Inventiones math. 16, 335–341 (1972)

Areas of Projections of Analytic Sets

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1.

If V is a pure 1-dimensional analytic subvariety of $\mathbf{C}^n$, then the area of V is the sum of the areas, counting multiplicity, of the projections of V onto the n coordinate lines [5, 6]. If V is a pure 1-dimensional variety in the unit ball in $\mathbf{C}^n$ which passes through the origin, then [5] the area of V is at least π. Together these theorems imply that a 1-variety through 0 in the unit ball has the property that the sum of the areas of its projections on the coordinate lines, taken with multiplicity, is bounded below by π. We shall generalize this result by showing that the conclusion is valid without counting multiplicities. A similar property of higher dimensional varieties will also be obtained. The proof depends upon a one variable result (Theorem 1) which relates the area of the image of a function analytic in the unit disc to its L^2-norm on the circle. As an application we extend a result of Nishino [4] to the effect that a family of varieties is locally finite if each family of coordinate slices is such. This in turn will be applied to complete a theorem of Rothstein [7] on characterizing analytic sets in $\mathbf{C}^n$ as those whose coordinate slices are analytic sets.

2.

We begin with the basic one variable result. Here D will denote the open unit disc $\{z\in\mathbf{C}: |z|<1\}$ and N the usual Nevanlinna space of analytic functions of bounded characteristic on D [2, p. 16]. For $f\in N$, f^* will denote the boundary function

$$f^*(e^{i\theta})=\lim_{r\to 1^-} f(r e^{i\theta})$$

defined for almost all θ.

Theorem 1. *Let $f\in N$ and $f(0)=0$. Then*

$$\tfrac{1}{2}\int_0^{2\pi} |f^*(e^{i\theta})|^2\, d\theta \leqq \operatorname{area}(f(D)).$$

This theorem follows by integrating, with respect to t, from 0 to $+\infty$ in the following inequality [3, pp. 421–422].

Lemma. *Let $f \in N$, $f(0)=0$. Then for $t>0$,*

$$t\,\ell\{e^{i\theta}: |f^*(e^{i\theta})|>t\} \leqq \ell(f(D)\cap\Gamma_t)$$

where $\Gamma_t=\{w: |w|=t\}$ and ℓ is arc length measure.

Proof. Fix r, $0<r<1$, and $t>0$. Put $E_t=\{e^{i\theta}: |f_r(e^{i\theta})|>t\}$ where $f_r(z)=f(rz)$. Now let $u(z)$ be the bounded harmonic function in D with boundary values equal to 0 if $e^{i\theta}\notin E_t$ and -1 if $e^{i\theta}\in E_t$. That is,

$$u(z)=\frac{-1}{2\pi}\int_{E_t}\operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta.$$

Note that $u(0)=\frac{-1}{2\pi}\ell(E_t)$, and that

$$|f_r(e^{i\theta})|<t \Rightarrow \lim_{z\to e^{i\theta}} u(z)=0. \tag{2.1}$$

Now, define $\varphi(w)$ for $|w|\leqq t$ by

$$\varphi(w)=\begin{cases}\min\{u(z): f_r(z)=w,\, z\in D\} & \text{if } w\in f_r(D)\\ 0 & \text{if } w\notin f_r(D).\end{cases}$$

We claim that φ is superharmonic in $|w|<t$.

We will assume that $E_t \neq \emptyset$, since otherwise (2.5) below is trivial. Therefore, since f_r is analytic on $|z|=1$, the set

$$F_t=\{e^{i\theta}: |f_r(e^{i\theta})|=t\}$$

is finite. Now φ is continuous on $|w|\leqq t$ except possibly for the finite set $f_r(F_t)\subset\{w: |w|=t\}$. Off the finite set X of critical values of f_r in $|w|\leqq t$, (2.1) implies that φ is locally the minimum of a finite number of harmonic functions, hence superharmonic off X. Since φ is continuous in $|w|<t$ and X is discrete, it follows that φ is superharmonic in $|w|<t$.

In particular, we have

$$\varphi(0)\geqq\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t\,e^{i\theta})\,d\theta. \tag{2.2}$$

Now, $\varphi(t\,e^{i\theta})=0$ unless $t\,e^{i\theta}\in\overline{f_r(D)}$ in which case $\varphi(t\,e^{i\theta})\geqq -1$ because $u(z)\geqq -1$ for $z\in D$. Thus, we get from (2.2),

$$\varphi(0)\geqq\frac{-1}{2\pi}\,\frac{\ell(\overline{f_r(D)}\cap\Gamma_t)}{t}. \tag{2.3}$$

On the other hand, since $f_r(0)=0$, we have

$$\varphi(0)\leqq u(0)=-\frac{1}{2\pi}\ell(E_t). \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$t\,\ell(E_t) \leqq \ell\left(\overline{f_r(D)} \cap \Gamma_t\right). \tag{2.5}$$

But, $\overline{f_r(D)} \subset f(D)$, so

$$t\,\ell\{e^{i\theta}: |f_r(e^{i\theta})| > t\} \leqq \ell\left(f(D) \cap \Gamma_t\right).$$

Letting $r \to 1^-$ gives the desired result because $f_r(e^{i\theta}) \to f^*(e^{i\theta})$ a.e., and therefore also in measure. Q.E.D.

Remark. The superharmonic function φ of the proof was essentially introduced by Nishino [4, p. 259] for a related problem.

In the next theorem, B denotes the unit ball in $\mathbf{C}^n$,

$$\left\{z = (z_1, \ldots, z_n): \sum_1^n |z_i|^2 < 1\right\}.$$

Theorem 2. *Let V be a pure 1-dimensional analytic subvariety of B containing the origin. Then the sum of the areas of the projections of V on the n coordinate lines is at least π.*

Proof. We may assume that V is irreducible. Let V^* be the normalization of V, τ: $V^* \to V$ the associated projection. Since V^* carries nonconstant bounded analytic functions, the uniformization theorem for Riemann surfaces implies that the universal covering surface of V^* is D, the unit disc. Let Ψ: $D \to V^*$ be the universal covering map, chosen so that $(\tau \circ \Psi)(0) = 0$. Now $\tau \circ \Psi$ is given by n analytic functions $f_1, \ldots, f_n$ on D with $f_j(0) = 0$ and $\sum_1^n |f_j(z)|^2 < 1$.

The f_i are bounded and have boundary functions f_i^*. We claim that: a.e. $d\theta$

$$\sum_1^n |f_j^*(e^{i\theta})|^2 = 1. \tag{2.6}$$

In fact, let $S = \{e^{i\theta}$: each f_j, $1 \leqq j \leqq n$, has a radial limit at $e^{i\theta}\}$. The complement of S is a null set as each f_j has a radial limit a.e. $d\theta$. We will show that (2.6) holds for $e^{i\theta} \in S$. Suppose not, then there is $e^{i\theta_0} \in S$ such that $\sum |f_i^*(e^{i\theta_0})|^2 < 1$. Then $\alpha = \left(f_1^*(e^{i\theta_0}), \ldots, f_n^*(e^{i\theta_0})\right) \in B$ and for $0 \leqq t \leqq 1$, $t \to (\tau \circ \Psi)(t\,e^{i\theta_0})$ is a path in V from 0 to α. It follows that for $0 \leqq t \leqq 1$, $t \mapsto \Psi(t\,e^{i\theta_0})$ is a path in V^* from $\Psi(0)$ to some point α^* with $\tau(\alpha^*) = \alpha$. This path, however, does not lift to a (compact) path in D, with initial point 0 and this contradicts the fact that Ψ is a covering projection. Thus (2.6) holds on S.

Now apply Theorem 1 to f_j:

$$\tfrac{1}{2} \int_0^{2\pi} |f_j^*(e^{i\theta})|^2\, d\theta \leqq \operatorname{area}\left(f_j(D)\right).$$

Adding for $1 \leqq j \leqq n$ and using (2.6) we get

$$\pi \leqq \sum_{j=1}^{n} \operatorname{area}(f_j(D)).$$

But $f_j(D)$ is the projection of V to the j-th coordinate line. Q.E.D.

Remark. An alternate proof of Theorem 2 in the context of polynomial convexity and based upon the elements of Banach algebras will appear in [1].

Next we want to generalize Theorem 2 to higher dimensional varieties. If α is a k-tuple of integers $(i_1, i_2, \ldots, i_k)$ with $1 \leqq i_j \leqq n$, we denote the length, k, of α by $|\alpha|$ and by π_α the projection $\mathbf{C}^n \to \mathbf{C}^k$: $\pi_\alpha(z_1, \ldots, z_n) = (z_{i_1}, z_{i_2}, \ldots, z_{i_k})$. It is known that if V is a pure k-dimensional analytic subvariety through 0 in the unit ball in $\mathbf{C}^n$, then the sum of the $2k$-dimensional measures of the sets $\pi_\alpha(V)$ counted with multiplicity and taken over increasing α's is bounded below by $\pi^k/k!$ = the $2k$-volume of the unit ball in $\mathbf{C}^k$. In fact, this sum is just the $2k$-volume of V [5]. We would like to show that the result remains valid without counting multiplicity. We have

Theorem 3. *There are positive constants C_k such that if V is a pure k-dimensional subvariety of the unit ball in $\mathbf{C}^n$ passing through 0 then*

$$\sum_{|\alpha|=k}' \lambda_{2k}(\pi_\alpha(V)) \geqq C_k$$

where the prime denote summation over increasing α's and λ_{2k} is Lebesque measure in $\mathbf{C}^k$.

Remark. The constants C_k are independent of n. A natural conjecture is that $\pi^k/k!$ can be chosen for C_k but we are unable to prove this unless $k=1$.

Proof. We proceed by induction on k, assuming the result for $j<k$ (for $k=1$ this is just Theorem 2). Let $B(p,r)$ be the Euclidean ball about p of radius r and let $W = V \cap B(0, \frac{1}{2})$. Since W contains a 1-dimensional variety through 0, we apply the 1-dimensional result in the ball of radius $\frac{1}{2}$ (necessitating a scale change) and conclude:

$$\sum_{j=1}^{n} \lambda_2(\pi_j(W)) \geqq (\tfrac{1}{2})^2 \pi. \tag{2.7}$$

Consider $\pi_n(W)$. If $\zeta \in \pi_n(W)$, there is $\zeta' \in \mathbf{C}^{n-1}$ such that $q = (\zeta', \zeta) \in V$ and $\|q\| < \frac{1}{2}$. Let $H_\zeta = \{z \in \mathbf{C}^n \colon \pi_n(z) = \zeta\}$ and consider the $(k-1)$-variety $H_\zeta \cap V$ (we may assume that V is irreducible so that $\dim H_\zeta \cap V < \dim V$). Now $q \in H_\zeta \cap V$ and we may apply the $(k-1)$-dimensional result to

$H_\zeta \cap V \cap B(q, \frac{1}{2})$ in $B(q, \frac{1}{2})$ and obtain:

$$\sum_{\substack{|\beta|=k-1 \\ n \notin \beta}}' \lambda_{2k-2}(\pi_\beta(H_\zeta \cap V)) \geqq (\tfrac{1}{2})^{2k-2} C_{k-1}.$$

This sum is over β with $n \notin \beta$ since $\pi_n \equiv \zeta$ on $H_\zeta \cap V$. Integrating over $\pi_n(W)$ we get

$$\sum_{\substack{|\beta|=k-1 \\ n \notin \beta}}' \int_{\pi_n(W)} \lambda_{2k-2}(\pi_\beta(H_\zeta \cap V)) \, d\lambda_2(\zeta) \geqq (\tfrac{1}{2})^{2k-2} C_{k-1} \lambda_2(\pi_n(W)).$$

Applying Fubini's theorem to the integrals on the left yields

$$\sum_{\substack{|\beta|=k-1 \\ n \notin \beta}}' \lambda_{2k}(\pi_{(\beta,n)}(V)) \geqq (\tfrac{1}{2})^{2k-2} C_{k-1} \lambda_2(\pi_n(W)). \tag{2.8}$$

Now the replacing in (2.8) of n by j, with $1 \leqq j \leqq n$, leads to n inequalities of the same type. Adding these and recalling (2.7) we get

$$\sum_{j=1}^{n} \sum_{\substack{|\beta|=k-1 \\ j \notin \beta}}' \lambda_{2k}(\pi_{(\beta,j)}(V)) \geqq (\tfrac{1}{2})^{2k} \pi C_{k-1}. \tag{2.9}$$

For a fixed α with $|\alpha| = k$ there are at most k (β, j)'s such that $(\beta, j) = \alpha$ as sets of k integers. Thus each $2k$-volume term in the left hand side of (2.9) is repeated at most k times. It follows that we may choose C_k to be $\pi C_{k-1}/k\,2^{2k}$. Q.E.D.

3.

As an application we have

Theorem 4. *Let $\mathscr{V}$ be a family of pure k-dimensional analytic subvarieties in an open subset Ω of $\mathbf{C}^n$. Suppose that for every coordinate $(n-k)$-plane H (= an affine linear subspace of $\mathbf{C}^n$ obtained by fixing k of the coordinates), the family $\mathscr{V}_H = \{V \cap H \colon V \in \mathscr{V}\}$ is locally finite. Then $\mathscr{V}$ is locally finite.*

Remark. This extends a result of Nishino [4] who treated the case of varieties of codimension 1.

Proof. We argue by contradiction. If $\mathscr{V}$ is not locally finite then there is $p \in \Omega$ and distinct $\{V_j\}_1^\infty \subseteq \mathscr{V}$ with $p_j \in V_j$ such that $p_j \to p$. Choose $r > 0$ such that $\overline{B(p,r)} \subseteq \Omega$. Let $W_j = V_j \cap B(p,r)$, and let α be a k-tuple. If $\zeta \in \mathbf{C}^k$, then $\pi_\alpha^{-1}(\zeta)$ is a coordinate $n-k$ plane and so $W_j \cap \pi_\alpha^{-1}(\zeta)$ is empty for large j by hypothesis. That is, the sets $\pi_\alpha(W_j)$ eventually omit every point in $B(\pi_\alpha(p), r)$. It follows that $\lambda_{2k}(\pi_\alpha(W_j)) \to 0$ for every α. But by Theorem 3 a finite sum of such terms is bounded below by a number approaching $r^{2k} C_k$ (since $p_j \to p$). This is a contradiction. Q.E.D.

Theorem 5. *Let A be a subset of an open set $\Omega \subseteq \mathbf{C}^n$. Let $r \geqq 1$ be an integer. Suppose that the intersection of A with every coordinate hyperplane (i.e. every coordinate $(n-1)$-plane) is a pure r-dimensional subvariety of Ω. Then A is a subvariety of Ω (of dimension $r+1$).*

Remark. Rothstein proved this proposition under the further assumption that A is a (relatively) closed subset of Ω in [7] and asked there if this restriction was really needed. What we shall prove here is simply that the hypotheses of the theorem force A to be a relatively closed subset of Ω.

Proof. In order to show that A is a relatively closed subset of Ω it is enough to show that if $0 \in \bar{A} \cap \Omega$ then $0 \in A$. We argue by contradiction and suppose otherwise; i.e. that $0 \in \bar{A} \cap \Omega$ but $0 \notin A$. Then if H is the coordinate hyperplane $\{z\colon \pi_n(z)=0\}$, $A \cap H$ is a subvariety of Ω and hence relatively closed. Consequently, as $0 \notin A$ there is $\delta > 0$ such that

$$\{z\colon |z_1| \leqq \delta, |z_2| \leqq \delta, \ldots, |z_{n-1}| \leqq \delta,\ z_n = 0\} \cap A = \emptyset \tag{3.1}$$

and by choosing δ small enough we may assume that

$$\{z\colon |z_j| \leqq \delta,\ 1 \leqq j \leqq n\} \subseteq \Omega.$$

As $0 \in \bar{A}$ there are $(c_j, c_j') \in (\mathbf{C}^{n-1} \times \mathbf{C}) \cap A$ such that $c_j' \to 0$, $c_j \to 0$ and $c_j = (c_1^j, \ldots, c_{n-1}^j) \in \mathbf{C}^{n-1}$ with each $|c_s^j| < \delta$ and $|c_j'| < \delta$.

Let $\Omega_1 = \{z \in \mathbf{C}^{n-1}\colon |z_s| < \delta,\ 1 \leqq s \leqq n-1\}$, let $\tilde{\pi}\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ be the projection $\tilde{\pi}(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1})$ and let

$$W_j = \tilde{\pi}(A \cap \{z\colon |z_1| < \delta, \ldots, |z_{n-1}| < \delta,\ z_n = c_j'\}).$$

Then W_j is a r-dimensional subvariety of Ω_1 and $\{W_j\}$ is not a locally finite family as $W_j \ni c_j \to 0$. Hence by Theorem 4, the intersections of the W_j's with some coordinate $(n-1)-r$ plane is not locally finite and consequently there is a coordinate $n-2$ plane K in $\mathbf{C}^{n-1}$ such that the sets $K \cap W_j$ are not eventually empty. Passing to a subsequence we may assume that $K \cap W_j$ is never empty. Without loss of generality, we may assume that $K = \{z \in \mathbf{C}^{n-1}\colon z_1 = \lambda\}$ where $|\lambda| < \delta$. This means that there are points $d_j = (d_1^j, \ldots, d_{n-2}^j) \in \mathbf{C}^{n-2}$ such that each $|d_s^j| < \delta$ and $(\lambda, d_j) \in W_j$. Hence $(\lambda, d_j, c_j') \in A$. By passing to a subsequence, we may assume that $d_s^j \to d^j$ with $|d^j| \leqq \delta$. Put $d = (d^1, d^2, \ldots, d^{n-2})$. Now $A \cap \{z\colon z_1 = \lambda\}$ is a relatively closed subset of Ω and contains the points (λ, d_j, c_j') which converge to $(\lambda, d, 0) \in A$. But this contradicts (3.1) and we conclude that A is relatively closed in Ω. Q.E.D.

References

1. Alexander, H.: Projections of polynomial hulls (to appear).
2. Duren, P.: Theory of H^p spaces. New York: Academic Press 1970.
3. Hewitt, E., Stromberg, K.: Real and abstract analysis. Berlin-Heidelberg-New York: Springer 1965.
4. Nishino, T.: Sur une propriété des familles de fonctions analytiques de deux variables complexes. J. Math. Kyoto Univ. 4-2, 255–282 (1965).
5. Stolzenberg, G.: Volumes, limits, and extensions of analytic varieties. Lecture Notes in mathematics **19**. Berlin-Heidelberg-New York: Springer 1966.
6. Rham, G. de: On the area of complex manifolds, global analysis (volume dedicated to K. Kodaira), pp. 141–148. Tokyo Univ. Press and Princeton Univ. Press, 1969.
7. Rothstein, W.: Zur Theorie der analytischen Mengen. Math. Ann. **174**, 8–32 (1967).

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(Received February 12, 1972)