Characterization of Families of Rank 3 Permutation Groups by the Subdegrees I

By

D. G. Higman

1. Introduction. The terminology and notation of [5] for rank 3 permutation groups are used throughout. We consider two cases of the problem of determining the rank 3 permutation groups for which the degree and subdegrees are specified in terms of a parameter. The results are as follows.

Theorem I. *If G is a rank 3 permutation group of degree m^2 with subdegrees $1, 2(m-1)$ and $(m-1)^2$, $m \geqq 2$, then G is isomorphic with a subgroup of the wreath product $\Sigma_m \wr \Sigma_2$ of the symmetric groups of degrees m and 2, in its usual action on m^2 letters.*

Theorem II. *If G is a rank 3 permutation group of degree $\binom{m}{2}$ with subdegrees 1, $2(m-2)$ and $\binom{m-2}{2}$, $m \geqq 5$, then G is isomorphic with a 4-fold transitive subgroup of Σ_m in its action on the 2-element subsets, unless one of the following holds:*

(a) $G \approx P\Gamma L_2(8)$,
(b) $\mu = 6$ *and* $m = 9, 17, 27$ *or* 57,
(c) $\mu = 7$ *and* $m = 51$, *or*
(d) $\mu = 8$ *and* $m = 28, 36, 325, 903$ *or* $8{,}128$.

Concerning the exceptional cases in Theorem II, $P\Gamma L_2(8)$ as a subgroup of Σ_9 is the only example of a subgroup of Σ_m, $m \geqq 4$, which is not 4-fold transitive but has rank 3 on the 2-element subsets (see Lemma 5). The case $\mu = 6$, $m = 9$ is realized by the group $G_2(2)$ and is known [4] to be the only such group. The remaining cases are undecided.

2. Rank 3 groups and strongly regular graphs. The proofs of Theorems I and II rest on the connection between rank 3 permutation groups of even order and strongly regular graphs [5, 10], and the known characterizations of the graphs of L_2-type [9] and triangular type [2, 3] and [6, 7].

The graphs considered in this paper are finite, undirected, and without loops. A graph $\mathscr{G}$ with n vertices is *strongly regular* [1] if there exist integers k, l, λ, μ such that

(1) each vertex is adjacent to exactly k vertices and non-adjacent to exactly l other vertices, k and l positive, and

(2) two adjacent vertices are both joined to exactly λ other vertices and two non-adjacent vertices are both adjacent to exactly μ vertices.

Assume that $\mathscr{G}$ is strongly regular. Then

(3) $\mu l = k(k - \lambda - 1)$,

(4) the minimum polynomial of the adjacency matrix A of $\mathscr{G}$ is

$$(x - k)(x^2 - (\lambda - \mu)x - (k - \mu)),$$

(5) A has k as eigenvalue with multiplicity 1, and the multiplicities f, g of the roots r, s of $x^2 - (\lambda - \mu)x - (k - \mu)$ as eigenvalues of A are respectively

$$f = \frac{(k + l)s + k}{s - r} \quad \text{and} \quad g = \frac{(k + l)r + k}{r - s}$$

with $f + g = k + l$, and

(6) one of the following holds

(a) $k = l, \mu = \lambda + 1 = k/2$ and $f = g = k$, or

(b) $d = (\lambda - \mu)^2 + 4(k - \mu)$ is a square.

The strongly regular graph $\mathscr{G}$ is connected if and only if $\mu > 0$, and its complement $\bar{\mathscr{G}}$ is connected if and only if $\mu < k$. We say that $\mathscr{G}$ is *primitive* if $\mathscr{G}$ and $\bar{\mathscr{G}}$ are connected, so that

(7) $\mathscr{G}$ is primitive if and only if $0 < \mu < k$.

If $\mu = 0$, then (identity) $\cup$ (adjacency) is an equivalence relation on the set of vertices, hence

(8) If $\mu = 0$, then $k + 1 \mid n$, and if $\mu = k$, then $l + 1 \mid n$.

If G is a rank 3 permutation group of even order on a finite set X, $|X| = n$, and if Δ and Γ are the nontrivial orbits of G in $X \times X$, then the graphs $\mathscr{G} = (X, \Delta)$ and $\bar{\mathscr{G}} = (X, \Gamma)$ are a complementary pair of strongly regular graphs, each admitting G as a rank 3 automorphism group, the parameters k, l and λ, μ being respectively the subdegrees (other than 1) and the intersection numbers of G [5, 10]. Primitivity of the permutation group G is equivalent to primitivity of the graph $\mathscr{G}$.

3. Proof of Theorem I. The wreath product $\Sigma_m \wr \Sigma_2$ of the symmetric group Σ_m of degree $m \geqq 2$ and the symmetric group Σ_2 of order 2 is constructed as the split extension of the group $N = \Sigma_m \times \Sigma_m$ by $\Sigma_2 = \langle \tau \rangle$, where τ acts on N according to $(\pi_1, \pi_2)^\tau = (\pi_2, \pi_1)$, $\pi_i \in \Sigma_m$. This group acts as a rank 3 permutation group on $X \times X$, $X = \{1, 2, \ldots, m\}$ according to

$$(i, j)^{(\pi_1, \pi_2)} = (i^{\pi_1}, j^{\pi_2}) \quad \text{and} \quad (i, j)^\tau = (j, i),$$

with $k = 2(m - 1)$, $l = (m - 1)^2$ and $\mu = 2$. The action is imprimitive for $m = 2$ and primitive for $m \geqq 3$. The graph $\mathscr{L}_m$ afforded by the suborbit of length $2(m - 1)$, in which vertices (a, b) and (c, d) are adjacent if and only if they are distinct and $a = c$ or $b = d$, is isomorphic with the line graph of the complete bipartite graph on $2m$ vertices, i.e. the strongly regular graph of L_2-type on $2m$ vertices.

Lemma 1. *$\Sigma_m \wr \Sigma_2$ is the full automorphism group of $\mathscr{L}_m$, $m \geqq 2$.*

Proof. Each vertex (a, b) is contained in exactly two maximal cliques, with vertices $\{(a, x) \mid x = 1, 2, \ldots, m\}$ and $\{(y, b) \mid y = 1, 2, \ldots, m\}$, respectively. If σ is an automorphism of $\mathscr{L}_m$, we have two possibilities.

Case 1. $(1, 1)^\sigma = (a_1, b_1)$ and $(1, 2)^\sigma = (a_1, b_2)$. In this case $(i, j)^\sigma = (a_1, j^\beta)$ and $(i, 1)^\sigma = (i^\alpha, b_1)$ with $\alpha, \beta \in \Sigma_m$ such that $1^\alpha = a_1$ and $1^\beta = b_1$. If i and j are both $\neq 1$, then (i, j), as the unique vertex $\neq (1, 1)$ adjacent to $(i, 1)$ and (i, j), must be mapped by σ onto the unique vertex $\neq (a_1, b_1)$ adjacent to (i^α, b_1) and (a_1, j^β), i.e., $(i, j)^\sigma = (i^\alpha, j^\beta)$. Hence $\sigma = (\alpha, \beta) \in N$.

Case 2. $(1, 1)^\sigma = (a_1, b_1)$ and $(1, 2)^\sigma = (a_2, b_1)$. In this case $(1, j)^\sigma = (j^\beta, b_1)$ and $(2, 1)^\sigma = (a_1, 2^\alpha)$ with $(\alpha, \beta) \in \Sigma_m$, so $(i, j)^\sigma = (j^\beta, i^\alpha) = (i, j)^{(\alpha, \beta)\tau}$. Hence $\sigma = (\alpha, \beta)\tau \in \Sigma_m \wr \Sigma_2$.

Lemma 2. *Let $\mathscr{G}$ be a strongly regular graph such that $k = 2(m-1)$ and $l = (m-1)^2$ for some $m \geqq 2$. Then $\mathscr{G}$ is isomorphic with $\mathscr{L}_m$ unless $m = 4$, in which case there is, up to isomorphism, exactly one exceptional graph.*

Proof. In view of [9], it suffices to prove that $\mu = 2$. By (8) of Section 2 we have $\mu > 0$. By (3) of Section 2 we have

$$\mu(m-1) = 2(2m - 3 - \lambda).$$

Assume first that μ is even, $\mu = 2\mu_0$. Then $\mu_0(m-1) = 2m - 3 - \lambda$, so that $(\mu_0 - 2)m = \mu_0 - 3 - \lambda$. We have $\lambda \geqq 0$, and since $0 < \mu \leqq k = 2(m-1)$, $0 < \mu_0 \leqq m - 1$. Hence $(\mu_0 - 2)m \leqq \mu_0 - 3 \leqq m - 4$, so that $(\mu_0 - 3)m \leqq -4$ and hence $\mu_0 \leqq 2$. If $\mu_0 = 2$, then $\lambda = -1$ which is impossible, hence $\mu_0 = 1$ and $\mu = 2$ as claimed.

Now assume that μ is odd, so that $2 \mid m - 1$ and we can write $m = 2t + 1$, $t \geqq 1$. Then $\mu t = 4t - 1 - \lambda$, that is, $\lambda = (4 - \mu)t - 1$, so that $\mu \leqq 3$. By (6) of Section 2 we have that $9t^2 + 4t$ or $t^2 + 8t + 4$ must be a square according as $\mu = 1$ or $\mu = 3$. But the non-negative integral solutions of the equations

$$y^2 = 9t^2 + 4t \quad \text{and} \quad y^2 = t^2 + 8t + 4$$

are respectively $y = 0$, $t = 0$ and $y = 2$, $t = 0$, contrary to $t \geqq 1$.

Since it is known [4] that the exceptional graph in case $m = 4$ of Lemma 2 does not admit a rank 3 automorphism group, we have Theorem I as an immediate Corollary to Lemmas 1 and 2.

If $H \leqq \Sigma_m$, $m \geqq 2$, is doubly transitive, then $H \wr \Sigma_2$ is a rank 3 subgroup of $\Sigma_m \wr \Sigma_2$ on the m^2 letters. There are rank 3 subgroups of $\Sigma_m \wr \Sigma_2$ not of this type and we make no attempt here to classify them.

Of course it should be noted that Lemma 2 is purely a result about graphs with no reference to groups. The same remark applies to Lemma 4 in the next section.

4. Proof of Theorem II. The symmetric group Σ_m on $X = \{1, 2, \ldots, m\}$, $m \geqq 3$, acts as a rank 3 group on the set of $\binom{m}{2}$ 2-element subsets of X, with subdegrees $k = 2(m-2)$, $l = \binom{m-2}{2}$, and $\mu = 4$. The action is imprimitive if $m = 4$ and primitive if $m \geqq 5$. The graph $\mathscr{J}_m$ afforded by the suborbit of length $2(m-2)$, in which two 2-element subsets S and T are adjacent if and only if $|S \cap T| = 1$, is isomorphic with the line graph of the complete graph on m vertices, i.e., the graph of triangular type on $\binom{m}{2}$ vertices.

Lemma 3. *Σ_m is the full automorphism group of $\mathscr{J}_m$, $m \geqq 5$, while Σ_4 has index 2 in the automorphism group of $\mathscr{J}_4$.*

Proof. The maximal cliques of $\mathscr{J}_m$ containing a vertex $\{a, b\}$ are of two types.

$$\text{(I)} \qquad \begin{cases} [a] = \{\{a, x\} \mid x \in X - \{a\}\}, \\ [b] = \{\{y, b\} \mid y \in X - \{b\}\} \end{cases}$$

and

$$\text{(II)} \qquad \{a, b\}, \quad \{b, z\} \quad \text{and} \quad \{a, z\}, \quad z \notin \{a, b\}.$$

If σ is an automorphism of $\mathscr{J}_m$ and $\{a, b\}^\sigma = \{c, d\}$, then, if $m \geqq 5$, $[a]^\sigma = [c]$ or $[d]$. Here we define $a^\pi = c$ if $[a]^\sigma = [c]$. Then it is easy to verify that $\pi \in \Sigma_m$ and π induces σ. In case $m = 4$ there are involutions in the automorphism group of $\mathscr{J}_4$ interchanging the two types (I) and (II) of maximal cliques, e.g., α mapping $\{1, 3\}$ onto $\{2, 4\}$ and fixing all other vertices. We see that $\langle \Sigma_4, \alpha \rangle$ is the full automorphism group of $\mathscr{J}_4$.

Lemma 4. *Let $\mathscr{G}$ be a strongly regular graph with $k = 2(m-2)$ and $l = \binom{m-2}{2}$, $m \geqq 4$. Then $\mu = 4$ unless*

$$\mu = 6 \quad \textit{and} \quad m = 7, 9, 17, 27 \textit{ or } 57,$$
$$\mu = 7 \quad \textit{and} \quad m = 51, \quad \textit{or}$$
$$\mu = 8 \quad \textit{and} \quad m = 28, 36, 325, 903 \textit{ or } 8{,}128.$$

Proof. If $\mu = 0$, then $m = 4$ by (8) Section 2 and hence $\lambda = 3$ by (3), which is impossible by (6). Hence $\mu > 0$. By (3) we have

$$(*) \qquad \mu(m-3) = 4(2m - 5 - \lambda).$$

We consider two cases according as μ is even or odd.

Case 1. $\mu = 2\mu_0$. In this case $\mu_0(m-3) = 2(2m-5-\lambda)$, so that $0 \leqq 2\lambda = (4-\mu_0)m + 3\mu_0 - 10$. Hence, since $\mu_0 \leqq (m-2)$, we have

$$(\mu_0 - 4)m \leqq 3\mu_0 - 10 \leqq 3(m-2) - 10 \leqq 3m - 16.$$

Hence $(\mu_0 - 7)m \leqq -16$ and $\mu_0 < 7$. If $\mu_0 = 5$ or 6, then $\lambda = \frac{5-m}{2}$ or $4 - m$ so that $m = 5$ or 4 respectively. In the first case, $\mu_0 = 5 > m - 2 = 3$ and in the second $\mu_0 = 6 > m - 2 = 2$, so both are ruled out. The remaining possibilities are:

μ_0	1	2	3	4
μ	2	4	6	8
λ	$\frac{3m-7}{2}$	$m-2$	$\frac{m-1}{2}$	1
d	$\frac{9m^2 - 34m + 25}{4}$	$(m-2)^2$	$\left(\frac{m+3}{2}\right)^2$	$8m+1$

By (6) of Section 2, the entries in the last row are squares.

For $\mu_0 = 1$ we have the equation $4y^2 = 9m^2 - 34m + 25$, for which the (non-negative integral) solutions are $y = 1$, $m = 3$ and $y = 0$, $m = 1$, both of which are impossible.

If $\mu_0 = 3$ we have by (4) and (5) of Section 2 that $r = \frac{m-5}{2}$ and $s = -4$, so that $f = \frac{4m(m-2)}{m+3}$. Since $(m+3, m) \mid 3$ and $(m+3, m-2) \mid 5$, we have that $m+3 \mid 4 \cdot 3 \cdot 5$ giving $m = 7, 9, 12, 17, 27$, or 57. If $m = 12$, then $\lambda = 13/2$, so this case is impossible.

If $\mu_0 = 4$, we have the equation $y^2 = 8m + 1$. Writing $y = 2w + 1$, we have $m = \binom{w+1}{2}$, $k = w^2 + w - 4$ and $l = \frac{1}{8}(w^2 + w - 6)k$, and since $\lambda = 1$ and $\mu = 8$, we get $r = w - 3$ and $s = -w - 4$ by (4) of Section 2. Hence by (5) of Section 2,

$$f = \frac{w(w+2)(w+3)(w^2+w-4)}{8(2w+1)}$$

and since $w \geqq 3$ and $(2w+1, w+2) \mid 3$, $(2w+1, w+3) \mid 5$ and $(2w+1, w^2+w-4) \mid 17$, we must have $2w + 1 \mid 3 \cdot 5 \cdot 17$, giving

$$w = 7,\ 8,\ 25,\ 42 \text{ or } 127\,,$$

i. e.,

$$m = 28,\ 36,\ 325,\ 903 \quad \text{or} \quad 8{,}128\,.$$

Case 2. $\mu \equiv 1$ (2). In this case (*) implies that $4 \mid m - 3$, and we write $m = 4t + 3$. Then by (*), $\lambda = (8 - \mu)t + 1$ and $\mu \leqq 9$. The possibilities are:

μ	1	3	5	7	9
λ	$7t+1$	$5t+1$	$3t+1$	$t+1$	0
d	$49t^2+32t+4$	$25t^2+12t$	$9t^2+8t+4$	$t^2+20t+16$	85

Again, the entries in the last row are squares by (6) of Section 2, so $\mu \neq 9$. The non-negative integral solutions of the equations $y^2 = 49t^2 + 32t + 4$, $y^2 = 25t^2 + 12t$ and $y^2 = 9t^2 + 8t + 4$ are $y = 2$, $t = 0$, $y = t = 0$ and $y = 2$, $t = 0$ respectively, so these cases are ruled out. Finally, the equation $y^2 = t^2 + 20t + 16$ has just the solutions $y = 4$, $t = 0$, which is ruled out, and $y = 20$, $t = 12$, giving $m = 51$.

Lemma 5. *If $G \leqq \Sigma_m$, $m \geqq 4$, has rank 3 on the 2-element subsets, then G is 4-fold transitive unless $m = 9$ and $G \approx P\Gamma L_2$ (8).*

Proof. The cases $m = 4$ and 5 follow from the fact that $\frac{nkl}{(k,l)}$ divides $|G|$. Assume that $m > 5$ and that G is not 4-fold transitive. Clearly G is transitive on the set of 4-element subsets, i.e., G is 4-homogeneous, and if S is a 2-element subset, then G_S is transitive on the 4-element sets containing S. Hence if Λ is a 4-element set, $G_\Lambda : G_{\Lambda,S} = 6$. Hence $G_\Lambda \mid \Lambda$ has order 12, and we have by Kantor's result [8] that possibilities are

m	G
6	$PSL_2(5)$, $P\Gamma L_2(5)$
9	$PSL_2(8)$, $P\Gamma L_2(8)$
33	$P\Gamma L_2(32)$.

The divisibility $\left.\frac{nkl}{(k,l)}\right| \, |G|$ rules out all but the case $m = 9$, $G = P\Gamma L_2$ (8).

Note. The result stated by KANTOR in [8] is proved there only under the additional assumption that $G_\Lambda | \Lambda$ has order $\neq 8$, which is quite sufficient for our purposes.

According to [2, 3] and [6, 7] if $\mathscr{G}$ is a strongly regular graph with $k = 2(m-2)$, $l = \binom{m-2}{2}$ and $\mu = 4$, then $\mathscr{G} \approx \mathscr{J}_m$ unless $m = 8$, in which case there are exactly three exceptions. Since none of these exceptional graphs admits a rank 3 automorphism group [4], and since the complement of $\mathscr{G}$ in the case $\mu = 6$, $m = 7$ has the parameters of $\mathscr{J}_7$, Theorem II follows by Lemmas 3, 4 and 5.

We remark that for $m = 4$, any rank 3 permutation group with subdegrees 1, 4, 1 is either isomorphic with Σ_4 or the full automorphism group of $\mathscr{J}_4$.

The exceptional case $\mu = 6$, $m = 9$ is realized by the group $G_2(2)$. It is in fact known that there is just one strongly regular graph with these parameters which admits a rank 3 automorphism group, and that $G_2(2)$ is its full automorphism group, no proper subgroup of which has rank 3 [4]. The question of the existence (even of graphs) for the remaining exceptional cases is undecided.

References

[1] R. C. BOSE, Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math. **13**, 389–419 (1963).

[2] L. C. CHANG, The uniqueness and nonuniqueness of the triangular association scheme. Sci. Record **3**, 604–613 (1959).

[3] L. C. CHANG, Association schemes of partially balanced block designs with parameters $v = 28$, $n_1 = 12$, $n_2 = 15$, $p_{11}^2 = 4$. Sci. Record **4**, 12–18 (1960).

[4] M. HESTENES, D. G. HIGMAN and C. C. SIMS, Rank 3 permutation groups of small degree (in preparation).

[5] D. G. HIGMAN, Finite permutation groups of rank 3. Math. Z. **86**, 145–156 (1964).

[6] A. J. HOFFMAN, On the uniqueness of the triangular association scheme. Ann. Math. Stat. **31**, 492–497 (1960).

[7] A. J. HOFFMAN, On the exceptional case in a characterization of the arcs of a complete graph. IBM J. Res. Develop. **4**, 487–496 (1961).

[8] W. M. KANTOR, 4-homogeneous groups. Math. Z. **103**, 67–68 (1968).

[9] S. S. SHRIKHANDE, The uniqueness of the L_2-association scheme. Ann. Math. Stat. **30**, 781–798 (1959).

[10] C. C. SIMS, On graphs with rank 3 automorphism groups. J. Comb. Theory (to appear).

Eingegangen am 18. 7. 1969

Anschrift des Autors:
D. G. Higman
Department of Mathematics
University of Michigan
Ann Arbor, Mich. 48104, USA