $\chi = \chi_{X+\mathfrak{g}^*_{x,-r_{i-1}}}$ be any character of $\mathscr{P}_{x,r_i}$ which occurs in the decomposition of W into irreducible $\mathscr{P}_{x,r_i}$-submodules. We claim that $X + \mathfrak{g}^*_{x,-r_{i-1}}$ contains no nilpotent elements and therefore χ is a minimal K-type. To prove the claim suppose $X + \mathfrak{g}^*_{x,-r_{i-1}}$ contains a nilpotent element. Then we may assume that X itself is nilpotent. According to Proposition 6.3, there is an optimal point $y \in \mathcal{O}$, $p \in \mathscr{P}_x$, and a $j \geqq 0$ such that $-s_j > -r_i$, i.e. $s_j < r_i$, and $X + \mathfrak{g}^*_{x,-r_{i-1}} \subset \mathfrak{g}^*_{p^{-1}y,-s_j}$. This implies that if $z = p^{-1}y$, $\mathfrak{g}^*_{x,-r_{i-1}} \subset \mathfrak{g}^*_{z,-s_j}$, from which we conclude by taking duals (see 3.5(*)) that $\mathfrak{g}_{x,r_i} \supset \mathfrak{g}_{z,s_{j+1}}$. From Proposition 6.5 we infer now that

$$\mathscr{P}_{x,r_i} \supset \mathscr{P}_{z,s_{j+1}}.$$

For $Y \in \mathfrak{g}_{z,s_{j+1}}$, we have $X(Y) \in \varpi\mathfrak{o}$ since $X \in \mathfrak{g}^*_{z,-s_j}$ (3.5(*)). So the restriction of χ to $\mathscr{P}_{z,s_{j+1}}$ is trivial. Hence $V_\pi^{\mathscr{P}_{z,s_{j+1}}}$, and therefore $V_\pi^{\mathscr{P}_{y,s_{j+1}}}$ also, is nontrivial. Now note that $\mathscr{P}_{y,s_{j+1}} = \mathscr{P}_{y,s_j^+}$ and as $s_j < r_i = r$, this contradicts the minimality of r. Hence any irreducible $\mathscr{P}_{x,r}$-submodule in W is a nondegenerate representation. This completes the proof of existence of a minimal K-type.

7.2. Associativity of minimal K*-types.* Suppose x and y are two points of the Bruhat–Tits building of G/k. Let χ be the representation of the group $\mathscr{P}_{x,r_i}$ on an irreducible $\mathscr{P}_{x,r_i}$-submodule V_χ of V_π. We assume that $(\mathscr{P}_{x,r_i}, \chi)$ is a minimal K-type. Let ξ be the representation of $\mathscr{P}_{y,s_j}$ on an irreducible $\mathscr{P}_{y,s_j}$-submodule V_ξ of V_π which is fixed pointwise by $\mathscr{P}_{y,s_{j+1}}$. Let E_ξ be an $\mathscr{P}_{y,s_j}$-equivariant projection of V_π onto V_ξ. Since V_π is irreducible, there is a $g \in G(k)$ so that

$$\varphi = E_\xi \pi(g^{-1})\colon V_\chi \to V_\xi$$

is nonzero. For $h \in \mathscr{P}_{x,r_i} \cap g\mathscr{P}_{y,s_j}g^{-1}$ we have

$$(*) \quad \varphi \circ \chi(h) = \xi(g^{-1}hg) \circ \varphi.$$

We shall consider now three cases according to whether r_i and s_j are both greater than zero, $r_i > 0$ and $s_j = 0$, or $r_i = s_j = 0$.

Case 1. Both $r_i, s_j > 0$. Let $X + \mathfrak{g}^*_{x,-r_{i-1}}$ and $Y + \mathfrak{g}^*_{y,-s_{j-1}}$ be the cosets which give the characters χ and ξ respectively. As χ is nondegenerate, $X + \mathfrak{g}^*_{x,-r_{i-1}}$ does not contain any nilpotent elements. By the intertwining principle $(*)$, the two characters $h \mapsto \chi(h)$ and $h \mapsto \xi(g^{-1}hg)$ agree on $\mathscr{P}_{x,r_i} \cap g\mathscr{P}_{y,s_j}g^{-1}$. This implies $(X - \mathrm{Ad}(g)Y)(Z) \in \varpi\mathfrak{o}$ for all $Z \in \mathfrak{g}_{x,r_i} \cap \mathrm{Ad}(g)\mathfrak{g}_{y,s_j}$, and so $X - \mathrm{Ad}(g)Y$ lies in $\mathfrak{g}^*_{x,-r_{i-1}} + \mathrm{Ad}(g)\mathfrak{g}^*_{y,-s_{j-1}}$. Hence, $X + \mathfrak{g}^*_{x,-r_{i-1}}$ and $\mathrm{Ad}(g)(Y + \mathfrak{g}^*_{y,-s_{j-1}})$ $(\subset \mathrm{Ad}(g)(\mathfrak{g}^*_{y,-s_j}) = \mathfrak{g}^*_{gy,-s_j})$ intersect. By Proposition 6.4, then $-s_j \leqq -r_i$, i.e. $s_j \geqq r_i$. In particular, if $(\mathscr{P}_{y,s_j}, \xi)$ is another minimal K-type in π, it follows that χ and ξ are associates.

Case 2. $r_i > 0, s_j = 0$. The representation ξ of $\mathscr{P}_y$ is the inflation of a representation of $\mathscr{P}_{y,0}/\mathscr{P}_{y,0^+}$. In particular, this means the trivial representation of $\mathscr{P}_{y,0^+}$, which corresponds to the coset $\mathfrak{g}^*_{y,0}$, occurs in π. By the same reasoning as in case 1, the two cosets $\mathfrak{g}^*_{gy,0}$ and $X + \mathfrak{g}^*_{x,-r_{i-1}}$ must intersect. However, $X + \mathfrak{g}^*_{x,-r_{i-1}}$ contains no nilpotent elements. This contradicts Proposition 6.4. Hence this case can not occur.

In the above argument, interchanging the roles of x and y, we conclude that if ξ is a minimal K-type and $r_i = 0$, then $s_j = 0$.

Case 3. $r_i = 0, s_j = 0$. In this case the only assertion which requires a proof is that when ξ is also a minimal K-type, then it is an associate of χ. So we assume ξ is also a minimal K-type. Then, χ (resp. ξ) is the inflation to $\mathscr{P}_{x,0}$(resp. $\mathscr{P}_{y,0}$) of a cuspidal representation of $\mathsf{M}_x(\mathfrak{f}) = \mathscr{P}_{x,0}/\mathscr{P}_{x,0^+}$ (resp. $\mathsf{M}_y(\mathfrak{f}) = \mathscr{P}_{y,0}/\mathscr{P}_{y,0^+}$). Observe that the image of $\mathscr{P}_x \cap \mathscr{P}_{gy}$ in $\mathsf{M}_x(\mathfrak{f})$ (resp. in $\mathsf{M}_{gy}(\mathfrak{f})$) is the group of $\mathfrak{f}$-rational points of a parabolic $\mathfrak{f}$-subgroup P_x (resp. P_{gy}) of M_x (resp. M_{gy}). If $\mathsf{P}_x = \mathsf{M}_x$ and $\mathsf{P}_{gy} = \mathsf{M}_{gy}$, then φ is an isomorphism of χ to $\mathrm{Ad}(g)\xi$. Thus, χ and ξ are associates. Suppose $\mathsf{P}_x \subsetneqq \mathsf{M}_x$. Using the fact that there is an apartment of the Bruhat–Tits building of G/K containing both x and gy and the description of parahoric subgroups given in 2.5, it is easy to check that the image of $\mathscr{P}_x \cap R_n(\mathscr{P}_{gy})$ in $\mathsf{M}_x(\mathfrak{f})$ contains the unipotent radical of $\mathsf{P}_x(\mathfrak{f})$. Let $\mathscr{U}$ be the inverse image in $\mathscr{P}_x \cap R_n(\mathscr{P}_{gy})$ of the unipotent radical of $\mathsf{P}_x(\mathfrak{f})$. As $\mathscr{U}$ is contained in $R_n(\mathscr{P}_{gy})$, the restriction of $\mathrm{Ad}(g)\xi$ to $\mathscr{U}$ is trivial. On the other hand, since χ is assumed to be cuspidal, the restriction of χ to $\mathscr{U}$ cannot contain the trivial representation of $\mathscr{U}$. This is a contradiction; therefore $\mathsf{P}_x = \mathsf{M}_x$. A similar argument shows that $\mathsf{P}_{gy} = \mathsf{M}_{gy}$.

7.3. Minimality of $\varrho(\pi)$. With the notation as in 7.1, we claim $\varrho(\pi) := r$ is the smallest nonnegative number such that there is a point y in the building with

$V_\pi^{\mathscr{P}_{y,\varrho(\pi)+}} \neq \{0\}$. By 7.1, $V_\pi^{\mathscr{P}_{x,\varrho(\pi)+}} \neq \{0\}$. Suppose by way of contradiction that there is a nonnegative real number $s < r$ and a point y in the building so that $V_\pi^{\mathscr{P}_{y,s+}} \neq \{0\}$. By case 2 in 7.2, $s > 0$. Let ξ be any character of $\mathscr{P}_{y,s}/\mathscr{P}_{y,s+}$ occurring in $V_\pi^{\mathscr{P}_{y,s+}}$. By case 1 in 7.2, $s \geqq r$, a contradiction.

7.4. It remains to show that if $\varrho(\pi) > 0$, and y is a point in the building of G/k with $V_\pi^{\mathscr{P}_{y,\varrho(\pi)+}} \neq \{0\}$, then any irreducible $\mathscr{P}_{y,\varrho(\pi)}$-constituent of $V_\pi^{\mathscr{P}_{y,\varrho(\pi)+}}$ is a minimal K-type. Let j be such that $s_j = \varrho(\pi)$ and let $Y + \mathfrak{g}^*_{y,-s_{j-1}}$ be the coset which represents such a constituent. If $Y + \mathfrak{g}^*_{y,-s_{j-1}}$ contains a nilpotent element, the argument of 7.1 shows that $\varrho(\pi)$ is not minimal, a contradiction. Thus the constituent must be a minimal K-type. This completes the proof of Theorem 5.2.

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