Inventiones math. 44, 87 – 102 (1978)

Inventiones mathematicae
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Conjugations of Arithmetic Automorphic Function Fields

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In [1], Doi and Naganuma showed that the conjugation of a Shimura curve is again a Shimura curve. The present paper deals with the generalization of their result.

Consider in general a reductive algebraic group G defined over $\mathbf{Q}$. Let G^u be the semi-simple part of G. Assume that $G^u_{\mathbf{R}}$ modulo a maximal compact subgroup defines a bounded symmetric domain $\mathscr{H}$, and that a system of canonical models (in the sense of Shimura [11, 2.13]) for the quotients of $\mathscr{H}$ by the arithmetic subgroups Γ_X of G exists. Let $\{V_X, \phi_X, J_{XW}(u)\}$ be such a system. Take an arbitrary automorphism τ of the complex number field $\mathbf{C}$, and conjugate all the V_X's and $J_{XW}(u)$'s by τ. Then one expects that $\{V_X^\tau, \tilde{\phi}_X, J_{XW}(u)^\tau\}$ with suitable $\tilde{\phi}_X$'s forms a system of canonical models for some reductive group G_1.

In this paper we show that this is the case if G is the type of groups investigated by Shimura in [11]. The corresponding group G_1 is defined explicitly in 1.2, and the precise statement of the result is given as Theorem 1.3.

In Shimura's construction, for special Γ_X the model V_X is first realized as a subvariety of a moduli-variety V_Ω for some PEL-type Ω. Consider V_X^τ as embedded in V_Ω^τ. It is known that the conjugate variety V_Ω^τ is isomorphic to the moduli-variety $V_{\Omega'}$ of another PEL-type Ω'. The relations between Ω and Ω' are provided by Shimura's work [6]. One of our main task then is to prove that the isomorphism of V_Ω^τ to $V_{\Omega'}$ induces an isomorphism of V_X^τ to V_{X_1} for some arithmetic subgroup Γ_{X_1} of G_1. This can be achieved by studying the isolated fixed points of $G_{\mathbf{Q}}$ and $G_{1\mathbf{Q}}$. We carry out these considerations in Sections 2 and 3.

The same functorial property also holds for the models constructed by Miyake in [3]. In Section 4 we deal with this case briefly. We shall not give the proof, because the argument is similar to, and actually simpler than, the one presented in this paper.

We assume the reader is familiar with Shimura's work [10] and [11], which will be quoted respectively as [A] and [C] hereafter.

* Partially supported by NSF grant MPS 75-07948

0020-9910/78/0044/0087/$03.20

Notations

We adopt the notations of [C], of which some are recalled here.

The multiplicative group of an associative ring S with identity is denoted by $S^\times$. For an algebraic group G defined over $\mathbf{Q}$, $G_\mathbf{A}$ denotes its adelization. The finite part and infinite part of $G_\mathbf{A}$ are denoted by G_0 and G_∞ respectively. Denote the identity component of G_∞ by $G_{\infty+}$, and put $G_{\mathbf{A}+}=G_0G_{\infty+}$.

For an algebraic number field F, $\mathfrak{r}_F$ denotes the ring of integers of F, F_{ab} the abelian closure of F, and F_c the closure of $F^\times F_{\infty+}^\times$ in the idele group $F_\mathbf{A}^\times$.

1. The Main Theorem

1.1. Let F be a totally real algebraic number field of degree g, and B a quaternion algebra over F. Let $\tau_1, \ldots, \tau_g$ be the g distinct isomorphisms of F into R arranged in such a way that B is unramified at $\tau_1, \ldots, \tau_r$, and ramified at all other infinite places. Denote the discriminant of B over F by $D(B/F)$.

Assume $r>0$. Define an algebraic group G over $\mathbf{Q}$ so that the $\mathbf{Q}$-rational points of G are

$$G_\mathbf{Q}=\{\alpha\in GL_n(B)\,|\,\alpha\cdot{}^t\alpha^\iota=\nu(\alpha)1_n \text{ with } \nu(\alpha)\in F^\times\},$$

where ι denotes the main involution of B and ${}^t\alpha$ the transpose of α. The semi-simple part of G is

$$G^u=\{\alpha\in G\,|\,\nu(\alpha)=1\},$$

and $G_\mathbf{R}^u$ modulo a maximal compact subgroup defines a bounded symmetric domain $\mathscr{H}$ which can be identified with $\mathscr{H}_n^r$, r copies of Siegel's upper half space $\mathscr{H}_n$ of degree n.

Consider a representation θ of F equivalent to $\sum_{\nu=1}^{r}\tau_\nu$. Let (F', θ') be the reflex of (F, θ) and $\lambda=\det\theta'$ (see [CI, § 1] for notation). Then λ is a homomorphism of $F'^\times$ to $F^\times$. Put

$$\bar{\mathscr{G}}_+=\{x\in G_{\mathbf{A}+}\,|\,\nu(x)\in\lambda(F_\mathbf{A}'^\times)F_c\}.$$

Define a group extension $\mathfrak{A}$ of $\mathfrak{A}^1=\bar{\mathscr{G}}_+/F_cG_{\infty+}$ and a subgroup A_+ of $\mathfrak{A}$ as in [CII, §4]. Let $\mathfrak{k}^*$ be the subfield of F'_{ab} defined in [CII, 3.10], and ρ the homomorphism from $\mathfrak{A}$ to $\mathrm{Gal}(\mathfrak{k}^*/F')$ defined in [CII, 4.7]. The kernel of ρ is the closure of A_+. Consider the set $\mathfrak{Z}$ of all open compact subgroup of $\mathfrak{A}$. For every $X\in\mathfrak{Z}$, $\Gamma_X=A_+\cap X$ acts on $\mathscr{H}$ properly discontinuously and $\mathscr{H}/\Gamma_X$ has finite measure. The subgroup $\rho(X)$ of $\mathrm{Gal}(\mathfrak{k}^*/F')$ corresponds to a subfield of $\mathfrak{k}^*$, which we denote by k_X.

As shown in [C], for every $X\in\mathfrak{Z}$ there is a model (V_X, ϕ_X) of $\mathscr{H}/\Gamma_X$ with V_X rational over k_X, and for $u\in\mathfrak{A}$, X, $W\in\mathfrak{Z}$ such that $uXu^{-1}\subset W$, there is a morphism $J_{WX}(u)$ of V_X onto $V_W^{\rho(u)}$ rational over k_X. Furthermore the system $\{V_X, \phi_X, J_{WX}(u), (X, W\in\mathfrak{Z};\, u\in\mathfrak{A})\}$ enjoys the properties stated in [CII, Theorem 5.2]. We call it a canonical system of models associated with G.

1.2. Let τ be an automorphism of $\mathbf{C}$. By a well-known Theorem of Hasse, there is a quaternion algebra B_1 over F, unique up to F-linear isomorphism, such that B_1 is unramified at $\tau_1\tau, \ldots, \tau_r\tau$, ramified at all other infinite places, and $D(B_1/F) = D(B/F)$. Consider the algebraic group G_1 over $\mathbf{Q}$ whose $\mathbf{Q}$-rational points are

$$G_{1\mathbf{Q}} = \{\alpha \in GL_n(B_1) \mid \alpha \cdot {}^t\alpha^\iota = \nu_1(\alpha) 1_n \text{ with } \nu_1(\alpha) \in F^\times\}.$$

Define the counterparts λ_1, $\mathfrak{A}_1$, $\mathfrak{k}_1^*$, ρ_1, $\mathfrak{Z}_1$, etc. of λ, $\mathfrak{A}$, $\mathfrak{k}^*$, ρ, $\mathfrak{Z}$, etc. Note that the representation θ_1 of F associated with B_1 is equivalent to $\sum_{\nu=1}^{r} \tau_\nu \tau \sim \theta^\tau$. Denote the reflex of (F, θ_1) by (F'_1, θ_1). Then it follows from Lemma 1.5 proved below that $F'_1 = F'^\tau$, $\lambda_1(x^\tau) = \lambda(x)$ for $x \in F'$, and $\mathfrak{k}_1^* = \mathfrak{k}^{*\tau}$. Let $\{V_{X_1}, \phi_{X_1}, J_{W_1 X_1}(u_1), (X_1, W_1 \in \mathfrak{Z}_1; u_1 \in \mathfrak{A}_1)\}$ be a system of canonical models associated with G_1.

1.3. **Theorem.** *There is a topological isomorphism* $\mathfrak{u}$ *of* $\mathfrak{A}$ *to* $\mathfrak{A}_1$, *which is locally algebraic, with the following properties. For* $X_1 \in \mathfrak{Z}_1$ *and* $u_1 \in \mathfrak{A}_1$, *put* $X = \mathfrak{u}^{-1}(X_1)$ *and* $u = \mathfrak{u}^{-1}(u_1)$. *Then* $k_{X_1} = k_X^\tau$, $\tau\rho_1(u_1) = \rho(u)\tau$ *on* $\mathfrak{k}^*$, *and for each* $X_1 \in \mathfrak{Z}_1$, *there is a biregular isomorphism* ψ_X *of* V_{X_1} *to* V_X^τ *rational over* k_X^τ *such that*

$$\psi_W^{\rho_1(u_1)} \circ J_{W_1 X_1}(u_1) = J_{WX}(u)^\tau \circ \psi_X \qquad (uXu^{-1} \subset W).$$

Remark. We have $\tau = \mathrm{id.}$ on F' if and only if $\{\tau_1, \ldots, \tau_r\} = \{\tau_1\tau, \ldots, \tau_r\tau\}$. In this case we have $B_1 = B$ and $G_1 = G$. Since $\rho\colon \mathfrak{A} \to \mathrm{Gal}(\mathfrak{k}^*/F')$ is surjective, there exists $u \in \mathfrak{A}$ such that $\rho(u) = \tau$ on $\mathfrak{k}^*$. Define $\mathfrak{u}(x) = u^{-1}xu$ for $x \in \mathfrak{A}$. Then $\mathfrak{u}$ satisfies the conditions of Theorem 1.3. Therefore the theorem is trivial in the case where $\tau = \mathrm{id.}$ on F'.

1.4. For $X \in \mathfrak{Z}$, let K_X be the field of all functions on V_X rational over k_X, and put

$$\mathscr{K}_X = \{f \circ \phi_X \mid f \in K_X\},$$
$$\mathscr{K} = \bigcup_{X \in \mathfrak{Z}} \mathscr{K}_X.$$

We call $\mathscr{K}$ the arithmetic automorphic function field associated with G. For $u \in \mathfrak{A}$ and $f \in K_X$, put

$$(f \circ \phi_X)^{\omega(u)} = f^{\rho(u)} \circ J_{XW}(u) \circ \phi_W,$$

where $W = u^{-1} X u$. Then ω is a homomorphism of $\mathfrak{A}$ to $\mathrm{Aut}(\mathscr{K}/F')$. Let $\mathscr{K}_1$ be the field of arithmetic automorphic functions associated with G_1, and $\omega_1\colon \mathfrak{A}_1 \to \mathrm{Aut}(\mathscr{K}_1/F'_1)$ the homomorphism corresponding to ω. In terms of arithmetic automorphic function fields, we can restate Theorem 1.3 as follows.

Theorem. *Let* $\mathfrak{u}\colon \mathfrak{A} \to \mathfrak{A}_1$ *be as in* Theorem 1.3. *Then there is an isomorphism* $\pi\colon \mathscr{K} \to \mathscr{K}_1$ *extending* $\tau\colon \mathfrak{k}^* \to \mathfrak{k}_1^*$ *such that* $\pi\omega_1(u_1) = \omega(u)\pi$ *for* $u \in \mathfrak{A}$ *and* $u_1 = \mathfrak{u}(u) \in \mathfrak{A}_1$.

1.5. **Lemma.** *Let E be an algebraic number field and Ψ a $\mathbf{Q}$-linear representation of E by complex matrices. For an automorphism τ of $\mathbf{C}$, let Ψ_1 be a representation of E equivalent to Ψ^τ. Denote the reflexes of (E, Ψ) and (E, Ψ_1) by (E', Ψ') and (E'_1, Ψ'_1) respectively. Then $E'_1 = (E')^\tau$ and Ψ'_1 is equivalent to the representation Φ of E'_1 given by $\Phi(x^\tau) = \Psi'(x)$ $(x \in E')$.*

Proof. By definition, E_1' is generated over $\mathbf{Q}$ by $\{\operatorname{tr}\Psi_1(x)|x\in E\}$. Since $\operatorname{tr}\Psi_1(x) = (\operatorname{tr}\Psi(x))^\tau$, we have $E_1'=(E')^\tau$. To show Ψ_1' is equivalent to Φ, take an (E, E')-module of type (E, Ψ) and of type (E', Ψ') [CI, 1.1]. Introduce an (E, E_1')-module structure on V as follows: For $a\in E$, $b\in E_1'$ and $x\in V$, put $(a\otimes b)x=axc$, where c is the element of E' such that $c^\tau=b$. Call this (E, E_1')-module V_1. Then V_1 is of type (E, Ψ_1) as well as of type (E_1', Φ). Therefore $\Psi_1'\sim\Phi$.

Remark. The above Lemma can be regarded as dual to Proposition 12 of [4].

1.6. Let B^* be the quaternion algebra over F^τ such that $D(B^*/F^\tau)=D(B/F)^\tau$ and such that B^* is unramified at $\tau^{-1}\tau_v\tau$ for $v=1,\dots,r$, and ramified for $v=r+1,\dots,g$. It is easy to see that the algebraic group defined in terms of $GL_n(B^*)$ is isomorphic to G_1 over $\mathbf{Q}$. Especially, if $F^\tau=F$, $D(B/F)^\tau=D(B/F)$ and $\{\tau^{-1}\tau_1\tau,\dots,\tau^{-1}\tau_r\tau\} =\{\tau_1,\dots,\tau_r\}$, then G_1 is isomorphic to G. Under this situation we have $F'^\tau=F'$. The isomorphism $\mathfrak{u}$ of Theorem 1.3 can be considered as an automorphism of $\mathfrak{A}$. If $\tau\neq\mathrm{id}.$ on F', then $\mathfrak{u}$ is not an inner automorphism. It would be interesting, as suggested by the Corollary in the introduction of [1], to see under what conditions we can reduce the fields of definition of V_X's. The author hopes to discuss this and other implications of the result on some other occasion.

2. Construction of $\mathfrak{u}$

2.1. Let K be a totally imaginary quadratic extension of F. For each $v=1,\dots,g$, fix an extension of τ_v to K and denote it again by τ_v. By Hasse's Theorem on central simple algebras, there is a quaternion algebra L over K which is isomorphic to both $B\otimes_F K$ and $B_1\otimes_F K$ over K. Denote the main involution of L by ι. Then there are a positive involution ρ on L and two invertible elements v and v_1 of L such that $v^\rho= -v$, $v_1^\rho=-v_1$, and

$$B\cong\{x\in L\,|\,x^\iota=vx^\rho v^{-1}\},$$
$$B_1\cong\{x\in L\,|\,x^\iota=v_1x^\rho v_1^{-1}\},$$

see [8, 7.2]. Via these isomorphisms we regard B and B_1 as F-subalgebras of L.

Put $T=v1_n$, $S=v^{-1}1_n$ and $\kappa=-(vv^\iota)^{-1}$. Then $T, S\in GL_n(L)$, $\kappa\in F^\times$ and

$$(2.1.1)\quad {}^tT^\rho=-T,\qquad S^{\iota\rho}S=\kappa 1_n,\qquad T\cdot{}^tS^\rho=-S^{\iota\rho}T^{\iota\rho}$$

[A, Prop. 6.2]. We assume Conditions (6.3.7) and (6.3.8) of [A] for κ and T. Similarly, put $T_1=v_1 1_n$, $S_1=v_1^{-1}1_n$ and $\kappa_1=-(v_1v_1^\iota)^{-1}$. We also assume for T_1 and κ_1 the conditions corresponding to [A, (6.3.7), (6.3.8)].

Let $\mathfrak{p}$ be a finite prime of F. Denote the completion of F at $\mathfrak{p}$ by $F_\mathfrak{p}$. For an algebra A over F, put $A_\mathfrak{p}=A\otimes_F F_\mathfrak{p}$.

2.2. **Proposition.** *For every finite prime $\mathfrak{p}$ of F, there is $a\in K_\mathfrak{p}^\times$ such that $\kappa/\kappa_1=aa^\rho$.*

Proof. Since $B_\mathfrak{p}$ and $B_{1\mathfrak{p}}$ are $F_\mathfrak{p}$-isomorphic, there is b in $L_\mathfrak{p}^\times$ such that $B_{1\mathfrak{p}}=bB_\mathfrak{p}b^{-1}$. It follows that

$$B_{1\mathfrak{p}}=\{x\in L_\mathfrak{p}\,|\,x^\iota=(bvb^\rho)x^\rho(bvb^\rho)^{-1}\}.$$

We also have $(bvb^\rho)^\rho = -bvb^\rho$. Hence $v_1 = cbvb^\rho$ for some $c \in F_{\mathfrak{p}}^\times$. Denote the reduced norm from L to K by N. Then it is easy to see that $\kappa = aa^\rho \kappa_1$ with $a = cN(b)$.

2.3. **Proposition.** *For every finite prime $\mathfrak{p}$ of F, there are $c \in F_{\mathfrak{p}}^\times$ and $\alpha \in GL_n(L_{\mathfrak{p}})$ such that $T_1 = c\alpha T \cdot {}^t\alpha^\rho$.*

Proof. Let the notation be as in the proof of Proposition 2.2. Put $\alpha = b1_n$. Then we have $T_1 = c\alpha T \cdot {}^t\alpha^\rho$.

2.4. Let

$$\mathscr{S} = \{z \in M_n(\mathbf{C}) \mid 1 - {}^t\bar{z}z \text{ is positive definite}\}$$

and

$$\mathscr{D}_n = \{z \in \mathscr{S} \mid {}^tz = z\}.$$

Define an algebraic group G^* over $\mathbf{Q}$ so that the $\mathbf{Q}$-rational points of G^* are the group of similitudes of T:

$$G^*_{\mathbf{Q}} = G(T)_{\mathbf{Q}} = \{\alpha \in GL_n(L) \mid aT \cdot {}^t\alpha^\rho = \nu(\alpha)T \text{with } \nu(\alpha) \in F^\times\}.$$

Then $G^*_{\mathbf{R}+}$ acts on $\mathscr{S}^r$ as described in [A, 6.5]. As shown in [A, 6.6] there are maps

$i\colon\ G \to G^*$, a $\mathbf{Q}$-rational injection,

$i'\colon\ B^n \to L^n$, a B-linear injection,

$j\colon\ \mathscr{H}_n^r \to \mathscr{D}_n^r \subset \mathscr{S}^r$, a holomorphic bijection,

such that $\nu(i(\alpha)) = \nu(\alpha)$, $j(\alpha(z)) = i(\alpha)(j(z))$, $i'(x\alpha) = i(x)i'(\alpha)$ for $\alpha \in G_{\mathbf{Q}+}$, $z \in \mathscr{H}_n^r$ and $x \in B^n$. The image of G under i is

$$i(G) = \{\alpha \in G^* \mid \alpha^{t\rho} S = S\alpha\}.$$

Using T_1 and S_1 instead of T and S, we define the counterparts of G^*, i, i' and j, and denote them by G_1^*, i_1, i_1' and j_1 respectively.

2.5. From B we can construct a representation Φ of $L_{\mathbf{R}}$ by complex matrices as in [A, 6.4]. Similarly, define a representation Φ_1 of $L_{\mathbf{R}}$, using B_1 instead. It is easy to see that Φ_1 is equivalent to Φ^τ. Let (H, Φ') be the reflex of (K, Φ). Choose an ample $\mathfrak{r}_F$-lattice $\mathfrak{m}$ in B^n in the sense of [A, 3.7], and put $\mathfrak{M} = \mathfrak{r}_K \cdot \mathfrak{m}$. Then $\mathfrak{M}$ is an $\mathfrak{r}_K$-lattice in L^n. Replacing v by cv for suitable $c \in F_+^\times$ if necessary, we can assume $\mathrm{tr}_{L/\mathbf{Q}}(T(\mathfrak{M}, \mathfrak{M})) = \mathbf{Z}$. Form the PEL-type $\Omega = (L, \Phi, \rho; T, \mathfrak{M})$. Then $\Omega^\tau = (L, \Phi_1, \rho; T^*, \mathfrak{M}^*)$, where $T^* \in M_n(L)$ and $\mathfrak{M}^*$ is a lattice in L^n. By [61, Proposition 1.11], for every finite prime $\mathfrak{p}$ of F, there are $c \in F_{\mathfrak{p}}^\times$ and $\alpha \in GL_n(L_{\mathfrak{p}})$ such that $T^* = c\alpha T \cdot {}^t\alpha^\rho$. (As already noted in [1], the original condition that $\tau = \mathrm{id}.$ on H is not necessary.)

Note that both T^* and T_1 have signature (n, n) at an infinite prime corresponding to $\tau_\nu\tau$ $(\nu = 1, \ldots, r)$, and $(2n, 0)$ at other infinite primes of F. Next, combining Proposition 2.3 with the observation made at the end of the last paragraph, we see that for every finite prime $\mathfrak{p}$ of F, there are $c_{\mathfrak{p}} \in F_{\mathfrak{p}}^\times$ and $\alpha_{\mathfrak{p}} \in GL_n(L_{\mathfrak{p}})$ such that $\alpha_{\mathfrak{p}} T^* \cdot {}^t\alpha_{\mathfrak{p}}^\rho = c_{\mathfrak{p}} T_1$. Hence by a Theorem of Landherr [2], there are $c \in F_+^\times$ and $\alpha \in GL_n(L)$ such that $\alpha T^* \cdot {}^t\alpha^\rho = cT_1$.

Replace v_1 by cv_1, hence T_1 by cT_1. Then $\Omega^\tau=(L,\Phi_1,\rho;T^*,\mathfrak{M}^*)$ is equivalent to $(L,\Phi_1,\rho;T_1,\mathfrak{M}^*\alpha^{-1})$. Put $\mathfrak{M}_1=\mathfrak{M}^*\alpha^{-1}$. Then we can assume

$$\Omega=(L,\Phi,\rho:T,\mathfrak{M}) \quad \text{and} \quad \Omega^\tau=(L,\Phi_1,\rho;T_1,\mathfrak{M}_1).$$

2.6. Take a totally imaginary quadratic extension P of F different from K such that $B\otimes_F P$ is isomorphic to $M_2(P)$. Put $Y=M_n(P)$. Denote the complex conjugation on P by ρ. For $a\in Y$, put $a^\delta={}^t a^\rho$. Then δ is a positive involution on Y. By [A, 4.7] there is an F-linear isomorphism f of Y into $M_n(B)$ such that $f(a^\delta)={}^t f(a)^\iota$ for all $a\in Y$. Let $z\in\mathscr{H}$ be the fixed point of (Y,P,δ,f) [A, (4.7.5)]. We use (Y,P,δ,f) to define a representation Ψ of P as in [A, 4.9]. The representation Ψ is equivalent to $\sum_{\nu=1}^{r}\chi_\nu$, where $\chi_\nu(\nu=1,\dots,r)$ is one of the extensions of τ_ν to P. For $\nu=r+1,\dots,g$, let χ_ν be any extension of τ_ν to P.

By Hasse's Theorem, $B_1\otimes_F P$ is also isomorphic to $M_2(P)$. Therefore there is an F-linear isomorphism f_1 of Y into $M_n(B_1)$ such that $f_1(a^\delta)={}^t f_1(a)^\iota$. Let $z_1\in\mathscr{H}$ be the fixed point of (Y,P,δ,f_1). Let Ψ_1 be the representation of P constructed from (Y,P,δ,f_1) as Ψ was from (Y,P,δ,f). Conjugating f_1 by a suitable element of $GL_n(B_1)$, we may assume that Ψ_1 is equivalent to Ψ^τ.

2.7. Let $\Sigma(\Omega)=\{\mathscr{Q}_w\,|\,w\in\mathscr{S}^r\}$ be an analytic family of PEL-structures of type Ω with parametrizing function $\mathfrak{y}(x,w)$, see [A, 6.4]. Let (Y,P,δ,f) be as in 2.6. Put $Z=L\otimes_F Y\cong M_{2n}(KP)$, and regard L^n as a Z-module by setting $x(a\otimes b)=ax(i(f(b)))$ for $a\in L$, $b\in Y$ and $x\in L^n$. Let $z\in\mathscr{H}$ be the fixed point of (Y,P,δ,f) and $y=j(z)\in\mathscr{S}^r$. Consider the structure $\mathscr{Q}_y=(A_y,\mathscr{C}_y,\theta_y)$. We can extend θ_y to an anti-isomorphism θ^* of Z into $\mathrm{End}_{\mathbf{Q}}(A_y)$ as in [A, 6.7–6.9]. Define a representation Ξ of Z so that

$$\Xi\sim 2\Psi+\sum_{\nu=r+1}^{g}(\chi_\nu+\chi_\nu\rho) \qquad \text{on } P,$$

$$\Xi\sim\sum_{\nu=1}^{r}(\tau_\nu+\tau_\nu\rho)+2\sum_{\nu=r+1}^{g}\tau_\nu \qquad \text{on } K,$$

see [A, 6.8, 6.9]. For $a,a'\in L^n$ put $U(a,a')=\mathrm{tr}_{L/\mathbf{Q}}(T(a,a'))$. Then $\mathscr{Q}^*=(A_y,\mathscr{C}_y,\theta^*)$ is of type $(Z,\Xi,\mathfrak{M},U)$ in the sense of [CI, 4.1].

Let (Y,P,δ,f_1) and z_1 be as in 2.6, and put $y_1=j_1(z_1)$. Denote $\Omega^\tau=(L,\Phi_1,\rho;T_1,\mathfrak{M}_1)$ by $\hat{\Omega}$. Let $\Sigma(\hat{\Omega})=\{\hat{\mathscr{Q}}_w|w\in\mathscr{S}^r\}$ be an analytic family of PEL-structures of type $\hat{\Omega}$ with parametrizing function $\mathfrak{y}_1(x,w)$. We can extend $\hat{\mathscr{Q}}_{y_1}=(\hat{A}_{y_1},\hat{\mathscr{C}}_{y_1},\hat{\theta}_{y_1})$ to a structure $\hat{\mathscr{Q}}^*=(\hat{A}_{y_1},\hat{\mathscr{C}}_{y_1},\hat{\theta}^*)$ of type $(Z,\Xi_1,\mathfrak{M}_1,U_1)$. Here we regard L^n as a Z-module by setting $x(a\otimes b)=ax(i_1(f_1(b)))$. Ξ_1 is a representation of Z defined in the same way as Ξ was defined in the last paragraph. And $U_1(a,a')=\mathrm{tr}_{L/\mathbf{Q}}(T_1(a,a'))$ for $a,a'\in L^n$. Comparing the restriction of Ξ_1 and Ξ^τ to both K and P, we see that Ξ_1 is equivalent to Ξ^τ.

We show that if the parametrizing function $\mathfrak{y}_1$ is chosen suitably, then we have

(2.7.1) $\quad \mathscr{Q}^{*\tau}\cong\hat{\mathscr{Q}}^*.$

Let u be a point of $\mathscr{S}^r$ such that $\mathscr{Q}_y^\tau$ is isomorphic to $\hat{\mathscr{Q}}_u=(\hat{A}_u,\hat{\mathscr{C}}_u,\hat{\theta}_u)$. There is an anti-isomorphism θ_u^* of Z into $\mathrm{End}_{\mathbf{Q}}(\hat{A}_u)$ which extends $\hat{\theta}_u$ and such that $(\hat{A}_u,\hat{\mathscr{C}}_u,\hat{\theta}_u^*)\cong\mathscr{Q}^{*\tau}$. Then $\hat{\theta}_u^*$ defines a K-linear embedding f^* of $K\otimes_F Y$ into $M_n(L)$.

Regard L^n as a Z-module by setting $x(a\otimes b)=ax(f^*(b))$. Then $\hat{\mathcal{Q}}_u^*=(\hat{A}_u, \mathscr{C}_u, \hat{\theta}_u^*)$ is of type (Z, Ξ^τ). Since $\hat{\mathcal{Q}}^*$ is also of type (Z, Ξ^τ), there is an isogeny λ of $\hat{\mathcal{Q}}^*$ to $\hat{\mathcal{Q}}_u^*$. Recall that for every $\alpha\in G(T_1)_{\mathbf{R}+}$ and $w\in\mathscr{S}^r$ there is a $\mathbf{C}$-linear automorphism $\Lambda_1(\alpha, w)$ of $\mathbf{C}^{4ng}$ such that

$$\Lambda_1(\alpha, w)\,\mathfrak{y}_1(x, w)=\mathfrak{y}_1(x\alpha, \alpha^{-1}(w))$$

for all $x\in L^n_{\mathbf{R}}$, see [8, 4.4]. The isogeny λ is induced by $\Lambda_1(\beta, y_1)$ for some $\beta\in G(T_1)_{\mathbf{Q}+}$ such that $\beta(u)=y_1$. Let ξ: $\mathbf{C}^{4ng}\to\hat{A}_u$ be a surjective holomorphic homomorphism with $\mathfrak{y}_1(\mathfrak{M}_1, u)$ as kernel. Then from the definitions of $\hat{\theta}^*$ and $\hat{\theta}_u^*$ we have

$$(2.7.2)\quad \xi(\mathfrak{y}_1(x i_1(f_1(b))\,\beta, u))=\xi(\mathfrak{y}_1(x\beta f^*(b)), u)$$

for all $x\in L^n_{\mathbf{R}}$ and $b\in Y$.

Consider the PEL-type $\Omega'=(L, \Phi_1, \rho; \mathfrak{M}_1\beta^{-1}, v(\beta)T_1)$ and let $\mathfrak{y}_1'(x, w)=\mathfrak{y}_1(x\beta, \beta^{-1}(w))$. Let $\Sigma(\Omega')=\{\mathcal{Q}_w'|w\in\mathscr{S}^r\}$ be the family of PEL-structures of type Ω' parametrized by $\mathfrak{y}_1'$. We have $\mathcal{Q}_w'\cong\hat{\mathcal{Q}}_v$, where $v=\beta^{-1}(w)$. Note that $\Lambda_1'(\alpha, w)=\Lambda_1(\beta^{-1}\alpha\beta, \beta^{-1}(w))$ has the property

$$\Lambda_1'(\alpha, w)\,\mathfrak{y}_1'(x, w)=\mathfrak{y}_1'(x\alpha, \alpha^{-1}(w))$$

for all $x\in L^n_{\mathbf{R}}$. Extend $\mathcal{Q}_{y_1}'$ to a structure $\mathcal{Q}'^*$ with the help of (Y, P, δ, f_1). Then it follows from (2.7.2) that $\mathcal{Q}'^*\cong\hat{\mathcal{Q}}_u^*\cong\mathcal{Q}^{*\tau}$.

Now replace $\mathfrak{M}_1$ by $\mathfrak{M}_1\beta^{-1}$, T_1 by $v(\beta)T_1$ and Ω' by $\hat{\Omega}$. Replace $\mathfrak{y}_1$ by $\mathfrak{y}_1'$, and let $\Sigma(\hat{\Omega})=\{\hat{\mathcal{Q}}_w|w\in\mathscr{S}^r\}$ stand for the family parametrized by the new $\mathfrak{y}_1$. Then the above reasoning shows that $\mathcal{Q}^{*\tau}\cong\hat{\mathcal{Q}}^*$. Especially $\mathcal{Q}_y^\tau$ is isomorphic to $\hat{\mathcal{Q}}_{y_1}$.

2.8. Now apply the results of [6 II, § 3] to $\mathcal{Q}^*=(A_y, \mathscr{C}_y, \theta^*)$ and τ. (We emphasize one more time that the results hold for any τ.) For every finite prime $\not{p}$ of F, there are a $Z_{\not{p}}$-linear automorphism $\alpha_{\not{p}}$ of $L^n_{\not{p}}$ and $c_{\not{p}}\in\mathbf{Q}_p^\times(\not{p}|p)$ such that $\mathfrak{M}_{\not{p}}\alpha_{\not{p}}=\mathfrak{M}_{1\not{p}}$ and $U(x, y)=c_{\not{p}}U_1(x\alpha_{\not{p}}, y\alpha_{\not{p}})$ for all $x, y\in L^n_{\not{p}}$. Furthermore, let s be any non-negative integer and $(t_1, \ldots, t_s)$(resp.$(t_{11}, \ldots, t_{1s})$) be an ordered set of elements from $L^n/\mathfrak{M}$ (resp. $L^n/\mathfrak{M}_1$). Then we have $t_{1i}\equiv t_i\alpha_{\not{p}}(\mathrm{mod}\,\mathfrak{M}_{1\not{p}})$ for all $i=1, \ldots, s$, provided the conjugation of $(Z, \Xi, \mathfrak{M}, U; t_1, \ldots, t_s)$ under τ is $(Z, \Xi_1, \mathfrak{M}_1, U_1; t_{11}, \ldots, t_{1s})$. From the relation between U and U_1, we obtain $T=c_{\not{p}}\alpha_{\not{p}}T_1\cdot{}^t\alpha_{\not{p}}^\rho$. Observe that $\alpha_{\not{p}}$ is a $Z_{\not{p}}$-automorphism means it is an $L_{\not{p}}$-automorphism and

$$(2.8.1)\quad \alpha_{\not{p}}^{-1}\,i(f(a))\,\alpha_{\not{p}}=i_1(f_1(a))\qquad \text{for all } a\in Y_{\not{p}}.$$

Let S and S_1 be as in 2.1. Then we have

$$(i(f(a)))^{t\rho}S=S\,i(f(a)),\qquad (i_1(f_1(a)))^{t\rho}S_1=S_1\,i_1(f_1(a))$$

for all $a\in Y_{\not{p}}$. On the other hand, it follows from (2.8.1) that

$$(i_1(f_1(a)))^{t\rho}[(\alpha_{\not{p}}^{-1})^{t\rho}S\alpha_{\not{p}}]=[(\alpha_{\not{p}}^{-1})^{t\rho}S\alpha_{\not{p}}](i_1(f_1(a))).$$

Hence $S_1^{-1}(\alpha_{\not{p}}^{-1})^{t\rho}S\alpha_{\not{p}}$ commutes with every element of $i_1(f_1(Y_{\not{p}}))$, i.e. it belongs to the commutor of $i_1(f_1(Y_{\not{p}}))$ in $M_n(L_{\not{p}})$, which is $K_{\not{p}}(i_1(f_1(P_{\not{p}})))$. Therefore there are $a\in K_{\not{p}}$ and $\lambda\in P_{\not{p}}$ such that

$$(2.8.2)\quad (\alpha_{\not{p}}^{-1})^{t\rho}S\alpha_{\not{p}}=aS_1\,i_1(f_1(\lambda)).$$

We show that λ is in $F_{\mathfrak{p}}$.

Combining $T=c_{\mathfrak{p}}\alpha_{\mathfrak{p}}T_1\cdot{}^t\alpha_{\mathfrak{p}}^{\rho}$ with (2.8.2), we have

$$T\cdot{}^tS^{\rho}=c_{\mathfrak{p}}a^{\rho}\alpha_{\mathfrak{p}}T_1\cdot{}^t(i_1(f_1(\lambda)))^{\rho}\cdot{}^tS_1^{\rho}\cdot{}^t\alpha_{\mathfrak{p}}^{\iota},$$

and

$$-S^{\iota\rho}T^{\iota\rho}=-c_{\mathfrak{p}}a^{\rho}\alpha_{\mathfrak{p}}S_1^{\iota\rho}(i_1(f_1(\lambda)))^{\iota\rho}T_1^{\iota\rho}\cdot{}^t\alpha_{\mathfrak{p}}^{\iota}.$$

Since $T\cdot{}^tS^{\rho}=-S^{\iota\rho}T^{\iota\rho}$, $T_1\cdot{}^tS_1^{\rho}=-S_1^{\iota\rho}T_1^{\iota\rho}$ (see (2.1.1)), this shows

(2.8.3) $\quad T_1\cdot{}^t(i_1(f_1(\lambda)))^{\rho}\cdot{}^tS_1^{\rho}=-S_1^{\iota\rho}(i_1(f_1(\lambda)))^{\iota\rho}T_1^{\iota\rho}.$

Because $f_1(\lambda)\in M_n(B_{1\mathfrak{p}})$, we have $S_1(i_1(f_1(\lambda)))=(i_1(f_1(\lambda)))^{\iota\rho}S_1$. So the left hand side of (2.8.3) is equal to

$$T_1^{\iota\rho}\cdot{}^t(i_1(f_1(\lambda)))^{\iota}=(i_1(f_1(\lambda)))^{\iota\rho}T_1^{\iota\rho}.$$

Applying $\iota\rho$ to the identity, we obtain

$$T_1\cdot{}^t(i_1(f_1(\lambda)))^{\rho}=i_1(f_1(\lambda))\,T_1.$$

On the other hand,

$$T_1\cdot{}^t(i_1(f_1(\lambda)))^{\rho}\,T_1^{-1}=i_1({}^tf_1(\lambda)^{\iota})=i_1(f_1(\lambda^{\delta})).$$

Therefore $\lambda=\lambda^{\delta}$, and hence $\lambda\in F_{\mathfrak{p}}$. This proves:

(2.8.4) $\quad (\alpha_{\mathfrak{p}}^{-1})^{\iota\rho}S\alpha_{\mathfrak{p}}=d_{\mathfrak{p}}S_1 \qquad$ for some $d_{\mathfrak{p}}\in K_{\mathfrak{p}}$.

Note that

(2.8.5) $\quad d_{\mathfrak{p}}d_{\mathfrak{p}}^{\rho}=\kappa/\kappa_1.$

This results from (2.8.4) and the identities $S^{\iota\rho}S=\kappa 1_n$, $S_1^{\iota\rho}S_1=\kappa_1 1_n$.

2.9. Define an isomorphism $\mathfrak{u}_{\mathfrak{p}}$ of $G_{\mathfrak{p}}^*$ onto $G_{1\mathfrak{p}}^*$ by $\mathfrak{u}_{\mathfrak{p}}(\alpha)=\alpha_{\mathfrak{p}}^{-1}\alpha\alpha_{\mathfrak{p}}$. Then (2.8.4) shows that $\mathfrak{u}_{\mathfrak{p}}$ actually sends $G_{\mathfrak{p}}$ isomorphically onto $G_{1\mathfrak{p}}$. Since $\mathfrak{M}_{\mathfrak{p}}=\mathfrak{M}_{1\mathfrak{p}}$ for almost all $\mathfrak{p}$, $\mathfrak{u}_{\mathfrak{p}}$'s can be put together to form an isomorphism $\mathfrak{u}_0$ of G_0 to G_{10}. For an infinite prime $\mathfrak{p}_{\lambda}$ corresponding to the embedding τ_{λ}, the signature of T at $\mathfrak{p}_{\lambda}$ is the same as that of T_1 at $\mathfrak{p}_{\mu}$ if $\tau_{\lambda}\tau=\tau_{\mu}$. Therefore we can define an isomorphism $\mathfrak{u}_{\lambda}$ of $G_{\mathfrak{p}_{\lambda}}^*$ onto $G_{1\mathfrak{p}_{\mu}}^*$ which sends $G_{\mathfrak{p}_{\lambda}}$ onto $G_{1\mathfrak{p}_{\mu}}$. Putting together all $\mathfrak{u}_{\lambda}$'s we get an isomorphism $\mathfrak{u}_{\infty}$ of $G_{\mathbf{R}+}^*$ onto $G_{1\mathbf{R}+}^*$ which sends $G_{\mathbf{R}+}$ onto $G_{1\mathbf{R}+}$. Finally by putting $\mathfrak{u}_0$ and $\mathfrak{u}_{\infty}$ together, we obtain an isomorphism $\mathfrak{u}$ of $G_{\mathbf{A}+}^*$ onto $G_{1\mathbf{A}+}^*$ that sends $G_{\mathbf{A}+}$ onto $G_{1\mathbf{A}+}$. By Lemma 1.5, we have $\mathfrak{u}(\bar{\mathscr{G}}_+)=\bar{\mathscr{G}}_{1+}$. Hence $\mathfrak{u}$ induces an isomorphism (again denoted by $\mathfrak{u}$) of $\mathfrak{A}^1$ onto $\mathfrak{A}_1^1$.

2.10. Now it is only a formality to extend $\mathfrak{u}$ to an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$. First observe that both $\mathfrak{A}/\mathfrak{A}^1$ and $\mathfrak{A}_1/\mathfrak{A}_1^1$ are canonically isomorphic to the finite group $\mathfrak{g}$ defined in [CII, 4.1]. Let the notation be as in Proposition 4.6 of [CII] and its proof. Write

$$A_+=\bigcup_{\alpha\in U}\alpha A_+^1 \quad\text{and}\quad A_{1+}=\bigcup_{\alpha_1\in U_1}\alpha_1 A_{1+}^1.$$

For $\alpha \in U$ put $\mathfrak{u}(\alpha)=\alpha_1 \in U_1$ if α and α_1 have the same image in $\mathfrak{g}$. Define $c_\alpha = \psi_1(\mathfrak{u}(\alpha))^{-1}\mathfrak{u}(\psi(\alpha)) \in \mathfrak{A}_1^1$, and for $(\alpha, x) \in U \times \mathfrak{A}^1 = \mathfrak{A}$ set

$$\mathfrak{u}(\alpha, x)=(\mathfrak{u}(\alpha), c_\alpha \mathfrak{u}(x)) \in U_1 \times \mathfrak{A}_1^1 = \mathfrak{A}_1.$$

Then a trivial computation shows that $\mathfrak{u}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$. In §3 we shall prove that $\mathfrak{u}$ satisfies the conditions of Theorem 1.3.

3. Proof of the Main Theorem

3.1. **Proposition.** *Let $\mathfrak{M}_1$ be determined as in 2.7. Then there are an ample $\mathfrak{r}_F$-lattice $\mathfrak{m}_1$ in B_1^n and an element f of $L_{\mathbf{A}}^\times$ such that $\mathfrak{M}_1 = f\mathfrak{r}_K \cdot i_1'(\mathfrak{m}_1)$.*

Proof. Let v and v_1 be as in 2.1. As we saw in the proof of Proposition 2.2, for every finite prime $\not{p}$ of F, there are $c_{\not{p}} \in F_{\not{p}}^\times$ and $b_{\not{p}} \in L_{\not{p}}^\times$ such that $v_1 = c_{\not{p}} b_{\not{p}} v b_{\not{p}}^\rho$. For $a_{\not{p}} = c_{\not{p}} N(b_{\not{p}}^\rho) \in K_{\not{p}}^\times$, we have $a_{\not{p}} a_{\not{p}}^\rho = \kappa/\kappa_1$, see Proposition 2.2. On the other hand, by (2.8.5) we also have $d_{\not{p}} d_{\not{p}}^\rho = \kappa/\kappa_1$, where $d_{\not{p}} \in K_{\not{p}}^\times$ satisfies (2.8.4). Thus $(d_{\not{p}}/a_{\not{p}})(d_{\not{p}}/a_{\not{p}})^\rho = 1$. Hence there is $e_{\not{p}} \in K_{\not{p}}^\times$ such that

$$d_{\not{p}}/a_{\not{p}} = e_{\not{p}}^\rho / e_{\not{p}}. \tag{3.1.1}$$

Put $f_{\not{p}} = e_{\not{p}} b_{\not{p}} \in L_{\not{p}}^\times$. For an infinite prime $\not{p}$, let $f_{\not{p}}$ be an arbitrary element of $L_{\not{p}}^\times$. It is easy to see that $f=(f_{\not{p}})$ belongs to $L_{\mathbf{A}}^\times$. Define $\mathfrak{m}_1$ by

$$i_1'(\mathfrak{m}_1) = f^{-1}\mathfrak{M}_1 \cap i_1'(B_1^n).$$

We show that for every finite $\not{p}$,

$$i_1'(B_{1\not{p}}^n) = f_{\not{p}} \cdot i'(B_{\not{p}}^n)\alpha_{\not{p}}. \tag{3.1.2}$$

In fact, we have

$$i'(B^n) = \{x \in L^n \mid x = v x^{\iota\rho} S\},$$

$$i_1'(B_1^n) = \{x \in L^n \mid x = v_1 x^{\iota\rho} S_1\}.$$

Therefore, for $x_{\not{p}} \in B_{\not{p}}^n$ we have

$$\begin{aligned} v_1(f_{\not{p}} \cdot i'(x_{\not{p}})\alpha_{\not{p}})^{\iota\rho} S_1 &= v_1 e_{\not{p}}^\rho b_{\not{p}}^{\iota\rho} \cdot i'(x_{\not{p}})^{\iota\rho} \alpha_{\not{p}}^{\iota\rho} S_1 \\ &= v_1 e_{\not{p}}^\rho b_{\not{p}}^{\iota\rho} \cdot i'(x_{\not{p}})^{\iota\rho} d_{\not{p}}^{-1} S\alpha_{\not{p}} = v_1 e_{\not{p}}^\rho d_{\not{p}}^{-1} b_{\not{p}}^{\iota\rho} v^{-1} i'(x_{\not{p}})\alpha_{\not{p}} \\ &= v_1 e_{\not{p}} a_{\not{p}}^{-1} b_{\not{p}}^{\iota\rho} v^{-1} i'(x_{\not{p}})\alpha_{\not{p}} = v_1 e_{\not{p}} c_{\not{p}}^{-1} N(b_{\not{p}}^\rho)^{-1} b_{\not{p}}^{\iota\rho} v^{-1} i'(x_{\not{p}})\alpha_{\not{p}} \\ &= e_{\not{p}} c_{\not{p}}^{-1} v_1 (b_{\not{p}}^{-1})^\rho v^{-1} i'(x_{\not{p}})\alpha_{\not{p}} = e_{\not{p}} b_{\not{p}} i'(x_{\not{p}})\alpha_{\not{p}} \\ &= f_{\not{p}} \cdot i'(x_{\not{p}})\alpha_{\not{p}}. \end{aligned}$$

This completes the proof of (3.1.2). Especially we have

$$\begin{aligned} i'(\mathfrak{m}_{1\not{p}}) &= f_{\not{p}}^{-1}\mathfrak{M}_{1\not{p}} \cap i_1'(B_{1\not{p}}^n) = f_{\not{p}}^{-1}(\mathfrak{M}_{\not{p}} \cap i'(B_{\not{p}}^n))\alpha_{\not{p}} \\ &= f_{\not{p}}^{-1} \cdot i'(\mathfrak{m}_{\not{p}})\alpha_{\not{p}}. \end{aligned} \tag{3.1.3}$$

Hence

$$\mathfrak{r}_{K_{\not p}} \cdot i'_1(\mathfrak{m}_{1\not p}) = f_{\not p}^{-1}\mathfrak{M}_{\not p}\alpha_{\not p} = f_{\not p}^{-1}\mathfrak{M}_{1\not p}.$$

This being true for all finite primes $\not p$, we have $\mathfrak{r}_K \cdot i'_1(\mathfrak{m}_1) = f^{-1}\mathfrak{M}_1$. In view of (3.1.3), $\mathfrak{m}_1$ is ample because $\mathfrak{m}$ is.

3.2. Put

$$\Gamma^* = \{\gamma \in G^*_{\mathbf{Q}} \mid \nu(\gamma) = 1,\ \mathfrak{M}\gamma = \mathfrak{M}\},$$
$$\Gamma^*_1 = \{\gamma \in G^*_{1\mathbf{Q}} \mid \nu_1(\gamma) = 1, \mathfrak{M}_1\gamma = \mathfrak{M}_1\}.$$

Let Ω and $\hat{\Omega} \sim \Omega^\tau$ be as in 2.7, and let (V, v, ϕ) be a moduli-system for $\Sigma(\Omega)$ in the sense of [7, 6.2]. Here v is an assignment which assigns a point $v(\mathcal{Q})$ of V to every PEL-structure $\mathcal{Q}$ of type Ω, ϕ is a holomorphic map from $\mathcal{S}^r$ onto V inducing a biregular isomorphism of $\mathcal{S}^r/\Gamma^*$ onto V, and $\phi(w) = v(\mathcal{Q}_w)$ for all $w \in \mathcal{S}^r$. Let (V_1, v_1, ϕ_1) be a moduli-system for $\Sigma(\hat{\Omega})$. Since $\hat{\Omega} \sim \Omega^\tau$, by [8, 4.23] there is a biregular isomorphism ψ of V_1 to V^τ such that $v^\tau = \psi \circ v_1$.

3.3. **Proposition.** *Let $z, z_1, y = j(z)$ and $y_1 = j_1(z_1)$ be the special points chosen in 2.6. Let $\beta \in G_{\mathbf{Q}+}$ and $w = j(\beta(z)) = i(\beta)(y)$. Then there is $\gamma \in G_{1\mathbf{Q}+}$ such that for $w_1 = j_1(\gamma(z_1))$ we have $\phi(w)^\tau = \psi(\phi_1(w_1))$.*

Proof. Let $\beta_1 = \mathfrak{u}(\beta) \in G_{1\mathbf{A}+}$. Then $\nu_1(\beta_1) = \nu(\beta) \in F^\times$. Hence by [A, 3.3], there is $\alpha \in G_{1\mathbf{Q}+}$ such that $\nu_1(\alpha) = \nu_1(\beta_1)$. Put $S_1 = \{x \in \mathcal{G}_{1+} \mid \mathfrak{m}_1 x = \mathfrak{m}_1\}$. Then $S_1 \in \mathfrak{Z}_1$. By the strong approximation theorem for G^u_1 (see [CI, 3.4]), there are $\alpha' \in G^u_{1\mathbf{Q}}$ and $s \in S_1 \cap G^u_{1\mathbf{A}}$ such that $\beta_1\alpha^{-1} = s\alpha'$. Put $\gamma = \alpha'\alpha \in G_{1\mathbf{Q}+}$. Then $\nu_1(\gamma) = \nu_1(\alpha) = \nu(\beta)$. We show that γ satisfies the condition of our proposition.

As in 2.7, we first consider L^n as a Z-module by setting $x(a \otimes b) = ax(i(f(b)))$. The extended structure $\mathcal{Q}^*$ of $\mathcal{Q}_y$ is then of type $(Z, \Xi, \mathfrak{M}, U)$. We see easily that $\mathcal{Q}_w$ can be extended to a structure $\mathcal{Q}^*_w$ of type $(Z, \Xi, \mathfrak{M}\beta, U_\beta)$, where $U_\beta(a, a') = \mathrm{tr}_{L/\mathbf{Q}}(\nu(\beta)^{-1}T(a, a'))$. Now regard L^n as a Z-module by setting $x(a \otimes b) = ax(i_1(f_1(b)))$ instead. Then $\hat{\mathcal{Q}}_{y_1}$ can be extended to a structure $\hat{\mathcal{Q}}^*$ of type $(Z, \Xi_1, \mathfrak{M}_1, U_1)$. Recall that we have $\mathcal{Q}^{*\tau} \cong \hat{\mathcal{Q}}^*$, see 2.7. We determine the type $(Z, \Xi_1, \mathfrak{M}^*, U^*)$ *of* $\mathcal{Q}^{*\tau}_w$.

We have $(Z, \Xi, \mathfrak{M}, U)^\tau \sim (Z, \Xi_1, \mathfrak{M}_1, U_1)$ and $(Z, \Xi, \mathfrak{M}\beta, U_\beta)^\tau \sim (Z, \Xi_1, \mathfrak{M}^*, U^*)$. Let $\not p$ be a finite prime of F. Then $\alpha_{\not p}$ is a $Z_{\not p}$-linear automorphism of $L^n_{\not p}$ such that $\mathfrak{M}_{1\not p} = \mathfrak{M}_{\not p}\alpha_{\not p}$ and $U(a, a') = c_{\not p} U_1(a\alpha_{\not p}, a'\alpha_{\not p})$ for some $c_{\not p} \in Q^\times_p$ $(\not p \mid p)$. Since $\mathcal{Q}^*$ and $\mathcal{Q}^*_w$ are isogenous, by a standard argument we can assume that we also have $\mathfrak{M}^*_{\not p} = (\mathfrak{M}\beta)_{\not p}\alpha_{\not p}$ and $U_\beta(a, a') = c_{\not p} U^*(a\alpha_{\not p}, a'\alpha_{\not p})$. Now $s_{\not p}\gamma = \beta_1 = \mathfrak{u}(\beta) = \alpha^{-1}_{\not p}\beta\alpha_{\not p}$. Therefore

$$\mathfrak{M}^*_{\not p} = (\mathfrak{M}\beta)_{\not p}\alpha_{\not p} = \mathfrak{M}_{\not p}\alpha_{\not p}s_{\not p}\gamma = \mathfrak{M}_{1\not p}s_{\not p}\gamma = \mathfrak{M}_{1\not p}\gamma.$$

This is true for all finite $\not p$. Hence $\mathfrak{M}^* = \mathfrak{M}_1\gamma$. Similarly, we have $U^* = U_{1\gamma}$, where $U_{1\gamma}(a, a') = \mathrm{tr}_{L/\mathbf{Q}}(\nu_1(\gamma)^{-1} T_1(a, a'))$.

So $\mathcal{Q}^{*\tau}_w$ is of type $(Z, \Xi_1, \mathfrak{M}_1\gamma, U_{1\gamma})$, which is the type of the extended structure of $\hat{\mathcal{Q}}_{w_1}$, where $w_1 = j_1(\gamma(z_1))$. Hence $\mathcal{Q}^\tau_w \cong \hat{\mathcal{Q}}_{w_1}$. Therefore, $v(\mathcal{Q}_w)^\tau = \psi(v_1(\mathcal{Q}^\tau_w)) = \psi(v_1(\hat{\mathcal{Q}}_{w_1}))$, or $\phi(w)^\tau = \psi(\phi_1(w_1))$.

3.4. For a positive integer b, define a member $S(\mathfrak{m}, b)$ (resp. $S_1(\mathfrak{m}_1, b)$) of $\mathscr{X}$ (resp. $\mathscr{X}_1$) as in [CI, 2.10]. Then it follows from Proposition 3.1 that $\mathfrak{u}(S(\mathfrak{m}, b)) = S_1(\mathfrak{m}_1, b)$. For any $\mathfrak{r}_K$-lattice $\mathfrak{N}$ in L^n, and any positive integer b, put

$$\Gamma^*(\mathfrak{N}, b) = \{\gamma \in G^*_{\mathbf{Q}} \,|\, \nu(\gamma) = 1,\ \mathfrak{N}\gamma = \mathfrak{N},\ \mathfrak{N}(1-\gamma) \subset b\mathfrak{N}\}$$

$$\Gamma_1^*(\mathfrak{N}, b) = \{\gamma \in G^*_{1\mathbf{Q}} \,|\, \nu_1(\gamma) = 1,\ \mathfrak{N}\gamma = \mathfrak{N},\ \mathfrak{N}(1-\gamma) \subset b\mathfrak{N}\}.$$

For a given positive integer a, choose b and c so that conditions (6.4.2)–(6.4.4) of [CI] are satisfied. Put

(3.4.1) $\quad S = S(\mathfrak{m}, c) \cdot \{x \in S(\mathfrak{m}, b) \,|\, \nu(x) = 1\}.$

Then

(3.4.2) $\quad \mathfrak{u}(S) = S_1 = S_1(\mathfrak{m}_1, c) \cdot \{x \in S_1(\mathfrak{m}_1, b) \,|\, \nu_1(x) = 1\}.$

We assume that b has been chosen in such a way that the following conditions corresponding to [CI, (6.4.3), (6.4.4)] also hold:

(3.4.3) *Put* $E = \mathfrak{r}_F^{\times}$. *Then, for every* $u_1 \in G_{1\mathbf{A}}$,

$$E \cdot \Gamma(u_1^{-1} S_1 u_1) = \{\alpha \in E \cdot \Gamma_1^*(\not{f}^{-1}\mathfrak{M}_1 u_1, b) \,|\, \alpha(\mathscr{D}_n^r) \cap \mathscr{D}_n^r \neq \emptyset\}.$$

(3.4.4) *For every* $u_1 \in G_{1\mathbf{A}}$, $\Gamma_1^*(\not{f}^{-1}\mathfrak{M}_1 u_1, b)$ *has no element of finite order other than the identity element.*

Note that

(3.4.5) $\quad \Gamma_1^*(\not{f}^{-1}\mathfrak{M}_1 u_1, b) = \Gamma_1^*(\mathfrak{M}_1 u_1, b).$

Let (H, Φ') (resp. (H_1, Φ_1')) be the reflex of (K, Φ) (resp. (K, Φ_1)). Then by Lemma 1.5, $H_1 = H^{\tau}$. Let M_c be the class field over H corresponding to the subgroup $H^{\times} \cdot \{h \in H_{\mathbf{A}}^{\times} \,|\, h \equiv 1 \bmod_0 (c)\}$. Define a class field M_{1c} over H_1 in the same way. Then $M_{1c} = M_c^{\tau}$.

3.5. Choose $q_1, \ldots, q_s \in L^n/\mathfrak{M}$ such that $b^{-1}\mathfrak{M}/\mathfrak{M} \cong \sum_{i=1}^{s} \mathbf{Z} q_i$. Consider the PEL-type

$$\Omega' = (L, \Phi, \rho; \mathfrak{M}, T; q_1, \ldots, q_s).$$

Fix $q_{11}, \ldots, q_{1s} \in L^n/\mathfrak{M}_1$ so that

$$\hat{\Omega}' = (L, \Phi_1, \rho; \mathfrak{M}_1, T_1; q_{11}, \ldots, q_{1s})$$

is equivalent to Ω'^{τ}. Then for every finite prime $\not{p}$ of F, we have $\mathfrak{M}_{1\not{p}} = \mathfrak{M}_{\not{p}}\alpha_{\not{p}}$ and $q_{1i} \equiv q_i \alpha_{\not{p}} (\bmod\ \mathfrak{M}_{1\not{p}})$. Therefore $b^{-1}\mathfrak{M}_1/\mathfrak{M}_1 \cong \sum_{i=1}^{s} \mathbf{Z} q_{1i}$. Let $k_{\Omega'}$ be the field of moduli of Ω'. Then $k_{\Omega'}$ is contained in M_c [CI. (6.6.1)]. The field of moduli of $\hat{\Omega}'$ is $k_{\Omega'}^{\tau}$, which is contained in M_{1c}.

Let $\mathfrak{y}(x, w)$ (resp. $\mathfrak{y}_1(x, w)$) be the parametrizing function for $\Sigma(\Omega)$ (resp. $\Sigma(\hat{\Omega})$) as given in 2.7. We can use the same $\mathfrak{y}(x, w)$ (resp. $\mathfrak{y}_1(x, w)$) to parametrize a family

$\Sigma(\Omega')=\{\mathscr{Q}'_w|w\in\mathscr{S}^r\}$ (resp. $\Sigma(\hat{\Omega}')=\{\hat{\mathscr{Q}}'_w|w\in\mathscr{S}^r\}$) of PEL-structures of type Ω' (resp. $\hat{\Omega}'$). Let (V', v', ϕ') (resp. (V_1, v_1, ϕ_1)) be a moduli-system for $\Sigma(\Omega')$ (resp. $\Sigma(\hat{\Omega}')$). We can identify V_S with the subvariety $\phi'(\mathscr{D}_n^r)$ of V' and take $\phi_S=\phi'\circ j$ [CI. 6.9]. Similarly, in view of (3.4.3)–(3.4.5), we can identify V_{S_1} with $\phi_1(\mathscr{D}_n^r)$ and take $\phi_{S_1}=\phi'_1\circ j_1$.

Since $\hat{\Omega}'\sim\Omega'^{\tau}$, there is a biregular isomorphism ψ_Ω of V'_1 to V'^{τ} over M_{1c} such that $v'^{\tau}=\psi\circ v'_1$. We show that ψ_Ω restricted to V_{S_1} defines an isomorphism of V_{S_1} to V_S^{τ} (over M_{1c}). Let $z\in\mathscr{H}$ be as in 3.3. Then $Z=\{\beta(z)|\beta\in G_{\mathbf{Q}+}\}$ is dense in $\mathscr{H}$. So it is sufficient to show that $(\phi'(j(Z)))^{\tau}$ is contained in $\psi_\Omega(V_{S_1})$, and is Zariski dense in there.

Let $\beta\in G_{\mathbf{Q}+}$ and $w=j(\beta(z))$. Let the notation be as in 3.2. Then Proposition 3.3 shows that $\phi(w)^{\tau}=\psi_\Omega(\phi_1(w_1))$ for some $w_1\in\mathscr{D}_n^r$. According to [9], $\mathscr{Q}_w^{\tau}\cong\hat{\mathscr{Q}}_{w_1}$ posesses a certain non-holomorphic endomorphism $\mathfrak{d}$. It follows that $\mathscr{Q}'^{\tau}_w$ also posesses this endomorphism $\mathfrak{d}$. Therefore $\mathscr{Q}'^{\tau}_w$ is isomorphic to $\hat{\mathscr{Q}}'_{v_1}$ for some $v_1\in\mathscr{D}_n^r$. Thus $\phi'(w)^{\tau}=\psi_\Omega(\phi'_1(v_1))\in\psi_\Omega(\phi'_1(j_1(\mathscr{H})))=\psi_\Omega(V_{S_1})$. This shows $(\phi'(j(Z)))^{\tau}$, and hence its Zariski closure V_S^{τ}, is contained in $\psi_\Omega(V_{S_1})$. Similarly, we show that $\psi_\Omega(V_{S_1})$ is contained in V_S^{τ}. Therefore the restriction ψ_S of ψ_Ω to V_{S_1} is a biregular isomorphism of V_{S_1} to V_S^{τ}.

Thus for S of the form (3.4.1) with sufficiently large b, and for $S_1=\mathfrak{u}(S)$, there is a biregular isomorphism ψ_S of V_{S_1} to V_S^{τ} rational over M_{1c}. Put $\tilde{\phi}_S=\psi_S\circ\phi_{S_1}$. Then $(V_S^{\tau}, \tilde{\phi}_S)$ is a model of $\mathscr{H}/\Gamma_{S_1}$ over k_S^{τ}.

3.6. For $x\in G_{\mathbf{A}+}$ and $x_1=\mathfrak{u}(x)\in G_{1\mathbf{A}+}$ we have $v_1(x_1)=v(x)$. Therefore, by the definitions of ρ and ρ_1 in 1.1, and by Lemma 1.5, we have $\tau\rho_1(x_1)=\rho(x)\tau$. Similarly, if $S\in\mathscr{Z}$ and $S_1=\mathfrak{u}(S)\in\mathscr{Z}_1$, then $k_S^{\tau}=k_{S_1}$.

3.7. Let $\mathscr{G}_+^*$ be the subgroup of $G_{\mathbf{A}}^*$ defined in [CI, 6.2]. Define a subgroup $\mathscr{G}_{1+}^*$ of $G_{1\mathbf{A}+}^*$ correspondingly. Then we have $\mathfrak{u}(\mathscr{G}_+^*)=\mathscr{G}_{1+}^*$ in view of Lemma 1.5. Let $\mathscr{G}_{H+}$, λ_H, etc. be as in [CI, 3.1], and define $\mathscr{G}_{1H_1+}$, λ_{H_1} in a similar way. Let S and S_1 be as in 3.4. For $x\in\mathscr{G}_{H+}$, $x_1=\mathfrak{u}(x)\in\mathscr{G}_{1H_1+}$, put $T=x^{-1}Sx$, and $T_1=x_1^{-1}S_1x_1=\mathfrak{u}(T)$. We show that V_T^{τ} and V_{T_1} are biregularly isomorphic over M_{1c}.

By [CI, (6.3.5)], there is $d\in H_{\mathbf{A}}^{\times}$ such that $v(x)/\lambda_H(d)\in F_+^{\times}F_{\infty+}^{\times}$ and $\pi(d)\,x\in\mathscr{G}_+^*$. (For the notation used here and in the following, see [CI, § 6].) Let $d_1=d^{\tau}\in H_{1\mathbf{A}}^{\times}$. Then $\lambda_{H_1}(d_1)=\lambda_H(d)$ and $\pi_1(d_1)=\pi(d)$. Hence

$$v_1(x_1)/\lambda_{H_1}(d_1)\in F_+^{\times}F_{\infty+}^{\times} \quad\text{and}\quad \pi_1(d_1)x_1\in\mathscr{G}_{1+}^*.$$

Let Ω', $\hat{\Omega}'\sim\Omega'^{\tau}$ be as in 3.5. Put $\sigma=[d^{-1}, H]$ and $\sigma_1=\tau^{-1}\sigma\tau=[d_1^{-1}, H_1]$. By [CI, (6.6.1)], Ω'^{σ} is equivalent to

$$\Omega''=(L, \Phi, \rho;\pi(d)\mathfrak{M}x, \mu(\pi(d)x)^{-1}T, \{\pi(d)q_ix\}),$$

while $\hat{\Omega}'^{\sigma_1}$ is equivalent to

$$\hat{\Omega}''=(L, \Phi_1, \rho;\pi(d)\,\mathfrak{M}_1x_1, \mu_1(\pi(d)x_1)^{-1}T_1, \{\pi(d)q_ix_1\}).$$

We have $\Omega''^{\tau}\sim\Omega'^{\sigma\tau}\sim\Omega'^{\tau\sigma_1}\sim\hat{\Omega}'^{\sigma_1}\sim\hat{\Omega}''$. Note that for every finite prime $\not{p}$ of F, we have $(\mathfrak{M}_1x_1)_{\not{p}}=(\mathfrak{M}x)_{\not{p}}\alpha_{\not{p}}$ and $\mu(\pi(d)x)^{-1}T=\mu_1(\pi(d)x_1)\alpha_{\not{p}}T_1\cdot{}^t\alpha_{\not{p}}^{\rho}$.

Let $\Sigma(\Omega'')=\{\mathscr{Q}''_w|w\in\mathscr{S}^r\}$ (resp. $\Sigma(\hat{\Omega}'')=\{\hat{\mathscr{Q}}''_w|w\in\mathscr{S}^r\}$) be the family of PEL-structures of type Ω'' (resp. $\hat{\Omega}''$) parameterized by $\mathfrak{y}(x, w)$ (resp. $\mathfrak{y}_1(x, w)$). Let

(V'', v'', ϕ'') (resp. (V_1'', v_1'', ϕ_1'')) be a moduli-system for $\Sigma(\Omega'')$ (resp. $\Sigma(\hat{\Omega}'')$). Since $\Omega''^{\tau} \sim \hat{\Omega}''$, there is a biregular isomorphism ψ'' of V_1'' to V''^{τ} such that $v''^{\tau} = \psi'' \circ v_1''$. Combining the arguments of [CI, 6.7] and 3.5, we see that ψ'' induces a biregular isomorphism ψ_T of V_{T_1} to V_T^{τ} over M_{1c}. Put $\tilde{\phi}_T = \psi_T \circ \phi_{T_1}$. Then $(V_T^{\tau}, \tilde{\phi}_T)$ is a model of $\mathscr{H}/\Gamma_{T_1}$ over k_T^{τ}.

3.8. Let S be as in 3.3. Consider $\mathscr{W} = \{x^{-1}Sx \mid x \in \mathscr{G}_{H+}\}$. Let $T = x^{-1}Sx \in \mathscr{W}$, $u \in \mathscr{G}_{H+}$ and $U = u^{-1}Tu$. Then $J_{UT}(u)$ is defined, and is a morphism of V_T onto V_U^{σ} over k_T, where $\sigma = \rho_H(u)$. Let

$$\Omega' = (L, \Phi, \rho;\ \mathfrak{M}, T;\ \{q_i\})$$

be as in 3.5. As observed in 3.7, there is a PEL-type Ω'' of the form

$$\Omega'' = (L, \Phi, \rho;\ k\mathfrak{M}x, \kappa T, \{kq_i x\}) \qquad (k \in K_{\mathbf{A}}^{\times},\ \kappa \in F_+^{\times})$$

such that if (V'', v'', ϕ'') is a moduli-system for $\Sigma(\Omega'')$, then V_T can be embedded in V'' in such a way that $\phi_T = \phi'' \circ j$. Let $d \in H_{\mathbf{A}}^{\times}$ be such that $v(u)/\lambda_H(d) \in F_+^{\times} F_{\infty+}^{\times}$. Then Ω''^{σ} is equivalent to

$$\Omega^* = (L, \Phi, \rho;\ \pi(d)k\mathfrak{M}x, \mu(\pi(d)x)^{-1}\kappa T, \{\pi(d)kq_i x\}).$$

Let (V^*, v^*, ϕ^*) be a moduli-system for $\Sigma(\Omega^*)$. Then V_U can be embedded in V^* in such a way that $\phi_U = \phi^* \circ j$. Since $\Omega''^{\sigma} \sim \Omega^*$, there is a biregular isomorphism J from V^* to V''^{σ} rational over M_c such that $v''^{\sigma} = J \circ v^*$. In view of the construction [CI, §6], we can identify $J_{UT}(u)$ with the restriction of J to V_T.

Put $x_1 = \mathrm{u}(x)$, $T_1 = x_1^{-1}S_1x_1$, $u_1 = \mathrm{u}(u)$ and $U_1 = \mathrm{u}(U)$. Also let $\sigma_1 = \rho_{H_1}(u_1)$. Then $U_1 = u_1^{-1}T_1u_1$ and $\sigma_1 = \tau^{-1}\sigma\tau$. Let

$$\hat{\Omega}' = (L, \Phi_1, \rho;\ \mathfrak{M}_1, T_1, \{q_{1i}\})$$

be as in 3.5. From 3.7 we know that Ω''^{τ} (resp. $\Omega^{*\tau}$) is equivalent to

$$\hat{\Omega}'' = (L, \Phi_1, \rho;\ k\mathfrak{M}_1x_1, \kappa_1 T_1, \{kq_{1i}x_1\})$$

$$(\text{resp. } \hat{\Omega}^* = (L, \Phi_1, \rho;\ \pi(d)\,k\mathfrak{M}_1x_1, \mu_1(\pi(d)\,x_1)^{-1}\kappa_1 T_1, \{\pi(d)\,kq_{1_i}x_1\}))$$

with a suitable $\kappa_1 \in F_+^{\times}$. Let (V_1'', v_1'', ϕ_1'') (resp. (V_1^*, v_1^*, ϕ_1^*)) be a moduli-system for $\Sigma(\hat{\Omega}'')$ (resp. $\Sigma(\hat{\Omega}^*)$). Embed V_{T_1} (resp. V_{U_1}) in V_1'' (resp. V_1^*) in such a way that $\phi_{T_1} = \phi_1'' \circ j_1$ (resp. $\phi_{U_1} = \phi_1^* \circ j_1$). We have a biregular isomorphism ψ'' (resp. ψ^*) of V_1'' (resp. V_1^*) onto V''^{τ} (resp. $V^{*\tau}$) over M_{1c} which induces the isomorphism ψ_T (resp. ψ_U) of V_{T_1} (resp. V_{U_1}) to V_T^{τ} (resp. V_U^{τ}).

We have $\hat{\Omega}^* \sim \Omega^{*\tau} \sim \Omega''^{\sigma\tau} \sim \Omega''^{\tau\sigma_1} \sim \Omega''^{\sigma_1}$. Therefore, there is a biregular isomorphism J_1 of V_1^* to $V_1''^{\sigma_1}$ such that $v_1''^{\sigma_1} = J_1 \circ v_1^*$. We can identify $J_{U_1T_1}(u_1)$ with the restriction of J_1 to V_{T_1}.

Now we have

$$J^{\tau} \circ \psi^* \circ v_1^* = J^{\tau} \circ v^{*\tau} = (J \circ v^*)^{\tau} = v''^{\sigma\tau}$$

on one hand, and

$$\psi''^{\sigma_1} \circ J_1 \circ v_1^* = \psi''^{\sigma_1} \circ v^{*\sigma_1} = (\psi'' \circ v^*)^{\sigma_1} = v''^{\tau\sigma_1} = v''^{\sigma\tau}$$

on the other. Therefore $J^{\tau}\circ\psi^{*}=\psi''^{\sigma_1}\circ J_1$. It follows that

(3.8.1) $J_{UT}(u)^{\tau}\circ\psi_T=\psi_U^{\sigma_1}\circ J_{U_1T_1}(u_1)$,

where $\sigma_1=\rho_{H_1}(u_1)$.

3.9. We prove that ψ_U is defined over $k_{U_1}H_1$. Let π_1 be an automorphism of M_{1c} over $k_{U_1}H_1$. By [CI. 3.3, 3.5], there is $u\in U\cap\mathscr{G}_{H+}$ such that $\rho_H(u)=\tau\pi_1\tau^{-1}$ on M_c. Put $u_1=\mathrm{u}(u)$. Then $\rho_{H_1}(u_1)=\pi_1$, $J_{UU}(u)=\mathrm{id.}$, and $J_{U_1U_1}(u_1)=\mathrm{id}$. Hence by (3.8.1) we have $\psi_U^{\pi_1}=\psi_U$. This being true for all $\pi_1\in\mathrm{Gal}(M_{1c}/k_{U_1}H_1)$, ψ_U is defined over $k_{U_1}H_1$.

Therefore, for all $U\in\mathscr{W}$ and $U_1=\mathrm{u}(U)$, we have a biregular isomorphism ψ_U: $V_{U_1}\to V_U^{\tau}$ over $k_{U_1}H_1$ satisfying (3.8.1), i.e. we have proved the main Theorem for $\{V_U, \phi_U, J_{UT}(u)\,|\,U, T\in\mathscr{W}, u\in\mathscr{G}_{H+}\}$ over H. Going through the reduction process of [CI, II], we conclude that u satisfies all the conditions of Theorem 1.3.

4. Conjugations of Miyake's Models

4.1. Let K be a totally imaginary quadratic extension of a totally real algebraic number field F of degree g, and B a central simple algebra over K with a positive involution ρ which coincides with the complex conjugation on K. Choose g embeddings $\tau_1,\ldots,\tau_g$ of K into $\mathbf{C}$ so that their restrictions to F are all distinct. Put $n^2=[B:K]$. Decompose $B_{\mathbf{R}}=B\otimes_{\mathbf{Q}}\mathbf{R}$ into the direct sum of simple algebras $B_1\oplus\cdots\oplus B_g$. Let ι_λ be the identity element of B_λ. For each $\lambda=1,\ldots,g$, there is an R-linear isomorphism ϕ_λ of B_λ onto $M_n(\mathbf{C})$ such that $\phi_\lambda(x^\rho)={}^t\overline{\phi_\lambda(x)}$ for all $x\in B_\lambda$. Fix ϕ_λ once and for all. For a ρ-hermitian element h of $B^\times$, i.e. an element h of $B^\times$ such that $h^\rho=h$, denote the signature of $\phi_\lambda(h)$ by $J_\lambda(h)$.

Let δ be an involution of B which coincides with ρ on K. Define an algebraic group G over $\mathbf{Q}$ so that the $\mathbf{Q}$-rational points of G are

$$G_{\mathbf{Q}}=\{\alpha\in B^\times\,|\,\alpha\alpha^\delta=\nu(\alpha)\in F^\times\}.$$

The semi-simple part of G is

$$G^u=\{\alpha\in G\,|\,\nu(\alpha)=1 \text{ and } N(\alpha)=1\},$$

where N denotes the reduced norm of B to K. Denote the homogeneous space $G^u_{\mathbf{R}}$ modulo a maximal compact subgroup by $\mathscr{H}$.

Fix a ρ-hermitian element $h\in B^\times$ such that $x^\delta=hx^\rho h^{-1}$ for all $x\in B$. Let $J_\lambda=J_\lambda(h)=(r(\lambda),s(\lambda))$. Put

$$J_{r(\lambda),s(\lambda)}=\begin{bmatrix}1_{r(\lambda)} & 0\\ 0 & -1_{s(\lambda)}\end{bmatrix}.$$

As in [3, 1.4], consider an element j of $B_{\mathbf{R}}$ such that $j^\delta=-j$, $j^2=-1$ and such that $\{u\in G^u_{\mathbf{R}}\,|\,uj=ju\}$ is a maximal compact subgroup of $G^u_{\mathbf{R}}$. By [3, Corollary 1, Prop. 2], for each $\lambda=1,\ldots,g$, there is an isomorphism ω_λ of B_λ onto $M_n(\mathbf{C})$ such that

$$\omega_\lambda(x^\delta)=J_{r(\lambda),s(\lambda)}\,{}^t\overline{\omega_\lambda(x)}\,J_{r(\lambda),s(\lambda)}\qquad\text{for all } x\in B_\lambda,$$

and

$$\omega_\lambda(j\,\iota_\lambda)=\sqrt{-1}\,J_{r(\lambda),s(\lambda)}.$$

Using ω_λ's, we can identify $\mathscr{H}$ with the product $\prod_{\lambda=1}^{g}\mathscr{H}_{r(\lambda),s(\lambda)}$, where $\mathscr{H}_{r,s}$ denotes the bounded symmetric domain consisting of all $r\times s$ complex matrices z such that $1_r - z\,{}^t\bar{z}$ is positive hermitian. In this way we have a bounded symmetric domain structure on $\mathscr{H}$. The main Theorem of [3] states that there is a canonical system of models for the quotients of $\mathscr{H}$ by arithmetic subgroups of G.

4.2. Let τ be an automorphism of $\mathbf{C}$. For each $\lambda=1,\dots,g$, there is a unique μ so that $\tau_\mu\tau=\tau_\lambda$ on F. Put $\sigma(\lambda)=\mu$. By a Theorem of Landherr [2] on hermitian forms over division algebras, there is a ρ-hermitian h_1 of $B^\times$ such that i) for every finite prime $\mathfrak{p}$ of F, there is $x_\mathfrak{p}\in B_\mathfrak{p}^\times$ such that $h_1=x_\mathfrak{p} h x_\mathfrak{p}^\rho$; and ii) $J_\lambda(h_1)=J_{\sigma(\lambda)}(h)$. Let δ_1 be the involution of B given by $x\mapsto h_1 x h_1^{-1}$. Then (B,δ_1) defines a reductive group G_1 as in 4.1.

Since $J_\lambda(h_1)=J_{\sigma(\lambda)}(h)$, there is an $\mathbf{R}$-linear isomorphism $\mathfrak{u}_\lambda$ of B_λ to $B_{\sigma(\lambda)}$ such that $\mathfrak{u}_\lambda(x^{\delta_1})=\mathfrak{u}_\lambda(x)^\delta$. Putting all $\mathfrak{u}_\lambda$'s together, we get an R-linear automorphism $\mathfrak{u}_\infty$ of $B_\mathbf{R}$ so that $\mathfrak{u}_\infty(x^{\delta_1})=\mathfrak{u}_\infty(x)^\delta$ for all $x\in B_\mathbf{R}$. The automorphism $\mathfrak{u}_\infty$ induces an isomorphism between $G_\mathbf{R}^u$ and $G_{1\mathbf{R}}^u$.

Put $\varepsilon_\lambda=1$ if $\tau_\lambda=\tau_{\sigma(\lambda)}\tau$ on K, and $\varepsilon_\lambda=-1$ if $\tau_\lambda=\tau_{\sigma(\lambda)}\tau\rho$ on K. Let $\varepsilon=\varepsilon_1\iota_1+\dots+\varepsilon_g\iota_g\in B_R$. Denote by j_1 the unique element of $B_\mathbf{R}$ such that $\mathfrak{u}_\infty(j_1\varepsilon)=j$. Then we have $j_1^{\delta_1}=-j_1, j_1^2=-1$ and $\{u_1\in G_{1\mathbf{R}}^u \mid u_1 j_1=j_1 u_1\}$ is a maximal compact subgroup of $G_{1\mathbf{R}}^u$. Hence j_1 defines an isomorphism $\omega_{1\lambda}$ of B_λ to $M_n(\mathbf{C})$ for each $\lambda=1,\dots,g$, as in 4.1. Using the $\omega_{1\lambda}$'s we can make the quotient of $G_{1\mathbf{R}}^u$ modulo a maximal compact subgroup into a bounded symmetric domain $\mathscr{H}_1$. Let $\{V_{X_1},\phi_{X_1},J_{W_1X_1}(u_1)\}$ be a system of canonical models for the quotients of $\mathscr{H}_1$ by arithmetic subgroups of G_1. Then the models $\{V_X,\phi_X,J_{WX}(u)\}$ associated with G and the models $\{V_{X_1},\phi_{X_1},J_{W_1X_1}(u_1)\}$ are related in the way described in Theorem 1.3. This fact can be proved in a similar way. Actually this is the easier case, because the bounded symmetric domains in question parametrize families of abelian varieties themselves.

4.3. We make some comments on the special case where τ is the complex conjugation. In this case we have $G_1=G$. Note that the domains $\mathscr{H}_1$ and $\mathscr{H}$ are not equivalent unless $r(\lambda)=s(\lambda)$ or $r(\lambda)\cdot s(\lambda)=0$ for each λ.

The isomorphism $\mathfrak{u}$ in the main Theorem can be given by $u\mapsto\alpha^{-1}u\alpha$ with a "negative" element α of G_Q in the sense of [4, §3]. This fact follows easily from the results of [4, §3]. The finite part α_0 of α belongs to the group $\mathscr{G}_{j+}$ defined in [3, 3.4] if and only if $r(\lambda)=s(\lambda)$ for all λ. When this happens we have $\overline{\phi_T(\alpha(\bar{z}))}=J_{TT_1}(\alpha_0)\circ\phi_{T_1}(z)$ for all $T\in\mathfrak{Z}$ and $T_1=\mathfrak{u}(T)\in\mathfrak{Z}_1$. The corresponding fact for Shimura's models is much harder to prove, see [4].

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Received January 6, 1977