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Rigidity of invariant convex sets in symmetric spaces*

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Abstract. The main result implies that a proper convex subset of an irreducible higher rank symmetric space cannot have Zariski dense stabilizer.

1. Introduction

In this paper we study convex subsets of symmetric spaces, and their stabilizers. The main results show that in the higher rank case convex sets are strongly restricted, and under mild assumptions can only arise from rank 1 constructions. This rigidity phenomenon for convex subsets is yet another example of a rigidity property enjoyed by higher rank symmetric spaces that has no analog for rank 1 symmetric spaces.

One can generate a supply of convex subsets of any Hadamard space by starting with geodesic segments, geodesic rays, complete geodesics, and horoballs, and then taking tubular neighborhoods and intersections. When X is a Hadamard manifold with pinched negative curvature convex subsets are abundant: by a theorem of Anderson [And83], any closed subset A of the geometric boundary $\partial_\infty X$ is the limit set of a closed convex subset $Y \subset X$. On the other hand, for general Hadamard spaces (or manifolds) it can be difficult to control the convex hull of even “small” subsets, like the union of three rays.

A group Γ of isometries of a Hadamard space X is *convex cocompact* if there is a Γ-invariant convex subset $C \subset X$ with compact quotient C/Γ. Discrete convex cocompact subgroups of the isometry group of hyperbolic 3-space are an important class in the theory of Kleinian groups; basic examples are uniform lattices, Schottky groups and quasi-Fuchsian groups. Analogous examples exist in $\mathrm{Isom}(H^n)$, as well as the isometry groups

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of other rank 1 symmetric spaces. In a higher rank symmetric space of noncompact type, one can produce examples by taking products of uniform lattices and rank 1 convex cocompact groups. In 1994, Corlette asked if this was essentially the only way to produce discrete convex cocompact groups. The answer is yes, see Theorem 1.3 below; in fact the theorem is proved by reducing it to the case of convex subsets with Zariski dense stabilizer:

Theorem 1.1. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type, and let $X = \mathbb{E}^n \times Y_1 \times Y_{\geq 2}$ denote the decomposition of X into the Euclidean factor, the product of the irreducible rank 1 factors, the product of the higher rank factors. Suppose $\Gamma \subset \mathrm{Isom}(X) = \mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(Y)$ is a subgroup whose projection to $\mathrm{Isom}(Y)$ is Zariski dense in the identity component $\mathrm{Isom}_o(Y)$, and whose projection to $\mathrm{Isom}(\mathbb{E}^n)$ does not preserve a proper affine subspace of $\mathbb{E}^n$. If $C \subset X := \mathbb{E}^n \times Y$ is a Γ-invariant closed convex set, then $C = \mathbb{E}^n \times C_1 \times Y_{\geq 2}$, where $C_1 \subset Y_1$ is a closed convex subset. Furthermore, for each de Rham factor X_i of Y_1, there is a Γ-invariant subset $\hat{C}_i \subset X_i$ such that*

- *$\hat{C}_i$ is the closed convex hull of its limit set.*
- *$|\partial_\infty \hat{C}_i| = \infty$,*
- *$\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$.*
- *$\partial_\infty \hat{C}_1 = \partial_\infty C_1$.*

We recall that by convention, a symmetric space of noncompact type has no Euclidean de Rham factor. Note that a subgroup of $\mathrm{Isom}_o(Y)$ is Zariski dense if and only if it neither fixes a point in the Tits boundary $\partial_T Y$ nor preserves a proper symmetric subspace of Y.

Corollary 1.2. *If X is a symmetric space of noncompact type with no rank 1 de Rham factors and $\Gamma \subset \mathrm{Isom}_o(X)$ is a Zariski dense subgroup, then X contains no proper closed Γ-invariant convex subsets.*

For discrete convex cocompact groups, we have the following structural result:

Theorem 1.3. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type. Suppose $\Gamma \subset \mathrm{Isom}(X) = \mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(Y)$ is a discrete subgroup acting cocompactly on a closed convex subset $C \subset X$, and assume Γ does not preserve any proper symmetric subspace of X. Then Γ projects to a subgroup of $\mathrm{Isom}(Y)$ which is Zariski dense in $\mathrm{Isom}_o(Y)$, and the conclusions of Theorem 1.1 apply to C.*

If a convex cocompact subgroup $\Gamma \subset \mathrm{Isom}(X)$ preserves a proper symmetric subspace $Z \subset X$, then it acts convex cocompactly on Z – just intersect a sufficiently big tubular neighborhood of a Γ-invariant convex set with Z. Therefore there is no loss of generality in assuming X contains no proper Γ-invariant symmetric subspace.

Corollary 1.4. *If X is a symmetric space of noncompact type with no rank 1 de Rham factors, and $\Gamma \subset \mathrm{Isom}(X)$ is a discrete subgroup acting cocompactly on a closed convex subset $C \subset X$, then either $C = X$ and Γ is a uniform lattice in $\mathrm{Isom}(X)$, or Γ preserves a proper symmetric subspace of X.*

We give a brief outline of the proof of Theorem 1.1 in the case the Euclidean factor is absent, and Y is an irreducible higher rank symmetric space. The first step is to apply a Theorem of Benoist [Ben97], which implies one may find an open neighborhood U of a pair of antipodal points ξ_1, ξ_2 in the Tits boundary $\partial_T X$, such that U is contained in the limit set of Γ. Applying a result from [KL97], we deduce that the geometric boundary of C is a top dimensional subbuilding B of the Tits boundary of X, which is a closed subset with respect to the topology of the geometric boundary $\partial_\infty X$. The main step in the paper, implemented in Theorem 3.1, is to show that any such building is contained in the geometric boundary of a proper symmetric subspace Y, unless it coincides with $\partial_T X$; the Zariski density assumption rules out the former possibility in the case at hand. We remark that Theorem 3.1 applies to products of symmetric spaces and Euclidean buildings, and may be of independent interest.

In view of the results in this paper one may wonder whether sufficiently large convex sets in symmetric spaces of noncompact type or in spherical buildings (such as Tits boundaries of symmetric spaces) are rigid.

Question 1.5. Suppose $C \subset B$ is a convex subset of a spherical building. If C does not have circumradius $\leq \frac{\pi}{2}$, must C itself be a spherical building?

It is unclear what one should expect here. A. Balser and A. Lytchak [BL] proved a partial result regarding convex subsets invariant under a group action, namely if $\dim(C) \leq 2$ and C is not a spherical building then $\mathrm{Isom}(C)$ has a fixed point in C.

After the first version of this paper was written, Quint informed the authors of very interesting related work [Qui04] on Zariski dense subgroups of semi-simple groups. His paper addresses an alternate definition of convex cocompact groups which is equivalent to the usual definition for rank 1 symmetric spaces but differs from ours in the higher rank case; for this reason it is difficult to make a direct comparison between the results of [Qui04] and the theorems above. We mention that his main result also applies to discrete subgroups of semi-simple p-adic groups.

In the last section of the paper, we discuss a variant of Theorem 1.1 for quasiconvex subsets.

The authors proved slightly weaker versions of Theorems 1.1 and 1.3 in 1998, and spoke publicly about them in the subsequent year.

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Contents

2. Preliminaries

2.1. Hadamard spaces. We recommend [Bal95,BH99,KL97] as references for Hadamard space facts.

The term *Hadamard space* is a synonym for a $CAT(0)$-space.

If X is a Hadamard space, we denote the geometric boundary by $\partial_\infty X$, the Tits boundary by $\partial_T X$, and the Tits angle between $\xi_1, \xi_2 \in \partial_T X$ by $\angle_T(\xi_1, \xi_2)$.

Recall that the set underlying $\partial_\infty X$ is the set of asymptote classes of geodesic rays, and that this may be identified with the set of rays leaving a given basepoint $p \in X$. If $x_1, x_2 \in X$, $Y \subset X$ is a subset, $y_i \in Y$ is a sequence with $\lim_{i\to\infty} d(y_i, p) = \infty$, then the segments $\overline{x_1 y_i}$ converge to a ray $\overline{x_1\xi}$ iff the segments $\overline{x_2 y_i}$ converge to a ray $\overline{x_2\xi}$. Thus the set of rays which can be obtained as limits in this fashion, as $\{y_i\}$ ranges over all such sequences, is a collection of asymptote classes and therefore determines a subset of $\partial_\infty X$, the *limit set of* Y, which we denote by $\Lambda(Y)$.

Lemma 2.1. *If $C \subset X$ is a closed convex subset, and $p \in C$, then every ray $\overline{p\xi}$ is contained in C, for $\xi \in \Lambda(C)$.*

Proof. This follows from the convexity of C and the definition of the limit set, since we are at liberty to select the basepoint. □

Definition 2.2. A subset Y of a $CAT(1)$ space Z is *convex* if it contains every segment of the form $\overline{\xi_1\xi_2}$, where $\xi_1, \xi_2 \in Y$ and $d_Z(\xi_1, \xi_2) < \pi$.

Lemma 2.3. *Let X be a proper Hadamard space, and let $C \subset X$ be a closed convex subset. Then the limit set of C in $\partial_\infty X$ determines a convex subset of $\partial_T X$, which is isometric to the Tits boundary of C, viewed as a Hadamard space.*

Proof. The isometric embedding $C \to X$ of Hadamard spaces induces an isometric embedding $\partial_T C \to \partial_T X$ of Tits boundaries. Since $\partial_T C$ is a $CAT(1)$ space, the image of the embedding is convex. □

Lemma 2.4. *If $\Gamma \curvearrowright X$ is a discrete, cocompact, isometric action on a Hadamard space X, and Γ fixes a point $\xi \in \partial_T X$, then there is a geodesic $\gamma \subset X$ such that $\xi \in \partial_T \gamma$ and the parallel set $\mathbb{P}(\gamma) \subset X$ is Γ-invariant.*

Proof. We may assume that X contains no proper, closed, convex, Γ-invariant nonempty subset, by applying Zorn's lemma.

Note that any element $g \in Z(\Gamma)$ is semi-simple and its minimum displacement set, $\min(g) \subset X$, is a closed, convex, and Γ-invariant subset; therefore by assumption we have $\min(g) = X$. Thus elliptic elements in $Z(\Gamma)$ act trivially on X and nonelliptic elements act by Clifford translations, i.e. they translate along the $\mathbb{R}$-factor of a product splitting $X = \mathbb{R} \times Z$. Hence X admits a product structure

$$X = \mathbb{E}^n \times Y \tag{2.5}$$

where $Z(\Gamma)$ acts cocompactly by translations on $\mathbb{E}^n$ and trivially on Y.

Pick a point $p \in X$, and a finite generating set $\Sigma \subset \Gamma$. Let $C := \max_{\sigma \in \Sigma} d(\sigma p, p)$. Note that the ray $\overline{p\xi} \subset X$ lies in the closed convex set

$$\Delta := \{x \in X \mid \text{For all } \sigma \in \Sigma, d(\sigma x, x) \leq C\},$$

since for all $g \in \Gamma$ and every $x \in \overline{p\xi}$, we have $d(gx, x) \leq d(gp, p)$ because $\overline{p\xi}$ and $\overline{(gp)\xi}$ are asymptotic rays. By a standard argument the centralizer, $Z(\Sigma) = Z(\Gamma)$, of the set Σ acts cocompactly on Δ, which implies that $\overline{p\xi}$ is contained in a finite tubular neighborhood of an n-flat $\mathbb{E}^n \times \{y\}$ of the product decomposition (2.5). Hence $\xi \in \partial_T \mathbb{E}^n$, and this implies the lemma. □

2.2. Affine and concave functions on convex sets

Lemma 2.6. *Let Z be a geodesic metric space with extendible geodesics. Then any concave function $Z \to [0, \infty)$ is constant.*

Proof. Trivial. □

Lemma 2.7. *Let Z be a CAT(-1)-space whose ideal boundary $\partial_\infty Z$ consists of at least two points. Suppose that there is no proper closed convex subset of Z whose ideal boundary equals $\partial_\infty Z$. Then any continuous concave function $f : Z \to [0, \infty)$ is constant.*

Proof. We first observe that f is constant along each complete geodesic. Furthermore, f is non-decreasing along each geodesic ray, and the restriction of f to a compact geodesic segment assumes its minimum at one of the endpoints.

Note that, by assumption, Z contains at least one complete geodesic. Let l be a complete geodesic and $z \in Z$ be an arbitrary point. Denote by $\rho_1, \rho_2 : [0, \infty) \to Z$ the rays emanating from z and asymptotic to the two ends of l. Then f is $\geq f(z)$ along each segment connecting $\rho_1(t)$ to $\rho_2(t)$

for $t \geq 0$. Since Z is CAT(-1) these segments converge to the line l. The continuity of f then implies that $f(l) \geq f(z)$. Thus f assumes on l its maximum which we denote by m.

It follows that f equals m on the union H_1 of all lines in Z. Consider the ascending sequence of subsets $H_n \subset Z$ defined inductively by requiring that H_{n+1} is the union of all segments with endpoints in H_n. Then the sequence of suprema $\sup(f|_{H_n})$ is non-decreasing. Hence $m = \sup(f|_{H_1}) \leq \sup(f|_{H_n}) \leq m$ and $f \equiv m$ on the closure of $\cup_{n\in\mathbb{N}} H_n$. This closure is a closed convex subset of Z with the same ideal boundary and, by assumption, equals Z. □

By an *affine* function on a geodesic metric space we mean a function whose restriction to each geodesic segment is an affine function.

Lemma 2.8. *Let Z be a CAT(-1)-space whose ideal boundary $\partial_\infty Z$ consists of at least three points. Then any affine continuous function $f : Z \to \mathbb{R}$ is constant.*

Proof. We first observe that the slope of f along a geodesic ray depends only on the ideal point represented by it. Indeed, let $\rho_1, \rho_2 : [0.\infty) \to Z$ be two rays parametrized by unit speed. Since the geodesic segments connecting $\rho_1(0)$ with $\rho_2(t)$ converge to the ray ρ_1 it follows using continuity that the slope of f along ρ_1 equals its slope along ρ_2.

Since any two ideal points in $\partial_\infty Z$ may be connected by a complete geodesic in Z it follows that the slopes of f at any two ideal points have opposite sign. Since $\partial_\infty Z$ contains at least three points the slopes of f must be zero at all ideal points, i.e. f is constant along every geodesic ray.

The same reasoning as in the proof of Lemma 2.7 above shows that for any point z and any complete geodesic l in Z we have $f(z) = f(l)$. Thus f is constant. □

The following observation is a special case of the main result in [Inn82] (which applies to complete Riemannian manifolds without any curvature assumption).

Lemma 2.9. *Let Z be a symmetric space of noncompact type and higher rank without Euclidean de Rham factor. Then any affine continuous function $f : Z \to \mathbb{R}$ is constant.*

Proof. We may apply Lemma 2.8 to (nonflat) totally geodesic subspaces of rank one and get that f is constant on any such subspace.

Let F be a maximal flat. Then $f|_F$ is affine. The previous remark implies that the gradient of $f|_F$ at a point $z \in F$ must be tangent to every singular hyperplane H through z because the lines in F perpendicular to H lie in a rank one subspace. Since Z has no Euclidean factor the intersection of all these hyperplanes H is just the point z. We conclude that f is constant along every maximal flat; since any two points lie in a maximal flat, this implies that f is constant on Z. □

2.3. Asymptotic slopes of convex functions. Let Z be a Hadamard space and $f : Z \to \mathbb{R}$ a continuous convex function. For a unit speed geodesic ray $\rho : [0, \infty) \to Z$ we define the *asymptotic slope* of f along ρ as $\mathrm{slope}_f(\rho) := \lim_{t\to\infty} \frac{f(\rho(t))}{t} \in \mathbb{R} \cup \{\infty\}$.

Lemma 2.10. *For any two asymptotic unit speed rays ρ_1 and ρ_2,* $\mathrm{slope}_f(\rho_1) = \mathrm{slope}_f(\rho_2)$.

Proof. Since the segments connecting $\rho_2(0)$ with $\rho_1(t)$ Hausdorff converge to ρ_2 one estimates using the continuity of f that $f(\rho_2(t)) \leq C + \mathrm{slope}_f(\rho_1) \cdot t$ for $t \geq 0$ and hence $\mathrm{slope}_f(\rho_2) \leq \mathrm{slope}_f(\rho_1)$. Symmetry implies equality. □

Thus we may speak of the asymptotic slope, $\mathrm{slope}_f(\xi)$, at an ideal point $\xi \in \partial_\infty Z$.

Lemma 2.11. $\mathrm{slope}_f : \partial_\infty Z \to \mathbb{R} \cup \{\infty\}$ *is lower semicontinuous with respect to the cone topology.*

Proof. Consider a sequence of unit speed rays ρ_n with same initial point which Hausdorff converges to the ray ρ. Since $\mathrm{slope}_f(\rho_n) \geq \frac{f(\rho_n(t))-f(\rho_n(0))}{t}$ for $t \geq 0$ we obtain

$$\liminf_{n\to\infty} \mathrm{slope}_f(\rho_n) \geq \frac{f(\rho(t)) - f(\rho(0))}{t} \xrightarrow{t \nearrow \infty} \mathrm{slope}_f(\rho).$$

□

As a consequence, slope_f attains a minimum if Z is locally compact.

Proposition 2.12. *If* $\mathrm{slope}_f : \partial_\infty Z \to \mathbb{R} \cup \{\infty\}$ *assumes negative values then it has a unique minimum.*

Proof. Let $\xi_1, \xi_2 \in \partial_\infty Z$ be ideal points with $\mathrm{slope}_f(\xi_i) \leq -a < 0$ and $\angle_T(\xi_1, \xi_2) \geq \epsilon > 0$. Let ρ_i be unit speed rays emanating from the same point $o \in Z$ and asymptotic to the ideal points ξ_i. For the midpoints $m(t)$ of the segments $\overline{\rho_1(t)\rho_2(t)}$ holds

$$\limsup_{t\to\infty} \frac{d(o, m(t))}{t} \leq \cos \frac{\angle_T(\xi_1, \xi_2)}{2} \leq \cos \frac{\epsilon}{2}.$$

On the other hand, $f(m(t)) - f(o) \leq -at$ for $t \geq 0$. Using the continuity of f in o, there exists $\delta > 0$ such that $f \geq f(o) - 1$ on the ball $B_\delta(o)$. By convexity, we have $f(m(t)) - f(o) \geq -\frac{1}{\delta} d(o, m(t))$ and so $d(o, m(t)) \geq \delta a t$. This implies that

$$\liminf_{t\to\infty} \frac{d(o, m(t))}{t} > 0$$

and hence $\angle_T(\xi_1, \xi_2) < \pi$. The segments $\overline{om(t)}$ Hausdorff converge to a ray asymptotic to the midpoint μ of $\overline{\xi_1\xi_2}$ which therefore satisfies $\mathrm{slope}_f(\mu) \leq -a(\cos \frac{\epsilon}{2})^{-1}$.

It follows that any sequence (ξ_n) in $\partial_\infty Z$ with $\text{slope}_f(\xi_n) \searrow \inf \text{slope}_f$ is a Cauchy sequence with respect to the Tits metric. Hence slope_f has a unique minimum on $\partial_\infty Z$. □

2.4. Spherical buildings. We refer the reader to [KL97,Ron89,Tit74] for further discussion of the material here.

We will be using the geometric definition of spherical buildings from [KL97], which we now recall.

Let (S, W) be a spherical Coxeter complex, so S is a Euclidean sphere and W is a finite group generated by reflections acting on S. A *spherical building modelled on* (S, W) is a $CAT(1)$-space B together with a collection $\mathcal{A}$ of isometric embeddings $\iota : S \to B$, called *charts*, which satisfies Properties SB1–2 described below and which is closed under precomposition with isometries in W. An *apartment* in B is the image of a chart $\iota : S \to B$; ι is a chart of the apartment $\iota(S)$.

SB1: Plenty of apartments. Any two points in B are contained in a common apartment.

Let ι_{A_1}, ι_{A_2} be charts for apartments A_1, A_2, and let $C = A_1 \cap A_2$, $C' = \iota_{A_2}^{-1}(C) \subset S$. The charts ι_{A_i} are *W-compatible* if $\iota_{A_1}^{-1} \circ \iota_{A_2}|_{C'}$ is the restriction of an isometry in W.

SB2: Compatible apartments. The charts are W-compatible.

2.5. Root groups. If B is a spherical building, and $a \subset B$ is a root, then the *root group of a* is the collection U_a of building automorphisms of B which fix a pointwise, as well as any chamber $\sigma \subset B$ such that $\sigma \cap a$ is a panel π which is not contained in the wall ∂a. The building B is *Moufang* if for every root $a \subset B$, the group U_a acts transitively on the set of roots opposite a.

Properties of root groups:

- When all the join factors of B have dimension at least 1, then U_a acts freely on the collection of roots opposite a.
- When X is a symmetric space of noncompact type and $B := \partial_T X$, then B is a Moufang building and $G := \text{Isom}_o(X)$ acts effectively on B by building automorphisms, so we may view G as a subgroup of $\text{Aut}(B)$. Each root group of B is contained in G, and is a unipotent subgroup [Tit74, pp. 77–78]. Furthermore, G is generated by the root groups of B.

2.6. Groups acting on symmetric spaces. Let X be a symmetric space of noncompact type, and let $G := \text{Isom}_o(X)$. We will require the following well known facts [Mos55,BT71]:

- A subgroup $H \subset G$ is Zariski dense if and only if H neither fixes a point in $\partial_T X$ nor preserves a proper symmetric subspace.

- A proper subgroup $H \subsetneq G$ with finitely many connected components is not Zariski dense; in particular H must either fix a point in $\partial_T X$ or preserve a proper symmetric subspace.

Remark 2.13. If a Zariski dense subgroup of a real simple group is not dense in the usual topology, then it must be discrete.

Lemma 2.14. *Let $F \subset X$ be a maximal flat, and suppose $K \subset G$ is a subgroup fixing $\partial_\infty F$ pointwise, and acting transitively on F. Then*

- *The fixed point set of K in $\partial_\infty X$ is precisely $\partial_\infty F$.*
- *If $\xi \in \partial_\infty X$, $\xi' \in \partial_T F$ is antipodal to ξ, and $\hat{\xi} \in \partial_T F$ is the antipode of ξ', then the closure of the K-orbit $K(\xi)$ contains $\hat{\xi}$.*

Proof. The first assertion is implied by the second.

First assume that ξ is regular. Then there is a geodesic γ with $\partial_\infty \gamma = \{\xi, \xi'\}$, and hence if $g \in K$ translates F in the direction $\hat{\xi}$, we get that $g^k \xi$ converges to $\hat{\xi}$ in the topology of $\partial_\infty X$.

In the general case, if $\sigma \subset \partial_T X$ is a chamber containing ξ, then there is a chamber $\sigma' \subset \partial_T F$ containing ξ' opposite to σ, and σ' is opposite a chamber $\hat{\sigma} \subset \partial_T F$. By the regular case, the K-orbit of σ accumulates on $\hat{\sigma}$, which implies that the K-orbit of ξ accumulates on $\hat{\xi}$. $\square$

3. Top dimensional subbuildings in the boundary of a symmetric space

In this section we prove:

Theorem 3.1. *Suppose*

$$X = X_1 \times \ldots \times X_k \tag{3.2}$$

is a product of irreducible symmetric spaces of noncompact type, irreducible Euclidean buildings with discrete affine Weyl groups, and Euclidean spaces. Let $B \subset \partial_T X$ be a top dimensional subbuilding which is closed with respect to the topology of the geometric boundary $\partial_\infty X$, and which is not contained in the boundary of any proper subspace $Y \subset X$ of the form $Y = Y_1 \times \ldots \times Y_k$, where $Y_i \subset X_i$ is either a totally geodesic subspace or a subbuilding, according to the type of X_i. Then there is a join decomposition

$$B = B_1 \circ \ldots \circ B_k$$

where $B_i := B \cap \partial_T X_i$, such that $B_i = \partial_T X_i$ unless X_i is an irreducible rank 1 symmetric space of noncompact type.

We begin the proof by observing that if there is more than one factor in the product decomposition (3.2), then by [KL97, Prop. 3.3.1], B and $\partial_T X$ will admit corresponding compatible join decompositions

$$B = B_1 \circ \ldots \circ B_k, \quad \partial_T X = \partial_T X_1 \circ \ldots \circ \partial_T X_k,$$

and hence it is sufficient to prove the theorem for the irreducible factors X_i separately. So henceforth we will assume that X is irreducible. If X is Euclidean, then $\partial_T X$ is the only top dimensional subbuilding of $\partial_T X$, and so this case is trivial.

3.1. The case when X is a Euclidean building. Let $Y \subset X$ be the union of the collection of apartments $A \subset X$ such that $\partial_T A \subset B$. By a chamber in X we mean a top-dimensional simplex with respect to the natural structure of the irreducible discrete Euclidean building X as a simplicial complex.

Lemma 3.3. *Any two chambers $\sigma_1, \sigma_2 \subset Y$ lie in an apartment $A \subset X$ which is entirely contained in Y.*

Proof. By the definition of Y, for $i = 1, 2$ there exists an apartment $A_i \subset X$ such that $\partial_T A_i \subset B$ and $\sigma_i \subset A_i$. For $i = 1, 2$, choose an interior point $p_i \in \sigma_i$, and consider the geodesic segment $\overline{p_1 p_2} \subset X$. By perturbing p_2 slightly, if necessary, we may assume that the Δ_{mod}-direction of $\overline{p_1 p_2}$ is regular. We may prolong $\overline{p_1 p_2}$ to a complete regular geodesic $\gamma \subset X$ by concatenating it with rays $\overline{p_1 \xi_1} \subset A_1$, $\overline{p_2 \xi_2} \subset A_2$. Since $\partial_T \gamma = \{\xi_1, \xi_2\}$ where $\xi_i \in \partial_T A_i \subset B$ are regular, there is a unique apartment $\partial_T A \subset \partial_T X$ containing $\partial_\infty \gamma$, and it is contained in B. Then by the definition of Y we have $A \subset Y$, and since γ is regular and $\partial_T \gamma \subset \partial_T A$, we get $\gamma \subset A$. This implies that $\sigma_i \subset A$, since $\sigma_i \cap A \supset \{p_i\} \neq \emptyset$ is a subcomplex of X. □

The lemma implies that Y is a subbuilding of X with Tits boundary B. By assumption we must therefore have $B = \partial_T X$, which proves Theorem 3.1 in this case.

3.2. X is an irreducible symmetric space of noncompact type. We will assume that X has rank at least two, since otherwise there is nothing to prove. The strategy of the proof is to use B to produce a subgroup $H \subset G$ which has no fixed point in $\partial_T X$, which can be used to tie B closely with X. When B is irreducible, H is generated using "restricted" root groups, and when B is reducible H is generated by transvections, and decomposes as a product.

We let W denote the Weyl group of X. Thus $\partial_T X$ is a spherical building modelled on a spherical Coxeter complex (S, W). We let $W_B \subset W$ denote the sub-Coxeter group defining a *thick* building structure on B, see [KL97, Sect. 3.7]; thus each W_B-wall in B lies in at least 3 roots (or half-apartments) of B.

Case 1. The subbuilding B is irreducible. Our first step is to show that the Moufang property restricts to top dimensional irreducible subbuildings. Let $a \subset B$ be a W_B-root in B. Let $U_a \subset \mathrm{Aut}(\partial_T X)$ denote the root group of a (see Sect. 2.5).

Definition 3.4. The *restricted root group* of a is defined to be the subgroup $U_a^B \subset U_a$ which preserves the subbuilding $B \subset \partial_T X$.

Lemma 3.5. *U_a^B acts transitively on the collection of roots in B opposite to a.*

Proof. Pick two W_B-roots $a_1, a_2 \subset B$ opposite a. Since $\partial_T X$ is Moufang, there is a unique $g \in U_a$ such that $g(a_1) = a_2$. Let $B' := B \cap g^{-1}(B)$. Note that $B' \subset B$ is a convex subset (see Definition 2.2) containing the apartment $a \cup a_1$; therefore by [KL97, Prop. 3.10.3], B' is a top dimensional subbuilding of B. Let $\sigma \subset a$ be a W-chamber disjoint from the boundary ∂a, and for $i = 1, 2$ let $\sigma_i \subset a_i$ be the chamber in a_i opposite σ; likewise, let $\pi \subset \sigma$ be a panel (a codimension 1 face) of σ, and for $i = 1, 2$ let $\pi_i \subset \sigma_i$ be the opposite panel in a_i. Now for each chamber $\sigma' \subset B$ incident to σ along π, for each $i = 1, 2$ there is a unique chamber σ_i' incident to σ_i along π_i, which corresponds to σ' under the correspondence of [KL97, Prop. 3.6.4]; clearly $g(\sigma_1) = \sigma_2$, and hence $g(\sigma_1') = \sigma_2'$. This implies that $\sigma_1' \subset B'$. Now we may argue as in the proof of [KL97, Prop. 3.12.2] to see that $B' = B$, and therefore $g(B) \subset B$; applying the same reasoning to g^{-1} we conclude that $g(B) = B$. Thus we have shown that U_a^B acts transitively on the roots in B opposite a. □

Now pick a W_B-wall $\omega \subset B$, and let $\partial_T X(\omega) \subset \partial_T X$ be the subbuilding consisting of the union of the apartments containing ω; similarly, let $B(\omega)$ be the subbuilding of B determined by ω. Thus if $F \subset X$ is a singular flat with $\partial_T F = \omega$, then the parallel set $\mathbb{P}(F)$ has Tits boundary $\partial_T X(\omega)$, the product splitting $\mathbb{P}(F) = F \times Y$ induces a join decomposition $\partial_T X(\omega) = \omega \circ \partial_T Y$, and $Y \subset X$ is a rank 1 symmetric subspace of dimension > 1. This join decomposition induces a join decomposition $B(\omega) = \omega \circ \Lambda$, where $\Lambda := \partial_T Y \cap B$.

Lemma 3.6. *Λ is a compact connected manifold of positive dimension, and each restricted root group U_a^B, when viewed as a subset of $G = \mathrm{Isom}_o(X)$, is connected.*

Proof. We observe that for each root $a \subset \partial_T X$ with $\partial a = \omega$, the root group U_a acts freely transitively by homeomorphisms on $\partial_\infty Y \setminus \{\xi\}$, where $a = \omega \circ \xi$. Thus if we choose $\xi' \in \partial_T Y \setminus \{\xi\}$ and let $a' := \omega \circ \xi'$, then the map $\phi : U_a \to \partial_\infty Y \setminus \{\xi\}$ defined by $\phi(g) := g\xi'$ is a continuous bijection between manifolds, and is therefore a homeomorphism. Now suppose $\xi, \xi' \in \Lambda$, so that $a, a' \subset B$. The restricted root group $U_a^B \subset U_a$ acts simply transitively on $\Lambda \setminus \{\xi\}$, so ϕ restricts to a homeomorphism $U_a^B \to \Lambda \setminus \{\xi\}$. Thus U_a^B is a closed subgroup of U_a, and is therefore a manifold, which means that $\Lambda \setminus \{\xi\}$ is also a manifold. Note that $|\Lambda| \geq 3$, since Λ is in bijection with the roots of B containing ω. Since $\xi \in \Lambda$ was chosen arbitrarily, it follows that the group generated by the collection of restricted root groups $\{U_a^B \mid a = \omega \circ \xi, \quad \xi \in \Lambda\}$, acts transitively on Λ. Thus Λ is a compact manifold.

Since U_a is unipotent, every $g \in U_a^B \setminus \{e\}$ has infinite order. This implies that Λ is an infinite set; being a compact manifold, it must have positive dimension.

If $\xi \in \Lambda$ and $a := \omega \circ \xi$, then U_a^B acts transitively on $\Lambda \setminus \{\xi\}$ while preserving the connected component of Λ containing ξ. It follows that Λ is connected.

For any W_B root $a = \omega \circ \xi$, the restricted root group U_a^B is homeomorphic to $\Lambda \setminus \{\xi\}$; since Λ is a compact connected manifold of dimension ≥ 1, this is obviously connected. $\square$

Let $H_\omega \subset G$ be the subgroup generated by the restricted root groups U_a^B, where $\partial a = \omega$. Since each U_a^B is connected, so is H_ω. As each restricted root group is unipotent, H_ω acts trivially on the flat factor F of $\mathbb{P}(F) = F \times Y$.

Lemma 3.7. *There is an H_ω-invariant symmetric subspace $Z \subset Y$ such that $\partial_T Z = \Lambda$, and the image of H_ω in* $\mathrm{Isom}(Z)$ *is the identity component of* $\mathrm{Isom}(Z)$. *Moreover, if $\xi_1, \xi_2 \in \Lambda$, and $\gamma \subset Y$ is the geodesic asymptotic to $\{\xi_1, \xi_2\}$, then there is a* 1*-parameter subgroup of H_ω which acts on $F \times \gamma$ by translating in the γ-direction.*

Proof. First observe that H_ω has no fixed points in $\partial_\infty Y$: if $\xi \in \Lambda$ and $a := \omega \circ \xi$, then U_a^B is a unipotent group whose only fixed point in $\partial_\infty Y$ is ξ.

Let $Z \subset X$ be a minimal H_ω-invariant symmetric subspace of Y. Clearly Z cannot be a single point, because it is invariant under the unipotent groups U_a^B. Since H_ω has a connected image $\bar{H}_\omega$ in $\mathrm{Isom}(Z)$, and no fixed points in $\partial_\infty Z$, it follows that $\dim Z > 1$, and hence by Sect. 2.6, $\bar{H}_\omega$ is the entire identity component of $\mathrm{Isom}(Z)$.

If $\xi \in \Lambda$, $a := \omega \circ \xi$, and $g \in U_a^B \setminus \{e\}$, then every orbit of $\{g^i\}$ in $\partial_\infty Y$ accumulates on ξ; since $\partial_\infty Z$ is closed and H_ω-invariant, it follows that $\xi \in \partial_\infty Z$. Both Λ and $\partial_\infty Z$ are H_ω-orbits, so $\Lambda = \partial_\infty Z$ as claimed.

The last statement follows immediately from the fact that $\Lambda = \partial_\infty Z$, and $\mathrm{Isom}(Z)$ contains the transvection along the geodesic γ. $\square$

Let $H \subset G$ be the subgroup generated by the restricted root groups U_a^B, where a ranges over all W_B-roots in B. H is a connected subgroup of the Lie group G since it is generated by connected subgroups.

For a maximal flat $F \subset X$ with $\partial_T F \subset B$ let $H_F \subset H$ be the subgroup of H which fixes $\partial_T F$ pointwise. Thus each $g \in H_F$ acts by a translation on F. By the previous lemma, for each W_B-wall $\omega \subset \partial_T F$, there is a 1-parameter subgroup of H_F which translates in the direction orthogonal to ω; as B is irreducible, these 1-parameter subgroups generate a subgroup of H_F which acts on F as the full translation group. Lemma 2.14 then implies that the fixed point set of H in $\partial_T X$ is contained in the intersection of the apartments of B, which is empty.

If H preserves a symmetric subspace $Y \subset X$, then $\partial_T Y \subset \partial_T X$ is a proper H-invariant subbuilding which defines a closed subset of $\partial_\infty X$. Each $\xi' \in B$ is opposite to some $\xi \in \partial_T Y$, and hence by Lemma 2.14, any antipode of ξ' in B belongs to $\partial_T Y$. Thus $B \subset \partial_T Y$, forcing $Y = X$.

Thus H is a connected subgroup of G which neither fixes a point in $\partial_T X$ nor preserves a proper symmetric subspace of X, and so we conclude that $H = G$, see Sect. 2.6. Therefore $B = \partial_T X$.

Case 2. The subbuilding B is reducible.

Lemma 3.8. *B cannot have a nontrivial spherical join factor.*

Proof. Let $S \subset B$ be a maximal spherical join factor of B, and let $F \subset X$ be a flat with $\partial_T F = S$. Then the boundary of the parallel set $\mathbb{P}(F)$ contains B. By our assumption we may conclude that $X = \mathbb{P}(F)$. However, X is an irreducible symmetric space of noncompact type, so this is a contradiction. □

Let

$$B = B_1 \circ \ldots \circ B_l$$

be the unique join decomposition of B into irreducible nonspherical join factors. By case 1 above we are done if there is only one factor, so we assume that $l > 1$.

For each i, we let $H_i \subset G$ be the closure of the subgroup generated by transvections along geodesics whose ideal endpoints lie in B_i. Note that H_i is connected. Since transvections along parallel geodesics coincide, and transvections along geodesics lying in a single flat commute, it follows that H_i commutes with H_j when $i \neq j$. Let $H := H_1 \times \ldots \times H_l$.

Lemma 3.9. *(i) H does not fix any point in $\partial_T X$.*
(ii) H preserves no proper symmetric subspace of X.

Proof. (i) Pick a maximal flat $F \subset X$ such that $\partial_T F \subset B$. As H contains the full transvection group of F, Lemma 2.14 implies that the fixed point set of H on $\partial_T X$ is contained in $\partial_T F$. This means that the fixed point set is contained in the intersection of the apartments of B; this intersection is empty since B has no spherical join factor.

(ii) Suppose that H preserves a symmetric subspace $Y \subset X$. For an apartment A in B consider the maximal flat F in X with $\partial_T F = A$. Since the whole group of transvections along F belongs to H the flat F has finite Hausdorff distance from Y and $A \subset \partial_T Y$. Hence $B \subset \partial_T Y$ and our assumption on B implies that $Y = X$. □

We must therefore have $H = H_1 \times \ldots \times H_l = G$, see Sect. 2.6. This contradicts the fact that G is a simple Lie group.

4. Convex sets preserved by Zariski dense groups

Theorem 4.1. *Let X be a symmetric space of noncompact type with de Rham decomposition $X = X_1 \times \ldots \times X_k$, let $\pi_i : X \to X_i$ be the projection* *map, and* $G = \mathrm{Isom}_o(X)$ *be the associated connected semi-simple Lie group. We denote by* $X = Y_1 \times Y_{\geq 2}$ *the decomposition of* X *into (the product of the) rank* 1 *and the higher rank factors. Suppose* $\Gamma \subset G$ *is a Zariski dense subgroup which preserves a closed convex subset* $C \subset X$*. Then* C *is of the form*

$$C_1 \times Y_{\geq 2}, \tag{4.2}$$

where $C_1 \subset Y_1$ *is closed convex. Furthermore, for each de Rham factor* X_i *of* Y_1*, there is a* Γ*-invariant subset* $\hat{C}_i \subset X_i$ *such that*

- $\hat{C}_i$ *is the closed convex hull of its limit set.*
- $|\partial_\infty \hat{C}_i| = \infty$,
- $\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$.
- $\partial_\infty \hat{C}_1 = \partial_\infty C_1$.

Proof. By Lemma 2.3, the limit set $\Lambda(C) = \partial_\infty C$ is a (cone topology) closed convex subset containing the limit set of Γ. By Benoist [Ben97], the limit set of Γ contains an open neighborhood (with respect to the topology of $\partial_T X$) of a pair of antipodal regular points $\xi, \hat{\xi} \in \partial_T X$. Hence $\partial_T C$ contains an apartment in $\partial_\infty X$. By [KL97, Prop. 3.10.3] it follows that $\partial_T C$ is a top dimensional subbuilding of $\partial_T X$.

Suppose $\partial_T C \subset \partial_T Y$ for some proper symmetric subspace $Y \subset X$. For every apartment $A \subset \partial_T C$, there is a unique maximal flat $F \subset X$ with $\partial_T F = A$, and so $F \subset Y$; likewise, we have $F \subset gY$ for all $g \in \Gamma$ which implies that $F \subset \cap_{g\in\Gamma} gY$. Since A was chosen arbitrarily, we conclude that $\cap_{g\in\Gamma} gY \subset X$ is a Γ-invariant proper symmetric subspace, which contradicts the Zariski density of Γ.

Theorem 3.1 applies, so the Tits boundary $\partial_T C$ splits as a join $\partial_T C = B_1 \circ \ldots \circ B_k$, where $B_i = \partial_T X_i$ when X_i has rank at least two, and $|B_i| = \infty$ for each i, by the Zariski density of Γ.

Applying Lemma 2.1, it follows that C splits as in (4.2).

Define $\hat{C}_i \subset X_i$ to be the closed convex hull of B_i; when $\mathrm{Rank}(X) \geq 2$ then $\hat{C}_i = X_i$. Applying Lemma 2.1, it follows that $\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$. □

5. Invariant convex subsets in symmetric spaces with Euclidean deRham factors

Theorem 5.1. *Let* Y *be a symmetric space of noncompact type without Euclidean de Rham factor, and suppose* $\Gamma \subset \mathrm{Isom}(Y) \times \mathrm{Isom}(\mathbb{E}^n)$ *is a subgroup whose projection* $\pi_Y(\Gamma) \subset \mathrm{Isom}(Y)$ *is Zariski dense in the identity component* $\mathrm{Isom}_o(Y)$. *If* $C \subset X := Y \times \mathbb{E}^n$ *is a* Γ*-invariant closed convex set, then either* $C = \pi_Y(C) \times \mathbb{E}^n$ *or there is a proper* Γ*-invariant affine subspace* $A \subset \mathbb{E}^n$ *(i.e.* A *is preserved by the induced action of* Γ *on* $\mathbb{E}^n$*).*

Proof. We denote by $\mathrm{Sh} := \pi_Y(C)$ the shadow of C in Y. For every point $y \in \mathrm{Sh}$ we consider the slice $(\{y\} \times \mathbb{E}^n) \cap C =: C_y$. Since C is closed, the boundary at infinity $\partial_T C_y$ does not depend on y and it is a closed convex subset D of the round $(n-1)$-sphere $\partial_T \mathbb{E}^n$. We may assume that it is a proper subset because otherwise $C = \mathrm{Sh} \times \mathbb{E}^n$ and we are done.

If the C_y split off an $\mathbb{R}^k$-factor, $1 \leq k < n$, then C itself splits off an $\mathbb{R}^k$-factor. If $E' \subset \mathbb{E}^n$ is the maximal Euclidean factor and $\mathbb{E}^n = E' \times E''$ a splitting then this splitting is preserved by Γ. We can therefore reduce to the case that the C_y have no Euclidean factor.

Case 1: The slices C_y are unbounded. The set $D \subset \partial_T \mathbb{E}^n$ has diameter $< \pi$ and hence a well-defined center ζ which must be fixed by Γ. Let b_ζ denote the Busemann function on X associated to ζ. For every $\gamma \in \Gamma$ the difference $b_\zeta(\gamma \cdot) - b_\zeta$ equals a constant $\rho(\gamma)$ and the map $\rho : \Gamma \to \mathbb{R}$ is a group homomorphism.

The restriction of b_ζ to C_y is bounded above and proper because $\partial_T C_y$ is contained in the open ball $B_{\frac{\pi}{2}}(\zeta)$. We may therefore assign to each $y \in \mathrm{Sh}$ the bottom height of the slice C_y in the direction ζ defined as $h(y) := \min(-b_\zeta|_{C_y})$. The function $h : \mathrm{Sh} \to \mathbb{R}$ is convex. We consider the asymptotic slope function $\mathrm{slope}_h : \partial_T \mathrm{Sh} \to \mathbb{R} \cup \{\infty\}$, see Sect. 2.3. It is Γ-invariant. If the homomorphism ρ is nontrivial then slope_h assumes also negative values, and by Proposition 2.12 it has a unique minimum. This minimum must be fixed by Γ, a contradiction to the Zariski density of $\pi_Y(\Gamma)$ in $\mathrm{Isom}(Y)$. Therefore ρ must be trivial, and the level sets of b_ζ yield Γ-invariant hyperplanes in $\mathbb{E}^n$.

Case 2: The slices C_y are bounded. We pick an ideal point $\zeta \in \partial_T E^n$. As above, measuring the height in the direction of ζ, we can consider the convex function $\mathrm{bot} : \mathrm{Sh} \to \mathbb{R}$ given by $\mathrm{bot}(y) := \min(-b_\zeta|_{C_y})$ and the concave function $\mathrm{top} : \mathrm{Sh} \to \mathbb{R}$ given by $\mathrm{top}(y) := \max(-b_\zeta|_{C_y})$. Both functions are continuous because C is closed.

We now use the structure Theorem 4.1 for convex sets invariant under a Zariski dense group. It implies that $\partial_T \mathrm{Sh}$ splits as the spherical join of the boundaries of the higher rank factors and of infinite subsets in the boundaries of the rank one factors. In particular, $\partial_\infty \mathrm{Sh}$ has a well-defined and therefore $\pi_Y(\Gamma)$-invariant convex hull $\mathrm{CH}(\partial_\infty \mathrm{Sh})$ in Y which is the product of the higher rank factors of Y with the closed convex hulls of the subsets in the boundaries of the rank one factors.

Lemma 2.6 applied to the higher rank factors and Lemma 2.7 applied to the rank one factors imply that the continuous concave function $\mathrm{top} - \mathrm{bot} : \mathrm{Sh} \to [0, \infty)$ is constant on $\mathrm{CH}(\partial_\infty \mathrm{Sh})$. It follows that the restrictions of top and bot to $\mathrm{CH}(\partial_\infty \mathrm{Sh})$ are affine. According to Lemmas 2.9 and 2.8 both functions are constant on $\mathrm{CH}(\partial_\infty \mathrm{Sh})$.

Since the values of $\mathrm{top}(y)$ (or $\mathrm{bot}(y)$) for all directions ζ determine the slice C_y it follows that the slices C_y equal the same compact set $B \subset E^n$ for all y in the $\pi_Y(\Gamma)$-invariant subset $\mathrm{CH}(\partial_\infty \mathrm{Sh})$. In particular, the action of Γ on $\mathbb{E}^n$ has bounded orbits and therefore a fixed point. $\square$

6. The convex cocompact case

In this section we prove:

Lemma 6.1. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type. If $\Gamma \subset \mathrm{Isom}(X)$ is a discrete convex cocompact group which does not preserve any proper symmetric subspace of X, then the fixed point set of Γ in $\partial_T X$ is contained in the Tits boundary of the Euclidean factor $\mathbb{E}^n$.*

Proof. Let C be a Γ-invariant closed convex set on which Γ acts cocompactly. Suppose Γ fixes a point $\xi \in \partial_\infty X \setminus \partial_\infty \mathbb{E}^n$. The Γ-action respects the join structure of $\partial_T X$, so we may assume without loss of generality that $\xi \in \partial_\infty Y$.

Recall that since Γ fixes ξ, the Γ-translates of the Busemann function b_ξ differ by a constant, and the map $\Gamma \ni g \mapsto g_*(b_\xi) - b_\xi$ defines a homomorphism $\rho : \Gamma \to \mathbb{R}$.

Suppose first that the homomorphism ρ is trivial, i.e. b_ξ is Γ-invariant. Then $b_\xi|_C$ is bounded and attains a minimum. The minimum set of $b_\xi|_C$ is a convex subset $C_1 \subset C$ lying in a horosphere. By triangle comparison one concludes that if $p_1, p_2 \in C_1$, then the ideal geodesic triangle $\overline{\xi p_1} \cup \overline{p_1 p_2} \cup \overline{p_2 \xi}$ bounds a flat half-strip. Thus C_1 is contained in the parallel set $\mathbb{P}(\gamma)$ of a geodesic $\gamma \subset \mathbb{E}^n \times Y$ which is parallel to the Y factor. Since C_1 is Γ-invariant it follows that Γ preserves a proper symmetric subspace of X, which is a contradiction. Therefore ρ is a nontrivial homomorphism and $b_\xi(C) = \mathbb{R}$.

Consider a group element $g \in \Gamma$ which translates the Busemann function b_ξ. We may assume that $b_\xi(gx) = b_\xi(x) - a$ for all $x \in X$ with $a > 0$. As the action is discrete, Γ acts on C by semi-simple isometries, and so g is an axial isometry. Pick a point $x_0 \in C$ and let $r : [0, \infty) \to X$ be the unit speed ray starting in x_0 and asymptotic to ξ. Then for $x_n = g^n x_0$ holds $b_\xi(x_n) = b_\xi(r(na))$. We obtain that $d(x_n, r(na)) \leq n\, d(x_1, r(a))$ and $\angle_{r(na)}(x_n, x_0) \geq \frac{\pi}{2}$. Triangle comparison implies for the forward ideal endpoint of the g-axes $\xi_1 := \lim_{n\to\infty} x_n$ that

$$\tan \angle_T(\xi_1, \xi) \leq \frac{d(x_1, r(a))}{a}$$

and thus $\angle_T(\xi_1, \xi) < \frac{\pi}{2}$.

Since $\xi_1 \in \partial_\infty C$ we have $\angle_T(\xi, \partial_\infty C) < \frac{\pi}{2}$ there is a unique $\eta \in \partial_\infty C$ at minimum Tits distance from ξ, and so η is fixed by Γ. As $\angle_T(\eta, \xi) < \frac{\pi}{2}$, it follows that η does not lie in $\partial_T \mathbb{E}^n$.

We now apply Lemma 2.4 to the convex set C. We obtain that the convex set C contains a Γ-invariant parallel set (with respect to C) $Z := \mathbb{P}(\gamma) \subset C$, where $\partial_T \gamma \ni \eta$. Therefore Γ preserves the parallel set of γ in X, which is a contradiction. $\square$

7. The proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. This follows immediately from Theorem 5.1 and Theorem 4.1. □

Proof of Theorem 1.3. By Lemma 6.1 and the fact that X contains no proper Γ-invariant symmetric subspace, the fixed point set of Γ is contained in $\mathbb{E}^n$. Therefore the projection of Γ to $\mathrm{Isom}(Y)$ is Zariski dense in $\mathrm{Isom}_o(Y)$, since otherwise it would preserve a proper symmetric subspace $Y' \subset Y$, contradicting our assumption on X. The theorem then follows from Theorem 1.1. □

8. Quasiconvex sets and their stabilizers

Definition 8.1. A subset Q of a Hadamard space X is *K-quasiconvex* if for every pair of points $x_1, x_2 \in X$, the segment $\overline{x_1x_2}$ is contained in the closed tubular neighborhood $\overline{N}_K(Q)$. We say that Q is *quasiconvex* if it is K-quasiconvex for some $K < \infty$.

Lemma 8.2. *Let Q be a K-quasiconvex subset of a Hadamard space X.*
1. *For all $r \geq 0$, the closed r-neighborhood $\overline{N}_r(C)$ is K-quasiconvex.*
2. *For all $p \in Q$, $\xi \in \Lambda(Q)$, the ray $\overline{p\xi}$ is contained in $\overline{N}_K(Q)$.*
3. *The limit set $\Lambda(Q) \subset \partial_\infty X$ is a closed subset with respect to the topology of $\partial_\infty X$ which defines a convex subset of $\partial_T X$.*

Proof. 1 and 2 follow immediately from triangle comparison. To see 3, pick $p \in Q$, $\xi_1, \xi_2 \in \Lambda(Q)$ with $\angle_T(\xi_1, \xi_2) < \pi$, and note that if $\eta \in \partial_T X$ lies on the segment $\overline{\xi_1\xi_2} \subset \partial_T X$, then the ray $\overline{p\eta}$ may be constructed as a limit of geodesic segments $\overline{px_j} \subset X$, where $x_j \in N_{2K}(Q)$; this clearly implies that $\eta \in \Lambda(Q)$. □

Using this lemma, we may adapt Theorem 4.1 to the quasiconvex case:

Theorem 8.3. *Let X be a symmetric space of noncompact type with de Rham decomposition $X = X_1 \times \ldots \times X_k$, let $\pi_i : X \to X_i$ be the projection map, and $G = \mathrm{Isom}_o(X)$ be the associated connected semi-simple Lie group. We denote by $X = Y_1 \times Y_{\geq 2}$ the decomposition of X into (the product of the) rank 1 and the higher rank factors. Suppose $\Gamma \subset G$ is a Zariski dense subgroup which preserves a closed K-quasiconvex subset $Q \subset X$. Then*

$$Q_1 \times Y_{\geq 2} \subset \overline{N}_K(Q), \tag{8.4}$$

where $Q_1 := \pi_{Y_1}(Q)$. Furthermore, for each de Rham factor X_i of Y_1, there is a Γ-invariant subset $\hat{C}_i \subset X_i$ such that

- *$\hat{C}_i$ is the closed convex hull of its limit set.*
- *$|\partial_\infty \hat{C}_i| = \infty$,*

- $\hat{C}_1 := \prod_i \hat{C}_i \subset N_{K'}(Q_1)$, *where K' depends only on K and X.*
- $\partial_\infty \hat{C}_1 = \Lambda(Q_1)$.

Proof. The proof is almost identical to the proof of Theorem 4.1, so we simply note the necessary changes. First, one uses Lemma 8.2 instead of Lemma 2.3 to see that $\Lambda(Q)$ defines a convex subset of $\partial_T X$. In the second to last paragraph of the proof, one invokes Lemma 8.2 again to obtain (8.4). In the last paragraph, one uses Lemma 8.2, together with the fact that every point in $\hat{C}_i$ lies within a uniformly bounded distance of a geodesic with ideal endpoints in $\partial_\infty \hat{C}_i$. □

Next, we adapt Theorem 5.1 to quasiconvex sets.

Theorem 8.5. *Let Y be a symmetric space of noncompact type without Euclidean de Rham factor, and suppose* $\Gamma \subset \mathrm{Isom}(Y) \times \mathrm{Isom}(\mathbb{E}^n)$ *is a subgroup whose projection* $\pi_Y(\Gamma) \subset \mathrm{Isom}(Y)$ *is Zariski dense in the identity component* $\mathrm{Isom}_o(Y)$. *If $Q \subset X := Y \times \mathbb{E}^n$ is a Γ-invariant closed K-quasiconvex set, then either $\pi_Y(Q) \times \mathbb{E}^n \subset \overline{N}_K(Q)$ or there is a proper Γ-invariant affine subspace $A \subset \mathbb{E}^n$.*

Proof. We assume that there exists no proper Γ-invariant affine subspace $A \subset \mathbb{E}^n$. By Lemma 8.2, it suffices to show that $\Lambda(Q)$ contains the subset $\partial_\infty \mathbb{E}^n$ of $\partial_\infty X$. The argument is very similar to the proof of Theorem 5.1.

As before, we define the shadow of Q as $\mathrm{Sh} := \pi_Y(Q)$. It is a K-quasiconvex subset of Y. For $R \geq 0$ and a point $y \in \overline{N}_R(\mathrm{Sh})$ we define the slice $Q_y^R := \overline{N}_{R+K}(Q) \cap \pi_Y^{-1}(y)$. It has limit set $\Lambda(Q) \cap \partial_\infty E^n$ independent of y and R.

Case 1: The slices Q_y^R are unbounded, i.e. $\Lambda(Q) \cap \partial_\infty E^n \neq \emptyset$. We may assume that $\Lambda(Q) \cap \partial_\infty E^n$ has diameter $< \pi$ because otherwise we could reduce the dimension of the Euclidean factor $\mathbb{E}^n$. Hence $\Lambda(Q) \cap \partial_\infty E^n$ has a well-defined center ζ. The height function $h(y) := \min(-b_\zeta|_{Q_y^K})$ defined on $\overline{N}_K(\mathrm{Sh})$ is merely quasiconvex in the sense that its supergraph is $2K$-quasiconvex. Thus the limit set of the supergraph is a convex subset of $\partial_T(Y \times \mathbb{R})$ by Lemma 8.2 and we can use it to define the asymptotic slope function slope_h on $\Lambda(\mathrm{Sh})$. As before we conclude that the homomorphism ρ must be trivial and obtain a contradiction.

Case 2: The slices Q_y^R are bounded, i.e. $\Lambda(Q) \cap \partial_\infty E^n = \emptyset$. We rework the argument from the convex case in a different language. Observe first that for $y_1, y_2 \in \overline{N}_R(\mathrm{Sh})$ and a point $y \in \overline{y_1 y_2}$ every segment $\overline{x_1 x_2}$ connecting points $x_i \in Q_{y_i}^R$ must intersect Q_y^{R+K}. This has the following implications. If $\rho : [0, \infty) \to Y$ is a ray asymptotic to $\Lambda(\mathrm{Sh})$ with initial point $\rho(0) \in \overline{N}_R(\mathrm{Sh})$ – and is therefore contained in $\overline{N}_{R+K}(\mathrm{Sh})$ – then for every $t \geq 0$

there is an isometric embedding

$$Q^R_{\rho(0)} \hookrightarrow Q^{R+K}_{\rho(t)} \tag{8.6}$$

induced by a translation. Here we regard the slices as subsets of $\mathbb{E}^n$ via the projection $\pi_{\mathbb{E}^n}$. In particular, for a complete geodesic $c : \mathbb{R} \to Y$ moving in $\overline{N}_R(\mathrm{Sh})$ every $Q^R_{c(t)}$ embeds into every $Q^{R+K}_{c(t')}$. It follows that $\mathrm{diam}(Q^R_{c(t)})$ is bounded uniformly in t. Furthermore, for $t_1 < 0 < t_2$ and $x_{t_1} \in Q^R_{c(t_1)}$ the cone consisting of all rays initiating in x_{t_1} and intersecting $Q^{R+K}_{c(0)}$ contains $Q^R_{c(t_2)}$. Letting $t_1 \to -\infty$ and $t_2 \to \infty$ we deduce that the limit $\lim_{t_1\to-\infty} x_{t_1}$ in $\partial_T X$ exists. Analogously, for any choice of points $x_{t_2} \in Q^R_{c(t_2)}$, $t_2 \geq 0$, the limit $\lim_{t_2\to\infty} x_{t_2}$ exists.

Note that each point $\eta \in \Lambda(\mathrm{Sh})$ has antipodes in $\Lambda(\mathrm{Sh})$, cf. Theorem 8.3, and thus is the ideal endpoint of a geodesic c running in some neighborhood $N_R(\mathrm{Sh})$. Our previous consideration yields more generally that for any $\eta \in \Lambda(\mathrm{Sh})$, any sequence of points $y_n \in \overline{N}_R(\mathrm{Sh})$ with $y_n \to \eta$ and points $x_n \in Q^R_{y_n}$ the limit $\lim_{n\to\infty} x_n =: \xi(\eta)$ exists and is an interior point of the hemisphere $\eta \circ \partial_T \mathbb{E}^n$. Hence $\Lambda(Q)$ is the image of the "section" $\xi : \Lambda(\mathrm{Sh}) \to \partial_T X - \partial_T \mathbb{E}^n$. We observe that ξ maps antipodes to antipodes. The arguments used to proving Lemmas 2.8 and 2.9 together with the convexity of $\Lambda(Q)$ show that $\Lambda(Q)$ must be horizontal, that is, $\Lambda(Q) \subset \partial_T Y$, where we regard $\partial_T Y$ as a subset of $\partial_T X$.

The embeddings of slices (8.6) are now induced by the identity, i.e. $\pi_{\mathbb{E}^n}(Q^R_{\rho(0)}) \subseteq \pi_{\mathbb{E}^n}(Q^{R+K}_{\rho(t)})$. For a geodesic c in $\overline{N}_R(\mathrm{Sh})$ we have that $Q^R_{c(t)} \subseteq Q^{R+K}_{c(t')}$ for all $t, t' \in \mathbb{R}$. If ρ is a ray strongly asymptotic to c, i.e. $\lim_{t\to\infty} d(\rho(t), c(t+a)) = 0$ for some $a \in \mathbb{R}$, one obtains that $Q^R_{\rho(0)} \subseteq Q^{R+2K}_{c(t)}$ for all t.

Invoking our structural result Theorem 8.3 and arguing in the spirit of Lemmas 2.6 and 2.7 we conclude that for sufficiently large R the union of slices $\cup_{y\in\hat{C}_1\times Y_{\geq 2}} \pi_{\mathbb{E}^n}(Q^R_y)$ is a bounded subset of $\mathbb{E}^n$. It follows that the action of Γ on $\mathbb{E}^n$ preserves a bounded subset, namely the set $\cup_{\gamma\in\Gamma} \pi_{\mathbb{E}^n}(Q^R_{\gamma y_0})$ with $y_0 \in \hat{C}_1 \times Y_{\geq 2}$, and hence fixes a point. This is a contradiction and concludes the proof of the theorem. $\square$

Combining Theorems 8.3 and 8.5, we obtain:

Theorem 8.7. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type, and let $X = \mathbb{E}^n \times Y_1 \times Y_{\geq 2}$ denote the decomposition of X into the Euclidean factor, the product of the irreducible rank 1 factors, the product of the higher rank factors. Suppose $\Gamma \subset \mathrm{Isom}(X) = \mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(Y)$ is a subgroup whose projection to $\mathrm{Isom}(Y)$ is Zariski dense in the identity component $\mathrm{Isom}_o(Y)$, and whose projection to $\mathrm{Isom}(\mathbb{E}^n)$ does not preserve a proper affine subspace of $\mathbb{E}^n$. If $Q \subset X := \mathbb{E}^n \times Y$ is a Γ-invariant closed K-quasiconvex set and $Q_1 := \pi_{Y_1}(Q)$, then $\mathbb{E}^n \times Q_1 \times Y_{\geq 2} \subset \overline{N}_K(Q)$. Fur-*

thermore, for each de Rham factor X_i of Y_1, there is a Γ-invariant subset $\hat{C}_i \subset X_i$ such that

- *$\hat{C}_i$ is the closed convex hull of its limit set.*
- $|\partial_\infty \hat{C}_i| = \infty$,
- *$\hat{C}_1 := \prod_i \hat{C}_i \subset N_{K'}(Q_1)$, where K' depends only on K and X.*
- $\partial_\infty \hat{C}_1 = \Lambda(Q_1)$.

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