Inventiones math. 43, 69–82 (1977)

Inventiones mathematicae

Exponential Sums with Multiplicative Coefficients*

H.L. Montgomery[1] and R.C. Vaughan[2]

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
[2] Imperial College of Science and Technology, Department of Mathematics, Queens Gate, London SW7 2BZ, United Kingdom

1. Statement of Results

Let A be an arbitrary constant with $A \geqq 1$, and let $\mathscr{F}$ be the class of all multiplicative functions f such that $|f(p)| \leqq A$ for all primes p and

$$\sum_{n=1}^{N} |f(n)|^2 \leqq A^2 N \tag{1}$$

for all natural numbers N. For $f \in \mathscr{F}$ and real α write

$$S(\alpha) = \sum_{n=1}^{N} f(n)\, e(n\alpha). \tag{2}$$

Daboussi [1] has shown that if $|\alpha - a/q| \leqq q^{-2}$, $(a,q)=1$, $3 \leqq q \leqq (N/\log N)^{\frac{1}{2}}$, then

$$S(\alpha) \ll_A N(\log\log q)^{-\frac{1}{2}} \tag{3}$$

uniformly for $f \in \mathscr{F}$.

By (1) and Cauchy's inequality,

$$\sum_{n=1}^{N} |f(n)| \leqq AN \tag{4}$$

so that $|S(\alpha)| \leqq AN$. Thus, when q is large the estimate (3) is non-trivial. As a consequence of (3), if α is irrational, then $S(\alpha) = o(N)$ as $N \to \infty$. Moreover, for almost all real α, including all real irrational algebraic α,

$$S(\alpha) \ll_A N(\log\log N)^{-\frac{1}{2}} \qquad (N > N_0(\alpha)). \tag{5}$$

The most striking aspect of Daboussi's estimate (3) is that it is uniform over $f \in \mathscr{F}$. Our main object here is to establish a sharp estimate of this kind.

* Work supported in part by National Science Foundation Grant MPS 75-07948

Theorem 1. *Suppose that $q \leqq N$ and $(a,q)=1$. Then*

$$S(a/q) \ll_A N((\log 2N)^{-1} + \phi(q)^{-\frac{1}{2}} + (q/N)^{\frac{1}{2}}(\log(2N/q))^{\frac{3}{2}}) \tag{6}$$

uniformly for $f \in \mathscr{F}$.

From this we deduce

Corollary 1. *Suppose that $|\alpha - a/q| \leqq q^{-2}$, $(a,q)=1$ and $2 \leqq R \leqq q \leqq N/R$. Then*

$$S(\alpha) \ll_A \frac{N}{\log N} + NR^{-\frac{1}{2}}(\log R)^{\frac{3}{2}}$$

uniformly for $f \in \mathscr{F}$.

Corollary 2. *For almost all α, including all real irrational algebraic α,*

$$S(\alpha) \ll_A \frac{N}{\log N} \qquad (N > N_0(\alpha)).$$

To demonstrate the precision of the above estimates, in §7 we establish the following simple propositions.

(i) For any real α and any $N \geqq 2$ there is an $f \in \mathscr{F}$ such that $|S(\alpha)| \gg N/\log N$.

(ii) If $q \leqq N^{\frac{1}{2}}$ and $(a,q)=1$, then there is an $f \in \mathscr{F}$ for which $|S(a/q)| \gg N q^{-\frac{1}{2}}$.

(iii) If $N(\log N)^{-3} \leqq Q \leqq N$, then there are a, q, f such that $Q - 3NQ^{-1} \leqq q \leqq Q$, $(a,q)=1$, $f \in \mathscr{F}$ and $|S(a/q)| \gg (Nq)^{\frac{1}{2}}$.

In fact, in each of the above the f we construct is totally multiplicative and satisfies $|f(n)| \leqq 1$ for every n.

Of course, for particular functions f, better estimates than that given by Theorem 1 can be obtained. For example, if f is identically 1, then $S(\alpha) \ll \min(N, \|\alpha\|^{-1})$ where $\|\alpha\|$ is the distance of α from the nearest integer. Also, in the case $f = \mu$, the Möbius function, Davenport [2] used methods of Vinogradov (see [11]) to show that $S(\alpha) \ll_B N(\log N)^{-B}$.

Nearly sixty years ago, Pólya [9] and Vinogradov [10] independently showed that if χ is a non-principal character modulo q, then for *any* x

$$\Big|\sum_{n \leqq x} \chi(n)\Big| < q^{\frac{1}{2}} \log q. \tag{7}$$

We use Theorem 1 to obtain a conditional improvement on this estimate.

By the Grand Riemann Hypothesis (GRH) we mean that no Dirichlet L-function $L(s,\chi)$ has any zeros ρ with $\operatorname{Re}\rho > \frac{1}{2}$.

Theorem 2. *Suppose that GRH is true. Then for any non-principal characte*r *χ modulo q and any x,*

$$\sum_{n \leqq x} \chi(n) \ll q^{\frac{1}{2}} \log\log q.$$

This estimate is essentially best possible, for Paley [8] has shown that there are infinitely many fundamental discriminants $D \equiv 1 \pmod 4$ for which

$$\max_x \left| \sum_{n \leqq x} \left(\frac{D}{n}\right) \right| > \frac{1}{7} D^{\frac{1}{2}} \log\log D.$$

We shall show elsewhere that this maximum is not usually so large. Nevertheless, it is easily seen, by applying Parseval's identity to the expansion of Lemma 1, that for any primitive character χ modulo q

$$\frac{1}{q} \int_0^q \left| \sum_{n \leqq x} \chi(n) \right|^2 dx > \frac{q}{20}.$$

Of course, when x is small compared with q, stronger conclusions than that of Theorem 2 are known. The unconditional estimates of Burgess are sharper when $x \ll q^{\frac{3}{8}-\varepsilon}$, and on GRH one has

$$\sum_{n \leqq x} \chi(n) \ll_\varepsilon x^{\frac{1}{2}} q^\varepsilon.$$

We devote §§2–6 to the proofs of Theorem 1 and the Corollaries. The general motivation of our argument lies in the methods of Vinogradov for the estimation of trigonometric sums, especially those of the kind $\sum_{p \leqq N} e(p\alpha)$.

We prove Theorem 2 in §§8, 9. The GRH is used to treat the 'major arcs' and Theorem 1 provides a suitable estimate on the 'minor arcs'.

2. Preliminary Reduction

In §§2–6, many implicit constants depend on A, which for brevity we do not indicate. Our first objective is to relate $S(\alpha)$ to a quadratic form of the kind $\sum_{mn \leqq N} f(m) f(n) \lambda_m e(mn\alpha)$. By Cauchy's inequality and (1)

$$\sum_{n \leqq N} f(n) e(na/q) \log N/n \ll \Big(\sum_{n \leqq N} (\log N/n)^2\Big)^{\frac{1}{2}} \Big(\sum_{n \leqq N} |f(n)|^2\Big)^{\frac{1}{2}} \ll N.$$

Thus

$$S(a/q) \log N \ll N + \Big| \sum_{n \leqq N} f(n) e(na/q) \log n \Big|.$$

Since $\log n = \sum_{m|n} \Lambda(m)$ it therefore suffices to show that

$$\sum_{mn \leqq N} f(mn) \Lambda(m) e(mna/q) \ll N + N\phi(q)^{-\frac{1}{2}} \log N + (Nq)^{\frac{1}{2}} (\log 2N/q)^{\frac{3}{2}} \log N. \tag{8}$$

In the expression on the left we seek to replace $f(mn)$ by $f(m) f(n)$. To this end we bound

$$T = \sum_{mn \leqq N} \Lambda(m) |f(mn) - f(m) f(n)|.$$

As f is multiplicative, so that $f(mn)=f(m)f(n)$ unless $(m,n)>1$, we can write

$$T \leqq \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{p,k\geqq 1} \sum_{\substack{n\leqq Np^{-k}\\ p|n}} (\log p)\,|f(p^k n)|,$$

$$\Sigma_2 = \sum_{p,k\geqq 1} (\log p)\,|f(p^k)| \sum_{j\geqq 1} |f(p^j)| \sum_{m\leqq Np^{-k-j}} |f(m)|.$$

By collecting together those terms in Σ_1 for which $p^k n$ is exactly divisible by p^j we obtain, by (4) and Cauchy's inequality,

$$\begin{aligned}\Sigma_1 &\leqq \sum_{p,j\geqq 2} (\log p)\,|f(p^j)|(j-1) \sum_{m\leqq Np^{-j}} |f(m)| \\ &\ll N \sum_{p,j\geqq 2} j p^{-j}\,|f(p^j)| \log p \\ &\ll N\Big(\sum_{p,j\geqq 2} j^2 p^{-3j/4} \log^2 p\Big)^{\frac{1}{2}} \Big(\sum_n n^{-\frac{5}{4}}\,|f(n)|^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{9}$$

The first sum is easily seen to be convergent, and that the second is also convergent follows from (1) by partial summation. Thus

$$\Sigma_1 \ll N.$$

The treatment of Σ_2 is similar. By (4),

$$\Sigma_2 \ll N \sum_{p,j\geqq 1,k\geqq 1} |f(p^j) f(p^k)|\, p^{-j-k} \log p.$$

On taking $\alpha_j = f(p^j)\,p^{-j/3}$ and observing that $|\alpha_j \alpha_k| \leqq |\alpha_j|^2 + |\alpha_k|^2$ we obtain

$$\begin{aligned}\Sigma_2 &\ll N \sum_{p,j\geqq 1} |f(p^j)|^2 p^{-4j/3} (\log p) \sum_{k\geqq 1} p^{-2k/3} \\ &\ll N \sum_n |f(n)|^2 n^{-\frac{4}{3}} \log n \\ &\ll N.\end{aligned}$$

Thus $T \ll N$, and we are left to estimate

$$\sum_{mn\leqq N} f(m) f(n) \Lambda(m)\, e(mna/q).$$

Those pairs m, n in which m is of the form p^k with $k\geqq 2$ contribute an amount to the sum which is bounded by

$$\sum_{p,k\geqq 2} |f(p^k)|\,(\log p) \sum_{n\leqq Np^{-k}} |f(n)|.$$

By (4) this is

$$\ll N \sum_{p,k\geqq 2} |f(p^k)|\, p^{-k}\log p$$

and this is majorized by the expression in (9). Therefore the conribution is $\ll N$. Hence to prove (8), and consequently Theorem 1, we need only show that

$$\sum_{pn\leqq N} f(p)f(n)\,e(pna/q)\log p \ll N + N\phi(q)^{-\frac{1}{2}}\log N + (Nq)^{\frac{1}{2}}(\log 2N/q)^{\frac{3}{2}}\log N. \quad (10)$$

3. The Partition into Rectangles

Our fundamental estimate, in §4, is for sums over pairs (p,n) lying in rectangular regions $(P',P'']\times(N',N'']$. Thus we partition the sum in (10) into many such regions, omitting only a few terms whose contribution can be estimated crudely.

We begin with the rectangles

$$\mathcal{R}_i=(0,2^i]\times(N2^{-i-1},N2^{-i}] \qquad (0\leqq i\leqq \log_2 N) \quad (11)$$

where $\log_2 N=(\log N)/(\log 2)$. Let

$$J_i=\min(i+1,[\log_2 N]-i+1,[\tfrac{1}{2}\log_2(64N/q)]). \quad (12)$$

In the remaining regions $\mathcal{S}_i$ defined by

$$\mathcal{S}_i=\{(x,y): xy\leqq N, x>2^i, N2^{-i-1}<y\leqq N2^{-i}\}$$

we place additional rectangles $\mathcal{R}_{ijk}$ for $j=1,2,\ldots,J_i$ where for each j, $2^{j-1}<k\leqq 2^j$. The $\mathcal{R}_{ijk}$ are defined in the following iterative manner. In $\mathcal{S}_i$ we put

$$\mathcal{R}_{i12}=(2^i,\tfrac{4}{3}2^i]\times(\tfrac{1}{2}N2^{-i},\tfrac{3}{4}N2^{-i}].$$

Left over in $\mathcal{S}_i$ are two regions of the same kind into each of which we place a further rectangle, and so on. Thus, on the j-th occasion we place 2^{j-1} rectangles $\mathcal{R}_{ijk}(2^{j-1}<k\leqq 2^j)$ where

$$\mathcal{R}_{ijk}=(2^{i+j}/k,2^{i+j+1}/(2k-1)]\times((k-1)N2^{-i-j},(2k-1)N2^{-i-j-1}]. \quad (13)$$

We do this for $j=1,2,\ldots,J_i$. The choice of J_i in (12) ensures that each $\mathcal{R}_{ijk}$ is a rectangle of the form $(P',P'']\times(N',N'']$ with

$$P''-P'\geqq\tfrac{1}{4}, \quad N''-N'\geqq\tfrac{1}{4}, \quad (P''-P')(N''-N')\gg q. \quad (14)$$

Let $\mathcal{E}$ denote the set of points (p,n) with $pn\leqq N$ which do not lie in any rectangle $\mathcal{R}_i$ or $\mathcal{R}_{ijk}$, and write $\mathcal{R}_i=\mathcal{P}_i\times\mathcal{N}_i$ and $\mathcal{H}_i=\{(p,n)\in\mathcal{E}: n\in\mathcal{N}_i\}$. Then $\mathcal{E}$ is the union of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, the unions of those $\mathcal{H}_i$ with $J_i>i+1$, $J_i=[\log N]-i+1$ and $J_i=[\frac{1}{2}\log_2(64N/q)]$ respectively. We now estimate the contribution in the sum on the left of (10) from the points (p,n) in $\mathcal{E}$.

Consider $\mathcal{E}_1$. For a given p, the number of n for which $(p,n)\in\mathcal{E}_1$ is $\ll Np^{-2}$, and for a given n, there are $\ll 1$ primes p for which $(p,n)\in\mathcal{E}_1$. Hence, by Cauchy's inequality,

$$\sum_{\mathscr{E}_1} |f(p)f(n)| \log p \ll \Big(\sum_{\mathscr{E}_1} |f(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{\mathscr{E}_1} (\log p)^2\Big)^{\frac{1}{2}}$$
$$\ll \Big(\sum_{n \leqq N} |f(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{p \leqq N} N p^{-2} (\log p)^2\Big)^{\frac{1}{2}}$$
$$\ll N.$$

For each pair $(p, n) \in \mathscr{E}_2$ we see that $n \leqq (2N)^{\frac{1}{2}}$, and for a given n the p with $(p, n) \in \mathscr{E}_2$ all lie in an interval of length $N n^{-2}$. Thus, by the Brun-Titchmarsh inequality [5; Theorem 3.7] the number of such p is $\ll N n^{-2} (\log 4 N n^{-2})^{-1}$. For a given p there are $\ll 1$ numbers n for which $(p, n) \in \mathscr{E}_2$. Thus

$$\sum_{\mathscr{E}_2} |f(p)f(n)| \log p \ll \Big(\sum_{\mathscr{E}_2} |f(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{\mathscr{E}_2} (\log p)^2\Big)^{\frac{1}{2}}$$
$$\ll \Big(\sum_{n \leqq \sqrt{(2N)}} |f(n)|^2 N n^{-2} (\log 4 N n^{-2})^{-1}\Big)^{\frac{1}{2}} \Big(\sum_{p \leqq N} (\log p)^2\Big)^{\frac{1}{2}}$$
$$\ll N.$$

When $(p, n) \in \mathscr{E}_3$ we have $(N/q)^{\frac{1}{2}} \leqq p \leqq (N q)^{\frac{1}{2}}$, and for each such p the number of n for which $(p, n) \in \mathscr{E}_3$ is $\ll (N q)^{\frac{1}{2}} p^{-1}$. For each n, the p for which $(p, n) \in \mathscr{E}_3$ lie in an interval of length $\ll (N q)^{\frac{1}{2}} n^{-1}$, so that, by the Brun-Titchmarsh theorem again, there are

$$\ll (N q)^{\frac{1}{2}} n^{-1} (\log 2 N q n^{-2})^{-1}$$

such p. Therefore

$$\sum_{\mathscr{E}_3} |f(p)f(n)| \log p \ll \Big(\sum_{\mathscr{E}_3} |f(n)|^2 \log 2N/n\Big)^{\frac{1}{2}} \Big(\sum_{\mathscr{E}_3} \log p\Big)^{\frac{1}{2}}$$
$$\ll (N q)^{\frac{1}{2}} \left(\sum_n |f(n)|^2 \frac{\log(2N/n)}{n \log(2 N q n^{-2})}\right)^{\frac{1}{2}} \left(\sum_p \frac{\log p}{p}\right)^{\frac{1}{2}}$$

where in each sum the variable indicated is restricted to the closed interval $[(N/q)^{\frac{1}{2}}, (N q)^{\frac{1}{2}}]$. The ratio of the logarithms in the first sum is less than $\log(4N/q)$. Thus

$$\sum_{\mathscr{E}_3} |f(p)f(n)| \log p \ll (N q)^{\frac{1}{2}} (\log 2N/q)^{\frac{1}{2}} \log q.$$

Combining the above estimates we obtain

$$\sum_{\mathscr{E}} f(p) f(n)\, e(p n a/q) \log p \ll N + (N q)^{\frac{1}{2}} (\log 2N/q)^{\frac{1}{2}} \log N. \tag{15}$$

4. The Fundamental Estimate

For $1 \leqq k \leqq K$, let $\mathscr{R}(k) = \mathscr{Q}(k) \times \mathscr{M}(k)$ be a rectangle with $\mathscr{Q}(k) = (Q'(k), Q''(k)]$, $\mathscr{M}(k) = (M'(k), M''(k)]$, and such that the $\mathscr{Q}(k)$ are disjoint, $\mathscr{Q}(k) \subset (0, Q]$, $Q''(k) - Q'(k) \leqq X$, the $\mathscr{M}(k)$ are disjoint, $\mathscr{M}(k) \subset (0, M]$, $M''(k) - M'(k) \leqq Y$,

$M''(k) \leqq 2M'(k)$. We show that if $(a,q)=1$ and $q \leqq XY$, then the sum

$$I=\sum_{k=1}^{K} \sum_{(p,n)\in\mathscr{R}(k)} f(p)f(n)\,e(pna/q)\log p$$

satisfies

$$I \ll (MQY\log 2Q + MQXY/\phi(q) + MQX + MQq\log 2XY/q)^{\frac{1}{2}}. \tag{16}$$

Let $\mathscr{R}=\mathscr{O}\times\mathscr{M}$ be one of the rectangles $\mathscr{R}(k)$. By Cauchy's inequality

$$\Big|\sum_{(p,n)\in\mathscr{R}} f(p)f(n)\,e(pna/q)\log p\Big|^2 \leqq \Big(\sum_{n\in\mathscr{M}}|f(n)|^2\Big)\Big(\sum_{n\in\mathscr{M}}\Big|\sum_{p\in\mathscr{Q}} f(p)\,e(pna/q)\log p\Big|^2\Big). \tag{17}$$

We could estimate the second term on the right by immediately squaring out the inner sum and interchanging the order of summation, the new inner sum then being easily estimated. However this apparently leads to a certain loss of precision. Instead, by introducing the smoothing factor $w(n)=\max(0,2-|2n-2M'-Y|Y^{-1})$ so that $w(n)\geqq 1$ for $n\in\mathscr{M}$, we arrange that the new inner sum behaves like the squared modulus of a sum. Thus the second factor on the right of (17) is

$$\begin{aligned}
&\ll \sum_{n} w(n)\Big|\sum_{p\in\mathscr{Q}} f(p)\,e(pna/q)\log p\Big|^2\\
&\quad= \sum_{p,p'\in\mathscr{Q}} f(p)\bar{f}(p')(\log p)(\log p')\sum_{n} w(n)\,e((p-p')na/q)\\
&\quad\ll (\log Q)^2 \sum_{p,p'\in\mathscr{Q}} \min(Y, \|(p-p')a/q\|^{-2}Y^{-1}).
\end{aligned}$$

Therefore, by Cauchy's inequality,

$$\begin{aligned}
I &\ll (\log Q)\Big(\sum_{k}\sum_{n\in\mathscr{M}(k)}|f(n)|^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{k}\sum_{p,\,p'\in\mathscr{Q}(k)}\min(Y,\|(p-p')a/q\|^{-2}Y^{-1})\Big)^{\frac{1}{2}}\\
&\ll (\log Q)\,M^{\frac{1}{2}}\Big(YQ(\log 2Q)^{-1} + \sum_{0<h\leqq X}\ \sum_{\substack{p\leqq Q\\ p+h=p'}}\min(Y,\|ha/q\|^{-2}Y^{-1})\Big)^{\frac{1}{2}}.
\end{aligned}$$

A standard sieve estimate [5, Theorem 3.11] asserts that the number of primes $p\leqq Q$ for which $p+h$ is prime is $\ll hQ(\log 2Q)^{-2}\phi(h)^{-1}$. Thus

$$I \ll (MQY\log Q + MQV)^{\frac{1}{2}} \tag{18}$$

where

$$V=\sum_{0<h\leqq X}(h/\phi(h))\min(Y,\|ha/q\|^{-2}Y^{-1}).$$

Hence, to prove (16) it now suffices to show that

$$V \ll XY\phi(q)^{-1} + X + Y\log 2X + q\log(2XY/q). \tag{19}$$

The quantity $h/\phi(h)$ has bounded mean value, but its presence here adds complications to what would otherwise be an easy estimation. We observe that

$$h/\phi(h) \ll \sum_{m|h} 1/m$$

so that

$$V \ll \sum_{m\leqq X} 1/m \sum_{n\leqq X/m} \min(Y, \|mna/q\|^{-2} Y^{-1}).$$

The innermost sum is of the form

$$W = \sum_{n\leqq Z} \min(Y, \|bn/r\|^{-2} Y^{-1})$$

with $r=q/(m,q)$ and $(b,r)=1$. This is easily seen to satisfy

$$W \ll \min(YZ, (Z+r)(Y+r)r^{-1}).$$

Therefore

$$V \ll \sum_{\substack{m\leqq X\\ (m,q)XY\leqq mq}} \frac{XY}{m^2} + \sum_{\substack{m\leqq X\\ (m,q)XY>mq}} \frac{1}{m}\left(\frac{XY}{mq}(m,q)+\frac{X}{m}+Y+\frac{q}{(m,q)}\right)$$

$$\ll \sum_{r|q}\sum_{s>XY/q} \frac{XY}{r^2 s^2} + \sum_{r|q}\sum_{s} \frac{XY}{rs^2 q} + X + Y\log 2X + \sum_{r|q}\sum_{s<XYr/q} \frac{q}{r^2 s}$$

and this gives (19) and hence, by (18), (16) as required.

5. Completion of the Proof of Theorem 1

We first apply (16) to the rectangle $\mathscr{R}_i$. We take $K=1$, $X=Q=2^i$, $Y=M=N2^{-i}$. Thus

$$\sum_{(p,n)\in\mathscr{R}_i} f(p)f(n)\,e(pna/q)\log p \ll N((i+1)2^{-i})^{\frac{1}{2}} + N\phi(q)^{-\frac{1}{2}}$$

$$+(N2^i)^{\frac{1}{2}} + (Nq\log 2N/q)^{\frac{1}{2}}. \tag{20}$$

Next, for each pair i, j with $1\leqq j\leqq J_i$ we apply (16) to the family of 2^{j-1} rectangles $\mathscr{R}_{ijk}$ with $2^{j-1}<k\leqq 2^j$. By (13) we may take $K=2^{j-1}$, $M=N2^{-i}$, $Q=2^{i+1}$, $X=2^{i-j+1}$, $Y=32N2^{-i-j}$. Thus, by (12), $XY\geqq q$, so that the conditions for (16) to hold are satisfied. Hence

$$\sum_{2^{j-1}<k\leqq 2^j}\sum_{(p,n)\in\mathscr{R}_{ijk}} f(p)f(n)\,e(pna/q)\log p$$

$$\ll N((i+1)2^{-i-j})^{\frac{1}{2}} + N2^{-j}\phi(q)^{-\frac{1}{2}} + (N2^{i-j})^{\frac{1}{2}} + (Nq\log 2N/q)^{\frac{1}{2}}.$$

By (12), $J_i \ll \log(2N/q)$. Hence, on summing over those j with $1\leqq j\leqq J_i$ we obtain

$$\sum_{1\leqq j\leqq J_i}\sum_{2^{j-1}<k\leqq 2^j}\sum_{(p,n)\in\mathscr{R}_{ijk}} f(p)f(n)\,e(pna/q)\log p$$

$$\ll N((i+1)2^{-i})^{\frac{1}{2}} + N\phi(q)^{-\frac{1}{2}} + (N2^i)^{\frac{1}{2}} + (Nq)^{\frac{1}{2}}(\log 2N/q)^{\frac{3}{2}}.$$

Therefore, by (20), on summing over the i, $0\leqq i\leqq \log_2 N$, we have

$$\sum_{\substack{pn\leqq N\\(p,n)\notin\mathscr{E}}} f(p)f(n)\,e(pna/q)\log p \ll N+N\phi(q)^{-\frac{1}{2}}\log N+(Nq)^{\frac{1}{2}}(\log 2N/q)^{\frac{3}{2}}\log N.$$

This with (15) gives (10) and thus Theorem 1.

6. Proofs of the Corollaries

Let $S(\alpha,u)=\sum_{n\leqq u} f(n)\,e(n\alpha)$. Then

$$S(\alpha)=e((\alpha-\beta)N)\,S(\beta,N)-2\pi i(\alpha-\beta)\int_1^N S(\beta,u)\,e((\alpha-\beta)u)\,du. \tag{21}$$

Suppose that $\beta=b/r$ with $(b,r)=1$ and $r\leqq N$. Then, on using (4) when $u\leqq r$ and Theorem 1 when $u>r$ we obtain

$$S(\alpha)\ll\left(\frac{N}{\log N}+N\phi(r)^{-\frac{1}{2}}+(Nr)^{\frac{1}{2}}\left(\log\frac{2N}{r}\right)^{\frac{3}{2}}\right)(1+N|\alpha-b/r|). \tag{22}$$

If $q>N^{\frac{1}{2}}$, then we take $b=a$, $r=q$ which gives Corollary 1 at once. If $q\leqq N^{\frac{1}{2}}$, then by Dirichlet's theorem there exist b,r such that $(b,r)=1$, $r\leqq 2N/q$ and $|\alpha-b/r|\leqq q/(2rN)$. Thus, either $r=q$ or $1\leqq|ar-bq|=rq|(\alpha-b/r)-(\alpha-a/q)|\leqq q^2/(2N)+r/q\leqq\frac{1}{2}+r/q$, whence in either case $r\geqq\frac{1}{2}q$. Therefore $|\alpha-b/r|\leqq N^{-1}$ and consequently, by (22),

$$S(\alpha)\ll\frac{N}{\log N}+Nq^{-\frac{1}{2}}(\log q)^{\frac{3}{2}}$$

from which Corollary 1 follows once more.

To establish Corollary 2, let $\mathscr{A}$ denote the set of real numbers α for which there is a $q_0(\alpha)$ such that

$$\|q\alpha\|\geqq\exp(-q^{\frac{1}{4}})$$

for all integers $q>q_0(\alpha)$. By a theorem of Khintchine (see [6, Theorem 198]), the complement of $\mathscr{A}$ has Lebesgue measure 0. By Liouville's theorem (see [6, Theorem 191]), all real irrational algebraic α are members of $\mathscr{A}$. We show that if $\alpha\in\mathscr{A}$, then there is an $N_0(\alpha)$ such that for any $N>N_0(\alpha)$ there exist a,q with $(a,q)=1$, $|\alpha-a/q|\leqq q^{-2}$, and $\frac{1}{2}(\log N)^3\leqq q\leqq N(\log N)^{-3}$. Then the desired bound follows from Corollary 1.

Let p_n/q_n be the n-th convergent to the continued fraction for α. Then $|\alpha-p_n/q_n|<q_n^{-2}$, $(p_n,q_n)=1$. Moreover $\|q_n\alpha\|<q_{n+1}^{-1}$, and hence $q_{n+1}<\exp q_n^{\frac{1}{4}}$ provided that $q_n>q_0(\alpha)$. Let n be the least value of n so that $q_{n+1}>N(\log N)^{-3}$. Then, for a suitable $N_0(\alpha)$, when $N>N_0(\alpha)$ we have $N(\log N)^{-3}\geqq q_n>(\log q_{n+1})^3>\frac{1}{2}(\log N)^3$.

7. Examples

We now construct three totally multiplicative functions f, each with $|f(n)| \leqq 1$ for all n, which satisfy (i), (ii) or (iii) in § 1.

(i) Let $f(p) = e(-\alpha p)$ for $\frac{1}{2}N < p \leqq N$, $f(p) = 0$ otherwise. Then $S(\alpha) = \sum_{\frac{1}{2}N < p \leqq N} 1 \gg N/\log N$. In fact, by considering

$$\sum_{n \leqq N} z^{\Omega(n)} e(n\alpha) + \sum_{\frac{1}{2}N < p \leqq N} (1 - z e(\alpha p))$$

and using the maximum modulus principle, it follows that there is a totally multiplicative f with $|f(n)| = 1$ for all n and $|S(\alpha)| \geqq \sum_{\frac{1}{2}N < p \leqq N} 1$.

(ii) Let $f(n) = \chi(n)$, where χ is a character modulo q induced by the primitive character χ^* modulo k, so that $k|q$. When $(a, q) = 1$ we have

$$\sum_{n=1}^{q} \chi(n) e(na/q) = \mu(q/k) \chi^*(q/k) \tau(\chi^*) \bar{\chi}(a),$$

whence

$$\sum_{n \leqq u} \chi(n) e(na/q) = u/q\, \mu(q/k) \chi^*(q/k) \tau(\chi^*) \bar{\chi}(a) + \theta q \tag{23}$$

where $|\theta| \leqq 1$. When $q \not\equiv 2 \pmod 4$ we choose χ so that $k = q$, but when $q \equiv 2 \pmod 4$ there are no primitive characters modulo q so we instead choose χ so that $k = \frac{1}{2}q$. In either case $|\tau(\chi^*)| = \sqrt{k}$, so that $S(a/q) \gg N q^{-\frac{1}{2}}$.

(iii) We can certainly suppose that N is large. Let $r = [3N/Q]$, choose b so that $(b, r) = 1$, and let $f = \chi$ where χ is a character modulo r constructed as in (ii). Then, by (21) and (23), $|S(\alpha)| \gg N r^{-\frac{1}{2}}$ uniformly for α with $|\alpha - b/r| \leqq \frac{1}{2}N^{-1}$. Let a/q be a neighbour of b/r among the Farey fractions of order $[Q]$. Then $q + r > Q$, so that $Q - 3N/Q < q \leqq Q$. Thus

$$\left|\frac{a}{q} - \frac{b}{r}\right| = \frac{1}{qr} < \frac{1}{2N}$$

and therefore

$$|S(a/q)| \gg N r^{-\frac{1}{2}} \gg (Nq)^{\frac{1}{2}}.$$

8. Preliminaries of the Proof of Theorem 2

Let χ be a character modulo q with conductor r induced by the character χ^* modulo r. Then $r|q$ and

$$\sum_{n \leqq x} \chi(n) = \sum_{\substack{n \leqq x \\ (n, q/r) = 1}} \chi^*(n)$$
$$= \sum_{d|q/r} \mu(d) \chi^*(d) \sum_{m \leqq x/d} \chi^*(m),$$

so that

$$\left|\sum_{n\leqq x}\chi(n)\right|\leqq 2^{\omega(q/r)}\max_{y}\left|\sum_{n\leqq y}\chi^*(n)\right|.$$

Therefore, as $2^{\omega(q/r)}\ll(q/r)^{\frac{1}{2}}$, to prove the theorem it suffices to consider primitive characters.

When χ is a primitive character, with modulus q, the sum $\sum_{n\leqq x}\chi(n)$ has a simple Fourier expansion which Pólya [9] put in the following quantitative form.

Lemma 1. *Let χ be a primitive character modulo q, $q>1$. Then*

$$\sum_{n\leqq x}\chi(n)=\frac{\tau(\chi)}{2\pi i}\sum_{\substack{h=-H\\ h\neq 0}}^{H}\frac{\bar{\chi}(h)}{h}(1-e(-hx/q))+O(1)+O(qH^{-1}\log q).$$

In this expansion $\tau(\chi)$, the Gaussian sum, satisfies $|\tau(\chi)|=\sqrt{q}$. Hence the constant term is

$$\tau(\chi)(1-\chi(-1))(2\pi i)^{-1}L(1,\chi)+O(q^{\frac{3}{2}}H^{-1}).$$

Littlewood [7] has shown that the estimate $L(1,\chi)\ll\log\log q$ is a consequence of GRH. Therefore, in order to prove Theorem 2 we need only show (assuming GRH) that for primitive χ modulo q with $q>1$ we have

$$\sum_{n=1}^{q}\frac{\chi(n)}{n}e(n\alpha)\ll\log\log q \tag{24}$$

uniformly in α.

Our further use of GRH takes the form of an appeal to

Lemma 2. *Let χ be a non-principal character modulo k, and suppose that $L(s,\chi)\neq 0$ for* $\operatorname{Re}s>\frac{1}{2}$. *Suppose further that* $(\log k)^4\leqq y\leqq k$ *and* $x\leqq k$. *Then*

$$\sum_{n\leqq x}\chi(n)=\sum_{\substack{n\leqq x\\ n\in\mathcal{N}}}\chi(n)+O(xy^{-\frac{1}{2}}(\log k)^4),$$

where $\mathcal{N}$ is the set of those natural numbers none of whose prime factors exceed y.

Proof. The hypothesis concerning $L(s,\chi)$ implies (see [3, § 19]) that

$$\sum_{p\leqq u}\chi(p)p^{-it}\ll u^{\frac{1}{2}}(\log(k(|t|+2)(u+2)))^2$$

and hence that

$$\sum_{p>y}\log((1-\chi(p)p^{-s})^{-1})\ll y^{-\frac{1}{2}}(\log k)^2 \tag{25}$$

uniformly for $\sigma\geqq 1$, $|t|\leqq k^2$, $y\leqq k$.

Let

$$M(s,\chi)=\prod_{p\leqq y}(1-\chi(p)p^{-s})^{-1}=\sum_{n\in\mathcal{N}}\chi(n)n^{-s}$$

and

$$N(s,\chi)=L(s,\chi)/M(s,\chi).$$

Then, uniformly for $\sigma\geqq 1$, t real, $y\leqq k$,

$$M(s,\chi)\ll\log k, \tag{26}$$

and, by (25), uniformly for $\sigma\geqq 1$, $|t|\leqq k^2$, $(\log k)^4\leqq y\leqq k$,

$$N(s,\chi)=\exp\Big(\sum_{p>y}\log((1-\chi(p)p^{-s})^{-1})\Big)=1+O(y^{-\frac{1}{2}}(\log k)^2). \tag{27}$$

Write $\theta=1+(\log k)^{-1}$, $T=k^2$. Then

$$\sum_{\substack{n\leqq x\\ n\notin\mathcal{N}}}\chi(n)=\frac{1}{2\pi i}\int_{\theta-iT}^{\theta+iT}M(s,\chi)(N(s,\chi)-1)\frac{x^s}{s}\,ds+O(1).$$

By (26) and (27) this is $\ll x y^{-\frac{1}{2}}(\log k)^4$ which gives the desired conclusion.

9. Proof of Theorem 2

Suppose that q is large, as we may, that $1\leqq u\leqq q$, and that

$$S(\alpha,u)=\sum_{n\leqq u}\chi(n)e(n\alpha)$$

where χ is a primitive character modulo q. If $u\leqq 500$, then it is completely trivial that

$$S(\alpha,u)\ll u/\log 2u. \tag{28}$$

If $u>500$, so that $u(\log u)^{-3}>1$, then by Dirichlet's theorem there exist b, r with $(b,r)=1$, $r\leqq u(\log u)^{-3}$ and

$$|\alpha-b/r|\leqq\frac{(\log u)^3}{ru}.$$

Let $y=(\log q)^{20}$. We now distinguish two cases. If $r\geqq(\log u)^3$, then by Corollary 1 we have (28) once more. Suppose, on the other hand, that $1\leqq r<(\log u)^3$. Let $\mathcal{N}$ be as in Lemma 2. Then, for $r\leqq y$ we have

$$\sum_{\substack{n\leqq v\\ n\notin\mathcal{N}}}\chi(n)e(nb/r)=\sum_{d\mid r}\frac{\chi(d)}{\phi(r/d)}\sum_{\psi \bmod r/d}\psi(b)\tau(\bar\psi)\sum_{\substack{m\leqq v/d\\ m\notin\mathcal{N}}}\psi\chi(m). \tag{29}$$

The character $\psi\chi$ is a non-principal character modulo qr/d, and since q is large the parameter y satisfies the condition $(\log(qr/d))^4\leqq y\leqq qr/d$. Hence, by Lemma 2, for $v\leqq q$ we have

$$\sum_{\substack{n \leqq v \\ n \notin \mathcal{N}}} \chi(n) e(nb/r) \ll \sum_{d \mid r} (r/d)^{\frac{1}{2}} (v/d)(\log q)^{-6}$$
$$\ll v(\log q)^{-4}.$$

Therefore, by (21),

$$\sum_{\substack{n \leqq u \\ n \notin \mathcal{N}}} \chi(n) e(n\alpha) \ll u(\log q)^{-4}(1+u|\alpha - b/r|)$$
$$\ll u/\log u.$$

Thus, in this case,

$$S(\alpha, u) = \sum_{\substack{n \leqq u \\ n \in \mathcal{N}}} \chi(n) e(n\alpha) + O(u/\log u).$$

Hence, by (28), we always have

$$S(\alpha, u) \ll (u/\log 2u) + \sum_{\substack{n \leqq u \\ n \in \mathcal{N}}} 1.$$

On a further summation by parts we find that

$$\sum_{n \leqq q} \frac{\chi(n)}{n} e(n\alpha) \ll \int_e^q \frac{du}{u \log u} + \sum_{n \in \mathcal{N}} 1/n$$
$$= \log\log q + \prod_{p \leqq y} (1-1/p)^{-1}$$
$$\ll \log\log q.$$

This gives (24) and completes the proof of the theorem.

We can also obtain hypothetical improvements on the Pólya-Vinogradov inequality (7) by arguing from bounds for $|L(s, \chi)|$. For example, suppose that

$$M(q) = \max |L(s, \chi)|$$

where the maximum is taken over $\sigma \geqq 1$, $|t| \leqq q^2$ and all non-principal characters χ to moduli not exceeding q^2. We may use ideas of G. Halász [4] to bound the sum $\sum_{m \leqq v/d} \psi \chi(m)$ of (29) in terms of $M(q)$, and proceed as above to show that

$$\sum_{n \leqq x} \chi(n) \ll q^{\frac{1}{2}} M(q)^{\frac{1}{3}} (\log q)^{\frac{2}{3}} \left(\log \frac{c \log q}{M(q)}\right)^{\frac{1}{3}}. \tag{30}$$

Since it is known that $M(q) \ll \log q$, any improvement in the bounds for $M(q)$ would give a corresponding sharpening of the Pólya-Vinogradov inequality.

References

1. Daboussi, H.: Fonctions multiplicatives presque périodiques B. D'après un travail commun avec Hubert Delange. Journées Arithmétique de Bordeaux (Conf. Univ. Bordeaux, 1974), 321–324. Astérisque, 24–25. Soc. Math France, Paris, 1975. See also Daboussi, H., Delange, H.: Quelques

propriétés des fonctions multiplicatives de module au plus égal à 1, C.R. Acad. Sci. Paris Ser. A **278**, 657–660 (1974)
2. Davenport, H.: On some infinite series involving arithmetical functions (II). Quart. J. Math. **8**, 313–320 (1937)
3. Davenport, H.: Multiplicative number theory. Chicago: Markham, 1966
4. Halász, G.: On the distribution of additive and the mean values of multiplicative arithmetic functions. Studia Sci. Math. Hungar. **6**, 211–233 (1971)
5. Halberstam, H., Richert, H.-E.: Sieve methods. London: Academic Press 1974
6. Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers, 4th ed. London: Oxford University Press 1962
7. Littlewood, J.E.: On the class number of the corpus $P(\sqrt{-k})$. Proc. London Math. Soc. **27**, 358–372 (1928)
8. Paley, R.E.A.C.: A theorem on characters. J. London Math. Soc. **7**, 28–32 (1932)
9. Pólya, G.: Über die Verteilung der quadratischen Reste und Nichtreste. Göttinger Nachrichten 1918, pp. 21–29
10. Vinogradov, I.M.: Über die Verteilung der quadratischen Reste und Nichtreste. J. Soc. Phys. Math. Univ. Permi **2**, 1–14 (1919)
11. Vinogradov, I.M.: The method of trigonometrical sums in the theory of numbers. New York: Interscience 1954

Received April 6, 1977