

On Tight Immersions of Maximal Codimension

JOHN A. LITTLE (Ann Arbor)* and WILLIAM F. POHL (Minneapolis)**

1. Introduction

Let $f: M^n \rightarrow E^m$ be an immersion of a compact differentiable manifold of dimension n into a Euclidean space of dimension m . The immersion f is called *tight* if there exists no immersion of M^n with smaller total Lipschitz-Killing curvature [3] in any Euclidean space. The immersion f is called *substantial* if $f(M^n)$ is not contained in any hyperplane of E^m . Kuiper [4] has shown that if f is both tight and substantial, then $m \leq N = \frac{1}{2}n(n+3)$. In case $m=N$ and $n=2$ (so that $N=5$), he has shown that if f is tight and substantial then M^2 must be diffeomorphic to the real projective plane and f must be an embedding onto a real algebraic variety, in fact onto a Veronese surface. In this paper we prove the corresponding result in higher dimensions.

Our hypothesis is, in fact, weaker. The immersion f is said to have the *two-piece property* if every hyperplane divides it into at most two pieces, or more exactly, if for every hyperplane $H \subset E^m$, $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are both connected sets, where H_1 and H_2 are the two open half-spaces which make up the complement of H in E^m . A tight immersion has the two-piece property, but not necessarily conversely (cf. [6, 9]). However, for the case of curves and surfaces the two properties are equivalent.

Let A be a real vector space of dimension $n+1$ and consider the map $v \mapsto v \otimes v$ from A to $A \otimes A$. Take a metric in A and restrict the map to the unit sphere centered at the origin. Since $(-v) \otimes (-v) = v \otimes v$, this map takes each pair of antipodal points to the same point. Hence it induces a map of the real projective n -space into $A \otimes A$. As we shall see, this last map is an embedding, and the image V lies substantially in an affine subspace of dimension $N = \frac{1}{2}n(n+3)$. We call any submanifold projectively equivalent to V and lying in an affine or projective space a *Veronese n -manifold*. Any Veronese manifold is tightly embedded [4]. We can now state our main result.

* This author's work was done during tenure of a Leverhulme Fellowship at the University of Liverpool.

** This author's work was supported by the N.S.F. under Grant GP-20871, and by the Netherlands Organization for the Advancement of Pure Research (Z. W.O.).

Theorem I. *Let M^n be compact, $n > 1$, and $f: M^n \rightarrow E^N$, $N = \frac{1}{2}n(n+3)$, an immersion which is differentiable of class C^4 , substantial, and which has the two-piece property. Then M^n is diffeomorphic to a real projective n -space, f is an embedding, and $f(M^n)$ is a Veronese n -manifold.*

Our proof of Theorem I depends on our main local result, Theorem II, the proof of which depends in turn on Theorem III. These last two theorems, which we will state in due course, generalize theorems of Segre characterizing the Veronese surface [10, 11]. A number of other characterizations of Veronese manifolds follow from Theorems II and III. For further information we refer the reader to [2] and [8].

We have tried to make the present paper essentially self-contained. For further information on the rich literature on tight immersions, most of it the work of Kuiper, we refer to [5] and [6]. The importance of the two-piece property seems first to have been realized by Banchoff [1].

The results of this paper were discovered and proved by the first-named author, who circulated a preprint "On smooth convex maps in the top dimension". Later, under the stimulation of Kuiper, both authors working together discovered the more geometric proof of Theorem II presented here. We now know a number of different proofs of these theorems, which we discuss in § 7.

2. The Veronese Manifold and the Curvature Indicatrix

Let A be as in the last § and let $g: A \rightarrow A \otimes A$ be the mapping $g(v) = v \otimes v$. Let g' denote the restriction of g to the unit sphere centered at the origin, S^n . The image of the map g , and therefore the Veronese manifold V defined by g , lies in the space of symmetric tensors in $A \otimes A$. If we take an orthonormal basis of A , e_1, \dots, e_{n+1} , and let

$$e_i \circ e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i),$$

we can write the map g as

$$g\left(\sum x_i e_i\right) = \sum_{i=1}^{n+1} x_i^2 e_i \circ e_i + 2 \sum_{1 \leq j < k \leq n+1} x_j x_k e_j \circ e_k, \tag{2.1}$$

so that by taking the $e_j \circ e_k$ as a basis for the space of symmetric tensors, we can write down the parametrization of the Veronese manifold as

$$(x_1, \dots, x_{n+1}) \rightarrow (x_1^2, \dots, x_{n+1}^2, 2x_1 x_2, \dots, 2x_1 x_{n+1}, 2x_2 x_3, \dots, 2x_n x_{n+1}), \tag{2.2}$$

where the x_i 's are subject to the relation

$$x_1^2 + \dots + x_{n+1}^2 = 1, \tag{2.3}$$

showing that V lies in a hyperplane in the space of symmetric tensors. It is useful to have a parametrization of a Veronese manifold in homogeneous coordinates. In fact, regarding x_1, \dots, x_{n+1} as homogeneous coordinates in the real projective n -space P^n , we get such a parametrization from (2.2), in terms of the homogeneous coordinates of P^N , $N = \frac{1}{2}n(n+3)$, where now we ignore (2.3). (Note that N is one less than the dimension of the space of symmetric tensors.) Call this map $g'' : P^n \rightarrow P^N$.

This homogeneous representation has several easy consequences. First, since the quadratic monomials in (2.2) satisfy no linear relation, the Veronese manifold lies substantially in P^N . Secondly, the Veronese manifold must have the two-piece property. For, the intersection of this Veronese manifold with a hyperplane in P^N is, by (2.2), a hyperquadric of P^n . But the complement of a hyperquadric in P^n has at most two connected components, as may be seen in the following way. Given three points in the complement, pass a plane through them. This plane meets the hyperquadric in a conic. Two of these points may certainly be joined by a curve in this plane which does not meet the conic.

Another consequence of the homogeneous representation is that a Veronese manifold is equivariantly embedded. We explain this as follows. If $R : A \rightarrow A$ is a linear transformation, then there is an induced linear transformation $R \otimes R : A \otimes A \rightarrow A \otimes A$ such that $R \otimes R(v \otimes w) = R(v) \otimes R(w)$. g is equivariant with respect to these maps, which is to say that $g \circ R = (R \otimes R) \circ g$. Since the linear transformations of A give the projective transformations of P^n , this implies that given any projective transformation R' of P^n , there is a projective transformation R'' of P^N such that $g'' \circ R' = R'' \circ g''$. In particular R'' takes the image of g'' onto itself. Using the equivariance it is easy to show that g'' is an embedding. For suppose $g'(v) = \lambda g'(w)$. We may assume that $v = e_1$. Then if $w = \sum x_i e_i$ it follows that

$$e_1 \otimes e_1 = g'(v) = \lambda g'(w) = \lambda \sum x_i x_j e_i \otimes e_j,$$

from which we conclude that $x_i = 0, i > 1$, so that $v \equiv w$. Hence g'' is one-to-one. Now by the equivariance, g'' must have constant rank, which can only be n . The equivariant embedding property implies, finally, that given points $p, q \in P^n$ and a frame in the tangent space at either point, there exists a projective transformation of P^N taking the Veronese manifold $g''(P^n)$ onto itself, taking p to q , and taking the given frame at p to the given frame at q .

The Veronese manifold enters into general differential-geometric considerations in the following way. Let $X : M^n \rightarrow E^m$, be any immersion of class C^2 of a differentiable manifold. Let $p \in M^n$, and u_1, \dots, u_n local coordinates valid in a neighborhood of p , with $u_1 = \dots = u_n = 0$ at p . Let

$$X_i = \frac{\partial X}{\partial u_i}, \quad X_{ij} = \frac{\partial^2 X}{\partial u_i \partial u_j},$$

the derivatives being evaluated at p . By the *osculating space of X at p* we mean the linear space through p spanned by the X_i and X_{ij} . If the osculating space at p has dimension $N = \frac{1}{2}n(n+3)$, the maximum possible, we say that X is *non-degenerate at p* . It is not hard to see that these notions are independent of the choice of the local coordinates u_1, \dots, u_n , and that they are of a projective nature in the geometry of the ambient space.

Let T_p and N_p denote respectively the tangent and normal spaces of X at p . For any vector V in E^n let V^\perp denote the orthogonal projection of V into N_p . Now let $x(t)$ be a curve on M^n such that $x(0) = p$. Then the orthogonal projection into N_p of the second derivative of $X(x(t))$ at 0 depends only on the first derivative of $X(x(t))$ at 0, as is well known (at least in the case of curves on surfaces in ordinary space, and as is proved in the same way in higher dimensions). Thus we have a map $\mathcal{N}: T_p \rightarrow N_p$ which assigns to each $v \in T_p$ the orthogonal projection in N_p of the second derivative vector of a curve on M^n through p whose tangent vector at p is v . To find an analytic expression for \mathcal{N} we consider the curves

$$u_i = x_i t, \quad 1 \leq i \leq n.$$

Then $X'(0) = \sum x_i X_i$, and $X''(0)^\perp = \sum_{i,j=1}^n x_i x_j X_{ij}^\perp$. Hence

$$\mathcal{N}(\sum x_i X_i) = \sum_{i,j=1}^n x_i x_j X_{ij}^\perp. \tag{2.4}$$

Now assume that X is non-degenerate at p . It follows that $X_{ij}^\perp, i \leq j$, are linearly independent, and we may identify X_{ij}^\perp with $X_i \circ X_j$. Thus \mathcal{N} is identified with the map g such that $g(v) = v \otimes v$. Letting S_p denote the sphere of tangent vectors of unit length to X at p , we find that $\mathcal{N}(S_p)$ is a Veronese $(n-1)$ -manifold, which we call the *curvature indicatrix*. It lies in a hyperplane in N_p (coming from the condition $x_1^2 + \dots + x_n^2 = 1$), which we call the *indicatrix plane, I_p* . The mapping \mathcal{N} is two-to-one (except at 0), identifying v and $-v$, and the image $\mathcal{N}(T_p)$ is a half cone consisting of rays from the origin through the points of the curvature indicatrix. We call this cone the *curvature indicatrix cone*. The indicatrix plane, I_p , does not contain the origin, so there is no immersed curve on M^n with zero curvature at p . Since $I_p \subset N_p$, I_p and T_p are in general position and their linear span is all of E^n .

A Veronese manifold is itself everywhere non-degenerate. To prove this it suffices to check it at any point ($x_1 = 1, x_2 = \dots = x_{n+1} = 0$ is convenient), since the Veronese manifold is equivariantly embedded and non-degeneracy is a projective notion. But we can avoid calculation by the following geometric argument.

Let U be a linear subspace of P^n of dimension r ; then its image under g^n is a Veronese r -manifold, as can be seen from (2.2) by taking homo-

geneous coordinates x_1, \dots, x_{n+1} in P^n such that U is defined by $x_{r+2} = \dots = x_{n+1} = 0$. Let $V = g''(P^n)$. We call $g''(U)$ a *Veronese submanifold* of V . Now a Veronese manifold of dimension one is just a non-singular conic. Since any two points of P^n may be joined by a line, any two points of a Veronese manifold may be joined by a conic lying in the Veronese manifold. This implies that V is non-degenerate; for if the osculating space at $p \in V$ were contained in a hyperplane H , then H would contain T_p and $\mathcal{N}(T_p)$, by (2.4), and hence the tangent and curvature vectors at p of all curves on V through p , and hence the planes of the one-dimensional Veronese submanifolds passing through p , and hence V itself. But this contradicts the already proven fact that V lies substantially in P^N .

In terms of the Euclidean representation (2.2), (2.3), a Veronese submanifold is defined in the following fashion. Take a linear subspace through the origin of A and intersect with the unit sphere S^n . The image of this intersection under the map g' is a Veronese submanifold.

Let $V^{n-1} \subset V^n \subset P^N$ be an $(n-1)$ -dimensional Veronese submanifold of a Veronese n -manifold. We claim that there is a unique hyperplane J of P^N such that $J \cap V^n = V^{n-1}$ and such that J is tangent to V^n at each point of V^{n-1} , which is to say that J contains the tangent space to V^n at each point of V^{n-1} . To show this it suffices to take V^{n-1} to be defined by $x_{n+1} = 0$, using our usual homogeneous coordinates, since V^n is equivariantly embedded. Referring to (2.2) we see that V^{n-1} then lies in the hyperplane J of P^N defined by setting the $(n+1)$ -st coordinate of P^N equal to zero, and any point of V^n for which this coordinate is zero must lie in V^{n-1} . This coordinate is x_{n+1}^2 , on V^n , so that V^n lies locally on one side of J . This implies that J is tangent to V^n along V^{n-1} , as required. This last determines J uniquely. For, referring to (2.1) for purposes of notation, we see by differentiating that the tangent space to g at e_i is spanned by $e_i \circ e_1, \dots, e_i \circ e_{n+1}$. Since J contains $g''(e_1), \dots, g''(e_n)$ it must contain all $e_i \circ e_j, 1 \leq i \leq n, 1 \leq j \leq n+1$. But these are sufficient to span J . Our claim is now established. We call J an *e.s. hyperplane* of V^n ("e.s." standing for "extremal support").

Let us note here that J must contain the curvature vectors of all curves on V^{n-1} . (We assume that $V \subset E^N$ for a moment.) Since J contains the tangent spaces to V^n at the points of V^{n-1} , it must contain $\mathcal{N}(v)$ for every v tangent to V^{n-1} , where \mathcal{N} is the curvature indicatrix cone for V^n .

Consider now an r -dimensional Veronese manifold W which happens to be contained in V^n . We claim that W is a Veronese submanifold of V^n in the sense defined above. For, choose a point p on W and let V^{n-1} be a Veronese submanifold of V^n , passing through p , whose tangent space at p contains the tangent space to W at p . Now V^{n-1} is contained in an e.s. hyperplane J . If we have any conic lying on W and passing through p its tangent vector v and its curvature vector must be contained in J ,

since J contains T_p and $\mathcal{N}(v)$. It follows that J contains the conic. And since any point of W may be joined to p by such a conic, $J \supset W$. But since $J \cap V^n = V^{n-1}$, we must have $W \subset V^{n-1}$. We can repeat this argument until we have a sequence of Veronese submanifolds

$$V^r \subset V^{r+1} \subset \dots \subset V^{n-1} \subset V^n$$

with $W = V^r$, which establishes the claim.

The other properties of Veronese manifolds which we need to prove could be established at this point by algebraic arguments. But it is more convenient to derive them as special cases of more general results which we prove later.

3. Consequences of the Two-Piece Property

Throughout this § M^n will denote a compact differentiable manifold of dimension n , and $f: M^n \rightarrow E^m$ a C^2 immersion having the two-piece property. It is easy to see that this implies that M^n is connected.

If $X \subset E^m$ is some subset, we say that a hyperplane $H \subset E^m$ is a *hyperplane of support* of X if X lies in one of the two closed half-spaces of E^m determined by H and if $X \cap H \neq \emptyset$. Note that if X is an immersed submanifold then H must contain the tangent space of X at each point of $X \cap H$. A hyperplane containing the tangent space at a point of an immersed submanifold is said to be *tangent* at that point. A smooth real-valued function F on M^n is said to have a *critical point* at $p \in M^n$ if $dF = 0$ at p , and a critical point p of F is said to be *non-degenerate* if the Hessian, d^2F , is non-degenerate at p . (Let us not confuse this notion with that of a non-degenerate immersion, defined in the last §.) If v is a vector in E^n , the inner product $v \cdot f: M^n \rightarrow R$ is called a *height function* on M^n .

Proposition 3.1. a) Let H be a hyperplane of E^m , $p \in M^n$, and U a neighborhood of p such that H supports $f(U)$ and $f(U) \cap H = f(p)$. Then H supports $f(M^n)$.

b) Any non-degenerate local maximum of a height function on M^n is a global maximum.

Proof. a) By making U smaller, if necessary, we can arrange that H does not meet $f(\partial U)$, where ∂U is the boundary of U . Let ε denote the distance in E^m from $f(\partial U)$ to H . Since M^n , and hence ∂U , are compact, $\varepsilon > 0$. If H were not a support hyperplane of $f(M^n)$ we could move H parallelly toward $f(\partial U)$ a distance of $\frac{1}{2}\varepsilon$ to obtain a hyperplane cutting M^n into at least three pieces, which would contradict the two-piece property. Hence H must support $f(M^n)$.

b) Let $v \cdot f$ be a height function with a non-degenerate local maximum at $p \in M^n$. Then, by Morse's lemma, we can find local coordinates x_1, \dots, x_n ,

valid in a neighborhood U of p , such that $v \cdot f = v \cdot f(0) - x_1^2 - x_2^2 - \dots - x_n^2$ in U . The hyperplane H through $f(p)$ perpendicular to v supports $f(U)$, and $f(U) \cap H = p$. Hence by Part a) H is a support plane of $f(M^n)$, which implies that $v \cdot f$ has a global maximum at p . This completes the proof.

We call a point $p \in M^n$ an *extreme point* of f if there exists a height function with a non-degenerate local maximum at p (and hence a global maximum at p).

Proposition 3.2. a) *The set of extreme points of f is open and non-empty.*

b) *Every extreme point is simple; i.e. if p is an extreme point and $f(p) = f(q)$, then $p = q$.*

Proof. a) Let $\pi: N \rightarrow M^n$ be the full normal bundle of f , and let $g: N \rightarrow E^m$ be the ‘‘Gauss map’’, i.e. the map which takes each normal vector to the same vector bound at the origin of E^m . It is well known that the critical points of a height function $v \cdot f$ are just the points $\pi(g^{-1}(v))$, and that if $v' \in g^{-1}(v)$ and $p = \pi(v')$, then $v \cdot f$ has a non-degenerate critical point at p if and only if g has maximal Jacobian rank at v' . Now by Sard’s theorem, the image under g of the critical locus of g has measure zero in E^m . But for every vector v in E^m the height function $v \cdot f$ attains its maximum value, since M^n is compact. Consequently there must be an abundance of height functions taking non-degenerate maxima, and hence an abundance of extreme points.

To show that the set of extreme points is open, suppose that p is an extreme point and let v be a vector such that $v \cdot f$ has a non-degenerate maximum at p . Then, regarding v as lying in N_p , g must have maximal rank at v , and hence must have maximal rank in some connected neighborhood U of v in N . The determinant of the Hessian matrix of $v' \cdot f$ cannot vanish at $\pi(v')$ for $v' \in U$; hence for each v' in U $v' \cdot f$ has a non-degenerate maximum at $\pi(v')$. Hence $\pi(U)$ is a neighborhood of p consisting of extreme points.

b) Suppose $p \in M^n$ is an extreme point of f and that $f(p) = f(q)$, $p \neq q$. Let v be a vector such that $v \cdot f$ has a non-degenerate maximum at p , and let H be the hyperplane of M^n through $f(p)$ and perpendicular to v . By Morse’s lemma, we can find a closed neighborhood U of p in M^n such that H supports $f(U)$ and $f^{-1}(H) \cap U = p$. Let ε denote the distance, in E^m , from H to $f(\partial U)$. Let H' be a hyperplane parallel to H at a distance $\frac{1}{2}\varepsilon$ toward $f(\partial U)$. Then H' divides M^n into at least three components, one containing p , one containing q , and one containing ∂U . Hence the assumption that $p \neq q$ is false, which completes the proof.

Proposition 3.3. *Let $p \in M^n$ be an extreme point and let H be a hyperplane in E^m tangent to f at p . Then H supports $f(M^n)$ if and only if $\mathcal{N}(S_p)$ lies in one of the closed half-spaces determined by H .*

Proof. Let v be a vector perpendicular to H . Then v is normal to f at p , so that the function $v \cdot f$ has a critical point at p . Let $x(t)$ be an immersed curve on M^n such that $x(0)=p$ and let w denote the tangent vector to x at 0. Then

$$\frac{d^2 v \cdot (f \circ x)}{dt^2}(0) = v \cdot \frac{d^2 (f \circ x)}{dt^2}(0) = v \cdot \mathcal{N}(w). \tag{3.1}$$

Now if $\mathcal{N}(S_p)$ lies on both sides of H , then (3.1) will be positive for some curves on M^n through p and negative for others. Hence $v \cdot f$ will have a non-degenerate local minimum along some curves at p and a local maximum along others. Hence H is not a support hyperplane of $f(M^n)$, and the Proposition is proved in one direction. Note that we have not yet used the two-piece property.

If H does not meet $\mathcal{N}(S_p)$, which implies that $\mathcal{N}(S_p)$ lies on one side of H , since $\mathcal{N}(S_p)$ is connected, (3.1) will have the same sign, no matter what curve $x(t)$ we take. Hence the Hessian of $v \cdot f$ is definite. Assume that v points into the half space determined by H and not containing $\mathcal{N}(S_p)$. Then $v \cdot f$ takes a non-degenerate local maximum at p . Hence by Proposition 3.1 b) it takes a global maximum at p , which implies that H supports $f(M^n)$.

Finally, suppose H meets and supports $\mathcal{N}(S_p)$. We may assume that v points into the half-space not containing $\mathcal{N}(S_p)$. Since p is an extreme point there exists a vector v' such that $v' \cdot f$ has a non-degenerate local maximum at p , and hence a global maximum at p . Hence the hyperplane H' through $f(p)$ and perpendicular to v' supports $f(M^n)$. Now H' cannot cut $\mathcal{N}(S_p)$, by the first paragraph of this proof. Hence $\mathcal{N}(S_p)$ lies in the intersection of two closed half-spaces, one determined by H and the other by H' . Now turn H' about $H' \cap H$ till it reaches H , so that all the intermediate hyperplanes do not meet $\mathcal{N}(S_p)$. By the last paragraph, these all support $f(M^n)$. Hence so does the limiting and final one, namely H . (To understand these last arguments, project everything orthogonally into the plane spanned by v and v' .) This concludes the proof.

Corollary 3.4. *Assume that f is differentiable of class C^3 . Then any hyperplane which supports $f(M^n)$ to the second-order supports $f(M^n)$ to the third order. By this we mean that if H is a hyperplane in E^m tangent to f at p which supports $\mathcal{N}(T_p)$, and if $X(t)=f(x(t))$ is any curve on M^n such that $x(0)=p$ and $X''(0)$ lies in H , then $X'''(0)$ lies in H .*

Proof. Suppose there is such a curve with $X'''(0)$ not in H . Expand $X(t)$ as a finite Taylor's series:

$$X(t) = X(0) + t X'(0) + \frac{1}{2} t^2 X''(0) + \frac{1}{6} t^3 (X'''(0) + t R(t)),$$

where $R(t)$ is continuous in t . If v is a non-zero vector in E^m normal to H , then

$$v \cdot (X(t) - X(0)) = \frac{1}{6} t^3 (v \cdot X'''(0) + t v \cdot R(t)),$$

which takes both positive and negative values near zero, so that H does not support $f(M^n)$. But this contradicts Proposition 3.3. Hence the corollary.

Proposition 3.5. *Let $p \in M^n$ be an extreme point. Then $f(M^n)$ is contained in the osculating space to f at p .*

Proof. If the osculating space to f at p is all of E^m , there is nothing to prove. Suppose then that the osculating space to f at p is a linear space J of dimension $r < m$. Since p is an extreme point, there is a vector v in E^m such that $v \cdot f$ has a non-degenerate maximum at p ; let H be the hyperplane through $f(p)$ perpendicular to v . Now H does not meet $\mathcal{N}(S_p)$; otherwise there would be a curve on M^n through p such that the second derivative of $v \cdot f$ along it would be zero, as we see from (3.1); but this contradicts the fact that the Hessian of $v \cdot f$ at p is negative definite. Since J contains $\mathcal{N}(S_p)$ by (2.4), it follows that H does not contain J , so that $H \cap J$ is a linear subspace of dimension $r - 1$; call it K . Since $\mathcal{N}(S_p)$ is connected and does not meet K , it must lie in one of the half-spaces of J determined by K , call it J^+ . Hence any hyperplane of E^m containing K but not J supports $\mathcal{N}(T_p)$, and hence, by Proposition 3.3, supports $f(M^n)$. It follows that $f(M^n)$ lies in the intersection of all closed halfspaces containing J^+ determined by hyperplanes cutting J in K . But this intersection is just J^+ . It follows that $f(M^n)$ lies in J .

Corollary 3.6. *If $f: M^n \rightarrow E^m$ is substantial, then $m \leq N = \frac{1}{2} n(n + 3)$.*

Proof. The osculating space at p has dimension $\leq N$.

Corollary 3.7. *If $f: M^n \rightarrow E^N$ is substantial, then f is non-degenerate at every extreme point.*

Proposition 3.8. *Let $f: M^n \rightarrow E^N$. Then every non-degenerate point is an extreme point.*

Proof. Let f be non-degenerate at p and let H be a hyperplane containing the tangent plane T_p and parallel to the indicatrix plane I_p . Let v be a non-zero vector perpendicular to H and pointing into the half-space determined by H and not containing I_p . Then $v \cdot f$ has a non-degenerate maximum at p , as we see from (3.1).

4. The Dual Manifold

The results of this § are of a local nature. Let M^n be an arbitrary connected differentiable manifold of dimension n , and let $f: M^n \rightarrow E^N$, $N = \frac{1}{2} n(n + 3)$, be an immersion which is differentiable of class C^3 , non-

degenerate, and which has the property that any hyperplane which supports $f(M^n)$ to the second order supports $f(M^n)$ to the third order. By this we mean, as we said before, that if H is any hyperplane tangent to f at p which supports $\mathcal{N}(T_p)$, and if $x(t)$ is any curve on M^n such that $x(0)=p$ and $X''(0)$ lies in H , then $X'''(0)$ lies in H . By Corollary 3.4, f has this property if it has the two-piece property. Veronese n -manifolds must therefore have this property, since they have the two-piece property. We now state the main local result of this paper. The first part of the proof will be given in this § and the completion in § 6.

Theorem II. *Let $n \geq 2$ and suppose $f: M^n \rightarrow E^N$ is differentiable of class C^4 , non-degenerate, and has the property that any hyperplane which supports $f(M^n)$ to the second order supports $f(M)$ to the third order. Then $f(M^n)$ is contained in a Veronese n -manifold.*

Remarks. The condition that any hyperplane which supports to the second order supports to the third is equivalent to the condition that "the characteristic curves are degenerate" [7]. Later on, following Proposition 4.9, we will put this condition in another form. Theorem II was proved by Segre [10, 11] for $n=2$, and from it follow many of the classical characterizations of the Veronese surface.

Let us consider a non-degenerate immersion $g: M^n \rightarrow E^N$, without assuming any other special properties. Let $p \in M^n$, and let H' be an e.s. hyperplane of $\mathcal{N}(S_p)$ in the indicatrix plane I_p . Since I_p and T_p are in general position, H' and T_p together span a hyperplane H , which we call an *e.s. hyperplane of g at p* . We claim that this agrees with the previous definition in case g is a Veronese manifold, at least for $n > 1$. For if $V \subset E^N$ is a Veronese manifold and H an e.s. hyperplane of V in the sense of § 2, with $p \in V \cap H$, then H supports V locally and hence globally, since V is connected. Therefore, by Proposition 3.3, H supports $\mathcal{N}(T_p)$. Now $V^{n-1} = V \cap H$ is a Veronese submanifold and hence H contains the curvature vectors of all curves on V^{n-1} . Since H contains T_p it must therefore contain $\mathcal{N}(S'_p)$, where $S'_p \subset S_p$ is the set of unit vectors at p tangent to V^{n-1} . It follows that $H \cap I_p$ supports $\mathcal{N}(S_p)$ and contains the Veronese submanifold $\mathcal{N}(S'_p)$, so that $H \cap I_p$ is an e.s. hyperplane of $\mathcal{N}(S_p)$ in I_p , from which it follows that H is an e.s. hyperplane in the sense of this §. On the other hand, if H is an e.s. hyperplane of V at p in the sense of this §, it must contain T_p and the Veronese submanifold $\mathcal{N}(S'_p)$, for some great $(n-2)$ -sphere S'_p of S_p . If V^{n-1} is a Veronese submanifold of V through p whose tangent space contains S'_p , then H must contain the tangent and curvature vectors at p of all curves on V^{n-1} through p , in particular those of the conics on V^{n-1} through p . Hence H contains all such conics, and since each point of V^{n-1} may be joined to p by such a conic, H contains V^{n-1} . Since H supports V , by Propo-

sition 3.3 (applicable in this case because V has the two-piece property) H must be an e.s. hyperplane of V in the sense of § 2. This establishes the claim.

An e.s. hyperplane at $p \in M^n$ is determined by giving a Veronese submanifold of dimension $n - 2$ of $\mathcal{N}(S_p)$. Such a Veronese submanifold is determined by giving a great $(n - 2)$ -sphere of S_p . This in turn may be determined by giving an $(n - 1)$ -plane through the origin of T_p . Note that an e.s. hyperplane H supports $\mathcal{N}(S_p)$ and contains $g(p)$, from which it follows that H supports the curvature cone $\mathcal{N}(T_p)$. If h is an $(n - 1)$ -plane through the origin of T_p such that $\mathcal{N}(h \cap S_p) = \mathcal{N}(S_p) \cap H$, then H must be tangent to $\mathcal{N}(T_p)$ along $\mathcal{N}(h)$.

Lemma 4.1. *Let $p \in M^n$, U a neighborhood of p in M^n , and H a hyperplane of E^N such that $g^{-1}(H) \cap U$ is an embedded submanifold of codimension one. Suppose that H contains T_p , and that H supports $\mathcal{N}(T_p)$ or that H supports $g(M^n)$; then H is an e.s. hyperplane of g at p .*

Proof. If H supports $g(M^n)$, H must support $\mathcal{N}(T_p)$, as can be seen from the proof of Proposition 3.3. Hence it suffices to assume that H supports $\mathcal{N}(T_p)$. Now H must contain all curvature vectors of curves on $g^{-1}(H) \cap U$. It follows that H contains the Veronese submanifold $\mathcal{N}(h_p)$, where $h_p \subset T_p$ is the tangent space to $g^{-1}(H) \cap U$. Hence $H \cap I_p$ is an e.s. hyperplane of $\mathcal{N}(S_p)$ containing $\mathcal{N}(h_p \cap S_p)$, from which the lemma follows.

These preliminaries out of the way, let us consider the map f again. The next considerations will lead to the definition of the dual manifold. Let $p \in M^n$, and $h \subset T_p$ an $(n - 1)$ -dimensional linear subspace. Let H' be the e.s. hyperplane, in I_p , of $\mathcal{N}(S_p)$ which contains $\mathcal{N}(h \cap S_p)$. Let $\varphi(h)$ be the e.s. hyperplane of f at p spanned by T_p and H' . Then φ is a mapping whose domain is the bundle of linear subspaces of dimension $n - 1$ of the tangent spaces of M^n . We denote this bundle by $\pi: G_{n-1} T(M^n) \rightarrow M^n$. The range of φ is the space of hyperplanes of E^N . It is convenient to add the hyperplane at infinity to this space, obtaining thereby the dual projective space P^{N*} . Thus $\varphi: G_{n-1} T(M^n) \rightarrow P^{N*}$. The mapping φ may also be defined in purely projective terms. In fact, $\varphi(h)$ is the unique hyperplane which supports f to the second order at $p \in M^n$ and which contains the osculating planes at p to all curves lying on M^n tangent to h at p .

Proposition 4.2. *φ has everywhere Jacobian rank n . The restriction of φ to any fibre $G_{n-1} T_p$ is one-to-one and has rank $n - 1$, i.e. is an embedding.*

Proof. We assume $n \geq 2$. The assertion that φ restricted to $G_{n-1} T_p$ is one-to-one is essentially the assertion that each e.s. hyperplane of the Veronese manifold $\mathcal{N}(S_p)$ meets $\mathcal{N}(S_p)$ in a unique Veronese $(n - 2)$ -

dimensional submanifold. But this we know already. From this, and the equivariant embedding property of $\mathcal{N}(S_p)$, it follows that φ restricted to $G_{n-1} T_p$ has constant rank $n-1$. We proceed, then, to show that φ has rank n , and we accomplish this by representing $\varphi(h)$ as an exterior N -vector, differentiating φ in $2n-1$ independent directions, and showing that the resulting N -vectors, together with $\varphi(h)$, span a space of dimension $n+1$.

Let $h \in G_{n-1} T(M^n)$ be arbitrary and let $p = \pi(h) \in M^n$. Introduce local coordinates u_1, \dots, u_n on M^n in a neighborhood of p such that h is spanned by $\partial/\partial u_1, \dots, \partial/\partial u_{n-1}$. Let $h_i(t)$ denote the linear span of

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{i-1}}, \frac{\partial}{\partial u_i} + t \frac{\partial}{\partial u_n}, \dots, \frac{\partial}{\partial u_{n-1}},$$

and let us compute $\varphi(h_i(t))$. We use the notation of § 2, so we will consider $f: M^n \rightarrow E^N$ as a position-vector function, which we write as $X: M^n \rightarrow E^N$. Now

$$\mathcal{N} \left(\sum x_j \frac{\partial}{\partial u_j} \right) = \sum x_k x_j X_{kj}^\perp, \tag{4.1}$$

so that

$$\frac{\partial \mathcal{N}}{\partial x_k} = 2 \sum x_j X_{kj}^\perp. \tag{4.2}$$

Hence the tangent space to the whole curvature cone \mathcal{N} at $\partial/\partial x_i$ is spanned by $X_{ki}^\perp, 1 \leq k \leq n$, and the tangent space to \mathcal{N} at $\partial/\partial u_i + t \partial/\partial u_n$ is spanned by $X_{ki}^\perp + t X_{kn}^\perp, 1 \leq k \leq n$. Hence $\varphi(h_i(t))$ must contain $X_1, \dots, X_n, X_{jk} \ (j \leq k, (j, k) \neq (i, n), (i, i), (n, n)), X_{ii} + t X_{in}$, and $X_{ni} + t X_{nn}$. These vectors are linearly independent, since f is non-degenerate, and they are sufficient in number to span $\varphi(h_i(t))$.

In order to represent $\varphi(h_i(t))$ as a multivector, we introduce homogeneous coordinates in E^N , say ξ_0, \dots, ξ_N , such that $y_j = \xi_j/\xi_0$ are ordinary coordinates in E^N . In these coordinates we may represent f by the vector function

$$Y = (1, y_1 \circ f, \dots, y_N \circ f),$$

and hence we may represent $\varphi(h_i(t))$ as

$$\varphi(h_i(t)) = Y \wedge Y_1 \wedge \dots \wedge Y_n \wedge \prod_{\substack{j \leq k \\ (j, k) \neq (i, n), (i, i), (n, n)}} Y_{jk} \wedge (Y_{ii} + t Y_{in}) \wedge (Y_{ni} + t Y_{nn}),$$

where

$$Y_j = \frac{\partial Y}{\partial u_j}, \quad Y_{jk} = \frac{\partial^2 Y}{\partial u_k \partial u_j}.$$

It is easy now to differentiate with respect to t . The derivative of $\varphi(h_i(t))$ at $t=0$, we call φ_i .

$$\varphi_i = \frac{d\varphi \circ h_i}{dt}(0) = Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge \prod_{\substack{j \leq k \\ (j,k) \neq (i,n), (i,i), (n,n)}} Y_{jk} \wedge Y_{ii} \wedge Y_{nn} = \pm \Omega_{in},$$

where by Ω_{jk} we understand the product of $Y, Y_1, Y_{lm}, 1 \leq l, m \leq n, l \leq m$, in some order with only Y_{jk} left out. Clearly $\varphi(h) = \Omega_{nn}, \varphi_i, 1 \leq i \leq n-1$, are linearly independent. The φ_i are directional derivatives of φ along a maximal independent set of fibre directions of $G_{n-1} T(M^n)$, which shows again that φ restricted to $G_{n-1} T_p$ has constant rank $n-1$.

To differentiate φ along the base directions, we write

$$\varphi(h) = \pm Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge \prod_{\substack{j \leq k \\ (j,k) \neq (n,n)}} Y_{jk} = \pm \Omega_{nn}.$$

Hence

$$\varphi_{u_i} \equiv \frac{\partial \varphi}{\partial u_i}(h) = Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge \sum_{\substack{l \leq m \\ (l,m) \neq (n,n)}} \pm Y_{ilm} \wedge \prod_{\substack{j \leq k \\ (j,k) \neq (n,n) \\ (j,k) \neq (l,m)}} Y_{jk},$$

for $i < n$, and

$$\begin{aligned} \varphi_{u_n} = \frac{\partial \varphi}{\partial u_n}(h) &= \pm Y \wedge Y_1 \wedge \cdots \wedge Y_{n-1} \wedge Y_{nn} \wedge \prod_{\substack{j \leq k \\ (j,k) \neq (n,n)}} Y_{jk} \\ &\quad \pm Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge (\cdots). \end{aligned}$$

Clearly $\varphi(h), \varphi_1, \dots, \varphi_{n-1}, \varphi_{u_n}$ are linearly independent. We claim that φ_{u_i} is a linear combination of $\varphi(h), \varphi_1, \dots, \varphi_{n-1}$, for all $i < n$.

To prove this claim we recall the elementary fact that any homogeneous cubic polynomial of several variables is a linear combination of cubes of homogeneous linear polynomials. From this it follows that any homogeneous linear differential operator of order three is a linear combination of cubes of linear differential operators of order one with constant coefficients. This implies that $Y_{ilm}(p)$, where $i, l, m < n$, is a linear combination of third derivatives of Y along curves on M^n through p tangent to h . But all such third derivatives must lie in $\varphi(h)$, by the hypothesis that a hyperplane supporting to the second order must support to the third order. Hence $Y_{ilm}(p), i, l, m < n$ is a linear combination of $Y, Y_1, \dots, Y_n, Y_{jk}, (j, k) \neq (n, n)$. Substituting these linear combinations into the expressions φ_{u_i} , we find that, for $i < n, \varphi_{u_i}$ is a linear combination of $\Omega_{jn}, j = 1, \dots, n$, and hence of $\varphi(h)$ and $\varphi_i, i < n$. This shows that all the directional derivatives of φ at h are linear combinations of $\varphi(h)$ and n such directional derivatives. Hence the rank of φ is n , which is what was to be proved.

It follows from this proposition that the image of φ is an immersed submanifold of dimension n in P^{N*} . More precisely, let us say that $x, y \in G_{n-1} T(M^n)$ are *equivalent* if there is a curve C joining x and y in $G_{n-1} T(M^n)$ such that $\varphi(C) = \varphi(x)$. The equivalence classes form a differentiable manifold M^* and φ induces an immersion $\varphi': M^* \rightarrow P^{N*}$. We call φ' , or M^* itself, the *dual manifold of $f: M^n \rightarrow E^N$* . Clearly the image of a fibre under φ , $\varphi(G_{n-1} T_p)$, is an algebraic variety; hence the dual manifold contains a family of algebraic varieties. We shall see later that these are in fact Veronese manifolds. But our immediate aim is to show that the dual manifold contains a family of conics.

Lemma 4.3. *Let P^a and P^b be linear subspaces of P^{a+b+1} which are in general position, i.e. which do not meet, and let S_t^{a-1} be a one-parameter family of hyperplanes of P^a which form a non-singular conic in the dual space P^{a*} . For each t , let S_t^{a+b} be the linear span of S_t^{a-1} and P^b . Then the family of hyperplanes S_t^{a+b} forms a non-singular conic in P^{a+b+1*} . If Q is the hyperquadric of P^a enveloped by S_t^{a-1} and Q' the hyperquadric of P^{a+b+1} enveloped by S_t^{a+b} , then Q' consists of linear spaces of dimension $b+1$, each of which is spanned by P^b and a point of Q , and every such linear space lies in Q' .*

Proof. Choose homogeneous coordinates x_0, \dots, x_{a+b+1} in P^{a+b+1} such that P^a is defined by $x_{a+1} = \dots = x_{a+b+1} = 0$, P^b is defined by $x_0 = \dots = x_a = 0$, and S_t^{a-1} consists of hyperplanes

$$\alpha_0 x_0 + \dots + \alpha_a x_a = 0$$

such that $-\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 0$ and $\alpha_3 = \dots = \alpha_a = 0$. Then the family S_t^{a+b} consists of hyperplanes

$$\alpha_0 x_0 + \dots + \alpha_{a+b+1} x_{a+b+1} = 0$$

such that $-\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 0$ and $\alpha_3 = \dots = \alpha_{a+b+1} = 0$. But this is clearly a non-singular conic in P^{a+b+1*} , which proves the first assertion of the lemma.

Now the equations of Q in P^{a+b+1} are

$$-x_0^2 + x_1^2 + x_2^2 = 0, \quad x_{a+1} = \dots = x_{a+b+1} = 0.$$

The equation of Q' is just $-x_0^2 + x_1^2 + x_2^2 = 0$, so that Q' consists exactly of the linear spans of P^b and points of Q . This completes the proof of the lemma.

Lemma 4.4. a) *Let $V^n \subset P^N$ be a Veronese manifold and $V^{n-2} \subset V^n$ a Veronese submanifold of codimension 2. Then the set of all e.s. hyperplanes of V^n which contain V^{n-2} is a one-parameter family S_t which forms a non-singular conic in the dual space P^{N*} . The hyperquadric enveloped by S_t contains V^n .*

b) Suppose $n \geq 2$ and $g: M^n \rightarrow E^N$ a non-degenerate C^2 immersion (we assume no other special properties). Let $p \in M^n$, l a linear subspace of dimension $n-2$ through the origin of T_p , and h_t the family of hyperplanes of T_p containing l . Let H_t denote the family of e.s. hyperplanes of g containing the $\mathcal{N}(h_t)$. Then H_t forms a non-singular conic in P^{N*} . H_t envelops a hyperquadric which contains all tangent and curvature vectors at p to curves on M^n through p .

Proof. a) We prove this by induction on n . For $n=1$, V^n is a non-singular conic and S_t is just the family of its tangent lines. The truth of Part a) is clear. Let us assume Part a) for $n-1 \geq 1$. Choose a point $p \in V^{n-2}$, and let m denote the tangent space to V^{n-2} at p . Then S_t consists of the linear spans of the tangent space to V^{n-1} at p , T_p , with the e.s. hyperplanes S'_t of $\mathcal{N}(S_p)$ containing $\mathcal{N}(m \cap S_p)$. By the induction hypothesis, S'_t forms a non-singular conic in the dual space of the indicatrix plane I_p . Hence by Lemma 4.3 S_t forms a non-singular conic in P^{N*} . Again by Lemma 4.3 the hyperquadric Q' enveloped by S_t contains the linear spans of T_p and the points of the hyperquadric Q' of I_p enveloped by S'_t . $Q' \supset \mathcal{N}(S_p)$, by the induction hypothesis. Hence Q contains all tangent and curvature vectors at p of curves on V^n through p . It follows that Q contains all conics on V^n passing through p . Since any point of V^n may be joined to p by such a conic, Q must contain V^n . This proves Part a).

The proof of Part b) is very simple. H_t consists of the linear spans of T_p and the e.s. hyperplanes of $\mathcal{N}(S_p)$ containing $\mathcal{N}(l \cap S_p)$. These last form a non-singular conic in the dual space of I_p , by Part a). Hence by Lemma 4.3 H_t forms a non-singular conic in P^{N*} . H_t envelops a hyperquadric which contains the linear spans of T_p with points of $\mathcal{N}(S_p)$. This completes the proof.

Let us suppose $n \geq 2$ and consider again the dual manifold $\varphi': M^* \rightarrow P^{N*}$. If $H \in \varphi'(M^*)$ and $H = \varphi(h)$, where $h \in G_{n-1} T_p$, then choose an $(n-2)$ -dimensional linear subspace $h' \subset h$ through the origin of T_p . By construction $\varphi'(M^*)$ contains the linear spans of T_p and the e.s. hyperplanes, in I_p , of $\mathcal{N}(S_p)$ which contain $\mathcal{N}(h' \cap S_p)$. But by Lemma 4.4b) these hyperplanes form a non-singular conic in P^{N*} . Hence given any point $x \in M^*$, there exists a closed embedded curve C on M^* passing through x such that $\varphi'(C)$ is a non-singular conic in P^{N*} . Such a curve we call an s -curve on M^* . Given $x \in M^*$, we call the set of all points of M^* which can be joined to x by an s -curve the *wedge* of x , W_x . We have just seen that W_x contains points of M^* other than x , for every x .

Proposition 4.5. *Let $n \geq 2$. For every $x \in M^*$ there is an $(n-1)$ -parameter family of s -curves passing through x . $W_x - \{x\}$ is open in M^* . The tangent lines at x to the s -curves through x fill out an open set in T_x .*

We first prove the converse of Lemma 4.1 under the assumption that every hyperplane which supports to the second order supports to the third order.

Lemma 4.6. *If $H \in P^{n*}$, then $\pi(\varphi^{-1}(H))$ is an embedded submanifold of codimension one in M^n . If $\varphi(h) = H$, $\pi(h) = p$, then $\pi(\varphi^{-1}(H))$ is tangent to h at p .*

Proof. Since φ has rank n , $\varphi^{-1}(H)$ is an embedded submanifold of $G_{n-1}T(M^n)$ of dimension $n-1$. $\varphi^{-1}(H)$ cannot have a tangent vector in common with any fibre of $G_{n-1}T(M^n)$, since φ restricted to any fibre has rank $n-1$. And $\varphi^{-1}(H)$ cannot meet any fibre in more than one point, since φ maps each fibre in a one-to-one fashion. It follows that $\pi(\varphi^{-1}(H))$ is an embedded submanifold of dimension $n-1$ of M^n . If $p \in \pi(\varphi^{-1}(H))$, then H contains the curvature vectors to all curves on $\pi(\varphi^{-1}(H))$ through p at p , since $H \supset \pi(\varphi^{-1}(H))$. It follows that $H \supset \mathcal{N}(h)$, where h is the tangent space to $\pi(\varphi^{-1}(H))$ at p . Hence $H = \varphi(h)$. This completes the proof.

Proof of Proposition 4.5. Let $x \in M^*$, $H = \varphi'(x)$. Let B denote the restriction of the bundle $G_{n-1}T(M^n)$ to $\pi(\varphi^{-1}(H))$; that is to say, let $B = \pi^{-1}(\pi(\varphi^{-1}(H)))$. Then by definition $\varphi'(W_x) = \varphi(B)$. To show that $W_x - \{x\}$ is open in M^* it suffices therefore to show that φ has rank n on $B - \varphi^{-1}(H)$.

To show this, let $y \in W_x$, $y \neq x$, be arbitrary, $L = \varphi'(y)$. Then there is an $l \in B$ such that $\varphi(l) = L$. Let $p = \pi(l)$, and let $h \subset T_p$ be the tangent space to $\pi(\varphi^{-1}(H))$ at p . Then $h \neq l$, since $\varphi(h) = H$ by Lemma 4.6 and $x \neq y$. Hence h and l intersect transversally. Choose local coordinates u_1, \dots, u_n on M^n in a neighborhood of p such that $\pi(\varphi^{-1}(H))$ is defined in that neighborhood by $u_{n-1} = 0$, (so that h is spanned by $\partial/\partial u_1, \dots, \partial/\partial u_{n-2}, \partial/\partial u_n$), and such that l is spanned by $\partial/\partial u_1, \dots, \partial/\partial u_{n-1}$. Let $h_i(t)$ denote the linear span of

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{i-1}}, \frac{\partial}{\partial u_i} + t \frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_{i+1}}, \dots, \frac{\partial}{\partial u_{n-1}},$$

the derivatives being evaluated at p . To show that φ has rank n on B at y it suffices to show that

$$\varphi(h), \frac{d\varphi \circ h_1}{dt}(0), \dots, \frac{d\varphi \circ h_{n-1}}{dt}(0), \frac{\partial \varphi}{\partial u_n}(p)$$

are linearly independent. But these were computed and shown to be linearly independent in the course of the proof of Proposition 4.2. Hence $W_x - \{x\}$ is open as asserted.

In order to show that the tangent lines to the s -curves through x fill out an open subset of T_x , we observe first that an s -curve through x is

constructed by choosing an $(n-2)$ -plane h' through the origin of T_p , for some point $p \in \pi(\varphi^{-1}(H))$, with $h' \subset h$, where h is such that $\varphi(h) = H$, and then taking the image under φ of all $h \supset h'$, $h \in B$. Let us call the set of all such h' , B' . So given such an h' , we compute the tangent line at h' of the corresponding s -curve, call it τ , differentiate the resulting expression along $n-1$ independent directions in B' , and show that τ and these derivatives are linearly independent.

Let h' be arbitrarily chosen as above, $h' \subset T_p$, $h \supset h'$ so that $\varphi(h) = H$. Choose local coordinates x_1, \dots, x_n on M^n in a neighborhood of p , so that $\pi(\varphi^{-1}(H))$ is defined in that neighborhood by $x_n = 0$ (so that h is spanned by $\partial/\partial x_1, \dots, \partial/\partial x_{n-1}$, by Lemma 4.6), and so that h' is spanned by $\partial/\partial x_1, \dots, \partial/\partial x_{n-2}$. Now let $h'_i(r)$ denote the linear span of

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{i-1}}, \frac{\partial}{\partial x_i} + r \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{n-2}},$$

and let $h_i(r, t)$ denote the linear span of

$$h'_i(r) \quad \text{and} \quad \frac{\partial}{\partial x_{n-1}} + t \frac{\partial}{\partial x_n}.$$

Note that $h'_i(r) \in B'$, and for fixed r , $h_i(r, t)$ is an s -curve through x with $\varphi(h_i(r, 0)) = H$. This s -curve corresponds to $h'_i(r)$. We call the curve $\varphi(h_i(r, t))$, for fixed r , C_{ir} . The tangent vector to C_{ir} at H , τ_{ir} , is spanned by $\varphi(h'_i(r, 0))$ and $d/dt \varphi(h_i(r, 0))$, which we proceed to compute.

From (4.1), (4.2), and using the notation of the proof of Proposition 4.2, we find that $\varphi(h_i(r, t))$ must contain

$$Y, Y_1, \dots, Y_n, Y_{jk}, \quad (j \neq i, n-1, n),$$

and

$$\begin{aligned} Y_{ii} + r Y_{i, n-1}, & \quad Y_{i, n-1} + t Y_{in}, \\ Y_{n-1i} + r Y_{n-1, n-1}, & \quad Y_{n-1, n-1} + t Y_{n-1, n}, \\ Y_{ni} + r Y_{n, n-1}, & \quad Y_{n, n-1} + t Y_{nn}. \end{aligned}$$

These vectors are linearly dependent. However, with allowance made for the symmetry $X_{jk} = X_{kj}$, if we leave out $Y_{n-1i} + r Y_{n-1, n-1}$, the remaining are linearly independent for $r, t = 0$, and hence certainly for small values of r, t . These vectors, with the indicated one left out, are sufficient in number to span $\varphi(h_i(r, t))$. If we multiply them together and calculate the derivative of the resulting expression, we obtain

$$\frac{d\varphi \circ h_i}{dt}(r, 0) = \pm \Omega_{n-1, n} \pm r \Omega_{in}.$$

We also find that

$$H = \varphi(h_i(r, 0)) = \pm \Omega_{nn}.$$

The tangent line of C_{ir} at $t=0$, τ_{ir} , is represented by the wedge product of these last two expressions in $A^2(A^N R^{N+1})$:

$$\tau_{ir} = \pm \Omega_{nn} \wedge \Omega_{n-1n} \pm r \Omega_{nn} \wedge \Omega_{in}.$$

Hence

$$\tau_i = \frac{d\tau_{ir}}{dr} (0) = \pm \Omega_{nn} \wedge \Omega_{in}.$$

Also, if τ denotes the tangent to the s -curve determined by h' , we have

$$\tau = \pm \Omega_{nn} \wedge \Omega_{n-1n}.$$

Hence

$$\begin{aligned} \frac{\partial \tau}{\partial x_{n-1}} &= \pm [Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge (\cdots)] \wedge \Omega_{n-1n} \\ &\quad \pm \Omega_{nn} \wedge [Y \wedge Y_1 \wedge \cdots \wedge Y_{n-1} \wedge Y_{n-1n} \wedge \prod_{\substack{j \leq k \\ (j,k) \neq (n-1,n)}} Y_{jk} \\ &\quad + Y \wedge Y_1 \wedge \cdots \wedge Y_n \wedge (\cdots)]. \end{aligned}$$

Hence $\tau, \tau_1, \dots, \tau_{n-2}, \partial\tau/\partial x_{n-1}$ are linearly independent, which shows that the tangent lines to s -curves at x fill out an open set in T_x . This completes the proof of the proposition.

Proposition 4.7. $\varphi': M^* \rightarrow P^{N*}$ is non-degenerate.

The proof depends on the following lemma and proposition.

Lemma 4.8. Let $X: M^n \rightarrow P^N$ be a non-degenerate immersion of class C^2 . Then there is no open set $U \subset M^n$ such that the tangent projective spaces to X at all points of U have a common point.

Proof. Suppose there is such an open set U . Let P be a point common to the tangent projective spaces at the points of U . Choose as hyperplane at infinity in P^N some hyperplane not containing P , and take P to be the origin of $E^N \subset P^N$. $X(U)$ is not contained in the hyperplane at infinity, since X is non-degenerate. Hence there is an open set $U' \subset U$ such that $X: U' \rightarrow E^N$, and the tangent spaces to X at the points of U' pass through the origin.

Let $U'' \subset U'$ be the domain of some local coordinate system x_1, \dots, x_n . There must exist smooth functions $\alpha_1, \dots, \alpha_n$ such that

$$X \equiv \alpha_1 X_1 + \cdots + \alpha_n X_n \quad \text{in } U'', \quad X_i = \frac{\partial X}{\partial x_i}.$$

Differentiating this with respect to x_i , we obtain

$$X_i = \frac{\partial \alpha_1}{\partial x_i} X_1 + \cdots + \frac{\partial \alpha_n}{\partial x_i} X_n + \alpha_1 X_{1i} + \cdots + \alpha_n X_{ni}.$$

But since X is non-degenerate, the X_i and X_{jk} , $j \leq k$, are linearly independent. Hence $\alpha_j \equiv 0$ for $j \neq i$. But this is true for any i ; hence $\alpha_j \equiv 0$ for all j . But this implies that $X \equiv 0$ in U'' , a contradiction.

Proposition 4.9. *Let $X: M^n \rightarrow P^N$ be a non-degenerate immersion of class C^2 . Then*

- a) *the e.s. hyperplanes of X have no common point in P^N ;*
- b) *if $p \in M^n$, the intersection of the e.s. hyperplanes of X at p is T_p .*

Proof. We proceed by induction. If $n=1$, we are concerned with a non-degenerate plane curve; the e.s. hyperplanes are just the tangent lines. In this case the proposition is well-known. Now assume the proposition for dimensions lower than n . Since the proposition is of a local and projective nature, we may assume $X: M^n \rightarrow E^N$ and apply our metric constructions. Consider the e.s. hyperplanes at p . Their intersection certainly contains T_p . If it is larger than T_p it meets the indicatrix plane I_p , and hence the e.s. hyperplanes of $\mathcal{N}(S_p)$ have a common point. But this is impossible by the induction hypothesis. This proves Part b).

To prove Part a) we observe that, by Part b), if all the e.s. hyperplanes of X have a common point, then all the tangent spaces of X have a common point. But this is impossible, by Lemma 4.8. This concludes the proof.

By Proposition 4.9 b), the condition that any hyperplane which supports to the second order supports to the third order may be put in the following form: if $p \in M^n$, $x(t)$ a curve on M^n through p with unit tangent vector v at p such that $x(0)=p$, then $X'''(0)$ lies in the linear span of T_p and the tangent space to \mathcal{N} at v , where $X(t)=f(x(t))$.

Proof of Proposition 4.7. Suppose ϕ' is degenerate at $x \in M^*$. Then there is a hyperplane $J \subset P^{N^*}$ containing all first and second derivatives of curves on M^* at x . J must then contain the images of all s -curves through x . Hence, by Proposition 4.5, J contains the image under ϕ of an open subset U of $G_{n-1}T(M^n)$. Assume U is the largest such open set. Now if U meets a fibre F , it meets F in an open set of F , and hence $F \subset U$, since ϕ restricted to F is real algebraic. It follows that all the e.s. hyperplanes of f at the points of $\pi(U)$, when viewed in P^{N^*} , lie in J . Dually, this says that all the e.s. hyperplanes of f at the points of $\pi(U)$ have a common point in E^N . But this is impossible by Proposition 4.9 a). This completes the proof.

5. Submanifolds of E^N Containing Many Plane Curves

In this § we prove the following.

Theorem III. *Let M be a connected differentiable manifold of dimension $n > 1$ and $f: M \rightarrow E^N$, $N = \frac{1}{2}n(n+3)$, a non-degenerate immersion dif-*

ferentiable of class C^2 . Suppose M contains a family of C^2 curves, which we call s -curves, such that each s -curve is mapped by f onto a plane curve in E^N . For each $x \in M$ let W_x denote the set of points of M which can be joined to x by s -curves. Suppose that $W_x - \{x\}$ is non-empty and open for each x and that the tangent lines to the s -curves at x fill out an open subset of the tangent space at x . Then $f(M)$ is contained in a Veronese n -manifold in P^N .

Before beginning the proof, we make a few remarks. By Proposition 4.5 the dual manifold of the last § satisfies the hypotheses of the theorem. Let us note that the hypotheses imply that if $y \in W_x$, then $x \in W_y$; and W_x has the property that if U is any neighborhood of x in M , then $U \cap W_x$ has a non-empty interior.

Proof of Theorem III. Let $x \in M$ and let j be a linear subspace of dimension $n - 2$ of T_x . By Lemma 4.4b) the e. s. hyperplanes of f at x which contain $\mathcal{N}(j)$ envelop a hyperquadric Q which contains the tangent and curvature vectors of all curves at x , in particular the s -curves through x . But the plane of an s -curve is spanned by the tangent and curvature vector at any point, because f is non-degenerate and therefore contains no curves with curvature vanishing anywhere. It follows that Q contains all s -curves on M through x and hence $Q \supset W_x$. In order to prove the theorem we will show that any point of M has a neighborhood whose image under f lies in a number of such hyperquadrics, which can be chosen in such a way that their intersection is a Veronese surface.

Let $y \in M$ be an arbitrary point. We work in a neighborhood of y which is embedded by f . We claim that there is some point $y_1 \in W_y$ such that $T_y \cap T_{y_1}$ is a linear space of dimension $< n - 1$ in E^N . For if not, let $\pi: E^N \rightarrow N_y$ denote orthogonal projection into the normal space at y . Then the rank of π on $f(W_y)$ must be ≤ 1 , so that $\pi \circ f(W_y)$ is a curve, possibly with singularities. Now if C is any s -curve on M through y , $\pi f(C)$ is a segment of a generating ray of the curvature cone $\mathcal{N}(T_y)$. Hence if $q \in \pi f(W_x)$, then the line segment joining q to y must also lie in $\pi f(W_x)$. Hence since $\pi f(W_x)$ is a curve, it must lie in a line. But this implies that $f(W_x)$ lies in the linear span of that line with T_y , contradicting the non-degeneracy of f . Let us note here that $\pi(W_y) \subset \mathcal{N}(T_y)$.

So let us choose $y_1 \in W_y$ such that $T_y \cap T_{y_1}$ has dimension less than $n - 1$. Next we choose $y_2 \in W_y \cap W_{y_1}$, $y_2 \neq y$, in such a way that $T_{y_2} \cap T_{y_1}$ has dimension $< n - 1$ and such that $y_2 \notin T_{y_1}$, and $y_1 \notin T_{y_2}$. This is possible since $W_{y_1} \cap W_{y_2}$ has a non-empty interior containing points arbitrarily close to y . We now choose $y_3, \dots, y_{n+1} \in M$, by a recursive process. We assume that y_1, \dots, y_i , $i < n + 1$, have been chosen in such a way that

$$1_i) \quad y, y_1, \dots, y_i \in W_y \cap W_{y_1} \cap \dots \cap W_{y_i}, f(y_j) \neq f(y), \text{ all } j \leq i;$$

2_i) if π_j denotes the projection into the normal space of f at y_j , then there exist $z_{j1}, \dots, z_{ji} \in T_{y_j}$ which are in general position in T_{y_j} and such that $\mathcal{N}(z_{jk}) = \pi_j f(y_k)$, for all $k \leq i$;

3_i) $T_{y_1} \cap \dots \cap T_{y_i}$ has dimension $\leq n - i$.

(Let us note that for $i=2$, the set $\{y_1, y_2\}$ already chosen satisfies conditions 1_i), 2_i), 3_i.) Let us now consider the interior, U , of $W_y \cap W_{y_1} \cap \dots \cap W_{y_i}$. If every point x of U has the property that $T_x \supset T_{y_1} \cap \dots \cap T_{y_i}$, then the assumed non-degeneracy and Lemma 4.8 give a contradiction. Hence there is some open set $U' \subset U$ with the property that if $x \in U'$, then $T_x \cap T_{y_1} \cap \dots \cap T_{y_i}$ has dimension $\leq n - i - 1$. Now let h_j denote the linear span, in T_{y_j} , of z_{j1}, \dots, z_{ji} . Let P_j denote the linear span of T_{y_j} and $\mathcal{N}(h_j \cap S_{y_j})$. Since $\mathcal{N}(h_j \cap S_{y_j})$ lies in a linear space of dimension $\frac{1}{2}(i-2)(i+1)$, P_j must have dimension $< N$. Hence, by the non-degeneracy, P_j can contain no open set of M . It follows that $U'' = U' - \bigcup P_j$ is non-empty, and so we choose $y_{i+1} \in U''$. Conditions 1_{i+1}), 2_{i+1}), and 3_{i+1}) are now easily verified, with the exception of 2_{i+1}) for the case $j=i+1$.

But suppose $z_{i+11}, \dots, z_{i+1i+1}$ are so chosen that $\mathcal{N}(z_{i+1k}) = \pi_{i+1} f(y_k)$. If these z 's are not in general position, then one of them lies in the linear span of the others, say z_{i+11} lies in the linear span of $z_{i+12}, \dots, z_{i+1i+1}$. Call this linear span Q . Let Q' denote the linear span of $T_{y_{i+1}}$ and $\mathcal{N}(Q \cap S_{y_{i+1}})$. Then Q' contains $f(y_1), \dots, f(y_{i+1})$. But Q' cannot contain an open set of M . And everything established so far remains true if we vary y_1 slightly. So vary y_1 slightly, to get it outside of Q' . And repeat this argument until $z_{i+11}, \dots, z_{i+1i+1}$ are in general position. This completes our account of the construction of y_1, \dots, y_{n+1} . Note that $T_{y_1} \cap \dots \cap T_{y_{n+1}}$ is empty.

Suppose we have an e.s. hyperplane H of f at y_j and suppose H contains $f(y_k)$. We claim that H is also an e.s. hyperplane of f at y_k . To show this, we observe first of all that since H supports $\mathcal{N}(T_{y_j})$ it supports any s -curve through y_j , and hence it supports W_{y_j} . Hence it supports $\mathcal{N}(T_{y_k})$ and contains T_{y_k} . Let C be an s -curve joining y_j and y_k . $\pi_j f(C)$ is a segment lying on the curvature cone $\mathcal{N}(T_{y_j})$, and this segment is contained in $\mathcal{N}(l)$, where l is the tangent line to C at y_j . Since H contains T_{y_j} and $f(y_k)$, H contains $f(C)$ and hence it contains this segment. Now by hypothesis the tangent lines to the s -curves through y_j fill out an open set in T_{y_j} . Since $H \cap \mathcal{N}(S_{y_j})$ is, by definition, a Veronese submanifold of dimension $(n-2)$ of $\mathcal{N}(S_{y_j})$, H must contain an $(n-2)$ -parameter family of s -curves containing C . This family sweeps out an $(n-1)$ -dimensional submanifold of M containing y_k , and hence, by Lemma 4.1, H is an e.s. hyperplane at y_k , which establishes the claim.

Let $i \leq n+1$, and pick $j \leq n+1, j \neq i$. Let h_{ji} denote the linear span of $z_{j1}, \dots, z_{ji-1}, z_{ji+1}, \dots, z_{jn+1}$. Let H_i be the e.s. hyperplane at y_i con-

taining $\mathcal{N}(h_{ji})$. Note that H_i contains $f(y_k)$ for all $k \neq i$ and is an e.s. hyperplane at each of them. Now for each $i, j \leq n+1, i \neq j$, choose $k \neq i, j, k \leq n+1$, and let h_{kij} denote the linear span of $\{z_{k1}, \dots, z_{kn+1}\} - \{z_{ki}, z_{kj}\}$. Consider the set Π_{ij} of e.s. hyperplanes at y_k containing h_{kij} . By property 2_{n+1}) above, and Lemma 4.4, Π_{ij} forms a non-singular conic in P^{N*} and envelops a hyperquadric containing W_{y_k} , and hence a neighborhood of y . The hyperplanes of Π_{ij} are e.s. hyperplanes at each $y_k, k \neq i, j$. Π_{ij} and Π_{il} have H_i in common.

Now let $k \leq n+1$ and consider the intersection of all hyperplanes of the families Π_{ij} such that $i, j \neq k$. We claim that this intersection is just T_{y_k} . This follows from the fact that if w_1, \dots, w_n are n points, lying on a Veronese manifold V^{n-1} , which are not contained in a proper Veronese submanifold, and if Π_{ij} denotes the family of e.s. hyperplanes of V^{n-1} which contain $\{w_1, \dots, w_n\} - \{w_i, w_j\}$, then the intersection of all the hyperplanes of the families Π_{ij} is void. This, in turn, is proved by induction on $n-1$, using an argument similar to that of Proposition 4.9. The essential step in the induction is the observation that if V^i is a Veronese manifold of dimension i and x_1, \dots, x_{i+1} are points on V^i not lying in a proper Veronese submanifold, then the tangent spaces to V^i at x_1, \dots, x_{i+1} have no common point. This, in turn, follows from the calculation near the end of § 2. Finally, since $T_{v_1} \cap \dots \cap T_{v_{n+1}} = 0$, there is no point common to all the hyperplanes of the families Π_{ij} .

Now draw the tangents to the conic Π_{ij} at H_i and H_j . They meet in a point H_{ij} , and the linear span of all the points H_i, H_{ij} is the same as the linear span of all the conics Π_{ij} . This has dimension $\leq n + \frac{1}{2}n(n+1) = N$. If it is strictly less than N , then all the Π_{ij} lie in a hyperplane of P^{N*} , which says that all the hyperplanes of the families Π_{ij} have a common point. Since this is impossible, the dimension above must be N , which says that the set of points $H_k, H_{ij}, i < j$, is in general position.

Let us introduce homogeneous coordinates in P^{N*} such that the coordinates of H_k or H_{ij} are all zero, but for one. For convenience we call these coordinates $\xi_i^*, \xi_{jk}^*, 1 \leq i \leq n+1, 1 \leq j < k \leq n+1$, and we require that

$$\begin{aligned} H_i \text{ is the point } \xi_i^* &= \delta_{i1}, & \xi_{jk}^* &= 0, \quad \text{all } j, k; \\ H_{jk} \text{ is the point } \xi_{mp}^* &= \delta_{jm} \delta_{kp}, & \xi_i^* &= 0, \quad \text{all } i. \end{aligned}$$

In these coordinates Π_{ij} has equations

$$a_{ij} \xi_i^* \xi_j^* + b_{ij} \xi_{ij}^{*2} = 0, \quad \xi_l^* = 0, \quad l \neq i, j, \quad \xi_{mp}^* = 0, \quad (m, p) \neq (i, j),$$

where $i < j$ and a_{ij} and b_{ij} are non-zero. We still have the freedom to normalize the coordinates, so we may incorporate the constants a_{ij}, b_{ij} in ξ_{ij}^* . The first equation becomes then

$$\varepsilon_{ij} \xi_i^* \xi_j^* - \xi_{ij}^{*2} = 0, \quad \varepsilon_{ij} = \pm 1.$$

If ξ_i, ξ_{jk} are the homogeneous coordinates in P^N dual to the ζ_i^*, ζ_{jk}^* , then the hyperquadric enveloped by Π_{ij} has equation

$$\varepsilon_{ij} \xi_i \xi_j - \frac{1}{4} \xi_{ij}^2 = 0.$$

Now each of these hyperquadrics, and hence the intersection of all of them, contains a neighborhood of y in M . In this neighborhood there must be a point whose first $n+1$ coordinates, ξ_1, \dots, ξ_{n+1} , are all different from zero. For otherwise this neighborhood would be contained in a finite union of hyperplanes, contradicting the non-degeneracy of f . By reversing the signs of the ξ_i , if necessary, we can arrange that ξ_1, \dots, ξ_{n+1} are all positive at some point in this neighborhood. We must then have all $\varepsilon_{ij} = 1$. Our equations thus become

$$\xi_i \xi_j - \frac{1}{4} \xi_{ij}^2 = 0.$$

In order to solve these equations, we set $x_i = |\xi_i|^{\frac{1}{2}}$. Then $\xi_{ij}^2 = 4 |\xi_i| |\xi_j|$, so

$$\xi_i = \pm x_i^2, \quad \xi_{ij} = \pm 2 x_i x_j.$$

This is a parametric representation of the various pieces of the intersection of our hyperquadrics, for various choices of the signs. But given any of these pieces, we can reverse the signs of ξ_{ij} and ξ_i as necessary to get it into the form

$$\xi_i = x_i^2, \quad \xi_{ij} = 2 x_i x_j.$$

But this we recognize as our parametrization of a Veronese manifold. Hence there is some neighborhood U of y in M such that $f(U)$ lies in a Veronese manifold.

Now $y \in M$ is an arbitrary point. Since M is connected, by analytic continuation we conclude that $f(M)$ lies on a Veronese n -manifold. This concludes the proof of Theorem III.

6. Proof of Theorems I and II

We prove Theorem II as follows. Let $f: M^n \rightarrow E^N$ be a non-degenerate immersion of class C^4 which has the property that any hyperplane which supports to the second order supports to the third. We may construct the dual manifold $\varphi': M^* \rightarrow P^{N*}$, which is differentiable of class C^2 and which, by Propositions 4.5 and 4.7, satisfies the hypotheses of Theorem III. Hence, by Theorem III, $\varphi'(M^*)$ is contained in a Veronese n -manifold V^* .

Now let V' be a Veronese manifold of dimension n lying in P^N . Since it has the two-piece property, by Corollary 3.4 it has the property that any hyperplane which supports to the second order supports to the third. Also V' is non-degenerate. Hence by the last paragraph its dual manifold $V'^* \rightarrow P^{N*}$ lies on a Veronese manifold. This last map is one-

to-one by § 2. And it is onto the Veronese manifold, because V' is compact and any Veronese manifold is connected. Now apply a projective transformation of P^{N*} which brings V'^* to V^* . Since the φ -mapping for V' may be constructed in a projectively invariant fashion, the adjoint projective transformation of P^N brings V' to a Veronese manifold V whose dual manifold is V^* .

Consider a point p of M^n or V , and consider the e. s. hyperplanes at p , $\varphi(G_{n-1} T_p)$. We claim that they form a Veronese $(n-1)$ -manifold in P^{N*} . For if we take homogeneous coordinates ξ_1, \dots, ξ_{N+1} in P^N such that T_p is defined by $\xi_{n+2} = \dots = \xi_{N+1} = 0$ and I_p , the indicatrix plane, by $\xi_1 = \dots = \xi_{n+1} = 0$, then if

$$\sum_{i=n+2}^{N+1} \alpha_i \xi_i = 0$$

is an e. s. hyperplane of $\mathcal{N}(S_p)$, the α_i must satisfy some system of equations defining a Veronese $(n-1)$ -manifold, since by the last paragraph the e. s. hyperplanes of the Veronese manifold $\mathcal{N}(S_p)$ form a Veronese $(n-1)$ -manifold. Now the e. s. hyperplanes $\varphi(G_{n-1} T_p)$ are just the linear spans of T_p and the e. s. hyperplanes of $\mathcal{N}(S_p)$. Hence these are hyperplanes

$$\sum_{i=1}^{N+1} \alpha_i \xi_i = 0,$$

such that $\alpha_1 = \dots = \alpha_{n+1} = 0$, and $\alpha_{n+2}, \dots, \alpha_{N+1}$ satisfy the equations mentioned above. But these together are the equations of a Veronese $(n-1)$ -manifold lying in P^{N*} , which proves the claim.

Now to each point $p \in M^n$ we can assign the Veronese $(n-1)$ -manifold $\Phi_M(p) = \varphi(G_{n-1} T_p)$, and to each point $q \in V$ the Veronese $(n-1)$ -manifold $\Phi_V(q) = \varphi(G_{n-1} T_q)$. The range space of Φ_M and Φ_V is the set of all Veronese $(n-1)$ -submanifolds of V^* , which may be identified with a projective space of dimension n , P^n . Now Φ_V must be onto and have constant rank, because V is equivariantly embedded and the mapping φ is projectively invariant. It follows that Φ_V is also one-to-one, since V and P^n are homeomorphic and the fundamental group is Z_2 .

Consider now the mapping $\Phi_V^{-1} \Phi_M: M^n \rightarrow V$. It is differentiable, by construction. If $\Phi_V^{-1} \Phi_M(p) = q$, then by Proposition 4.9 $T_p = T_q$. If we represent $f: M \rightarrow E^N$ as a position vector function X , and V by a position vector function Y , then we can write

$$Y \circ \Phi_V^{-1} \Phi_M = X + \alpha_1 X_1 + \dots + \alpha_n X_n,$$

where x_1, \dots, x_n are local coordinates on M^n and $X_i = \partial/\partial x_i X$. Differentiating, we obtain

$$(Y \circ \Phi_V^{-1} \Phi_M)_i = X_i + \sum \frac{\partial \alpha_i}{\partial x_j} X_j + \sum \alpha_j X_{ij}.$$

But since $T_p = T_q$, and X_{ij}, X_i are linearly independent, we must have $\alpha_j = 0$, $1 \leq j \leq n$. But this implies that $f(M^n) \subset V$. This completes the proof of Theorem II.

Finally we prove Theorem I. Let M^n be compact and let $f: M^n \rightarrow E^N$ be a C^4 immersion which is substantial and which has the two-piece property. It is trivial to show that M^n must be connected. Consider the set of extreme points of f . By Proposition 3.2 this set is non-empty and open. Let M_0 be one of its connected components. Then by Corollaries 3.7 and 3.4, f restricted to M_0 is non-degenerate and has the property that any hyperplane which supports to the second order supports to the third order. Hence, by Theorem II, $f(M_0)$ lies on a Veronese n -manifold $V \subset P^N$. Now if M_0 had a boundary point p , then f would be degenerate at p , by Proposition 3.8. By continuity, $p \in V$ and V would be degenerate at p . But this is impossible. Hence M_0 has no boundary points, so $M_0 = M^n$, and $f(M^n) = V$. Finally, f must be an embedding, by Proposition 3.2. This completes the proof.

7. Alternative Proofs

We think it might be worthwhile to indicate some other proofs of Theorem I.

1) The original proof of Little is essentially that of the present paper, up to Corollary 3.4. The chief difference between his paper and the present one lies in the proof of Theorem II, which he carried out by using the given conditions on the third derivatives to introduce local coordinates on the submanifold in which the second derivatives of the immersion map vanish identically. This proof is in some sense more straightforward than that presented here, but the calculations involved are very complicated. The deduction of Theorem I from Theorem II is essentially that given in the present paper.

2) A simplified proof of Theorem I, but one which does not yield Theorem II, may be given as follows. Proposition 3.3 may be interpreted as saying that if we project $f(M^n)$ orthogonally into the normal space at an extreme point p , then the image will lie inside the convex hull of $\mathcal{N}(T_p)$. On the other hand, using the map φ , it may be seen that each e.s. hyperplane at p meets M^n in a submanifold of codimension one through p , and it can be shown that these submanifolds fill up a neighborhood of p . But when an e.s. of f at p is projected into the normal space at p , it goes onto an e.s. hyperplane of $\mathcal{N}(S_p)$. It follows that the orthogonal projection of some neighborhood of p must lie on $\mathcal{N}(T_p)$. We can now modify the argument of § 5 to prove that some neighborhood of p is a Veronese manifold, and then use the concluding arguments of § 6 to obtain Theorem I. This approach avoids considering the dual

manifolds as such, and moreover requires only C^3 differentiability for f . But because of other applications of Theorem II we have gone the present route.

3) Kuiper's proof [4] of Theorem I for surfaces uses Morse theory to show that a tight surface in E^5 carries a two-parameter family of ellipses. It then appeals, in effect, to what we have proved here as Theorem III, which was proved for surfaces by Segre [10, 11].

4) An heuristic proof of Theorem I for surfaces may be given in the following way. By an argument essentially due to Banchoff [1], any e.s. hyperplane at an extreme point p of an immersed surface in E^5 having the two-piece property must meet the surface in more than one point. Let us accept that it is a curve. This curve is a top set in the sense of Kuiper [4], so it must be a plane convex curve. Accepting that such curves fill out a neighborhood of p , it follows that orthogonal projection into the normal space at p sends this neighborhood into $\mathcal{N}(T_p)$. We now proceed as in § 5.

References

1. Banchoff, T. F.: The spherical two-piece property and tight surfaces in spheres. *J. Diff. Geom.* **4**, 193–205 (1970).
2. Bertini, E.: *Einführung in die projektive Geometrie mehrdimensionaler Räume.* Vienna: L. W. Seidel und Sohn, 1924.
3. Chern, S. S., Lashof, R. K.: On the total curvature of immersed manifolds, I. *Amer. Journal of Math.* **78**, 306–313 (1957).
4. Kuiper, N. H.: On convex maps. *Nieuw Archief voor Wisk.* **10**, 147–164 (1962).
5. — Der Satz von Gauss-Bonnet für Abbildungen im E^N . *Jahr.-Ber. DMV Bd.* **69**, 77–88 (1967).
6. — Minimal total absolute curvature for immersions. *Inventiones Math.* **10**, 209–238 (1970).
7. Lane, E. P.: *Projective differential geometry of curves and surfaces.* Chicago: University of Chicago Press, 1932.
8. Little, J. A.: On singularities of submanifolds of higher dimensional Euclidean spaces. *Annali di Mat. pura ed appl. (IV)* **83**, 261–336 (1969).
9. Morse, M.: The existence of polar non-degenerate functions on differentiable manifolds. *Ann. of Math.* **71**, 352–383 (1960).
10. Segre, C.: Su un classe di superficie degl' iperspazi legata colle equazioni lineari alle derivate parziali di 2° ordine. *Torino Atti* **42**, 1047 (1907).
11. — Le linee principali di una superficie di S_5 e una proprietà caratteristica della superficie di Veronese. *Rom. Acc. L. Rend. (5)* **30**, 200 and 227 (1921).

J. A. Little
 Department of Mathematics
 University of Michigan
 Ann Arbor, Michigan 48104
 USA

W. F. Pohl
 Department of Mathematics
 University of Minnesota
 Minneapolis, Minn. 55455
 USA

(Received February 24, 1971)